The variety of modal $\mathbf{FL}_{ew}$-algebras is generated by its finite simple members

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ABSTRACT. In this paper, we prove that the variety of modal $\mathbf{FL}_{ew}$-algebras is generated by its finite simple members. The result is obtained by showing that every *free* modal $\mathbf{FL}_{ew}$-algebra is semisimple and then showing that every variety generated by a simple modal $\mathbf{FL}_{ew}$-algebra is generated by a set of finite simple modal $\mathbf{FL}_{ew}$-algebras.

1 Introduction

In [7], the authors show that the variety of $\mathbf{FL}_{ew}$-algebras is generated by its finite simple members. The result is obtained by first showing that every *free* $\mathbf{FL}_{ew}$-algebra is semisimple and then showing that every variety generated by a simple $\mathbf{FL}_{ew}$-algebra is generated by a set of finite simple $\mathbf{FL}_{ew}$-algebras. To show the former, based on Grišin's idea in [4] authors introduced a sequent system SFL^{+}_{ew} such that

1. algebras for SFL^{+}_{ew} are exactly equal to $\mathbf{FL}_{ew}$-algebras,
2. cut elimination theorem holds for SFL^{+}_{ew}.

Then, using proof-theoretic properties of SFL^{+}_{ew}, the semisimplicity of free $\mathbf{FL}_{ew}$-algebras is obtained. Moreover, they show that the finite embeddability property holds for simple $\mathbf{FL}_{ew}$-algebras. Finally, they have the variety of $\mathbf{FL}_{ew}$-algebras is generated by its finite simple members.

In this paper, their proof works well also for the variety of modal $\mathbf{FL}_{ew}$-algebras with some modification. We assume a familiarity with the paper [7].

2 Modal $\mathbf{FL}_{ew}$-algebras

We give a precise definition of modal $\mathbf{FL}_{ew}$-algebras introduced by Ono [10], which is $\mathbf{FL}_{ew}$-algebras with S4-like modality. Note that an $\mathbf{FL}_{ew}$-algebra is a kind of residuated lattice. It is known that residuated lattices are algebraic semantics for substructural logics. For more information, see [6, 5].

An algebraic structure $\mathbf{A} = \langle A, \cdot, \to, \wedge, \vee, \Box, 0, 1\rangle$ is called a modal $\mathbf{FL}_{ew}$-algebra, if it satisfies the following:

(1)$\langle A, \wedge, \vee, 0, 1\rangle$ is a bounded lattice with the greatest element 1, the least element 0,
(2)$\langle A, \cdot, 1\rangle$ is a commutative monoid, and

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(3) $x \cdot y \leq z \Leftrightarrow y \leq x \to z$, for any $x, y, z \in A$,
(4) $\Box$ is a unary operation on A satisfying:

(a) $1 \leq \Box 1$, (b) $\Box x \cdot \Box y \leq \Box(x \cdot y)$, ($c$) $\Box x \leq x$, (d) $\Box x \leq \Box\Box x$,
(e) if $x \leq y$ then $\Box x \leq \Box y$ for any $x, y \in A$.

For simplicity, we write xy instead of $x \cdot y$.

DEFINITION 1 (Congruence filter). A subset F of A is a *congruence filter* (called simply, a filter) of $\mathbf{A}$ if it satisfies that
(1) $1 \in F$,
(2) If $x, x \to y \in F$ then $y \in F$, and
(3) If $x \in F$ then $\Box x \in F$.

It is easy to see that a congruence filter characterizes congruences in a modal $\mathbf{FL}_{ew}$-algebra $\mathbf{A}$. Suppose F is a congruence filter of $\mathbf{A}$. We define a binary relation θ on $\mathbf{A}$, putting $x\theta y$ if the terms $x \to y$ and $y \to x$ are both in F. Then θ is a congruence relation with respect to the operations of modal $\mathbf{FL}_{ew}$-algebra. Conversely, suppose θ is a congruence in a modal $\mathbf{FL}_{ew}$-algebra. Then $F_\theta = \{x \in \mathbf{A} : 1\theta x\}$ is a congruence filter of $\mathbf{A}$.

Next, suppose $x \in F$ and $x \leq y$, we have $1 \leq x \to y$. Thus $x \to y \in F$ by (1). By (2), we have $y \in F$. Thus, a filter F is a upward closed set, i.e., if $x \in F$ and $x \leq y$ then $y \in F$. We can also show that if $x, y \in F$ then $xy \in F$. Moreover it is easy to see that a nonempty subset F of a modal $\mathbf{FL}_{ew}$-algebra is a filter if and only if it satisfies: (a) if $x \in F$ and $x \leq y$ then $y \in F$, (b) if $x, y \in F$ then $xy \in F$, and (3).

