On the intermediate logic of open subsets of metric spaces

Timofei Shatrov

ABSTRACT. In this paper we study the intermediate logic $ML_{O(\mathcal{X})}$ of open subsets of a metric space $\mathcal{X}$. This logic is closely related to Medvedev's logic of finite problems ML. We prove several facts about this logic: its inclusion in ML, impossibility of its finite axiomatization and indistinguishability from ML within some large class of propositional formulas.

Keywords: intermediate logic, Medvedev's logic, axiomatization, indistinguishability

1 Introduction

In this paper we introduce and study a new intermediate logic $ML_{O(\mathcal{X})}$, which we first define using Kripke semantics and then we will try to rationalize this definition using the concepts from prior research in this field.

An (intuitionistic) *Kripke frame* is a partially ordered set $(F, \leq)$. A *Kripke model* is a Kripke frame with a valuation θ (a function which maps propositional variables to upward-closed subsets of the Kripke frame). The notion of a propositional formula being true at some point $w \in F$ of a model $M = (F, \theta)$ is defined recursively as follows:

$$\text{For propositional letter } p_i,\ M, w \vDash p_i \text{ iff } w \in \theta(p_i)$$
$$M, w \vDash \psi \wedge \chi \text{ iff } M, w \vDash \psi \text{ and } M, w \vDash \chi$$
$$M, w \vDash \psi \vee \chi \text{ iff } M, w \vDash \psi \text{ or } M, w \vDash \chi$$
$$M, w \vDash \psi \to \chi \text{ iff for any } w' \geq w,\ M, w \vDash \psi \text{ implies } M, w' \vDash \chi$$
$$M, w \nvDash \bot$$

A formula is true in a model M if it is true in each point of M. A formula is valid in a frame F if it is true in all models based on that frame.

The set of formulas which are valid on every Kripke frame is *intuitionistic logic* **Int**. An *intermediate logic* is an extension of **Int** closed under modus ponens and substitution. Every consistent intermediate logic is contained in *classical logic* **CL**.

The set $L(F)$ of all formulas valid in a given frame F forms an intermediate logic, which allows us to define a logic just by constructing its Kripke frame (although not every intermediate logic can be constructed in this way).

For example, the logics ML and ML_1, which will be often mentioned in this paper are defined as follows:

DEFINITION 1. Let X be a set, then $P_1(X)$ is the Kripke frame $(2^X \setminus \emptyset, \supseteq)$ (non-empty subsets of X ordered by converse inclusion). *Medvedev's logic of finite problems* ML is $\bigcap\{L(P_1(X))|X \text{ is finite}\}$ [1] and the *logic of infinite problems* ML_1 is $L(P_1(\omega))$ (or $L(P_1(X))$ for any infinite X) [6].

It is well-known that $ML_1 \subseteq ML$, but it is an open problem whether $ML_1 = ML$ or not.

Let $\mathcal{X}$ be a topological space, and $O(\mathcal{X})$ a set of non-empty open sets in $\mathcal{X}$. Then $(O(\mathcal{X}), \supseteq)$ is a Kripke frame with the least element $\mathcal{X}$. Logic $ML_{O(\mathcal{X})}$ is defined as the logic of this frame:

DEFINITION 2. $ML_{O(\mathcal{X})} = L(O(\mathcal{X}), \supseteq)$. We will often use $O(\mathcal{X})$ to denote the frame $(O(\mathcal{X}), \supseteq)$ as a simple notation.

Finally, a few words on why this logic is interesting. We can understand *information* as a subset of some basic set Ω. An *information type* is an arbitrary set of informations (these notions were introduced by Yuri Medvedev in [2]). A type σ is called regular iff

$$\forall E_1 \forall E_2 \; E_1 \in \sigma \;\&\; E_2 \subseteq E_1 \Rightarrow E_2 \in \sigma$$

As described in [5] regular information types form a structure that is naturally connected to Medvedev's logic of finite problems ML (in case of finite Ω), or logic of infinite problems ML_1 in case of infinite Ω. Namely, the set of all regular types $I(\Omega)$ ordered by inclusion is a Heyting algebra, and it is isomorphic to the Heyting algebra of upward-closed subsets of $P_1(\Omega)$. Thus, the logic of Heyting algebra $I(\Omega)$ is $L(P_1(\Omega))$.

