Modal logics for mereotopological relations

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abstract. We present a complete axiomatization of a logic denoted by MTML (Mereo-Topological Modal Logic) based on the following set of mereotopological relations: *part-of, overlap, underlap, contact, dual contact* and *interior part-of*. We prove completeness theorems for MTML with respect to several classes of models including the standard topological models over the set of regular-closed subsets of arbitrary topological spaces. We show that MTML possesses fmp with respect to a class of non-standard models, which implies its decidability. In this way we propose also a solution of the main open problem, formulated in [17] to find a decidable modal logic for topological relations.

Keywords: mereotopology, spatial relations, spatial reasoning, modal logic, completeness theorems, decidability.

Introduction

This paper can be considered as an application of modal logic to mereotopology. Mereotopology is an extension of mereology with some relations of topological nature. Mereology is an ontological discipline which can be characterized shortly as a theory of "Parts and Wholes" (see [21] for a general reference to mereology). Typical in mereology are the relations "part-of", "overlap" and "underlap". One of the basic mereological systems is Lesnewski's mereology, but as Tarski showed, the mathematical equivalent of mereology are complete Boolean algebras (see [21] for this fact). In Boolean formulation the part-off relation coincides with the Boolean ordering $x \leq y$, the overlap relation xOy can be defined by $x.y \neq 0$ (where "." is the Boolean multiplication) and the underlap (dual overlap) $x\widehat{O}y$ is defined by $x\widehat{O}y$ iff $x^*.y^* \neq 0$ iff $x + y \neq 1$ (x^* is the Boolean complement of x). Mereology, however, is not capable for describing some relations between individuals as, for instance, one individual to be in a contact with another one. Adding to mereology contact-like relations goes back to de Laguna [8] and Whitehead [26]. The intention of de Laguna and Whitehead was to use mereology for building of a new, point-free theory of space as an extension of mereology with the relation of contact (or "connection" in Whitehead terminology). The primitive objects of the new theory of space are called regions and it is called "point-free", because points are not taken as primitives but are definable by means of regions, contact and some mereological relations. As Tarski showed (see [21]) standard point models of the new theory of space are regular closed (or open) sets of some topological spaces with a topological definition of the contact relation. This motivates some authors to call the extension of mereology with the contact relation (or some of its derivatives) "mereotopology". Mereotopology is often called also a "region-based theory of space", because it is on the base of the Whitehedian approach to the theory of space. Since mereology can be identified in some sense with the theory of Boolean algebras, mereotopology can be identified with the theory of *contact algebras*, which are Boolean algebras with an additional relation C called contact (see [10]). The reader can find more about mereotopology, region-based theory of space and the related logics in the papers [2, 3, 17, 19, 25, 27]. This field of research is closely related to some applied areas as Qualitative Spatial Reasoning (QSR), Knowledge Representation (KR) and Geographical Information Systems (GIS). A survey on the research in QSR and related subareas in KR and GIS can be find in [6, 7].

The main aim of the present paper is to build a multimodal logic interpreted in frames related to mereotopology. The standard frames for such a logic will be in the form $(W, R_1, \ldots, R_n)$, where W is a nonempty set of regular closed sets of a given topological space and the relations R_i are certain mereotopological relations between regions. Logics of such kind have been considered for the first time by Lutz and Wolter in [17]. The relations which Lutz and Wolter considered are the well-known 8 Egenhofer-Franzosa RCC-8 topological relations between regions [13]. However, all considered logics in [17] are undecidable and one of the main open problems formulated in [17] was to find decidable modal logics based on a reasonable set of mereotopological relations. We present such a logic, based on the following mereotopological relations:

(I) the mereological relations: overlap O, underlap (the dual overalap) $\widehat{O}$, part-of $\leq$ and converse part-of $\geq$,

(II) the mereotopological relations: contact C, dual contact $\widehat{C}$, interior part-of $\ll$ and its converse $\gg$.

Frames, based on such kind of relations, are called in this paper *mereotopological structures*. The modal logic corresponding to the class of all mereotopological structures is called mereotopological modal logic and is denoted by MTML. We denote box and diamond modalities of MTML by $[R]$ and $\langle R \rangle$, where $R \in \{O, \widehat{O}, \leq, \geq, C, \widehat{C}, \ll, \gg\}$. Additionally we include the universal modality $[U]$.

Motivations to chose mereotopological structures as a semantical basis of MTML are, among others, the following. The relations from group (I) are the most typical mereological relations. Moreover the corresponding modal logic (introduced in [23] under the name *modal logic of set relations*) was decidable and our aim was to extend it with some mereotopological relations, preserving the completeness theorem and decidability. The first attempt was by adding the contact relation and this was done in Nenov's master thesis [18]. Still the obtained logic was complete with respect to its intended topological semantics and decidable. Then, we decided to extend further the language by modalities, corresponding to dual contact $\widehat{C}$, interior part-of $\ll$ and its converse $\gg$. Note that these relations are definable in contact algebras: $x\widehat{C}y \leftrightarrow x^*Cy^*$ and $x \ll y \leftrightarrow x\overline{C}y^*$. This fact shows that all relations of mereotopological structures are definable in contact algebras, which makes possible to use the corresponding representation theory developed in [10, 12]. Let us note that RCC-8 relations - the semantic base of Lutz-Wolter modal logic ([17]) (LW-logic for short and for later references), are definable in our mereotopological structures, while the converse is not true: for instance, dual overlap is not definable in RCC-8 (this will be discussed with more details in the main text). This does not imply, however, that all modalities from LW-logic are definable in MTML. In fact MTML and LW-logic are incomparable in the sense that neither of the two can be considered as a part of the other. But LW-logic is much more expressive: it possesses difference modality, and hence definable nominals. Moreover, since the relations in RCC-8 are jointly exhaustive and pairwise disjoint (JEPD), all Boolean combinations of them are expressible by sums of the basic 8 relations, and hence their corresponding modalities are definable. So in LZ-logic one can work with quite enough different modalities. In LMTM we have 9 basic modalities and also we may define new modalities by the sums and compositions but not by complements and intersections of the base relations. Maybe just the closure with complements and intersections of the basic modalities of the LZ-logic is one of the reasons of its undecidability. We can see later that if we can allow modalities of MTML corresponding to Boolean combinations of the base modalities, we can interpret LW-logic in MTML and obtain in this way that the resulting extension is undecidable.

Let us now discuss what kind of reasoning can be expressed in MTML. Note that MTML and LW-logic are similar as logical formalisms: both are modal logics over frames which elements are spatial regions. Since the propositional variables in modal logics are interpreted by subsets of a given frame, in general, these two logics propose reasoning for sets of regions. For instance, the formula $[U](p \Rightarrow \langle C\rangle q)$ expresses the fact that each region from the set p is in a contact with some region from the set q. Another example: the formula $[U](p \Rightarrow [\geq]p)$ expresses the fact that the set of regions p is closed with respect to part-of relation: if $x \in p$ and y is a part of x, then y is in p. The frame condition (Con) $xOx \wedge yOy \rightarrow x\widehat{O}y \vee xCy$ is true in the frame of all closed regions in a topological space iff the space is connected. This condition is modally definable in MTML by the modal formula $\langle O\rangle([\widehat{O}]p \wedge [C]q) \Rightarrow [U]([O](p \vee q)$, which distinguished connected from non-connected topological spaces. This is an example of a property expressible in MTML but not in the LW-logic.

As it was mentioned in [17], LW-logic is similar to the Halpern and Shoham's temporal logic [16], based semantically on the Allen's 13 relations between time intervals. The same can be said also for MTML. Allen's relations are relations not between time points but between time intervals, which over the real line are closed regions. The interpretation of RCC-8 and our mereotopological structures over the real line gives a temporal meaning of LW-logic and MTML. Let us mention yet another logic with a similar nature: the hyperboolean modal logics introduced in [15]. The frames of this logics are arbitrary Boolean algebras and Boolean algebras of sets, which relates these logics to LW-logic and MTML.

The main aim of this paper is to give a finite normal and complete axiomatization of MTML and to prove its decidability. The axiomatization goes through several steps. We first give an abstract characterization of the relations $\leq, O, \widehat{O}, C, \widehat{C}, \ll$ by means of a finite set of first-order sentences, introducing in this way an abstract, point-free semantics for MTML. We prove that each abstract mereotopological structure $\underline{W}$ is representable in a contact algebra, and then, applying the topological representation theory of contact algebras developed in [10], we show that $\underline{W}$ can be isomorphically embedded into the contact algebra of regular closed subsets of some topological space. The method of the proof of this characterization is based on a considerable generalization of the Stone representation theory of distributive lattices (see [1]). The obtained results for mereotopological structures have also some independent interest for mereotopology: they can be considered as a kind of first-order logic for mereotopological relations disregarding the Boolean structure of regions. The obtained abstract semantics of MTML cannot give, however, a direct axiomatization of the logic, because one of its axioms is not modally definable. That is why we introduce a nonstandard semantics of MTML which leads to an easy and complete axiomatization. Then, by using the Segerberg's bulldozer techniques [20], we prove the equivalence of the standard and nonstandard semantics for MTML. Finally, applying the method of filtration to the non-standard models of MTML we prove its decidability. We show, however, that MTML does not possess fmp with respect to its standard semantics.

We propose as standard reference books: [4, 5, 20] for modal logic, [14] for topology, and [1] for Stone representation theory.

1 The first-order logic of mereotopological structures

1.1 Contact algebras, topological and relational representation

DEFINITION 1. [10] By a *Contact Algebra* (CA) we will mean any system $\underline{B} = (B, C) = (B, 0, 1, ., +, *, C)$, where $(B, 0, 1, ., +, *)$ is a non-degenerate Boolean algebra with a complement denoted by "$*$" and C – a binary relation in B, called *contact* and satisfying the following axioms:

$(C1)$ $xCy \rightarrow x, y \neq 0$, $(C2)$ $xCy \rightarrow yCx$,

$(C3)$ $xC(y+z) \leftrightarrow xCy$ or xCz, $(C4)$ $x.y \neq 0 \rightarrow xCy$.

The algebra $\underline{B}$ is *connected* if it satisfies the axiom of connectedness

(Con) $x \neq 0$, $y \neq 0$ and $x + y = 1 \rightarrow xCy$.

The complement of C is denoted by $\overline{C}$.