Using this fact, we can have the following representation of the filter generated by a given nonempty subset S of a modal $\mathbf{FL}_{ew}$-algebra $\mathbf{A}$. Let H_S be $\{x \in \mathbf{A} : \Box s_1 \cdots \Box s_k \leq x \text{ for some } s_1, \cdots s_k \in S\}$. Then we have the following characterization of a filter generated a given set by a similar argument of [5, 12, 3].

LEMMA 2. *For each nonempty subset S of a modal $\mathbf{FL}_{ew}$-algebra $\mathbf{A}$, the filter generated by S is equal to H_S.*

When S is a singleton set $\{a\}$, the filter generated by $\{a\}$ is of the form $\{x \in \mathbf{A} : (\Box(a))^n \leq x \text{ for some } n \in \mathbb{N}\}$.

We say that a modal $\mathbf{FL}_{ew}$-algebra $\mathbf{A}$ is *semisimple* if it has a subdirect representation with simple factors. Let Φ be the set of all maximal filters of a modal $\mathbf{FL}_{ew}$-algebra $\mathbf{A}$. Define the radical $Rad_{\mathbf{A}}$ of $\mathbf{A}$ by $Rad_{\mathbf{A}} = \bigcap_{F \in \Phi} F$. Then, the following can be easily shown.

LEMMA 3. *For any modal $\mathbf{FL}_{ew}$-algebras, $\mathbf{A}$ is semisimple if and only if $Rad_{\mathbf{A}} = \{1\}$.*

Corresponding to Theorem 2.3 of [7], we can show the following result, which gives a sufficient and necessary condition for $x \in \mathbf{A}$ to be a member of $Rad_{\mathbf{A}}$.

PROPOSITION 4. *Let* $\mathbf{A}$ *be a modal* $\mathbf{FL}_{ew}$*-algebra. An element* $x \in \mathbf{A}$ *is in* $Rad_{\mathbf{A}}$ *if and only if for any* $n \geq 1$ *there exists* $m \in \mathbb{N}$ *such that* $(\Box\neg((\Box x)^n))^m = 0$*, where* $\neg x$ *stands for* $x \to 0$*.*

Proof. Assume that for any $n \geq 1$ there exists $m \in \mathbb{N}$ such that $(\Box\neg((\Box x)^n))^m = 0$. Suppose x is not in $Rad_{\mathbf{A}}$. Then there is a maximal filter F with $x \notin F$. Since F is maximal, there is a $k \geq 1$ such that $\neg((\Box x)^k) \in F$. Thus $\Box\neg((\Box x)^k) \in F$. By assumption, there is $m \geq 1$ for which $(\Box\neg((\Box x)^k))^m = 0$. This implies $0 \in F$, which contradicts the fact that F is proper.

Conversely, suppose that there exists $n \geq 1$ such that $(\Box\neg((\Box x)^n))^m \neq 0$ for any m. Then $(\Box\neg((\Box x)^n))^m > 0$ for any m. Put $z = \neg((\Box x)^n)$ and H_z a filter generated by $\{z\}$. Clearly, H_z is proper. By Zorn's lemma, there is a maximal filter G such that $H_z \subseteq G$. Now suppose $x \in G$, then $(\Box x)^n \in G$. However this is a contradiction, since $z \in G$. Therefore $x \notin G$ and so $x \notin Rad_{\mathbf{A}}$.

COROLLARY 5. *A modal* $\mathbf{FL}_{ew}$*-algebra* $\mathbf{A}$ *is semisimple if and only if for every* $x \in \mathbf{A}\backslash\{1\}$ *there is an* $n \geq 1$ *such that for any* $m \geq 1$*, we have* $(\Box\neg((\Box x)^n))^m \neq 0$*.*

3 Semisimplicity of free modal $\mathbf{FL}_{ew}$-algebras

In this section we show that every free modal $\mathbf{FL}_{ew}$-algebra is semisimple, using the sequent system $\Box FL^+_{ew}$ introduced below. Our proof proceeds similarly to Grišin [4], Kowalski and Ono [7].

Similarly to the sequent system SFL_{ew} introduced in [7], we introduce a sequent system, which we call $\Box FL^+_{ew}$ as follows. A sequent is of the form $\Gamma \Rightarrow \alpha$ where Γ is a possibly empty multiset of formulas:

1. initial sequents

   (1) $p, \Gamma \Rightarrow p$ where p is a propositional variable,
   (2) $0, \Gamma \Rightarrow \alpha$.