Now, for a topological space $\mathcal{X}$ it may be more natural to understand information as an open subset of $\mathcal{X}$, and not an arbitrary one. For example, every measurement in a physical experiment has a margin of error, so the only kind of information we can get about the measured value is that it is within some interval (open subset of $\mathbb{R}$). In this case regular information types correspond to upward-closed subsets of $O(\mathcal{X})$ and thus form a Heyting algebra of $ML_{O(\mathcal{X})}$.

2 Useful facts and definitions

In this section we briefly recall facts related to Kripke semantics of intermediate logics.

DEFINITION 3. Let $u \in F$ be a point of a Kripke frame $(F, \leq)$. Then the Kripke frame $F^u = \{v \in F | u \leq v\}$ with the ordering $\leq$ is called a *cone*.

DEFINITION 4. The surjective mapping h from frame F to frame G is called a *p-morphism* iff $\forall u \in F \; h(F^u) = G^{h(u)}$, that is

1. $\forall u, v \in F \; (u \leq v) \Rightarrow h(u) \leq h(v)$

2. $\forall u \in F, w \in G \ (h(u) \leq w) \Rightarrow \exists v \geq u : h(v) = w$

If there is a p-morphism from F to G, this is denoted by $F \twoheadrightarrow G$.

LEMMA 5.

1. *$L(F) \subseteq L(F^u)$.*
2. *If $F \twoheadrightarrow G$ then $L(F) \subseteq L(G)$.*

Let F be a finite Kripke frame with the least element (called the root and denoted by 0_F). Then it is possible to construct a propositional formula $X(F)$, called *Jankov (characteristic) formula* with the following properties:

LEMMA 6.

1. *For any propositional formula A, $X(F) \in (\mathbf{Int} + A) \iff F \not\models A$*
2. *For any frame G, $G \not\models X(F) \iff \exists u \in G : G^u \twoheadrightarrow F$*

3 $ML_{O(\mathcal{X})}$ as a subset of Medvedev's logic

In this section we construct a p-morphism from the Kripke frame of open subsets of $\mathcal{X}$ to the Kripke frame of ML_1, thus proving that $ML_{O(\mathcal{X})}$ is a subset of ML_1 and ML.

THEOREM 7. *Let $\mathcal{X}$ be an infinite metric space. Then $ML_{O(\mathcal{X})} \subseteq ML_1$.*

Proof. To prove this we construct a p-morphism $f : O(\mathcal{X}) \twoheadrightarrow P_1(\omega)$. $\mathcal{X}$ can be split into 3 disjoint subsets: $\mathcal{X} = \mathcal{X}_1 \cup \mathcal{X}_2 \cup \mathcal{X}_3$.
$\mathcal{X}_1 = \{x \in \mathcal{X} | \exists \varepsilon \forall y \in \mathcal{X}, y \neq x \Rightarrow \rho(x,y) > \varepsilon\}$ contains all isolated points.
$\mathcal{X}_2 = \{x \in \mathcal{X} | \exists \{x_n\}_{n=1}^{\infty} \subset \mathcal{X}_1, x_n \to x, n \to \infty\} \setminus \mathcal{X}_1$ contains the limits of isolated points.
$\mathcal{X}_3 = (\mathcal{X} \setminus \mathcal{X}_1) \setminus \mathcal{X}_2$ contains the limit points which are not the limits of isolated points. A point of $\mathcal{X}_3$ is always a limit of other points in $\mathcal{X}_3$, thus $\mathcal{X}_3$ is either empty or infinite.