EXAMPLES 2. **Examples of contact algebras.**

• **(1). Topological example: the CA of regular closed sets.** Let X be an arbitrary topological space. A subset a of X is *regular closed* if

$a = Cl(Int(a))$, where Cl and Int are the standard topological closure and interior operations in X. The set of all regular closed subsets of X will be denoted by $RC(X)$. It is a well-known fact that regular closed sets with the operations

$$a + b = a \cup b,\ a.b = Cl(Int(a \cap b)),\ a^* = Cl(X \setminus a),\ 0 = \varnothing \text{ and } 1 = X$$

form a Boolean algebra. If we define the contact by $a\, C_X\, b$ iff $a \cap b \neq \varnothing$, then $RC(X)$ with the above contact is a contact algebra. If X is a connected space then $RC(X)$ is a connected contact algebra. The following representation theorem is a special case of Theorem 5.1 from [10].

THEOREM 3. *Every (connected) contact algebra $\underline{B}$ can be isomorphically embedded into the contact algebra $RC(X)$ over some (connected) topological space X.*

• **(2). Non-topological example, related to Kripke semantics of modal logic.** Let (X, R) be a reflexive and symmetric modal frame and let $B(X)$ be the Boolean algebra of all subsets of X. Define a contact C_R between two subsets $a, b \in B(X)$ by aC_Rb iff $(\exists x \in a)(\exists y \in b)(xRy)$. Then we have that $B(X)$ equipped with the contact C_R is a contact algebra, called the contact algebra over the frame (W, R) [12, 9]. If (W, R) is connected in a graph sense (every two points are connected by an R-sequence), then $B(X)$ is a connected contact algebra [12]. Moreover the following representation theorem is true:

THEOREM 4. *[12] Every contact algebra can be isomorphically embedded into the contact algebra of some reflexive and symmetric frame (W, R).*

1.2 Mereotopological structures

DEFINITION 5. Let (B, C) be a contact algebra. We define in B the following relations:

• *part-of* $\quad a \leq b$ iff $a.b^* = 0$ ($\leq$ is the standard Boolean ordering) we denote the converse of $\leq$ by $\geq$.

• *overlap* $\quad aOb$ iff $a.b \neq 0$,

• *underlap* (dual overlap) $\quad a\widehat{O}b$ iff a^*Ob^* iff $a + b \neq 1$,

• *dual contact* $\quad a\widehat{C}b$ iff a^*Cb^*,

• *interior part-of* $\quad a \ll b$ iff $a\overline{C}b^*$. The converse of $\ll$ is denoted by $\gg$.

The complements of the above relations are denoted by $\not\leq, \not\geq, \overline{O}, \overline{\widehat{O}}, \overline{\widehat{C}}\ \not\ll, \not\gg$.

The proof of the following lemma is straightforward.

LEMMA 6. *The relations $\leq, O, \widehat{O}, C, \widehat{C}, \ll$ satisfy the following first-order conditions:*

(≤ 0)	$a \leq b$ *and* $b \leq a \rightarrow a = b$,	(≤ 1)	$a \leq a$,
(≤ 2)	$a \leq b$ *and* $b \leq c \rightarrow a \leq c$,		
$(O1)$	$aOb \rightarrow bOa$,	$(\widehat{O}1)$	$a\widehat{O}b \rightarrow b\widehat{O}a$,
$(O2)$	$aOb \rightarrow aOa$,	$(\widehat{O}2)$	$a\widehat{O}b \rightarrow a\widehat{O}a$,
$(\overline{O} \leq)$	$a\overline{O}a \rightarrow a \leq b$,	$(\overline{\widehat{O}} \leq)$,	$b\overline{\widehat{O}}b \rightarrow a \leq b$,
$(O \leq)$	aOb *and* $b \leq c \rightarrow aOc$,	$(\widehat{O} \leq)$	$c \leq a$ *and* $a\widehat{O}b \rightarrow c\widehat{O}b$,
$(O\widehat{O})$	aOa *or* $a\widehat{O}a$,	$(\leq O\widehat{O})$	$c\overline{O}a$ *and* $c\overline{\widehat{O}}b \rightarrow a \leq b$,
(C)	$aCb \rightarrow bCa$,	$(\widehat{C})$	$a\widehat{C}b \rightarrow b\widehat{C}a$,
$(CO1)$	$aOb \rightarrow aCb$,	$(\widehat{C}\widehat{O}1)$	$a\widehat{O}b \rightarrow a\widehat{C}b$,
$(CO2)$	$aCb \rightarrow aOa$,	$(\widehat{C}\widehat{O}2)$	$a\widehat{C}b \rightarrow a\widehat{O}a$,
$(C \leq)$	aCb *and* $b \leq c \rightarrow aCc$,	$(\widehat{C} \leq)$	$a\widehat{C}b$ *and* $c \leq b \rightarrow a\widehat{C}c$,
$(\ll\leq 1)$	$a \ll b \rightarrow a \leq b$,		
$(\ll\leq 2)$	$a \leq b$ *and* $b \ll c \rightarrow a \ll c$,	$(\ll\leq 3)$	$a \ll b$ *and* $b \leq c \rightarrow a \ll c$,
$(\ll O)$	$a\overline{O}a \rightarrow a \ll b$,	$(\ll \widehat{O})$	$b\overline{\widehat{O}}b \rightarrow a \ll b$,
$(\ll CO)$	aCb *and* $b \ll c \rightarrow aOc$,	$(\ll \widehat{C}\widehat{O})$	$c \ll a$ *and* $a\widehat{C}b \rightarrow c\widehat{O}b$,
$(\ll C\widehat{O})$	$c\overline{C}a$ *and* $c\overline{\widehat{O}}b \rightarrow a \ll b$,	$(\ll \widehat{C}O)$	$c\overline{O}a$ *and* $c\overline{\widehat{C}}b \rightarrow a \ll b$.

DEFINITION 7. Let $\underline{W} = (W, \leq, O, \widehat{O}, C, \widehat{C}, \ll)$, $W \neq \varnothing$, be a relational system. Then $\underline{W}$ is called a *mereotopological structure* if it satisfies the first-order conditions of lemma 6; $\underline{W}$ is called a *standard mereotopological structure* if there exists a contact algebra (B, C) such that $W \subseteq B$ and the relations $\leq, O, \widehat{O}, C, \widehat{C}, \ll$ coincide with those that are defined in Definition 5; $\underline{W}$ is called *completely standard* if the algebra (B, C) is the contact algebra of regular closed subsets of some topological space; if in addition $W = B$, then the (standard, completely standard) mereotopological structure is called *full*.

The following lemma is an easy consequence of Theorem 3.

LEMMA 8. *A mereotopological structure is standard iff it is completely standard.*

In the next section we will show that each mereotopological structure is a standard one, and in view of Lemma 8 that it is completely standard.

REMARKS 9. (1) Let us note that the axioms $(\overline{O} \leq)$, $(\overline{\widehat{O}} \leq)$, $(O2)$ and $(\widehat{O}2)$ follow from the remaining and can be skipped. We preserve them in the definition, because they are part of an important subset of the axioms characterizing mereological relations.

(2) We can establish some duality between the relations in a mereotopological structure and their axioms. We divide the relations in a dual pairs as follows: $(\leq - \geq)$, $(O - \widehat{O})$, $(C - \widehat{C})$ and $(\ll - \gg)$. Note also that the set of axioms is closed with respect to this duality and very often we may skip some proofs which are "dual" to given ones.

(3) We adopt the standard definitions of isomorphism and embedding between mereotopological structures and two isomorphic structures are treated as identical. Thus, for instance, a structure which is isomorphic to a standard structure will be called also a standard structure. We say that a mereotopological structure $\underline{W}$ is embeddable into a contact algebra (B, C) if there exists an isomorphic embedding of $\underline{W}$ into the full mereotopological structure over (B, C).

(4) It can be seen that the axiom of connectedness for contact algebras can be expressed by the following axiom in the language of mereotopological relations

(Con) $aOa \wedge bOb \rightarrow a\widehat{O}b \vee aCb$.

It is natural to call a mereotopological structure *connected* if it satisfies the axiom (Con). Note that mereotopological structures over connected topological spaces are connected. Another non-topological example can be obtained from connected contact algebras over frames (W, R) with a reflexive, symmetric and connected relation R (see Examples 2(2)).

The following lemma lists some easy consequences of the axioms of mereotopological structures which sometimes we will use later on without explicit reference.

LEMMA 10. *(COO) $aCb \rightarrow aOa$ and bOb, $(\widehat{C}\widehat{O}\widehat{O})$ $a\widehat{C}b \rightarrow a\widehat{O}a$ and $b\widehat{O}b$, $(\leq\leq O)$ $a \leq a'$, $b \leq b'$, $aOb \rightarrow a'Ob'$, $(\leq\leq C)$ $a \leq a'$, $b \leq b'$, $aCb \rightarrow a'Cb'$, $(\geq\geq \widehat{O})$ $a \geq a'$, $b \geq b'$, $a\widehat{O}b \rightarrow a'\widehat{O}b'$, $(\geq\geq \widehat{C})$ $a \geq a'$, $b \geq b'$, $a\widehat{C}b \rightarrow a'\widehat{C}b'$, $(\leq\ll\leq)$ $a \leq a'$, $a' \ll b'$, $b' \leq b \rightarrow a \ll b$.*

1.3 Representation theory for mereotopological structures

In this section we will develop a representation theory for mereotopological structures by a generalization of the representation theory for distributive lattices. First we will do this for a subsystem of mereotopological structures which we call *mereological structures.*

Mereological structures and a characterization of mereological relations $\leq, O, \widehat{O}$.

DEFINITION 11. A system $\underline{W} = (W, \leq, O, \widehat{O})$ is called a *mereological structure* if it satisfies the axioms of mereotopological structure containing only the relations $\leq, O$ and $\widehat{O}$.

Obviously every mereotopological structure is a mereological structure. Mereological structures was introduced and studied in another context, name and notations in [22] from which we will use some results.

DEFINITION 12. [22] Let $\underline{W}$ be a mereological structure and A be a subset of W.

- A is called a $\leq$*-set* if $(\forall x, y \in W)(x \in A$ and $x \leq y \to y \in A)$,
- A is called a $\geq$*-set* if $(\forall x, y \in W)(x \in A$ and $x \geq y \to y \in A)$,
- A is a *filter* if A is a $\leq$-set and $(\forall x, y \in A)(xOy)$,
- A is an *ideal* if A is a $\geq$-set and $(\forall x, y \in A)(x\widehat{O}y)$.
- A is a *good filter* if A is a filter and $(\forall x, y \notin A)(x\widehat{O}y)$,
- A is a *good ideal* if A is an ideal and $(\forall x, y \notin A)(xOy)$.

We denote by $GF(\underline{W})$ the set of good filters of $\underline{W}$. Similarly $GI(\underline{W})$ will denote the set of good ideals of $\underline{W}$.