2. rules of inference

$$\frac{\Gamma \Rightarrow \alpha \quad \alpha, \Sigma \Rightarrow \theta}{\Gamma, \Sigma \Rightarrow \theta}\ (cut)$$

$$\frac{\alpha, \Gamma \Rightarrow \theta \quad \beta, \Gamma \Rightarrow \theta}{\alpha \vee \beta, \Gamma \Rightarrow \theta}\ (\vee \Rightarrow) \qquad \frac{\Gamma \Rightarrow \alpha}{\Gamma \Rightarrow \alpha \vee \beta}\ (\Rightarrow \vee 1) \quad \frac{\Gamma \Rightarrow \beta}{\Gamma \Rightarrow \alpha \vee \beta}\ (\Rightarrow \vee 2)$$

$$\frac{\Gamma \Rightarrow \alpha \quad \Gamma \Rightarrow \beta}{\Gamma \Rightarrow \alpha \wedge \beta}\ (\Rightarrow \wedge) \qquad \frac{\alpha, \Gamma \Rightarrow \theta}{\alpha \wedge \beta, \Gamma \Rightarrow \theta}\ (\wedge 1 \Rightarrow) \quad \frac{\beta, \Gamma \Rightarrow \theta}{\alpha \wedge \beta, \Gamma \Rightarrow \theta}\ (\wedge 2 \Rightarrow)$$

$$\frac{\Gamma \Rightarrow \alpha \quad \beta, \Sigma \Rightarrow \theta}{\Gamma, \alpha \to \beta, \Sigma \Rightarrow \theta} \ (\to\Rightarrow) \qquad \frac{\alpha, \Gamma \Rightarrow \beta}{\Gamma \Rightarrow \alpha \to \beta} \ (\Rightarrow\to)$$

$$\frac{\Box\Gamma \Rightarrow \alpha}{\Box\Gamma \Rightarrow \Box\alpha} \ (\Rightarrow \Box) \qquad \frac{\alpha, \Gamma \Rightarrow \beta}{\Box\alpha, \Gamma \Rightarrow \beta} \ (\Box \Rightarrow)$$

$$\frac{\Gamma_1 \Rightarrow \alpha_1, \cdots, \Gamma_m \Rightarrow \alpha_m}{\Gamma_1, \cdots, \Gamma_m \Rightarrow \alpha_1 * \cdots * \alpha_m} \ (\Rightarrow *) \qquad \frac{\Gamma, \alpha_1, \cdots, \alpha_m, \Delta \Rightarrow \theta}{\Gamma, \alpha_1 * \cdots * \alpha_m, \Delta \Rightarrow \theta} \ (* \Rightarrow)$$

Here, we assume that in each application of rules $(\Rightarrow *)$ and $(* \Rightarrow)$, none of α_i must be fusion formulas, i.e., formulas whose outermost logical connective is the fusion $*$. The reason why we need the system $\Box FL^+_{ew}$ will be explained by the following Lemma, which can be shown in the usual way. In [7], they proved the cut elimination holds for SFL^+_{ew} (the proof from [9] works with obvious modifications). Their system SFL^+_{ew} is equal to the sequent system deleting $\Box$ rule from our $\Box FL^+_{ew}$. Thus, the important new case for $\Box FL^+_{ew}$ is as follows.

$$\frac{\dfrac{\Box\Gamma \Rightarrow A}{\Box\Gamma \Rightarrow \Box A} \quad \dfrac{\Delta, A \Rightarrow B}{\Delta, \Box A \Rightarrow B}}{\Box\Gamma, \Delta \Rightarrow B} \ (cut)$$

This cut can be replaced by the following, which has the same end-sequent, but the size of the proof above the cut-formula becomes smaller by one;

$$\frac{\Box\Gamma \Rightarrow A \quad \Delta, A \Rightarrow B}{\Box\Gamma, \Delta \Rightarrow B} \ (cut)$$

LEMMA 6.
1. Cut elimination holds for $\Box FL^+_{ew}$.
2. Free modal $\mathbf{FL}_{ew}$*-algebras are precisely Lindenbaum-Tarski algebras of* $\Box FL^+_{ew}$.