If $\mathcal{X}_3$ is empty then $\mathcal{X}_1$ cannot be finite (since otherwise $\mathcal{X}_2$ is empty, and $\mathcal{X}$ is finite). We can choose a sequence $x_n \in \mathcal{X}_1$, $n = 0,1,2,\dots$ where all x_n are different. Let $D_i = \{x_i\}$, $i = 1,2,3,\dots$, $D_0 = \{x_0\} \cup \{x \in \mathcal{X}_1 | \forall n \ x \neq x_n\}$. Each of D_i is open, since each x_i is an isolated point. Let $G \subset \mathcal{X}$ be a nonempty open subset. If $G \cap D_i = \emptyset$, $i = 0,1,2,\dots$ then $G \cap \mathcal{X}_1 = \emptyset$, so $G \subseteq \mathcal{X}_2$, since $\mathcal{X}_3$ is empty. But since each point in $\mathcal{X}_2$ is a limit of isolated points from $\mathcal{X}_1$, G has to contain some, which is a contradiction. Thus, there always exists D_n which intersects G. We define $f(G) = \{n \in \omega | G \cap D_n \neq \emptyset\}$, which is easily checked to be p-morphism.

Let $\mathcal{X}_3$ be nonempty. It is open, since its complement $\mathcal{X} \setminus \mathcal{X}_3 = \mathcal{X}_1 \cup \mathcal{X}_2$ is clearly closed (a limit of points in $\mathcal{X}_2$ is also in $\mathcal{X}_2$). Choose an arbitrary $x_0 \in \mathcal{X}_3$ and let $D_n = \{x \in \mathcal{X}_3 | \frac{1}{n+1} < \rho(x,x_0) < \frac{1}{n}\}$, $n = 2,3,\dots$. If we exclude empty D_n, there would be still an infinite number of them left (since

x_0 is a limit point of $\mathcal{X}_3$). Also define $D_1 = \{x \in \mathcal{X}_3 | \rho(x, x_0) > \frac{1}{2}\}$, $D_0 = \mathcal{X}_1$. Each open subset $G \subset \mathcal{X}$ intersects one of D_n. After renumbering D_n such that all of them are non-empty we can define $f(G) = \{n \in \omega | G \cap D_n \neq \emptyset\}$, which is a p-morphism. ■

It should be noted that by definition $ML_1 = ML_{O(\omega)}$. For a metric space $\mathcal{X}$, $ML_{O(\mathcal{X})}$ would be equal to ML_1 if it is possible to construct a p-morphism from $P_1(Y)$ to $O(\mathcal{X})$ for some infinite set Y.

For example, consider $\mathcal{X} = \{0\} \cup \{1/n | n \in \mathbb{N}\}$ with the usual metric and $Y = \mathbb{N} \times \{0, 1\}$. Open sets of $\mathcal{X}$ are either arbitrary subsets of $\mathcal{X} \setminus 0$, or cofinite subsets of $\mathcal{X}$. We define $h : P_1(Y) \to O(\mathcal{X})$ as follows:

- $h(E) = \{1/n | \exists j \ (n, j) \in E\}$ if $|\mathcal{X} \setminus E| = \omega$
- $h(E) = \{1/n | \exists j \ (n, j) \in E\} \cup \{0 | \exists n \ (n, 1) \in E\}$ if $|\mathcal{X} \setminus E| < \omega$

It is easy to check that h is a p-morphism. The following lemma is a generalization of this construction.

LEMMA 8. *Suppose $\mathcal{X}$ is an infinite metric space consisting only of isolated points and limits of isolated points ($\mathcal{X} = \mathcal{X}_1 \cup \mathcal{X}_2$ using the notation from the proof of Theorem 7). Then $ML_1 = ML_{O(\mathcal{X})}$.*