The given definitions of a filter, good filter, ideal and a good ideal are generalizations of the standard notions of a filter, prime filter, ideal and a prime ideal from the theory of distributive lattices (see [1]).

Note that $\varnothing$ and W are both $\leq$- and $\geq$-sets. Define for $x \in W$:
$[x) = \{y \in W : x \leq y\}$, $(x] = \{y \in W : x \geq y\}$.

LEMMA 13. *[22] (i) The set $[x)$ (the set $(x]$) is the smallest $\leq$-set ($\geq$-set) containing x.*

(ii) If A, B are $\leq$-sets ($\geq$-sets) then $A \cup B$ and $A \cap B$ are $\leq$-sets ($\geq$-sets). If A is a $\leq$-set ($\geq$-set) then $-A = W \setminus A$ is a $\geq$-set ($\leq$-set).

(iii) Let A be a $\leq$-set ($\geq$-set). Then $A \cup [x)$ ($A \cup (x]$) is the smallest $\leq$-set ($\geq$-set) containing A and x. In particular the set $[x) \cup [y)$ (the set $(x] \cup (y]$) is the smallest $\leq$-set ($\geq$-set) containing x and y.

(iv) The set $[x)$ is a filter iff xOx (The set $(x]$ is an ideal iff $x\widehat{O}x$).

(v) Let A be a filter (ideal). Then $A \cup [x)$ is a filter iff xOx and $(\forall y \in A)(xOy)$, ($A \cup (x]$ is an ideal iff $x\widehat{O}x$ and $(\forall y \in A)(x\widehat{O}y)$).

Let $A \neq \varnothing$ be a filter (ideal). Then $A \cup [x)$ is a filter iff $(\forall y \in A)(xOy)$, ($A \cup (x]$ is an ideal iff $(\forall y \in A)(x\widehat{O}y)$).

(vi) The set $[x) \cup [y)$ is a filter iff xOy. The set $(x] \cup (y]$ is an ideal iff $x\widehat{O}y$.

(vii) Let $\{A_i : i \in I\}$ be a non-empty family of filters (ideals), linearly ordered by set-inclusion. Then the set $A = \bigcup_{i \in I} A_i$ is a filter (ideal).

(viii) Let A be a filter (ideal). Then A is a good filter (good ideal) iff $-A = W \setminus A$ is an ideal (filter).

DEFINITION 14. Let $\underline{W}$ be a mereological structure. A pair $\Gamma = (A, B)$ of subsets of W is called a filter-ideal pair, if A is a filter, B is an ideal and $A \cap B = \varnothing$). Γ is called a good filter-ideal pair if A is a good filter and B is a good ideal. Γ is called a complete filter-ideal pair if $A \cup B = W$. Obviously every complete pair is a good pair. If Γ denotes a filter-ideal pair, then Γ_1 will denote its filter part and Γ_2 will denote its ideal part. If Γ, Δ are filter-ideal pairs, we will define the ordering relation $\Gamma \subseteq \Delta$ iff $\Gamma_i \subseteq \Delta_i$, $i = 1, 2$.

LEMMA 15. *Let (F, I) be a filter-ideal pair. Then for every $x \in W$ either (1) $F \cup [x)$ is a filter and $(F \cup [x)) \cap I = \varnothing$ or (2) $I \cup (x]$ is an ideal and $F \cap (I \cup (x]) = \varnothing$.*

Proof. Suppose that the assumptions of the lemma are fulfilled and that neither (1) nor (2) are true, so we have $\neg(1)$ and $\neg(2)$. Note that $\neg(1)$ is equivalent to: (a) $F\cup[x)$ is not a filter or (a') $(F\cup[x))\cap I\neq\varnothing$. By Lemma 13(v) (a) is equivalent to the disjunction: (a1) $x\overline{O}x$ or (a2) $(\exists y\in F)(x\overline{O}y)$. It is easy to see that (a') is equivalent to (a3) $x\in I$.

In a similar way $\neg(2)$ is equivalent to: (b) $I\cup(x]$ is not an ideal or (b') $F\cap(I\cup(x])\neq\varnothing$. By Lemma 13(v) (b) is equivalent to the disjunction: (b1) $x\overline{\widehat{O}}x$ or (b2) $(\exists z\in I)(x\overline{\widehat{O}}z)$. Also, it is easy to see that (b') is equivalent to: (b3) $x\in F$.

So $\neg(1)\leftrightarrow(a1)$ or $(a2)$ or $(a3)$ and $\neg(2)\leftrightarrow(b1)$ or $(b2)$ or $(b3)$ and we have to consider all combinations $(ai)(bj)$ for $i\neq j, i,j=1,2,3$ and in each case to obtain a contradiction.

Case (a1)(b1): $x\overline{O}x$ and $x\overline{\widehat{O}}x$. This contradicts axiom $(O\widehat{O})$.

Case (a1)(b2): $x\overline{O}x$ and $x\overline{\widehat{O}}z$, $z\in I$. From $x\overline{O}x$ by $(\overline{O}\leq)$ we get $x\leq z$ and consequently $x\in I$. From $x,z\in I$ we obtain $x\widehat{O}z$ which contradicts $x\overline{\widehat{O}}z$.

Case (a1)(b3): $x\overline{O}x$ and $x\in F$, which implies xOx - a contradiction.

The cases **(a2)(b1)** and **(a3)(b1)** can be considered in a dual way.

Case (a2)(b2): $y\in F$, $x\overline{O}y$, $z\in I$ and $x\overline{\widehat{O}}z$. From $x\overline{O}y$ and $x\overline{\widehat{O}}z$ we get by axiom $(\leq O\widehat{O})$ that $y\leq z$. Conditions $z\in I$ and $y\leq z$ imply $y\in I$. But $y\in F$ and $y\in I$ imply that $F\cap I\neq\varnothing$ - a contradiction.

Case (a2)(b3): $y\in F$, $x\overline{O}y$ and $x\in F$. Conditions $x\in F$ and $y\in F$ imply xOy, which contradicts $x\overline{O}y$.

In a similar (dual) way one can consider the **case (a3)(b2)**.

Case (a3)(b3): $x\in I$, $x\in F$. This case contradicts the condition $F\cap I=\varnothing$. ■

The following lemma generalizes the separation lemma for filters and ideals from the theory of distributive lattices (see [1]).

LEMMA 16. *[22]* **Separation Lemma.** *Let (F_0,I_0) be a filter-ideal pair. Then there exists a complete (and consequently a good) filter-ideal pair (F,I) extending the pair (F_0,I_0).*

Proof. Let $M=\{(F,I):(F_0,I_0)\subseteq(F,I)\}$. It follows by Lemma 13(vii) that the conditions of the Zorn Lemma for the set M ordered by the relation $\subseteq$ are fulfilled, and hence M has a maximal element (F,I). Applying Lemma 15 to (F,I) we obtain that (F,I) is a complete filter-ideal pair. This implies that $F=-I$ and $I=-F$, which by Lemma 13 (viii) implies that F is a good filter and that I is a good ideal. ■

LEMMA 17. *[22]* **Characterization Lemma for the relations $\leq, O, \widehat{O}$.** *Let $\underline{W}$ be a mereological structure. Then for all $x,y\in W$ we have:*

(i) $x\leq y\leftrightarrow(\forall A\in GF(W))(x\in A\rightarrow y\in A)$,

(ii) $xOy\leftrightarrow(\exists A\in GF(W))(x\in A$ *and* $y\in A)$,

(iii) $x\widehat{O}y\leftrightarrow(\exists A\in GF(W))(x\notin A$ *and* $y\notin A)$.

Proof. (i) $(\rightarrow)$ – obvious.

($\leftarrow$) We will reason by contraposition. Let $x \not\leq y$. Then $[x) \cap (y] = \varnothing$. By ($\overline{O} \leq$), and ($\overline{\widehat{O}} \leq$) and $x \not\leq y$ we obtain xOx and $y\widehat{O}y$. Then by lemma 13 (iv) $[x)$ is a filter and $(y]$ is an ideal. Since $[x) \cap (y] = \varnothing$, by the Separation Lemma there exist a good filter F and a good ideal I such that $[x) \subseteq F$, $(y] \subseteq I$ and $F \cap I = \varnothing$. It follows from these conditions that $x \in F$ and $y \notin F$.

(ii) ($\leftarrow$) – obvious.

($\rightarrow$) Suppose xOy. Then by Lemma 13 (vi) the set $[x) \cup [y)$ is a filter. Since $\varnothing$ is an ideal, then $([x) \cup [y)) \cap \varnothing = \varnothing$, and by the Separation Lemma there exist a good filter F such that $[x) \cup [y) \subseteq F$, which implies that $x, y \in F$.

(iii) ($\leftarrow$) is the obvious part.

($\rightarrow$) Suppose $x\widehat{O}y$. Then, as in (ii) but reasoning in a dual way, we can obtain a good ideal I such that $x, y \in I$. Then putting $F = -I$ we find a good filter F such that $x, y \notin F$. ∎

A characterization of mereotopological relations $C, \widehat{C}$ and $\ll$.

DEFINITION 18. Let $\underline{W}$ be a mereotopological structure and A, B be subsets of W and R be any of the relations $\not\ll, C$ and $\widehat{C}$. We define the following three relations between such subsets:

$A\rho_R B$ iff $(\forall x \in A, \forall y \in B)(xRy)$.

We define the following relation ρ in the set of all filter-ideal pairs:

$\Gamma\rho\Delta$ iff $\Gamma_1\rho_C\Delta_1$ and $\Gamma_2\rho_{\widehat{C}}\Delta_2$ and $\Gamma_1\rho_{\not\ll}\Delta_2$ and $\Delta_1\rho_{\not\ll}\Gamma_2$.

LEMMA 19. (*i*) *In the set of filters of* $\underline{W}$, ρ_C *is a reflexive and symmetric relation.*

(*ii*) *In the set of ideals of* $\underline{W}$, $\rho_{\widehat{C}}$ *is a reflexive and a symmetric relation.*

(*iii*) *If* Γ *is a filter-ideal pair, then* $\Gamma_1\rho_{\not\ll}\Gamma_2$.

(*iv*) *The relation* ρ *in the set of filter-ideal pairs is a reflexive and a symmetric relation.*

(*v*) *If* Γ *and* Δ *are filters and* $\Gamma\rho_C\Delta$, *then* $(\Gamma, \varnothing)\rho(\Delta, \varnothing)$.