Here, we briefly describe the Lindenbaum-Tarski algebra for the sequent system $\Box FL^+_{ew}$. We say that the formula algebra of $\Box FL^+_{ew}$ is the algebra whose elements are precisely the formulas of $\Box FL^+_{ew}$. Now we define the binary relation $\sim$ on the formula algebra, putting $\alpha \sim \beta$ if the sequents $\alpha \Rightarrow \beta$ and $\beta \Rightarrow \alpha$ are both provable in $\Box FL^+_{ew}$. It is easy to see that $\sim$ is a congruence relation. Then we have the quotient algebra $\Box FL^+_{ew}/\sim$ which is known as the Lindenbaum-Tarski algebra for $\Box FL^+_{ew}$. For more precise information, see [8].

Let a formula α be given. In the following, $\neg\alpha$ denotes $\alpha \to 0$. For each formula α, let $\ell(\alpha)$ denote the length of α as a sequence of symbols.

For a sequence Γ of formulas $\alpha_1, \cdots, \alpha_m$, the length $\ell(\Gamma)$ is defined by $\ell(\Gamma) = \ell(\alpha_1) + \cdots + \ell(\alpha_m)$. Also we need to introduce some notations for our main lemma. The expression $\{\alpha^N\}^m$ stands for the sequence $\alpha^N, \cdots, \alpha^N$ with m times, where α^N is of the form $\alpha * \cdots * \alpha$ (N times).

To show that any free modal $\mathbf{FL}_{ew}$-algebra $\mathbf{A}$ is semisimple, Lemma 3 says that it is enough to show that the radical $Rad_{\mathbf{A}}$ of any Lindenbaum-Tarski algebra of $\Box FL_{ew}^{+}$ is equal to $\{1\}$. By Proposition 4, this follows from the following lemma.

LEMMA 7. *Suppose that a formula α is not provable in $\Box FL_{ew}^{+}$ and that $n > \ell(\alpha)$. For any sequent $\Gamma \Rightarrow \sigma$ such that $\ell(\Gamma, \sigma) \leq \ell(\alpha)$, and any m, if $\{\Box\neg((\Box\alpha)^n)\}^m, \Gamma \Rightarrow \sigma$ is provable in $\Box FL_{ew}^{+}$ then $\Gamma \Rightarrow \sigma$ is provable in $\Box FL_{ew}^{+}$.*

Proof. The proof will be given by double induction on $(m, \ell(\Gamma, \sigma))$, where m is the number of formulas come from the set $\{\Box\neg((\Box\alpha)^n)\}^m$ which contains $\neg((\Box\alpha)^n)$ as a subformula, not the number of formulas of the form $\Box\neg((\Box\alpha)^n)$. Thus, we assume that the lemma holds for $m' < m$. More precisely we assume that the lemma holds for $\{\Box\neg((\Box\alpha)^n)\}^{m_1}, \{\neg((\Box\alpha)^n)\}^{n_1}, \Gamma \Rightarrow \sigma$, where $m_1 + n_1 = m'$ and it also holds for $(m, \ell(\Delta, \delta))$, whenever $\ell(\Gamma, \sigma)) < \ell(\Delta, \delta)$.

Suppose α is not provable but $\{\Box\neg((\Box\alpha)^n)\}^m, \Gamma \Rightarrow \sigma$ is provable in $\Box FL_{ew}^{+}$.

(1) Suppose that the given sequent $\{\Box\neg((\Box\alpha)^n)\}^m, \Gamma \Rightarrow \sigma$ is an initial sequent. Then, either σ is a propositional variable which occurs also in Γ, or 0 occurs in Γ. It is obvious that $\Gamma \Rightarrow \sigma$ is provable in either case.

(2) Next suppose that the sequent $\{\Box\neg((\Box\alpha)^n)\}^m, \Gamma \Rightarrow \sigma$ is the lower sequent of an inference rule I.

(i) First assume that the principal formula of I is either in Γ or in σ. Then (each of) the upper sequent(s) of I is of the form $\{\Box\neg((\Box\alpha)^n)\}^{m_i}, \Delta_i \Rightarrow \delta_i$ with $m_1 < m$ and $\ell(\Delta_i, \delta_i) < \ell(\Gamma, \sigma)$. Therefore, by hypothesis of induction $\Delta_i \Rightarrow \delta_i$ is provable. Then $\Gamma \Rightarrow \sigma$ is also provable by applying the same inference rule.