Proof. Let $Y = \mathcal{X}_1 \times (\mathcal{X}_2 \cup \{x_0\})$ ($x_0 \in \mathcal{X}_1$), and for $y = (y_1, y_2) \in Y$ $\phi(y) = y_1$, $\psi(y) = y_2$. Define $h : P_1(Y) \to O(\mathcal{X})$ as follows:

$$h(E) = \text{int}(\phi(E) \cup (\mathcal{X}_2 \cap \psi(E)))$$

$\phi(E)$ is always in $h(E)$, and we only add appropriate limit points from $\psi(E)$ so that the result is an open set. If $G \in O(\mathcal{X})$ then $h(E) = G$ for $E = H(G) = (G \cap \mathcal{X}_1) \times (\{x_0\} \cup (G \cap \mathcal{X}_2))$ (every open subset of $\mathcal{X}$ contains some points from $\mathcal{X}_1$, so $G \cap \mathcal{X}_1 \neq \emptyset$). So, h is surjective. If $E' \subseteq E$ then of course $h(E') \subseteq h(E)$. If $G \subseteq h(E)$ then $h(E') = G$ for $E' = E \cap H(G)$. So, h is a p-morphism, and that means that $ML_1 \subseteq ML_{O(\mathcal{X})}$. Combining this with the result of Theorem 7 we conclude that $ML_1 = ML_{O(\mathcal{X})}$. ■

On the other hand, if $\mathcal{X}_3 \neq \emptyset$, there is no p-morphism from $P_1(Y)$ to $O(\mathcal{X})$. Suppose h is such p-morphism and $h(Y') = \mathcal{X}_3$ ($\mathcal{X}_3$ is an open set, so such $Y' \subseteq Y$ must exist). Then for $y \in Y'$ $h(\{y\})$ must be a singleton (isolated point), but there are no such open subsets of $\mathcal{X}_3$.

It is unknown whether $ML_{O(\mathcal{X})}$ is always equal to ML_1, although we will prove later that they are quite close to each other.

4 $ML_{O(\mathcal{X})}$ is not finitely axiomatizable

Let $\mathcal{X}$ be a metric space, F is a finite Kripke frame and suppose there exists a p-morphism $h : O(\mathcal{X}) \twoheadrightarrow F$. We will construct a mapping $\overline{h}$, which is an extension of h to some other (non-open) subsets of $\mathcal{X}$.

Let $A \subset \mathcal{X}$ be a non-empty subset of $\mathcal{X}$, and let $\{A_n\}$ be a sequence of open subsets of $\mathcal{X}$ such that $A_{i+1} \subseteq A_i$ and $\bigcap_{i=1}^{\infty} A_i = A$. Then we define $\overline{h}(A) = \sup_{\{A_i\}} \lim_i h(A_i)$. Note that for every sequence A_i with these properties the limit does exist. This follows from the fact that F is finite and $h(A_i) \leq h(A_{i+1})$. If A_i and B_i are sequences which yield different limits, then the sequence $A_i \cap B_i$ yields a limit that is greater or equal to both of these limits ($h(A_i \cap B_i) \geq h(A_i)$). Thus, the supremum always exists as long as at least one such sequence exists.

Since $\mathcal{X}$ is a metric space, $\overline{h}$ is defined for every non-empty finite subset of $\mathcal{X}$ (for $\{x_1, \ldots, x_n\}$ there exists a sequence $A_i = \bigcup_{1 \leq j \leq n} \{x | \rho(x, x_j) < \frac{1}{i}\}$).

For $u \in F$ we denote the set of its immediate successors by $br(u)$; the maximum length of an increasing chain starting at u is denoted by $d(u)$.

LEMMA 9. *Let $\mathcal{X}$ be a metric space and let F be a finite frame with the least element, and $|br(u)| \neq 1$ for any $u \in F$. If there exists a p-morphism $h : O(\mathcal{X}) \twoheadrightarrow F$, then for any $u \in F$ there exists $E \subset \mathcal{X}$ such that $\overline{h}(E) = u$ and $|E| < 2^{d(u)}$.*

Proof. If $d(u) = 1$ then $\exists G \in O(\mathcal{X}) : h(G) = u$ and $\forall G' \subset G\ h(G') = u$. Thus any $E = \{g\}, g \in G$ satisfies $\overline{h}(E) = u$, as there is a sequence of neighborhoods of g which yields a limit u, but since u is a maximal point of F, $\overline{h}(E)$ is also u.