(*vi*) *If* Γ *and* Δ *are ideals and* $\Gamma\rho_{\widehat{C}}\Delta$, *then* $(\varnothing, \Gamma)\rho(\varnothing, \Delta)$.

(*vii*) *If* Γ *is a filter,* Δ *is an ideal and* $\Gamma\rho_{\not\ll}\Delta$, *then* $(\Gamma, \varnothing)\rho(\varnothing, \Delta)$ *and* $(\varnothing, \Delta)\rho(\Gamma, \varnothing)$.

Proof. (i) The statement follows from axiom (CO1) and (C).

(ii) The statement follows from axiom ($\widehat{C}\widehat{O}1$) and ($\widehat{C}$).

(iii) Suppose that Γ is a filter-ideal pair and that for some $x \in \Gamma_1$ and $y \in \Gamma_2$ we have $x \ll y$. Then by axiom ($\ll\leq 1$) we have $x \leq y$, which implies that $y \in \Gamma_1$. This contradicts the fact that $\Gamma_1 \cap \Gamma_2 \neq \varnothing$.

(iv) follows from (i),(ii) and (iii).

(v), (vi) and (vii) follow just from the definition of the ρ-relation between filter-ideal pairs. ∎

LEMMA 20. (*i*) *If* xCy, *then there exist filter-ideal pairs* Γ, Δ *such that* $\Gamma\rho\Delta$ *and* $x \in \Gamma_1$ *and* $y \in \Delta_1$.

(ii) If $x\widehat{C}y$, then there exist filter-ideal pairs Γ, Δ such that $\Gamma\rho\Delta$ and $x \in \Gamma_2$ and $y \in \Delta_2$.

(iii) If $x \not\ll y$, then there exists a filter-ideal pairs Γ, Δ such that $\Gamma\rho\Delta$, $x \in \Gamma_1$ and $y \in \Delta_2$.

Proof. (i) Let xCy. Then by Lemma 10 we have xOx and yOy, so by Lemma 13 $[x)$ and $[y)$ are filters. We shall show that $[x)\rho_C[y)$. Let $x' \in [x)$ and $y' \in [y)$. Then $x \leq x'$ and $y \leq y'$ and by xCy this implies by Lemma 10 that $x'Cy'$. Now by Lemma 19 we have $([x), \varnothing)\rho([y), \varnothing)$ which proves the statement.

(ii) The proof is similar (dual) to that of (i).

(iii) Let $x \not\ll y$. Then by axioms ($\ll$ O) and ($\ll \widehat{O}$) we obtain xOx and $y\widehat{O}y$ and by Lemma 13 $[x)$ is a filter and $(y]$ is an ideal. We will show that $[x)\rho_{\not\ll}(y]$. Let $x \leq x'$ and $y' \leq y$. Since $x \not\ll y$, we have by Lemma 10 that $x' \not\ll y'$ which proves $[x)\rho_{\not\ll}(y]$. Now by Lemma 19 we obtain $([x), \varnothing)\rho(\varnothing, (y])$. ■

LEMMA 21. **Point extension Lemma for filter-ideal pairs.** *Let Γ, Δ be a filter-ideal pairs and let $\Gamma\rho\Delta$. Then for any $x \in W$: either* (1) $\Delta_1 \cup [x)$ *is a filter and* $\Gamma\rho(\Delta_1 \cup [x), \Delta_2)$ *or* (2) $\Delta_2 \cup (x]$ *is an ideal and* $\Gamma\rho(\Delta_1, \Delta_2 \cup (x])$.

Proof. Suppose $\Gamma\rho\Delta$ and that we have $\neg(1)$ and $\neg(2)$. Due to the assumption $\Gamma\rho\Delta$ we obtain that $\neg(1)$ is equivalent to the disjunction of the following conditions:

$\neg(1) \equiv$ ($\Delta_1 \cup [x)$ is a not a filter) or ($\Gamma_1\overline{\rho_C}(\Delta_1 \cup [x))$ or ($(\Delta_1 \cup [x))\overline{\rho_{\not\ll}}\Gamma_2$).

It is easy to see that ($\Gamma_1\overline{\rho_C}(\Delta_1 \cup [x))$ is equivalent to $(\exists z_1 \in \Gamma_1)(z_1\overline{C}x)$.

Similarly $(\Delta_1 \cup [x))\overline{\rho_{\not\ll}}\Gamma_2)$ is equivalent to $(\exists t_1 \in \Gamma_2)(x \ll t_1)$. Having in mind these equivalencies and Lemma 13 (v) we obtain that $\neg(1)$ is equivalent to the following disjunction:

$\neg(1) \equiv$ (11) $x\overline{O}x$ or (12) $(\exists y_1 \in \Delta_1)(x\overline{O}y_1)$ or (13) $(\exists z_1 \in \Gamma_1)(z_1\overline{C}x)$ or (14) $(\exists t_1 \in \Gamma_2)(x \ll t_1)$.

In a similar way we can see that $\neg(2)$ is equivalent to the following disjunction:

$\neg(2) \equiv$ (21) $x\overline{\widehat{O}}x$ or (22) $(\exists y_2 \in \Delta_2)(x\overline{\widehat{O}}y_2)$ or (23) $(\exists z_2 \in \Gamma_1)(z_2 \ll x)$ or (24) $(\exists t_2 \in \Gamma_2)(t_2\overline{\widehat{C}}x)$.

We have to combine all conditions (1i) with (2j) for $i, j = 1, 2, 3, 4$ and in all 16 cases to obtain a contradiction.

Case (11)(21): $x\overline{O}x$ and $x\overline{\widehat{O}}x$ – this contradicts axiom $(O\overline{\widehat{O}})$.

Case (11)(22): $x\overline{O}x$ and $(\exists y_2 \in \Delta_2)(x\overline{\widehat{O}}y_2)$. From $x\overline{O}x$ we get by $(\overline{O} \leq)$ that $x \leq y_2$. From here and $y_2 \in \Delta_2$ we obtain $x \in \Delta_2$, because Δ_2 is an ideal. Also from $x, y_2 \in \Delta_2$ we obtain $x\widehat{O}y_2$ – a contradiction.

In a similar way we can treat the **cases (11)(23)** and **(11)(24)** and reasoning by duality – the cases **(12)(21), (13)(21), (14)(21)**.

Case (12)(22): $(\exists y_1 \in \Delta_1)(x\overline{O}y_1)$, $(\exists y_2 \in \Delta_2)(x\overline{\widehat{O}}y_2)$. From $x\overline{O}y_1$ and $x\overline{\widehat{O}}y_2$ we get by axiom $(\leq O\widehat{O})$ that $y_1 \leq y_2$. From this and $y_1 \in \Delta_1$ we obtain $y_2 \in \Delta_1$. Since $y_2 \in \Delta_2$ we obtain that $\Delta_1 \cap \Delta_2 \neq \varnothing$ - a contradiction.

Case (12)(23): $(\exists y_1 \in \Delta_1)(x\overline{O}y_1)$, $(\exists z_2 \in \Gamma_1)(z_2 \ll x)$. From $y_1 \in \Delta_1$ and $z_2 \in \Gamma_1$ we get y_1Cz_2. From here and $z_2 \ll x$ we obtain xOy_1 which contradicts $x\overline{O}y_1$.

Case (12)(24) $(\exists y_1 \in \Delta_1)(x\overline{O}y_1)$, $(\exists t_2 \in \Gamma_2)(t_2\overline{\widehat{C}}x)$. From $\Gamma\rho\Delta$ we get $\Delta_1\rho_{\not\ll}\Gamma_2$ and from here that $y_1 \not\ll t_2$. From $x\overline{O}y_1$ and $t_2\overline{\widehat{C}}x$, by $(\widehat{C})$ and $(\ll \widehat{C}O)$ we obtain $y_1 \ll t_2$ - a contradiction.

Case (13)(22): $(\exists z_1 \in \Gamma_1)(z_1\overline{C}x)$, $(\exists y_2 \in \Delta_2)(x\overline{\widehat{O}}y_2)$. From $\Gamma\rho\Delta$ we get $\Gamma_1\rho_{\not\ll}\Delta_2$ and from here – $z_1 \not\ll y_2$. From $z_1\overline{C}x$ and $x\overline{\widehat{O}}y_2$ we obtain $z_1 \ll y_2$ – a contradiction.

Case (13)(23): $(\exists z_1 \in \Gamma_1)(z_1\overline{C}x)$, $(\exists z_2 \in \Gamma_1)(z_2 \ll x)$. From $z_1 \in \Gamma_1$ and $z_2 \in \Gamma_1$ we get z_1Oz_2. $z_2 \ll x$ implies $z_2 \leq x$. Conditions z_1Oz_2 and $z_2 \leq x$ imply z_1Ox which implies z_1Cx - a contradiction.

Case (13)(24): $(\exists z_1 \in \Gamma_1)(z_1\overline{C}x)$, $(\exists t_2 \in \Gamma_2)(t_2\overline{\widehat{C}}x)$. From $z_1 \in \Gamma_1$ and $t_2 \in \Gamma_2$ we get by Lemma 19 (iii) that $z_1 \not\ll t_2$. From $z_1\overline{C}x$ and $t_2\overline{\widehat{C}}x$ we obtain by $(CO1)$ and $(\ll \widehat{C}O)$ that $z_1 \ll t_2$ – a contradiction.

Case (14)(22) $(\exists t_1 \in \Gamma_2)(x \ll t_1)$, $(\exists y_2 \in \Delta_2)(x\overline{\widehat{O}}y_2)$. From $t_1 \in \Gamma_2$ and $y_2 \in \Delta_2$ we get $t_1\widehat{C}y_2$. This with $x \ll t_1$ implies $x\widehat{O}y_2$ which contradicts $x\overline{\widehat{O}}y_2$.

Case (14)(23): $(\exists t_1 \in \Gamma_2)(x \ll t_1)$, $(\exists z_2 \in \Gamma_1)(z_2 \ll x)$. From $x \ll t_1$ and $z_2 \ll x$ we obtain $z_2 \leq t_1$ and consequently – $t_1 \in \Gamma_1$. This contradicts the fact that $\Gamma_1 \cap \Gamma_2 = \varnothing$.