(ii) Next suppose that the principal formula of I is one of $\Box\neg((\Box\alpha)^n)$. Then the inference rule I is of the form that;

$$\frac{\{\Box\neg((\Box\alpha)^n)\}^{m-1}, \neg((\Box\alpha)^n), \Gamma \Rightarrow \sigma}{\{\Box\neg((\Box\alpha)^n)\}^m, \Gamma \Rightarrow \sigma}\ (\Box \Rightarrow)$$

In this inference I, the degree of m and $\ell(\Gamma, \sigma)$ of the upper sequent is the same as the lower sequent. Consider the next inference rule J to the upper sequent

$$\{\Box\neg((\Box\alpha)^n)\}^{m-1}, \neg((\Box\alpha)^n), \Gamma \Rightarrow \sigma.$$

There are three possibilities.

(a) First case is that the principal formula of J is one of $\Box\neg((\Box\alpha)^n)$. This is not the essential case of the induction, since we at last reach a case which is essentially equivalent to the either (b) or (c) mentioned below by tracing back the proof. In this case the degree of m and $\ell(\Gamma,\sigma)$ of the upper sequent is the same as the lower sequent. Thus, we have to continue to trace back the proof. Consider the next inference rule there are also three possibilities. First case is the same case as (a) except the numbers of $\Box\neg((\Box\alpha)^n)$. Note that the degree of m and $\ell(\Gamma,\sigma)$ of the induction is never changed during tracing back, while the inference rules in the proof are in the case of (a). Moreover the number of $\Box\neg((\Box\alpha)^n)$ in the given sequent is m, so we cannot continue this case at most m-times applying the given provable sequent. Therefore we sometimes reach the essentially the same as the other two cases are mentioned below.

Note that applying the inference rules in the case of (a) after i-th times, the given sequent will be of the form

$$\{\Box\neg((\Box\alpha)^n)\}^{m_1}, \{\neg((\Box\alpha)^n)\}^{n_1}, \Gamma \Rightarrow \sigma$$

where $m_1 + n_1 = m$. If the next principal formula of the inference rule is one of $\neg((\Box\alpha)^n)$ then we will replace the sequent

$$\{\Box\neg((\Box\alpha)^n)\}^{m-1}, \neg((\Box\alpha)^n), \Gamma \Rightarrow \sigma$$

to

$$\{\Box\neg((\Box\alpha)^n)\}^{m_1}, \{\neg((\Box\alpha)^n)\}^{n_1}, \Gamma \Rightarrow \sigma$$

in the case (c) below and applying the same argument. The same modification is also needed in the case of (b) below after applying the case (a) i-th times.

(b) Second case is that the principal formula of J is either in Γ or in σ. In this case, we apply the same argument in (2)-(i) then we have $\Gamma \Rightarrow \sigma$ as desired.

(c) Third case is that the principal formula of J is $\neg((\Box\alpha)^n)$. In this case, the inference rule is of the form that;

$$\frac{\{\Box\neg((\Box\alpha)^n)\}^{m_2}, \Pi_2 \Rightarrow (\Box\alpha)^n \quad 0, \{\Box\neg((\Box\alpha)^n)\}^{m_1}, \Pi_1 \Rightarrow \sigma}{\{\Box\neg((\Box\alpha)^n)\}^{m-1}, \neg((\Box\alpha)^n), \Gamma \Rightarrow \sigma}\ (\neg \Rightarrow)$$

where Π_1, Π_2 are equal to Γ.

Taking the left upper sequent $\{\Box\neg((\Box\alpha)^n)\}^{m_1}, \Pi_2 \Rightarrow (\Box\alpha)^n$ and the provable sequent $(\Box\alpha)^n \Rightarrow \alpha^n$, and applying the cut rule. Then we have $\{\Box\neg((\Box\alpha)^n)\}^{m_1}, \Pi_2 \Rightarrow \alpha^n$. Consider the proof R of this sequent. We will trace back branches of R, which consist of sequents having α^n in the conclusion, to the places where this α^n is introduced. Note that α^n is introduced at one place in each branch of R. It is easy to see that each α^n is introduced either as an initial sequent, or by $(\Rightarrow *)$ rule. We will show that any α^n is introduced only as an initial sequent. Suppose that at least one place, α^n is introduced by $(\Rightarrow *)$, whose lower sequent is of the form $\Delta \Rightarrow \alpha^n$. We assume here that α is of the from $D_1 * \cdots * D_w$ and none of D_j are fusion-formulas. Then, I must have $n \cdot w$ upper sequents, each of which is of the form $\Delta_i \Rightarrow D_{n_i}$, where $1 \leq n_i \leq w$ and the list $\Delta_1, \cdots, \Delta_{n\cdot w}$ is equal to Δ. For each j such that $1 \leq j \leq w$, there exist exactly n sequents with the conclusion D_j among those sequents. We enumerate them as $S_1^j, \cdots, S_n^j$. Next, for each h such that $1 \leq h \leq n$, take $S_h^1, \cdots, S_h^w$ for upper sequent and apply $(\Rightarrow *)$ rule to them. Then we can have a sequent of the form $\Sigma_h \Rightarrow \alpha$ for $1 \leq h \leq n$ and the list of $\Sigma_1, \cdots, \Sigma_n$ is equal to Δ. Now $\ell(\Delta) \leq \ell(\Pi_2) \leq \ell(\Gamma, \sigma) \leq \ell(\alpha) < n$. If we assume that $\ell(\Sigma_i) > 0$ for any i such that $1 \leq i \leq n$ then $\ell(\Delta) \geq n$, which is a contradiction. Therefore, Σ_i must be empty for some i. But this means that $\Rightarrow \Box\alpha$ is provable. This contradicts the assumption that α is unprovable. Hence, we conclude that at any place α^n is introduced as an initial sequent of the form $0, \Lambda \Rightarrow \alpha^n$.