Now suppose $d(u) = n > 1$ and we have already proved the statement of the lemma for $d(u) < n$ (and for any appropriate $\mathcal{X}$ and F). Choose $G \in O(\mathcal{X}) : h(G) = u$. There exist at least 2 immediate successors of u in F, which we denote by v_1 and v_2. $\exists G_1, G_2 :\ G' = G_1 \cup G_2 \subset G, \overline{h}(G_1) = v_1$, $\overline{h}(G_2) = v_2$, $|G_1| < 2^{d(u)-1}$, $|G_2| < 2^{d(u)-1}$. This follows from the induction hypothesis for $\mathcal{X} := G, F := F^u$, $h = h|_{O(G)}$. Then $|G'| < 2^{d(u)}$ and $\overline{h}(G') = u$. This completes the induction step. ■

We will call a finite subset $A = \{a_1, a_2, \ldots, a_n\}$ *stable* under $\overline{h}$ iff

$$\exists \varepsilon \forall A' = \{a'_1, a'_2, \ldots, a'_n\}\ \forall i \leq n\ \rho(a_i, a'_i) < \varepsilon \Rightarrow \overline{h}(A') = \overline{h}(A)$$

If every non-empty subset of A is also stable, we say that A is *hereditarily stable* under $\overline{h}$.

LEMMA 10. *Let $E = \{e_1, e_2, \ldots, e_n\}$ be a subset constructed by applying the previous lemma. Then it is stable under $\overline{h}$.*

Proof. Note that every point $e_i \in E$ is chosen at the induction base as an arbitrary point in an open set G_i. Since E is finite, we can choose ε such every $e_i \in E$ will remain in its corresponding set G_i if we move it no farther than ε. Thus E is stable. ■

LEMMA 11. *If a finite set $E = \{e_1, e_2, \ldots, e_n\}$ is stable under $\overline{h}$ then $\forall \varepsilon \exists E' = \{e'_1, e'_2, \ldots, e'_n\} : \forall i \leq n\ \rho(e_i, e'_i) < \varepsilon$ such that $\overline{h}(E') = \overline{h}(E)$ and E' is hereditarily stable.*

Proof. Let δ be a positive number which is smaller than ε, $\min_{i \neq j} \rho(e_i, e_j)$ and $\forall E' = \{e'_1, e'_2, \ldots, e'_n\}\ \forall i \leq n\ \rho(e_i, e'_i) < \delta \Rightarrow \overline{h}(E') = \overline{h}(E)$. Such δ exists since E is stable.

A point $x \in \mathcal{X}^n$ with distinct coordinates can be regarded as n-element subset of $\mathcal{X}$. $x^0 = (e_1, e_2, \ldots, e_n)$ corresponds to E and points in δ-neighborhood of x^0 correspond to possible E' (by choice of δ, all such points have distinct coordinates and $\overline{h}(E') = \overline{h}(E)$).

$k = 2^n - 1$ is the number of non-empty subsets of $I = \{1, \ldots, n\}$, and we will denote $J \subseteq I$ by $I(\sum_{j \in J} 2^{j-1})$ (so that every non-empty subset is numbered from 1 to k).

We define the mapping $H : B_\delta(x^0) \to F^k$ as follows: $H_i(x) = \overline{h}(E_i(x))$, $i = 1, \ldots, k$, where $E_i(x) = \{x_j | j \in I(i)\}$. The frame F^k is ordered as follows:

$$(f_1, \ldots, f_k) \leq (g_1, \ldots, g_k) \rightleftharpoons f_1 \leq g_1 \& \ldots \& f_k \leq g_k$$

It can be easily shown that under that ordering H has a local minimum in each point x from δ-neighborhood of x^0. Indeed, if $H_i(x) = u$ then there is open $G \supset E_i(x)$ such that $h(G) = u$ and

$$\forall x' \; E_i(x') \subset G \Rightarrow H_i(x') = \overline{h}(E_i(x')) \geq h(G) = u$$

