Modality, Paraconsistency and Paracompleteness

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ABSTRACT. We present here a modal logic that is both paraconsistent and paracomplete. The purpose of such endeavor is twofold. First we want to contribute to the understanding of the relations which some theoreticians have recently claimed that modal logic holds with paraconsistent and paracomplete logics. This will be achieved through the analysis of the formal features of our logic in comparison to traditional (normal) modal logic. We have made our logic to depart as less as possible from traditional modal logic, as well as provided some formal results such as a translation from one logic into another. Secondly, we try to show how situations which require a formal analysis of the notion of plausibility inside the context of what we called the skeptical and credulous approaches to induction might require a logic like the one we present here.

1 Introduction

From a point of view of truthfulness and falsehood, the principles of non-contradiction and excluded middle are the dual of each other. Roughly speaking, a paraconsistent logic [7] is a logic rejecting the principle of non-contradiction. Putting these two facts together, some theoreticians labeled the term "paracomplete logic" to designate those logics that do not satisfy the principle of excluded middle [15]. Intuitionist logic will obviously be classified as a paracomplete logic [2]. Following the same notational standard, logics that are both paraconsistent and paracomplete are called paranormal (or non-alethic) logics[1].

A more precise characterization of paraconsistent logic and paracomplete logic would go as follows. A paraconsistent negation is a unary operator that does not satisfy the principle of explosion (for any $\alpha, \{\alpha, \neg\alpha\} \vdash \beta$ for any β) [2] and has enough properties to be called a negation; and a paracomplete negation is a unary operator that does not satisfy the excluded middle principle ($\vdash \alpha \vee \neg\alpha$) and has enough properties to be called a negation. A paraconsistent logic then is a logic having a paraconsistent negation, and a

[1] This "para" notation is all due to Miro Quesada [2].

[2] The need to consider the *ex-contradictio sequitur quod libet* principle instead of an internal principle of non contradiction is of course due to the non-equivalence between these two principles, but above all to the fact that the real novelty of paraconsistent logic is to be able to formalize inconsistent but non-trivial theories [6]

paracomplete logic is one having a paracomplete negation[3].

The relation between paraconsistent and paracomplete logic and modal logic is not new. After all, the first known system of paraconsistent logic, Jaskowski's discussive logic, was defined as a fragment of S5 [13]. Others have proposed semantics for paraconsistent and paracomplete logics which strongly resembles a Kripkean possible-world semantics [17, 18, 20]. More significant for us is the fact that intuitionistic negation can be interpreted (as it is commonly so) in terms of a certain translation into S4 that interprets this negation by $\Box\neg$. According to our tentative definition of paracompleteness, the existence of this derived negation inside S4 by itself allows us to classify S4 as a paracomplete logic. The same can be done to obtain a paraconsistent negation inside S4 or any other normal modal logic: just define $\sim$ as $\Diamond\neg$ and you will have a unary operator that does not satisfy the principle of explosion ($\sim\alpha, \alpha \vdash \beta$) and has enough properties to be called a negation [3]. This idea is of course not new: it has been deeply explored in between the mid-70s and the 80s [8, 23]. However, only recently we have seen the appearance of foundational works which try to relate in a more general way these two fields [1, 3, 4, 16], some of which with bold titles such as "S5 is a paraconsistent logic and so is first-order classical logic" [3] or "Nearly every normal modal logic is paranormal" [16].

At the heart of all this is the semantic structure of modal logic explored in the interpretation of $\Box$ and $\Diamond$. The fact that "α is possible" and "$\neg\alpha$ is possible" can coexist without trivializing the theory embodies a subtler sort of paraconsistency which some have named "hertian" [2] or conceptual [22] paraconsistency. Similarly, the fact that there might be a model where neither $\Box\neg\alpha$ nor $\Box\alpha$ are true entitles us to speak of a "hertian" or conceptual paracompleteness. That this conceptual paraconsistency, for example, cannot be turned into a truly or formal one by the use of $\sim$ can be easily checked. Even though regarding non-modal formulas $\sim$ will behave paraconsistently, with respect to modal formulas it is as classical as $\neg$: $\sim\Diamond\alpha, \Diamond\alpha \vdash \beta$, for any β, for example. (This or course might be different if we consider normal logics weaker than S5. Also, basing the truth-values of formulas of a non-modal language into a possible world semantics[4] might allow us to obtain a formal paraconsistent or paracomplete negation.)