Case (14)(24): $(\exists t_1 \in \Gamma_2)(x \ll t_1)$, $(\exists t_2 \in \Gamma_2)(t_2\overline{\widehat{C}}x)$. From $t_1, t_2 \in \Gamma_2$ we obtain $t_2\widehat{C}t_1$. From $x \ll t_1$ and $t_2\overline{\widehat{C}}x$ we obtain $t_2\overline{\widehat{C}}t_1$ - a contradiction. ■

LEMMA 22. **ρ-extension Lemma.** *Let Γ_0, Δ_0 be filter-ideal pairs and let $\Gamma_0\rho\Delta_0$. Then Γ_0 and Δ_0 can be extended correspondingly into complete pairs Γ and Δ such that $\Gamma\rho\Delta$.*

Proof. Let $\Gamma_0\rho\Delta_0$. By an application of the Zorn Lemma and Lemma 21 we can find a complete pair Δ such that $\Delta_0 \subseteq \Delta$ and $\Gamma_0\rho\Delta$. By the symmetry of ρ we obtain $\Delta\rho\Gamma_0$. Then in the same way we can find a complete pair Γ such that $\Gamma_0 \subseteq \Gamma$ and $\Delta\rho\Gamma$. By symmetry of ρ we obtain $\Gamma\rho\Delta$ and the proof is finished. ■

DEFINITION 23. Let $\underline{W}$ be a mereotopological structure. We define the following relation R in the set $GF(\underline{W})$ of good filters:

$\Gamma R\Delta$ iff $(\Gamma, -\Gamma)\rho(\Delta, -\Delta)$ where $-\Gamma = W \setminus \Gamma$ and $-\Delta = W \setminus \Delta$.

The relational system $(GF(\underline{W}), R)$ will be called the canonical system of $\underline{W}$.

LEMMA 24. *If Γ, Δ are complete pairs then $\Gamma\rho\Delta$ iff $\Gamma_1 R\Delta_1$. R is a reflexive and symmetric relation.*

Proof. The proof follows from the definition of R and the fact that for a complete pair Γ we have $\Gamma_2 = -\Gamma_1$. ■

LEMMA 25. **Good-filter characterization of C, $\widehat{C}$ and $\ll$.** *Let $\underline{W}$ be a mereotopological structure and let $GF(\underline{W})$ be the set of good filters of $\underline{W}$. Then for any $x, y \in W$ we have:*

(i) xCy iff $(\exists \Gamma, \Delta \in GF(\underline{W}))(\Gamma R \Delta$, $x \in \Gamma$ and $y \in \Delta)$.

(ii) $x\widehat{C}y$ iff $(\exists \Gamma, \Delta \in GF(\underline{W}))(\Gamma R \Delta$, $x \notin \Gamma$ and $y \notin \Delta)$.

(iii) $x \not\ll y$ iff $(\exists \Gamma, \Delta \in GF(\underline{W}))(\Gamma R \Delta$, $x \in \Gamma$ and $y \notin \Delta)$.

Proof. (i) $(\rightarrow)$ Suppose xCy. Then by Lemma 20 (i) there exist filter-ideal pairs Γ', Δ' such that $\Gamma' \rho \Delta'$, $x \in \Gamma'_1$ and $y \in \Delta'_1$. Then by the ρ-extension Lemma 22 we can extend Γ' and Δ' into complete pairs Γ'' and Δ'' such that $\Gamma'' \rho \Delta''$. Let $\Gamma = \Gamma''_1$, $\Delta = \Delta''_1$. Then we have $x \in \Gamma$, $y \in \Delta$ and by Lemma 24 that $\Gamma R \Delta$.

$(\leftarrow)$ Let $\Gamma R \Delta$, $x \in \Gamma$ and $y \in \Delta$. Then by the definition of R we have $(\Gamma, -\Gamma)\rho(\Delta, -\Delta)$. From here we obtain $\Gamma \rho_C \Delta$ which implies xCy.

(ii) The proof of (ii) is similar (dual) to that of (i).

(iii) $(\rightarrow)$ Suppose $x \not\ll y$. Then by lemma 20 (iii) there exist filter-ideal pairs Γ', Δ' such that $\Gamma' \rho \Delta'$, $x \in \Gamma'_1$ and $y \in \Delta'_2$. Then by the ρ-extension Lemma 22 we can extend Γ' and Δ' into complete pairs Γ'' and Δ'' such that $\Gamma'' \rho \Delta''$. Let $\Gamma = \Gamma''_1$, $\Delta = \Delta''_1$. Then we have $x \in \Gamma$, $y \notin \Delta$ (because $y \in \Delta''_2 = -\Delta''_1 = -\Delta$) and by Lemma 24 that $\Gamma R \Delta$.

$(\leftarrow)$ The proof is similar to the corresponding proof of (i). ■

Now we are ready to prove a representation theorem for mereotopological structures. To each mereotopological structure $\underline{W}$ we associate its canonical system $(GF(\underline{W}), R)$. Since R is a reflexive and symmetric relation in $GF(\underline{W})$, then by the construction of non-topological example of contact algebra in Examples 2 (2) we associate to $(GF(\underline{W}), R)$ a contact algebra consisting of all subsets of $GF(\underline{W})$ with the standard Boolean operations and a contact C_R between any subsets $a, b \subseteq GF(\underline{W})$ defined by: aC_Rb iff $(\exists \Gamma \in a, \exists \Delta \in b)(\Gamma R \Delta)$.

THEOREM 26. **Representation Theorem for mereotopological structures.** *Let $\underline{W}$ be a mereotopological structure, $(GF(\underline{W}), R)$ be the corresponding canonical structure and $(B(GF(\underline{W})), C_R)$ be the contact algebra over $(GF(\underline{W}), R)$. For $x \in W$ define $h(x) = \{\Gamma \in GF(\underline{W} : x \in \Gamma\}$. Then h is an isomorphic embedding of $\underline{W}$ into the contact algebra $(B(GF(\underline{W})), C_R)$.*

Proof. The proof follows from the following equivalencies.

- $x \leq y$ iff (by Lemma 17 (i)) $(\forall \Gamma \in GF(W))(x \in \Gamma \rightarrow y \in \Gamma)$ iff $(\forall \Gamma \in GF(W))(\Gamma \in h(x) \rightarrow \Gamma \in h(y))$ iff $h(x) \subseteq h(y)$ iff $h(x) \leq h(y)$.
- xOy iff (by Lemma 17 (ii) $(\exists \Gamma \in GF(W))(x \in \Gamma$ and $y \in \Gamma)$ iff $(\exists \Gamma \in GF(W))(\Gamma \in h(x)$ and $\Gamma \in h(y))$ iff $h(x) \cap h(y) \neq \varnothing$ iff $h(x)Oh(y)$.
- $x\widehat{O}y$ iff (by Lemma 17 (iii) $(\exists \Gamma \in GF(W))(x \notin \Gamma$ and $y \notin \Gamma)$ iff $(\exists \Gamma \in GF(W))(\Gamma \notin h(x)$ and $\Gamma \notin h(y))$ iff $h(x) \cup h(y) \neq GF(W)$ iff $h(x)\widehat{O}h(y)$.
- xCy iff (by Lemma 25 (i)) $(\exists \Gamma, \Delta \in GF(\underline{W}))(\Gamma R \Delta$, $x \in \Gamma$ and $y \in \Delta)$ iff $(\exists \Gamma, \Delta \in GF(\underline{W}))(\Gamma R \Delta$, $\Gamma \in h(x)$ and $\Delta \in h(y))$ iff $h(x)C_Rh(y)$.

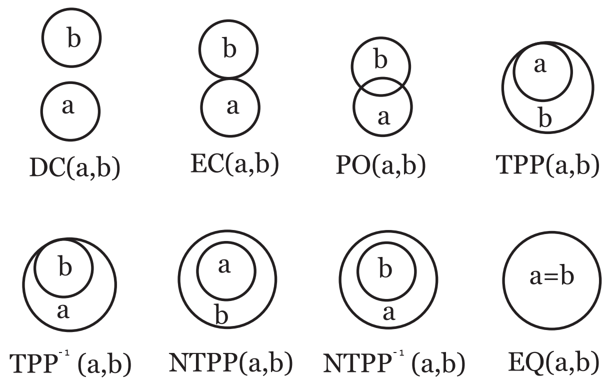
- $x\widehat{C}y$ iff (by Lemma 25 (ii)) $(\exists\Gamma,\Delta \in GF(\underline{W}))(\Gamma R\Delta,\ x \notin \Gamma)$ and $y \notin \Delta$ iff $(\exists\Gamma,\Delta \in GF(\underline{W}))(\Gamma R\Delta,\ \Gamma \notin h(x)$ and $\Delta \notin h(y))$ iff $-h(x)C_R - h(y)$ iff $h(x)\widehat{C}h(y)$.
- $x \not\ll y$ iff (by Lemma 25 (iii)) $(\exists\Gamma,\Delta \in GF(\underline{W}))(\Gamma R\Delta,\ x \in \Gamma$ and $y \notin \Delta)$ iff $(\exists\Gamma,\Delta \in GF(\underline{W}))(\Gamma R\Delta, \Gamma \in h(x)$ and $\Delta \notin h(y))$ iff $h(x)C_R{-}h(y)$ iff $h(x) \not\ll h(y)$. ■

COROLLARY 27. *Every mereotopological structure is completely standard.*

Proof. By Theorem 26 every mereotopological structure $\underline{W}$ is a standard one and by Lemma 8 $\underline{W}$ is also a completely standard. ■

REMARK 28. Note that Theorem 26 generalizes considerably Theorem 4 from [12] and Corollary 27 extends the topological representation theory of contact algebras by regular closed sets from [10]. If we consider the axiomatic definition of mereotopological structures as their *point-free* formulation, then the representation process can be considered as the *Whiteheadian process* of defining points. The first kind of points are the *good filters*, but they are not enough, because they allow only a non-topological representation in which regions are arbitrary sets and the mereotopological relations between them are defined by a binary relation between points. The second kind of points are introduced in the second phase of the representation, where we apply the topological representation theorem (Theorem 3) to the obtained discrete contact algebra. The definition and the theory of this second kind of points (called clans) can be found in [10]. Note also that the first phase of the representation introduces not only the first kind of points, but also extends the mereotopological structures with the Boolean operations between regions, which are necessary for introducing the second kind of points.