(this follows from the definition of $\overline{h}$). Note that a hereditarily stable set corresponds to a point in $\mathcal{X}^n$ such that H is constant in some neighborhood of it. If H is not constant near x^0 then in $\frac{\delta}{2}$-neighborhood of x^0 $\exists x^1 : H(x^1) > H(x^0)$. Likewise, if H is not locally constant in x^1, then in $\frac{\delta}{4}$-neighborhood of x^1 $\exists x^2 : H(x^2) > H(x^1)$. F^k is finite and hence has no infinite increasing sequence, so we will stop at some point. That is, there exists x^m where H is locally constant and it is within δ-neighborhood of x^0. A set E' which corresponds to x^m is hereditarily stable, and satisfies the requirement of being close enough to E. ■

LEMMA 12. *Under the assumptions of Lemma 9,* $\forall u \in F \; |br(u)| < 2^{d(u)}$.

Proof. Choose a finite set E using Lemma 9 such that $\overline{h}(E) = u$. It is stable and thus by Lemma 11 we can construct E' which is hereditarily stable and $\overline{h}(E') = u$. Choose $E_0 \subseteq E'$ such that $\overline{h}(E_0) = u$ and $\forall E_1 \subset E_0 \; \overline{h}(E_1) > u$. Assume that $|E_0| = n$, $E_0 = \{e_1, e_2, \ldots, e_n\}$. We also introduce the following open sets: $A_{ij} = \{x \in \mathcal{X} : \rho(x, e_j) < 1/i\}$, $A_i = \bigcup_{j=1}^n A_{ij}$. By the definition of $\overline{h}$, $\lim h(A_i) = u$. There exists i_0 such that for $i > i_0$ $A_{ij_1} \cap A_{ij_2} = \emptyset$ if $j_1 \neq j_2$. In the remaining part of the proof we assume that $i > i_0$.

We assume that $br(u)$ is non-empty since otherwise the statement is trivial. Select $v \in br(u)$ and for each A_i construct $G_i \subset A_i : |G_i| < 2^{d(u)-1}$, $\overline{h}(G_i) = v$ by Lemma 9. $K_i = \{j : G_i \cap A_{ij} \neq \emptyset\}$ is a sequence of subsets of $\{1, \ldots, n\}$. Since there is only a finite number of such subsets, there is a subset K, which is equal to infinitely many of K_i, say $K = K_{i_m}$, $m \in \mathbb{N}$, $i_{m+1} > i_m$. We can now define the following sets: $A'_m = \bigcup_{j \in K} A_{i_m j}$, $E_1 = \{e_j | j \in K\}$. Note that $\overline{h}(E_1) = \lim_i h(A_i)$.

If $E_1 = E_0$ then $G_{i_m} \subseteq A'_m = A_{i_m}$ and $\forall j \; G_{i_m} \cap A_{i_m j} \neq \emptyset$. Since E' is hereditarily stable, E_0 is stable. Thus for a sufficiently large m

($\frac{1}{i_m} < \varepsilon$) we can select $E^* = \{e_j^* | e_j^* \in G_{i_m} \cap A_{i_m j}, j = 1, \dots, n\}$ such that $\overline{h}(E^*) = \overline{h}(E_0) = u$. But since $E^* \subset G_{i_m}$, $\overline{h}(E^*) \geq v$, which leads to a contradiction ($u < v$).

So, E_1 is a proper subset of E_0. $\overline{h}(E_1) > u$ (by the choice of E_0) and $\overline{h}(E_1) \leq v$, since each A'_m contains G_{i_m} and $h(G_{i_m}) = v$, so $h(A'_m) \leq v$. Thus $h(E_1) = v$. Choose $E_2 : E_1 \subseteq E_2 \subset E_0$, $|E_2| = n - 1$. $\overline{h}(E_2) = v$ (again, by the choice of E_0). There are only n possible subsets of E_0 of this cardinality, thus $|br(u)| \leq n \leq |E'| = |E| < 2^{d(u)}$. QED. ■

LEMMA 13. *Under the assumptions of Lemma 9, if E is a hereditarily stable subset under $\overline{h}$, and $\overline{h}(E) = u \in F$ then $\overline{h}|_{P_1(E)}$ is a p-morphism from $P_1(E)$ to F^u.*

Proof. We will prove that $\overline{h}(P_1(E)) = F^u$ using induction by $|E|$. If $|E| = 1$, then $|br(u)| = 0$, and $F^u = \overline{h}(P_1(E)) = \{u\}$.