We will modify the proof R of $\{\Box\neg((\Box\alpha)^n)\}^{m_2}, \Pi_2 \Rightarrow \alpha^n$ as follows. We replace every sequent $\Theta \Rightarrow \alpha^n$ in a branch which we have traced in R, including an initial sequent of the form $0, \Lambda \Rightarrow \alpha^n$ mentioned above, by the sequent $\Pi_1, \Theta \Rightarrow \sigma$. Then we will have the proof whose end sequent is $\{\Box\neg((\Box\alpha)^n)\}^{m_2}, \Gamma, \Rightarrow \sigma$. Note that $m_2 \leq m-1 < m$. Hence, by hypothesis of induction, we conclude that $\Gamma \Rightarrow \sigma$ is provable. This completes the proof.

Clearly, the sequent system $\Box FL_{ew}^+$ is consistent, i.e., the sequent $\Rightarrow 0$ is not provable. Let m be a positive integer. If the sequent $\{\Box\neg(\Box\alpha)^n\}^m \Rightarrow 0$ is provable in $\Box FL_{ew}^+$ then it follows immediately from Lemma 7 by taking the sequent $\Rightarrow 0$ for $\Gamma \Rightarrow \sigma$. Thus, we have $\Rightarrow 0$ is provable in $\Box FL_{ew}^+$, which is a contradiction. Therefore, we have the following.

PROPOSITION 8. *Let α be any formula which is not provable in $\Box FL_{ew}^+$. Then there exists a natural number $n \geq 1$ such that $\{\Box\neg(\Box x)^n\}^m \Rightarrow 0$ is not provable in $\Box FL_{ew}^+$ for any m.*

In terms of the Lindenbaum-Tarski algebra $\mathbf{A}$ of $\Box FL_{ew}^+$, the above proposition says that if $[\alpha] \neq [1]$ in $\mathbf{A}$ then there exists $n \geq 1$, $[0] \neq [\Box\neg((\Box a)^n)^m]$ for any m, where $[\gamma]$ denotes the equivalence class, to which a given formula γ belongs. Thus, using Lemma 3, Proposition 4 and Corollary 5, we have the following.

THEOREM 9 (Semisimplicity of free modal $\mathbf{FL}_{ew}$-algebras). *Every free modal $\mathbf{FL}_{ew}$-algebra is semisimple.*

COROLLARY 10. *The variety of modal $\mathbf{FL}_{ew}$-algebras is generated by its simple members.*

4 Finite embeddability property of simple modal $\mathbf{FL}_{ew}$- algebras

In this section we show that the class of simple modal $\mathbf{FL}_{ew}$- algebras has the finite embeddability property (FEP).

Given an algebra $\mathbf{A} = \langle A, \langle f_i^{\mathbf{A}} : i \in I \rangle\rangle$ of finite type and any nonempty subset $B \subseteq A$, the partial subalgebra $\mathbf{B}$ of $\mathbf{A}$ with domain B is the partial algebra $\langle B, \langle f_i^{\mathbf{B}} : i \in I \rangle\rangle$, where for $i \in I$, f_i is an n-ary function symbol, and $b_1, \cdots, b_n \in B$

$$f_i^{\mathbf{B}}(b_1, \cdots, b_n) = \begin{cases} f_i^{\mathbf{A}}(b_1, \cdots, b_n) & f_i^{\mathbf{A}}(b_1, \cdots, b_n) \in B \\ undefined & f_i^{\mathbf{A}}(b_1, \cdots, b_n) \notin B \end{cases}$$

A class $\mathcal{K}$ of algebras has the finite embeddability property if every finite partial subalgebra of a member of $\mathcal{K}$ can be embedded into a finite member of $\mathcal{K}$.