1.4 RCC-8 and mereotopological structures



RCC-8 relations

One of the most popular systems of topological relations in the community of QSR is RCC-8. Probably this was one of the main motivations this system to be taken by Lutz and Wolter as a semantical base of the modal logic of topological relations [17]. The system RCC-8 was introduced for the first time by Egenhofer and Franzosa in [13]. It consists of 8 JEPD relations between non-empty regular closed subsets of arbitrary topological space. Having in mind the topological representation of contact algebras, it was given in [25] an equivalent definition of RCC-8 in the language of contact algebras:

DEFINITION 29. The system **RCC-8.**

- **disconnected** – $\mathbf{DC}(a,b)$**:** $a\overline{C}b$,
- **external contact** – $\mathbf{EC}(a,b)$**:** aCb and $a\overline{O}b$,
- **partial overlap** – $\mathbf{PO}(a,b)$**:** aOb and $a \not\leq b$ and $b \not\leq a$,
- **tangential proper part** – $\mathbf{TPP}(a,b)$**:** $a \leq b$ and $a \not\ll b$ and $b \not\leq a$,
- **tangential proper part**$^{-1}$ – $\mathbf{TPP}^{-1}(a,b)$**:** $b \leq a$ and $b \not\ll a$ and $a \not\leq b$,
- **nontangential proper part** $\mathbf{NTPP}(a,b)$**:** $a \ll b$ and $a \neq b$,
- **nontangential proper part**$^{-1}$ – $\mathbf{NTPP}^{-1}(a,b)$**:** $b \ll a$ and $a \neq b$,
- **equal** – $\mathbf{EQ}(a,b)$**:** $a = b$.

Looking at the above definitions we see that they can be repeated in the language of mereotopological structures. The following lemma represents some relationships between RCC-8 and mereotopological structures.

LEMMA 30. *Let* $\underline{W} = (W, \leq, O, \widehat{O}, C, \widehat{C}, \ll)$ *be a mereotopological structure and let* $W^- = \{a \in W : aOa\}$*. Then:*

(i) The system of relations in W^- *given as in Definition 29 represents an equivalent definition of RCC-8 relations. So RCC-8 is definable in the system of mereotopological relations.*

(ii) The following equivalencies are true in W^-*:*

$a \leq b$ *iff* $TPP(a,b) \vee NTPP(a,b) \vee a = b$,

$a\overline{O}b$ *iff* $DC(a,b) \vee EC(a,b)$

aCb *iff* $\neg DC(a,b)$,

Hence the relations $\leq, \geq, O$ *and* C *are definable in RCC-8.*

(iii) The relation $\widehat{O}$ *is not definable in RCC-8. Hence the system of mereotopological structures is more rich than RCC-8.*

Proof. (i) follows from the topological representation of mereotopological structures and Definition 29. (ii) follows from (i) and the axioms of mereotopological relations. (iii) follows from a result in [11] where a system, called RCC-10 is introduced as an extension of RCC-8. The definitions of the new relations in RCC-10 are given by means of the relation $\widehat{O}$. It follows from this fact that the relation $\widehat{O}$ is not definable in RCC-8. ■

2 A modal logic for mereotopological structures

In this section we introduce a poly-modal logic based on mereotopological structures, denoted by MTML (Mereo-Topological Modal Logic). MTML has the following modal box operators: $[\leq], [\geq], [\ll], [\gg], [O], [\widehat{O}], [C], [\widehat{C}]$, $[U]$, where $[U]$ is the universal modality. The corresponding diamond modality is denoted by $\langle R\rangle$ and defined as $\neg[R]\neg$. We adopt standard notations for Boolean connectives. The semantics of this language is the Kripke semantic over mereotopological structures. If $\underline{W}$ is a mereotopological structure and v is a valuation of the propositional variables in W, then the pair $M = (\underline{W}, v)$ is called, as usual, a model over $\underline{W}$. The fact that a formula A is true (false) at a point $x \in W$ will be denoted by $v(x, A) = 1$ ($v(x, A) = 0$). We adopt the standard semantical definitions of truth of a formula in a model, in a Kripke structure , etc. Let us note that all conditions of mereotopological structure except (≤ 0) are modally definable in this language by Sahlqvist formulas which then can be taken as axioms of the corresponding axiomatic system. So, in order to obtain a complete axiomatization of MTML we introduce another, non-standard semantics, which consists of a class of relational structures in which the non-definable axiom (≤ 0) is replaced by several modally definable consequences. This class admits an easy and straightforward modal axiomatization by means of generated canonical models. By using p-morphism techniques, we prove that generalized models of MTML are equivalent to the standard ones, which yields the completeness with respect to the standard semantics of the logic.

Comparing MTML and the modal logic of topological relations introduced by Lutz and Wolter in [17] (LW-logic), we can see that, on the base of Lemma 30, our modalities $[\leq], [\geq], [O]$ and $[C]$ are definable in LW-logic while the modality $[\widehat{O}]$ is not definable. Conversely, all basic modalities of LW-logic are not definable in MTML. If however we extend the language of MTML including modalities corresponding to Boolean combinations of the basic relations, then all modalities of LW-logic will be definable in this extended version of MTML, which will imply its undecidability.

2.1 Generalized mereotopological structures and the bulldozer construction

DEFINITION 31. A generalized mereotopological structure is a generalization of the notion of a mereotopological structure by dropping the axiom (≤ 0) and by adding the following additional axioms:

$(=_1)$ $a\overline{O}a \wedge\ b \leq a \to a = b$, $(=_2)$ $a\overline{\widehat{O}}a \wedge a \leq b \to a = b$,

$(=_3)$ $a\overline{O}c \wedge b\overline{\widehat{O}}c \wedge b \leq a \to a = b$.

It can easily be seen that the above three conditions hold in mereotopological structures, so we have the following lemma.

LEMMA 32. *Each mereotopological structure is a generalized mereotopological structure.*

Now we shall show that each generalized mereotopological structure is a p-morphic image of a mereotopological structure. The construction is similar to the given one in [24] for a similar logic and is an adaptation of the Segerberg's bulldozer construction from [20]. To this end we first introduce in a given generalized mereotopological structure $\underline{W}$ the following equivalence relation. For $x, y \in W$, $x \equiv y$ iff $x \leq y$ and $y \leq x$. We denote by $\equiv(x) = \{y : x \equiv y\}$ the equivalence class generated by x and call such sets clusters. If $\equiv(x) = \{x\}$, then $\equiv(x)$ is called degenerated cluster. The following lemma states some easy properties of degenerated clusters.

LEMMA 33. *Let $\underline{W}$ be a generalized mereotopological structure. Then:*
(*i*) *If $x\overline{O}x$, then $\equiv(x)$ is a degenerated cluster.*
(*ii*) *If $x\overline{\widehat{O}}x$, then $\equiv(x)$ is a degenerated cluster.*
(*iii*) *If $x\overline{O}z$ and $x\overline{\widehat{O}}z$, then $\equiv(x)$ is a degenerate cluster.*

Proof. We will give a proof of (iii). Let $y \in \equiv(x)$. Then we have $x \leq y$ and $y \leq x$. From $x\overline{\widehat{O}}z$ and $x \leq y$ we get $y\overline{\widehat{O}}z$. Then $x\overline{O}z$, $y\overline{\widehat{O}}z$ and $y \leq x$ imply by $(=_3)$ that $x = y$, which shows that $\equiv(x)$ is a degenerate cluster. In a similar way, making use of the conditions $(=_1)$ and $(=_2)$, one can prove (i) and (ii). ■

DEFINITION 34. Let $\underline{W} = (W, \leq, O, \widehat{O}, \ll, C, \widehat{C})$ be a generalized mereotopological structure. We say that the structure $\underline{W}' = (W', \leq', O', \widehat{O}', \ll', C', \widehat{C}')$ is obtained from the structure $\underline{W}$ by the **bulldozer construction** if the following constructions hold.

Let $Z = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$ be the set of integers and ∞ be a symbol such that $\infty \notin W \cup Z$. For $f \in Z$ and $x \in W$ define

$$f(x) = \begin{cases} (x, \infty) & \text{if } \equiv(x) \text{ is a degenerate cluster} \\ (x, f) & \text{otherwise.} \end{cases}$$

Define $W' = \{f(x) : x \in W, f \in Z\}$. For $R \in \{O, \widehat{O}, C, \widehat{C}\}$ define $f(x)R'g(y)$ iff xRy. For $\leq'$ and $\ll'$ we have the following definitions:

$$f(x) \leq' g(y) \leftrightarrow \begin{cases} x \leq y & \text{if } x \not\equiv y \text{ or } f(x) = (x, \infty) \\ f < g \text{ or } (f = g \text{ and } x = y) & \text{otherwise.} \end{cases}$$

$f(x) \geq' g(y)$ iff $g(y) \leq' f(x)$, $f(x) \ll' g(y)$ iff $x \ll y$ and $f(x) \leq' g(y)$, and $f(x) \gg' g(y)$ iff $g(y) \ll' f(x)$. Define the mapping $P : W' \to W$ as follows: $P(f(x)) = x$, for every $f(x) \in W'$.

LEMMA 35. **Bulldozer Lemma.** *Let $\underline{W}$ be a generalized mereotopological structure and let $\underline{W}'$ be obtained from $\underline{W}$ by the bulldozer construction. Then:*
(*i*) *$\underline{W}'$ is a mereotopological structure.*
(*ii*) *The mapping P is a p-morphism from $\underline{W}'$ onto $\underline{W}$.*

Proof. The proof that P is a p-morphism from $\underline{W}'$ onto $\underline{W}$ is straightforward. The proof that $\underline{W}'$ is a mereotopological structure is long because requires verification of a great number of axioms. For the most of the axioms this is a quite easy exercise. We will illustrate this giving several proofs only for the more difficult axioms. Note that for all relations R we have the following: if $f(x)R'g(y)$, then xRy, for all $x, y \in W$, and we will use this without explicit reference.

• **Axiom** (≤ 0) $\quad f(x) \leq' g(y)$ and $g(y) \leq' f(x) \to f(x) = g(y)$.

Suppose $f(x) \leq' g(y)$. Then $x \leq y$ and $y \leq x$ and hence $x \equiv y$ and $\equiv(x) = \equiv(y)$.

Case 1: $\equiv(x)$ is a degenerate cluster. Then $\equiv(x) = \{x\}$, $\equiv(y) = \{y\}$ and consequently $x = y$. We have in this case $f(x) = (x, \infty)$, $g(y) = (y, \infty)$ and hence $f(x) = g(y)$.

Case 2: $\equiv(x)$ is not a degenerate cluster. Then we have $f(x) = (x, f)$ and $g(y) = (y, g)$ and we are in the second case of the definition of $\leq'$. Then we have: $(f < g$ or $f = g \& x = y)$ and $(g < f$ or $g = f \& y = x)$. This implies $f = g$ and $x = y$ which again gives $f(x) = g(y)$.

• **Axiom** (≤ 2)**:** Transitivity of $\leq'$. Suppose $f(x) \leq' g(y)$, $g(y) \leq' h(z)$. then we have $x \leq y$ and $y \leq z$ which implies $x \leq z$. We have to show that $f(x) \leq' h(z)$.