Now consider E such that $|E| > 1$. For any $E' \subseteq E$ $\overline{h}(E') \geq u$, so $\overline{h}(P_1(E)) \subseteq F^u$. For any $w \in F^u \setminus \{u\}$ there is $v \in br(u)$ such that $u \leq v \leq w$, and there is $E_1 \subseteq E$ such that $\overline{h}(E_1) = v$ (as demonstrated in the proof of Lemma 12). By induction hypothesis, $\overline{h}(P_1(E_1)) = F^{\overline{h}(E_1)}$, so $\exists E_2 \subseteq E_1$ $\overline{h}(E_2) = w$. Thus, $\overline{h}(P_1(E)) = F^u$. ■

The remaining part of the proof is almost identical to the proof of Theorem 5.5 in [5]. This method was introduced in [4], where it was used to prove that ML is not finitely axiomatizable.

We will use the family of finite frames $\Phi(k, m)$ ($k, m > 0$). $\Phi(k, m)$ is the set of pairs (i, j) such that either $0 \leq i \leq k$ and $0 \leq j \leq 1$, or $i = k + 1$ and $1 \leq j \leq m$, or $i = k + 2$ and $j = 0$. The ordering is defined as follows: $(i, j) < (i', j')$ iff $(i > i')$. We will also use the frames $\Phi^i(k, m) \rightleftharpoons \Phi(k, m) \setminus \{(i, 1)\}$, where $0 \leq i \leq k$.

LEMMA 14. *Let $\mathcal{X}$ be an infinite metric space.*

1. $X(\Phi(k, 2^{k+3})) \in ML_{O(\mathcal{X})}$
2. $X(\Phi^i(k, m) \notin ML_{O(\mathcal{X})}$

Proof.

1. By Lemma 6 we need to prove that there doesn't exist a p-morphism from some cone of $O(\mathcal{X})$ (that is $O(Y)$ where $Y \subseteq \mathcal{X}$) to $\Phi(k, 2^{k+3})$. But if such p-morphism exists, by Lemma 12 we have $|br(k + 2, 0)| < 2^{d(k+2,0)}$, which is false, since $|br(k+2, 0)| = 2^{k+3}$ and $d(k + 2, 0) = k + 3$.

2. To prove this we can either construct a p-morphism from $ML_{O(\mathcal{X})}$ to $\Phi^i(k, m)$, which is easy, or use a result from [5] (Lemma 5.3), stating that $X(\Phi^i(k, m) \notin ML$, and we already know that $ML_{O(\mathcal{X})} \subseteq ML$.

■

DEFINITION 15. A *k-formula* is a formula that does not contain propositional letters besides $p_1, \ldots, p_k$.

LEMMA 16.

If A is a k-formula and $\Phi(k,m) \not\models A$ then there exists $i \leq k$ such that $\Phi^i(k,m) \not\models A$. Or, in terms of Jankov formulas, if $X(\Phi(k,m)) \in (\mathbf{Int}+A)$ then $X(\Phi^i(k,m)) \in (\mathbf{Int}+A)$ for some $i \leq k$.

Proof. Let $(\Phi(k,m),\theta)$ be a model refuting A. Since each $\theta(p_r)$ is an upwards closed set, for every i, except maybe a single one, it either contains all (i,j) or none. Since there are only k variables, and $k+1$ possible values for i, there has to be a level i^* on which $(i^*,0)$ and $(i^*,1)$ have the same valuations. Clearly, the model $(\Phi^{i^*}(k,m),\theta)$ also refutes A. ■

THEOREM 17. *Let $\mathcal{X}$ be an infinite metric space. Then $ML_{O(\mathcal{X})}$ is not axiomatizable in finite number of variables, that is for any $k \in \mathbb{N}$ $ML_{O(\mathcal{X})}$ is not axiomatizable by any set of k-formulas.*