Next, we introduce a construction method to prove the FEP of the variety of $\mathbf{FL}_{ew}$-algebra which is originally proposed by Blok and van Alten [2]. Later, Amano proposed a natural extension for the modal operator of this construction to prove the FEP of the variety of modal $\mathbf{FL}_{ew}$-algebra [1].

Let $\mathbf{A}$ be a modal $\mathbf{FL}_{ew}$-algebra and $\mathbf{B}$ be a finite partial subalgebra of $\mathbf{A}$. We shall sometimes omit the multiplicative symbol $\cdot$. Let $\mathbf{M}$ be the submonoid of $\mathbf{A}$ generated by $\mathbf{B}$. For $X, Y \subseteq M$ and $a \in M$, put $XY = \{ab : a \in X, b \in Y\}$ and $Xa = X\{a\}$.

For each $x \in M$ and $b \in B$, define

$$(x \rightsquigarrow b] = \{c \in M : xc \leq b\} = \{c \in M : c \leq x \to b\}.$$

The set $(x \rightsquigarrow b]$ is a downward closed subset of M. Note that $(1 \rightsquigarrow b] = \{a \in M : a \leq b\}$. Write the sets $\overline{D} = \{(x \rightsquigarrow b] : x \in M, b \in B\}$ and $D = \{\bigcap \Xi : \Xi \subseteq \overline{D}\}$. Let C be the closure operator on the set of subsets of M associated with D, i.e., for $X \subseteq M$, let

$$\mathsf{C}(X) = \bigcap \{(x \rightsquigarrow b] \in \overline{D} : X \subseteq (x \rightsquigarrow b]\}.$$

Define for all $X, Y \subseteq M$ and $X_i \subseteq M$, $i \in I$,

$$X \cdot^D Y = \mathsf{C}(XY)$$
$$X \to^D Y = \{a \in M : Xa \subseteq Y\}$$
$$\bigwedge_{i\in I}^{D} X_i = \mathsf{C}(\bigcap_{i \in I} X_i)$$
$$\bigvee_{i\in I}^{D} X_i = \mathsf{C}(\bigcup_{i \in I} X_i)$$
$$1^D = M$$
$$0^D = \bigcap \overline{D}$$
$$\Box^D X = \mathsf{C}(\{a \in X : \Box a = a\}).$$

The definition of modality follows van Alten [13]. Then, we have the following three results which are proved by Amano [1].

THEOREM 11. *The structure* $\mathbf{D}(A,B) = \langle D, \cdot^D, \to^D, \wedge^D, \vee^D, 1^D, 0^D, \Box^D \rangle$ *is a modal* $\mathbf{FL}_{ew}$*-algebra .*

THEOREM 12. *If* $\mathbf{A}$ *is a modal* $\mathbf{FL}_{ew}$*-algebra and* $\mathbf{B}$ *is a partial subalgebra of* $\mathbf{A}$*, then* $\mathbf{B}$ *can be embedded, as a partial subalgebra, into the modal* $\mathbf{FL}_{ew}$*-algebra* $\mathbf{D}(A,B)$*.*

Proof. We define an embedding from $\mathbf{B}$ to $\mathbf{D}(A,B)$ by $b \mapsto (1 \rightsquigarrow b]$. It suffices that all operations $\to, \wedge, \vee, \cdot, \Box$ and constants $1, 0$ in $\mathbf{B}$ are preserved $\mathbf{D}(A,B)$ by this embedding. It is routine and we omit the detail.

Now we have the algebra $\mathbf{D}(A,B)$ into which we can embed $\mathbf{B}$. The proof of the finiteness of $\mathbf{D}(A,B)$ is exactly the same as in [2].

THEOREM 13. *The variety of modal* $\mathbf{FL}_{ew}$*-algebras has the FEP.*

Next, we will show that the class of simple modal $\mathbf{FL}_{ew}$-algebras has the FEP. Recall, we say that an algebra is *simple* if it has only two congruences. In the case of modal $\mathbf{FL}_{ew}$-algebras, simple algebras are characterized by the following lemma.

LEMMA 14. *A modal* $\mathbf{FL}_{ew}$*-algebra* $\mathbf{A}$ *is simple if and only if for any* $x(\neq 1)$ *in* A *there exists a positive integer* m *such that* $(\Box x)^m = 0$*.*

Using this characterization, we can prove that the FEP for simple modal $\mathbf{FL}_{ew}$-algebras.