Case 1: $\equiv(x)$ is a degenerate cluster or $x \not\equiv z$. Since $x \leq z$ we obtain $f(x) \leq' h(z)$.

Case 2: $\equiv(x)$ is a not a degenerate cluster and $x \equiv z$. Then we obtain $x \leq z$ and $z \leq x$. Then from $z \leq x$ and $x \leq y$ we obtain $z \leq y$. From $y \leq z$ and $z \leq y$ we get $\equiv(y) = \equiv(z)$, and hence $\equiv(x) = \equiv(y) = \equiv(z)$. From here we obtain that $\equiv(y)$ and $\equiv(z)$ are not degenerate clusters. So for $f(x) \leq' g(y)$ and $g(y) \leq' h(z)$ we are in the second case of the definition of $\leq'$. This yields: $(f < g$ or $f = g \& x = y)$ and $(g < h$ or $g = h \& y = z)$. From here we obtain $(f < h$ or $f = h \& x = z)$ which gives $f(x) \leq' h(z)$.

• **Axiom** $(\leq O\widehat{O})$**:** $h(z)\overline{O'}f(x)$ and $h(z)\overline{\widehat{O}'}g(y) \to f(x) \leq' g(y)$.

Suppose $h(z)\overline{O'}f(x)$ and $h(z)\overline{\widehat{O}'}g(y)$ Then we have $z\overline{O}x$ and $z\overline{\widehat{O}}y$ which implies $x \leq y$.

Case 1: $x \not\equiv y$ or $\equiv(x)$ is a degenerate cluster. In this case we have (by $x \leq y$) that $f(x) \leq' g(y)$.

Case 2: $x \equiv y$ and $\equiv(x)$ is not a degenerate cluster. From $z\overline{\widehat{O}}y$ and $x \equiv y$ we obtain $z\overline{\widehat{O}}x$. Then from $z\overline{O}x$, $z\overline{\widehat{O}}x$ and Lemma 33 (iii) we get that $\equiv(x)$ is a degenerate cluster, which shows that this case is impossible.

• **Axiom** $(\ll C\widehat{O})$**:** $h(z)\overline{C'}f(x)$ and $h(z)\overline{\widehat{O}'}g(y) \to f(x) \ll' g(y)$.

Suppose $h(z)\overline{C'}f(x)$ and $h(z)\overline{\widehat{O}'}g(y)$. This implies $z\overline{C}x$ and $z\overline{\widehat{O}}y$, which yield $x \ll y$. Condition $h(z)\overline{C'}f(x)$ implies $h(z)\overline{O'}f(x)$. This, together with $h(z)\overline{\widehat{O}'}g(y)$ implies (as we have just proved) $f(x) \leq' g(y)$. Conditions $x \ll y$ and $f(x) \leq' g(y)$ imply $f(x) \ll' g(y)$.

• **Axiom** $(=_1)$ $f(x)\overline{O'}f(x)$ and $g(y) \leq' f(x) \to f(x) = g(y)$.

Suppose $f(x)\overline{O'}f(x)$ and $g(y) \leq' f(x)$. This implies $x\overline{O}x$ and $y \leq x$. From $x\overline{O}x$ we obtain $x \leq y$ which with $y \leq x$ implies $x \equiv y$ and hence $\equiv(x) = \equiv(y)$. Condition $x\overline{O}x$ implies by Lemma 33 (i) that $\equiv(x)$ is a degenerate cluster. Then also $\equiv (y)$ is a degenerate cluster and hence $\equiv (x) = \{x\}$ and $\equiv (y) = \{y\}$ and hence $x = y$. In this case we have $f(x) = (x, \infty)$ and $g(y) = (y, \infty)$. Consequently $f(x) = g(y)$.

We expect that the above examples will show the reader how to verify the remaining axioms of generalized mereotopological structures. ■

2.2 Axiomatization and completeness theorem

We adopt the following system of axiom schemes and rules for MTML. All axioms are just the Sahlqvist modal equivalents of the axioms of generalized mereotopological structures.

Axiom Schemes

$(Bool)$ All boolean tautologies
(K) $[R](A \Rightarrow B) \Rightarrow ([R]A \Rightarrow [R]B)$,
(A_0) $\langle\leq\rangle[\geq]A \Rightarrow A$, $\langle\geq\rangle[\leq]A \Rightarrow A$, $\langle\ll\rangle[\gg]A \Rightarrow A$, $\langle\gg\rangle[\ll]A \Rightarrow A$,
$[U]A \Rightarrow A, \langle U\rangle[U]A \Rightarrow A$, $[U]A \Rightarrow [U][U]A$, $[R]A \Rightarrow [U]A$,
$(A_{\leq 1})$ $[\leq]A \Rightarrow A$, $(A_{\leq 2})$ $[\leq]A \Rightarrow [\leq][\leq]A$, (A_{O1}) $\langle O\rangle[O]A \Rightarrow A$,
$(A_{\widehat{O}1})$ $\langle\widehat{O}\rangle[\widehat{O}]A \Rightarrow A$, $(A_{O\leq})$ $[O]A \Rightarrow [O][\leq]A$,
$(A_{\widehat{O}\leq})$ $[\widehat{O}]A \Rightarrow [\widehat{O}][\geq]A$, $(A_{O\widehat{O}})$ $([O]A \Rightarrow A) \vee ([\widehat{O}]B \Rightarrow B)$,
$(A_{\leq O\widehat{O}})$ $[O]A \wedge [\widehat{O}]B \wedge \langle U\rangle([\leq]C \wedge \neg A) \Rightarrow [U](B \vee C)$,
(A_C) $\langle C\rangle[C]A \Rightarrow A$, $(A_{\widehat{C}})$ $\langle\widehat{C}\rangle[\widehat{C}]A \Rightarrow A$,
(A_{CO1}) $[C]A \Rightarrow [O]A$, $(A_{\widehat{C}\widehat{O}1})$ $[\widehat{C}]A \Rightarrow [\widehat{O}]A$,
(A_{CO2}) $\langle C\rangle\top \wedge [O]A \Rightarrow A$, $(A_{\widehat{C}\widehat{O}2})$ $\langle\widehat{C}\rangle\top \wedge [\widehat{O}]A \Rightarrow A$,
$(A_{C\leq})$ $[C]A \Rightarrow [C][\leq]A$, $(A_{\widehat{C}\leq})$ $[\widehat{C}]A \Rightarrow [\widehat{C}][\geq]A$,
$(A_{\ll\leq 1})$ $[\leq]A \Rightarrow [\ll]A$, $(A_{\ll\leq 2})$ $[\ll]A \Rightarrow [\leq][\ll]A$,
$(A_{\ll\leq 3})$ $[\ll]A \Rightarrow [\ll][\leq]A$, $(A_{\ll O})$ $\neg A \wedge [O]A \wedge [\ll]B \Rightarrow [U]B$,
$(A_{\ll\widehat{O}})$ $\neg A \wedge [\widehat{O}]A \wedge [\gg]B \Rightarrow [U]B$,
$(A_{\ll CO})$ $[O]A \Rightarrow [C][\ll]A$, $(A_{\ll\widehat{C}\widehat{O}})$ $[\widehat{O}]A \Rightarrow [\widehat{C}][\gg]A$,
$(A_{\ll C\widehat{O}})$ $[C]A \wedge [\widehat{O}]B \wedge \langle U\rangle([\ll]C \wedge \neg A) \Rightarrow [U](B \vee C)$,
$(A_{\ll\widehat{C}O})$ $[O]A \wedge [\widehat{C}]B \wedge \langle U\rangle([\ll]C \wedge \neg A) \Rightarrow [U](B \vee C)$,
$(A_{=_1})$ $\langle\leq\rangle([O]A \wedge \neg A \wedge B) \Rightarrow B$, $(A_{=_2})$ $\langle\geq\rangle([\widehat{O}]A \wedge \neg A \wedge B) \Rightarrow B$,
$(A_{=_3})$ $\langle U\rangle(B \wedge \neg C \wedge \langle\leq\rangle(A \wedge C)) \Rightarrow \langle O\rangle A \vee \langle\widehat{O}\rangle B$.

Rules of inference:
Modus Ponens(MP) $A, A \Rightarrow B \vdash B$,
Necessitation (N) $A \vdash [R]A$ for $R \in \{\leq, \geq, \ll, \gg, O, \widehat{O}, C, \widehat{C}, U\}$.

THEOREM 36. **Completeness theorem for MTML.** *The following conditions are equivalent for any formula A of MTML:*
(i) A is a theorem of MTML,
(ii) A is true in all generalized mereotopological structures,
(iii) A is true in all mereotopological structures,
(iv) A is true in all standard and completely standard mereotopological structures.

Proof. The implications $(i) \to (ii) \to (iii) \to (iv)$ form the soundness part of the theorem and are straightforward. The implication $(iv) \to (iii)$ follows by Corollary 27. $(iii) \to (ii)$ is true by the Bulldozer Lemma 35. And finally the implication $(ii) \to (i)$ can be proved by using the standard techniques of generated canonical models (see [4, 5]). ■

2.3 Filtration

LEMMA 37. *MTML do not possess fmp with respect to its standard semantics.*

Proof. It is easy to see that the Grzegorczyk formula

$$[\leq]([\leq](p \Rightarrow [\leq]p) \Rightarrow p) \Rightarrow p$$

is true in all finite mereotopological structures (because they are finite partial orderings with respect to $\leq$) but that it is falsified in the generalized mereotopological structure $W = \{a, b\}$ in which the relations $\leq, O, \widehat{O}, C, \widehat{C}$ and $\ll$ coincide with W^2, which proves the lemma. ■

LEMMA 38. **Filtration Lemma for MTML.** *MTML admits filtration with respect to its nonstandard semantics and hence is decidable.*

Proof. The next definition presents the relevant constructions of the filtration.

DEFINITION 39. **Filtration for MTML**. Let $M = (\underline{W}, v)$ be a model over a generalized mereotopological structure and A_0 be a formula. Let Γ be the smallest set of formulas closed under sub-formulas, containing A_0 and satisfying the following closure conditions:

(Γ1) $\langle O \rangle \top$ and $\langle \widehat{O} \rangle \top$ are in Γ,

(Γ2) if $[R]A \in \Gamma$ for some $R \in \{O, \widehat{O}, \leq, \geq, \ll, \gg, C, \widehat{C}\}$, then $[R]A \in \Gamma$ for all $R \in \{O, \widehat{O}, \leq, \geq, \ll, \gg, C, \widehat{C}\}$.

We define an equivalence relation $\sim$ in W as follows:

$(\forall x, y \in W)(x \sim y \leftrightarrow (\forall A \in \Gamma)(v(x, A) = v(y, A)))$.