Proof. Let Σ be a set of k-formulas and suppose that Σ axiomatizes $ML_{O(\mathcal{X})}$. Then $X(\Phi(k,2^{k+3}))$ could be derived from some finite number of axioms $A_1,\ldots,A_n \in \Sigma$. But then Lemma 16 with $A = A_1 \wedge \ldots \wedge A_n$ provides a contradiction, since by Lemma 14 $X(\Phi^i(k,m))$ is never in $ML_{O(\mathcal{X})}$. ■

COROLLARY 18. *$ML_{O(\mathcal{X})}$ is not finitely axiomatizable.*

Finally, let us show that $ML_{O(\mathcal{X})}$ is not distinguishable from ML within the class of characteristic formulas of finite frames, to which Lemma 9 is applicable.

THEOREM 19. *Let $\mathcal{X}$ be an infinite metric space and F a finite frame with the least element, and $|br(u)| \neq 1$ for any $u \in F$. Then*

$$X(F) \in ML_{O(\mathcal{X})} \iff X(F) \in ML$$

Proof. If $X(F) \in ML_{O(\mathcal{X})}$ then it is also in ML because $ML_{O(\mathcal{X})} \subseteq ML$.

If $X(F) \notin ML_{O(\mathcal{X})}$ then by Lemma 6 $\exists U \subseteq \mathcal{X}\ \exists h : O(U) \twoheadrightarrow F$. By Lemmas 9 and 11 we can find a finite $E \subset U$ such that $\overline{h}(E) = 0_F$ and E is hereditarily stable. $\overline{h} : P_1(E) \to F$ is a p-morphism (see Lemma 13), hence $X(F) \notin L(P_1(E))$ and thus $X(F) \notin ML$. ■

COROLLARY 20. *Let L be an intermediate logic, and $ML_{O(\mathcal{X})} \subseteq L \subseteq ML$ for some infinite metric space $\mathcal{X}$. Then L is not axiomatizable in finite number of variables.*

Proof. By Theorem 19 L has the same formulas of the form $X(F)$ as $ML_{O(\mathcal{X})}$ or ML. Therefore Lemma 14 applies to L, and so does the rest of the proof of Theorem 17. ■

5 Conclusion

To summarise the results of this paper, we have discovered that logics $ML_{O(\mathcal{X})}$ (for different infinite metric spaces $\mathcal{X}$) are subsets of ML, are not finitely axiomatizable and we cannot distinguish them from ML or each other by using formulas of the form $X(F)$, where F is a finite frame with a particular property. Although these results were stated only for metric spaces, they could easily be extended to various topological spaces as well.

Many questions still remain open. Here are some of them:

1. Is (for example) $ML_{O(\mathbb{R})}$ different from ML_1 and ML?
2. Is $ML_{O(\mathbb{R})}$ recursively axiomatizable?
3. Is $ML_{O(\mathbb{R})}$ decidable?
4. What is the intersection of the logics $ML_{O(\mathcal{X})}$? Does there exist a metric space $\mathcal{X}_0$ such that $ML_{O(\mathcal{X}_0)} \subseteq ML_{O(\mathcal{X})}$ for all metric spaces $\mathcal{X}$?
5. [5] introduces a sequence of logics $ML_n = L(P_n(\omega))$, where $P_n(X) = (\{E \subseteq X \| E| \geq n\}, \supseteq)$. $ML_i \subseteq ML_j$ if $i \leq j$. Is it true that $ML_2 \subseteq ML_{O(\mathcal{X})}$?

Instead of open sets we can choose another class of sets to build a Kripke frame. For example closed sets generate a logic between ML_1 and ML [5]. [3] shows that many families of sets such as n-balls in $\mathbb{R}^n$, connected compacts and convex compacts generate intuitionistic logic **Int**.

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Timofei Shatrov

119234, Russia, Moscow, ul. Leninskiye Gory, 1b-1421

timofei.shatrov@gmail.com