THEOREM 15 (The FEP of simple modal $\mathbf{FL}_{ew}$-algebras). *The class of simple modal* $\mathbf{FL}_{ew}$*-algebras has the FEP.*

Proof. Let $\mathbf{A}$ be a simple modal $\mathbf{FL}_{ew}$-algebra and $\mathbf{B}$ a finite partial subalgebra of $\mathbf{A}$. Construct the structure $\mathbf{D}(A,B)$ then $\mathbf{D}(A,B)$ is a finite modal $\mathbf{FL}_{ew}$-algebra by Theorem 11. Thus, we need to show that $\mathbf{D}(A,B)$ is also a simple modal $\mathbf{FL}_{ew}$-algebra, i.e., for any $X(\neq 1^D) \in \mathbf{D}(A,B)$ there exists n with $(\Box X)^n = 0^D$. By Lemma 14, for each $x \in A$, there exists $n \in \mathbb{N}$ such that $(\Box x)^n = 0$. Let $\mathbf{M}$ be the submonoid of $\mathbf{A}$ generated by $\mathbf{B}$. Recall that an element of $\mathbf{D}(A,B)$ is a downward closed subset of $\mathbf{M}$.

Let $F(k)$ be the free commutative monoid on k generators $\{x_1, \cdots, x_k\}$. An element of $F(k)$ can be considered as the products $x_1^{n_1} \cdots x_k^{n_k}$ where $n_i < \omega$, $i = 1, \cdots, k$ and defined $x_i^0 = 1, i = 1, \cdots, k$. We can define an order $\leq$ on $F(k)$ by $x_1^{n_1} \cdots x_k^{n_k} \leq x_1^{m_1} \cdots x_k^{m_k}$ if and only if $n_i \geq m_i$ for each $i \in \{1, \cdots, k\}$. It is easy to see that the order is a well-quasi order. Note that for any $A \subseteq F(k)$, the set $\mathrm{Max}(A)$ of maximal elements of A is an antichain. Since the order on $F(k)$ is a well-quasi order, $\mathrm{Max}(A)$ is a finite set.

Now we consider a monoid homomorphism $F(k)$ to $\mathbf{M}$ defined by each generator x_i of $F(k)$ to an element of $\mathbf{B}$. Note that $\mathbf{M}$ is the submonoid of $\mathbf{A}$ generated by $\mathbf{B}$. Then for any $X \subseteq \mathbf{M}$, the set Max(X) of maximal elements of X is also finite.

Put Y is the set $\{a \in X : \Box a = a\}$. Then $(\Box X)^n = \mathsf{C}(\underbrace{\mathsf{C}(Y) \cdots \mathsf{C}(Y)}_{n}) \subseteq \mathsf{C}(Y^n)$. Now we write $k = |\mathrm{Max}(Y)|$. Note that for all $x_i \in \mathrm{Max}(Y)$ there exists $n_i \in \mathbb{N}$ such that $(\Box x_i)^{n_i} = 0$. Since Max(Y) is finite there exists $max\{n_i\}$. Therefore $(\Box X)^{k \cdot max\{n_i\}} \subseteq \mathsf{C}(Y^{k \cdot max\{n_i\}}) = \mathsf{C}(\{0\}) = 0^D$ for any X. We conclude that $\mathbf{D}(A, B)$ is also simple.

5 Conclusion and remarks

THEOREM 16 (Main theorem). *The variety of modal* $\mathbf{FL}_{ew}$*-algebras is generated by its finite simple members.*

Proof. The proof goes essentially the same as in [7]. It suffices to show that for any non theorem α of $\Box\mathbf{FL}_{ew}$ there is a finite simple modal $\mathbf{FL}_{ew}$-algebra $\mathbf{A}$ falsifying α. It immediately follows from Theorem 9, Corollary 10 and Theorem 15.

To finish off this paper, we will list some open questions about the topic.

(Q1) Is the variety of $\mathbf{FL}_{ew}$-algebras with K-like (or other types of) modality generated by its finite simple members?

In [11], the author shows that every free $\mathbf{FL}_w$-algebra is semisimple by using Grišin's idea and Kowalski and Ono's technique.

(Q2) Is a free modal $\mathbf{FL}_w$-algebra semisimple?
(Q3) Is the variety of modal $\mathbf{FL}_w$-algebras generated by its finite simple members?

There are many other interesting problems in residuated lattices with modalities. We expect that many researchers are getting interested in this area.

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