Further we define $|x| = \{y : x \sim y\}$ and $W' = \{|x| : x \in W\}$.

The valuation v' in W' is defined as follows: for $|x| \in W'$ and for propositional variable p we put $v'(|x|, p) = 1$ iff $v(x, p) = 1$.

We define the relational structure $\underline{W}' = (W', O', \widehat{O}', \leq', \geq', \ll', \gg', C', \widehat{C}')$ over $\underline{W}$ by specifying the relations $O', \widehat{O}', \leq', \geq', \ll', \gg', C', \widehat{C}'$ as follows. For any $|x|, |y| \in W'$ we define:

- $|x| \leq' |y|$ iff $(\forall [\leq]A \in \Gamma)$ $((v(x,[\leq]A) = 1 \rightarrow v(y,[\leq]A) = 1)$ & $(v(y,[\geq]A) = 1 \rightarrow v(x,[\geq]A) = 1)$ & $(v(x,[\ll]A) = 1 \rightarrow v(y,[\ll]A) = 1)$ & $(v(y,[\gg]A) = 1 \rightarrow v(x,[\gg]A) = 1)$ & $(v(y,[O]A) = 1 \rightarrow v(x,[O]A) = 1)$ & $(v(x,[\widehat{O}]A) = 1 \rightarrow v(y,[\widehat{O}]A) = 1)$ & $(v(y,[C]A) = 1 \rightarrow v(x,[C]A) = 1)$ & $(v(x,[\widehat{C}]A) = 1 \rightarrow v(y,[\widehat{C}]A) = 1)$ & $(v(x,\langle O\rangle\top) = 1 \rightarrow v(y,\langle O\rangle\top) = 1)$ & $(v(y,\langle\widehat{O}\rangle\top) = 1 \rightarrow v(x,\langle\widehat{O}\rangle\top) = 1))$,
- $|x| \geq' |y|$ iff $|y| \leq |x|$,
- $|x| \ll |y|$ iff $(\forall [\ll]A \in \Gamma)$ $((v(x,[\ll]A) = 1 \rightarrow v(y,[\leq]A) = 1)$ & $(v(y,[\gg]A) = 1 \rightarrow v(x,[\geq]A) = 1)$ & $(v(y,[O]A) = 1 \rightarrow v(x,[C]A) = 1)$ & $(v(x,[\widehat{O}]A) = 1 \rightarrow v(y,[\widehat{C}]A) = 1)$ & $(v(x,\langle O\rangle\top) = 1 \rightarrow v(y,\langle O\rangle\top) = 1)$ & $(v(y,\langle\widehat{O}\rangle\top) = 1 \rightarrow v(x,\langle\widehat{O}\rangle\top) = 1))$,
- $|x| \gg' |y|$ iff $|y| \ll' |x|$,
- $|x|O'|y|$ iff $(\forall [O]A \in \Gamma)$ $((v(x,[O]A) = 1 \rightarrow v(y,[\leq]A) = 1)$ & $(v(y,[O]A) = 1 \rightarrow v(x,[\leq]A) = 1)$ & $(v(x,\langle O\rangle\top) = 1$ & $v(y,\langle O\rangle\top) = 1))$,
- $|x|\widehat{O}'|y|$ iff $(\forall [\widehat{O}]A \in \Gamma)$ $((v(x,[\widehat{O}]A) = 1 \rightarrow v(y,[\geq]A) = 1)$ & $(v(y,[\widehat{O}]A) = 1 \rightarrow v(x,[\geq]A) = 1)$ & $(v(x,\langle\widehat{O}\rangle\top) = 1$ & $v(y,\langle\widehat{O}\rangle\top) = 1))$,
- $|x|C'|y|$ iff $(\forall [C]A \in \Gamma)$ $((v(x,[C]A) = 1 \rightarrow v(y,[\leq]A) = 1)$ & $(v(y,[C]A) = 1 \rightarrow v(x,[\leq]A) = 1)$ & $(v(x,[O]A) = 1 \rightarrow v(y,[\ll]A) = 1)$ & $(v(y,[O]A) = 1 \rightarrow v(x,[\ll]A) = 1)$ & $(v(x,\langle O\rangle\top) = 1$ & $v(y,\langle O\rangle\top) = 1))$,
- $|x|\widehat{C}'|y|$ iff $(\forall [\widehat{C}]A \in \Gamma)$ $((v(x,[\widehat{C}]A) = 1 \rightarrow v(y,[\geq]A) = 1)$ & $(v(y,[\widehat{C}]A) = 1 \rightarrow v(x,[\geq]A) = 1)$ & $(v(x,[\widehat{O}]A) = 1 \rightarrow v(y,[\gg]A) = 1)$ & $(v(y,[\widehat{O}]A) = 1 \rightarrow v(x,[\gg]A) = 1)$ & $(v(x,\langle\widehat{O}\rangle\top) = 1$ & $v(y,\langle\widehat{O}\rangle\top) = 1))$.

We have to prove two things. First that the new model $(\underline{W}', v')$ satisfies the two conditions of filtration for each relation R, namely for all $x, y \in W$

(F1) If xRy, then $|x|R'|y|$, and

(F2) If $|x|R'|y|$, then $(\forall [R]A \in \Gamma)(v(x,[R]A) = 1 \rightarrow v(y,A) = 1)$.

And second, to show that the new structure $\underline{W}'$ is a finite generalized mereotopological structure.

The finiteness of $\underline{W}'$ follows by the fact that Γ is a finite set – the closure conditions for Γ do not make it infinite.

The most tedious part of the proof is the verification of the conditions (F1) and (F2) – it is quite long but in each case easy. As an example we will verify the conditions (F1) and (F2) for the relation O'.

(F1,O) If xOy then $|x|O'|y|$.

Suppose xOy and let $[O]A \in \Gamma$. We have to verify the following conditions corresponding to the clauses of the definition of O':

(a) $v(x,[O]A) = 1 \to v(y,[\leq]A) = 1$,

(b) $v(y,[O]A) = 1 \to v(x,[\leq]A) = 1$,

(c) $v(x,\langle O\rangle\top) = 1$,

(d) $v(y,\langle O\rangle\top) = 1$.

Proof of (a). Suppose $v(x,[O]A) = 1$. To prove $v(y,[\leq]A) = 1$ suppose $y \leq z$. Then xOy and $y \leq z$ imply xOz and since $v(x,[O]A) = 1$ we obtain $v(z,A) = 1$. In a similar way we prove (b).

Proof of (c). From xOy we get xOx and since $v(x,\top) = 1$ we obtain $v(x,\langle O\rangle\top) = 1$. In the same way we verify (d).

Condition (F2) for O' can be verified rather easy. Suppose $|x|O'|y|$, $[O]A \in \Gamma$ and $v(x,[O]A) = 1$. By the first line of the definition of O' we obtain $v(y,[\leq]A) = 1$, Since $y \leq y$ we get $v(y,A) = 1$.

We left to the reader the verification of the conditions (F1) and (F2) for the other relations.

The verification of the axioms of generalized mereotopological structure is also quite long but in each case it is easy. We will demonstrate proofs only for some examples.

• **Axiom** (≤ 1) $|x| \leq' |x|$. By (≤ 1) we have $x \leq x$. Then by (F1) we obtain $|x| \leq' |x|$.

• **Axiom** (≤ 2) $|x| \leq' |y|$ and $|y| \leq' |z| \to |x| \leq' |z|$.

Suppose $|x| \leq' |y|$ and $|y| \leq' |z|$ and proceed to show $|x| \leq' |z|$. We have to verify the 10 clauses of the definition of $\leq'$ for $|x| \leq' |z|$. Let us demonstrate the clause for O:

$v(z,[O]A) = 1 \to v(x,[O]A) = 1$.

Suppose $v(z,[O]A) = 1$. Since $|y| \leq' |z|$ we get $v(y,[O]A) = 1$. This and $|x| \leq' |y|$ imply $v(x,[O]A) = 1$.

• **Axiom** $(O1)$ $|x|O'|y| \to |y|O'|x|$. The axiom follows from the fact that the definition of O' is symmetric with respect to its arguments.

• **Axiom** $(O2)$ $|x|O'|y| \to |x|O'|x|$.

Suppose $|x|O'|y|$ and proceed to verify $|x|O'|x|$. The first two conditions of the definition of O' for $|x|O'|x|$ are equal and easy to proof. The third condition $v(x,\langle O\rangle\top) = 1$ follows from the assumption $|x|O'|y|$. The forth condition is equal to the third one.

Most of the other axioms can be treated in a similar way. Since the axioms $(=_1)$, $(=_2)$ and $(=_3)$ present some difficulties we will consider one of them, say $(=_3)$ (the other two can be treated similarly).

• **Axiom** $(=_3)$ $|x|\overline{O'}|z|$, $|y|\overline{\widehat{O}'}|z|$ and $|y| \leq' |x| \to |x| = |y|$.

Suppose $|x|\overline{O'}|z|$, $|y|\overline{\widehat{O'}}|z|$ and $|y| \leq' |x|$. By (F1) we get $x\overline{O}z$ and $y\overline{\widehat{O}}z$. We shall show that $x = y$ which automatically implies $|x| = |y|$. Suppose that $x \neq y$. Then By axiom $(=_3)$ (and $x\overline{O}z$ and $y\overline{\widehat{O}}z$) we obtain $y \not\leq x$. Then by axiom $(\leq O\widehat{O})$ we obtain yOz or $x\widehat{O}z$. By (F1) we obtain $|y|O'|z|$ or $|x|\widehat{O}|z|$. We shall show that both alternatives yield a contradiction.

(a) $|y|O'|z|$ and $|y| \leq' |x|$ imply $|x|O'|z|$ which contradicts $|x|\overline{O'}|z|$.

(b) $|x|\widehat{O'}|z|$ and $|y| \leq' |x|$ imply $|y|\widehat{O'}|z|$ which contradicts $|y|\overline{\widehat{O'}}|z|$.

∎

3 Concluding remarks

We conclude the paper by formulating some open problems.

The first open problem concerns the completeness theorem of an extension of MTML over mereotopological structures satisfying the axiom of connectedness (see Remarks 9 (4)). The standard models of this extension are over connected topological spaces, for instance, models over R^n. Let us note that the techniques of the representation theorem for mereotopological structures, used in this paper, do not hold in the presence of this axiom. So one has to invent some new techniques. Another reasonable problem is to look for possible extensions of MTML with some new modalities, preserving decidability. And the last problem is the complexity of the satisfiability of MTML.

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