But if the core of paraconsistency and paracompleteness of traditional modal logic (be it the conceptual or the formal one through $\sim$) is the behavior of $\Box$ and $\Diamond$ with respect to the negation, we can wonder: (1) What if

[3]All this should be seen as a rough characterization of paraconsistency and paracompleteness. Despite the vagueness of the positive side of our characterizations of paraconsistent and paracomplete negations, we think it fulfills its purpose of giving us something to start with. After all, one will hardly find something much superior in the literature, for the question of what is a paraconsistent (and paracomplete) negation is still something quite controversial. See for instance [2].

[4]Something like $V(p) = 1$ iff $v_w(p) = 1$ for all worlds w, where p is a propositional symbol, V is the valuation function which will give recursively the truth-value of every formula, and for each world w there is a function v_w that gives the truth-value of each propositional symbol in this world [5].

instead of a classical negation we had a truly paraconsistent and paracomplete negation in which $\neg\Diamond\alpha, \Diamond\alpha \nvdash \beta$, for some β, and $\nvdash \neg\Box\alpha \vee \Box\alpha$? (2) Will in this case our understanding of the mentioned connection between paraconsistent and paracomplete logic and modal logic somewhat improve? (3) If so, besides this theoretical interest, what other advantages, perhaps in terms of applications, the resulting logic would have? (4) And how much should it depart from traditional modal logic, or how much would it differ from the existing paraconsistent and paracomplete logics?

Our purpose in this paper is to contribute to the understanding of the relations that exist between paraconsistent and paracomplete logic and modal logic by trying to develop such a modal logic where $\Box$ and $\Diamond$ have a truly paracomplete and paraconsistent behavior, respectively. We will call such a logic paranormal modal logic. We acknowledge that this term might be misleading, for normal modal logic [5] have already an established meaning in modal logic and the "para" in paranormal does not apply to it. However, unfortunately the other term used to designate logics that are both paraconsistent and paracomplete, "non-alethic," also has a well established meaning in modal logic. In the lack of a better term then, we shall use the first one. In the next section we will set the basic assumption that will guide us in the construction of such a logic, partially answering question (4). In Section 3 we address (3) and try to say why it is worthy to have such a logic. In Section 4 we present the language and calculus of our logic and in Section 5 the semantics. Side by side to our exposition in these sections we will try to advance our answer to question (4) and perhaps also to (2). In Section 6 we present a formal comparison between paranormal modal logic and traditional normal modal logic, giving a big step, we believe, in answering (2), and finally, in Section 7, we try to summarize our contribution.

2 Paranormal Modal Logic

Here is the main prescription that will guide us in our task of developing our system: we want a paraconsistent and paracomplete logic in the way described in the previous section which departs the least possible from traditional modal logic. The rationale behind this is of course our foundational analysis: besides letting clear which formal and conceptual aspects we should give up and which ones we can retain when transforming modal logic into paranormal modal logic, this will make a technical comparison between the two logics easier. This principles of course the same as one of the conditions which according to da Costa any paraconsistent logic should fulfill [7], only now applied to classical modal logic instead of classical first-order logic. We will now proceed, based on this principle, to figure out which features of traditional modal logic we are willing to keep and which ones we should abandon.

First of all, due to the very task we set in the introduction, paranormal

[5] A normal modal logic is modal logic which has K ($\Box(\alpha \rightarrow \beta) \rightarrow (\Box\alpha\, rightarrow \Box\beta)$) as a valid principle and *modus ponens*, generalization (from α conclude $\Box\alpha$) and the rule of uniform substitution as valid inference rules [12].

modal logic should have two dual modal operators like $\Box$ and $\Diamond$. Just to avoid confusion, we shall use two different symbols -! and ?- in a post-fixed notation to represent our operators. ! should correspond to $\Box$ and ? to $\Diamond$. Another desirable feature is to have the same semantic structure used in traditional modal logic for the evaluation of formulas. Putting these things together, it seems that if we really want ! and ? to behave as much as possible like $\Box$ and $\Diamond$, respectively, then $\alpha?$ should be true iff α is true in at least one world, and $\alpha!$ iff α is true in all worlds [6]. Since our real concern is the negation, we should try to leave all else intact. Aside that, as far as possible we shall try to follow the standard presentation style of modal logics found in classical modal logic textbooks. We shall have a basic paranormal modal system, which we might call $K_?$, from which by adding new axioms, on the syntactic side, or by restricting the model structure, on the semantic one, we obtain several other paranormal modal systems, say, $T_?$, $D_?$, $S4_?$, $S5_?$, and so on.

About the distinguishing features of paranormal modal logic, ? and ! should have a paracomplete and a paraconsistent behavior, respectively. Given two formulae α and β, it may be that $\alpha?$ and $\neg(\alpha?)$ are both true, and that $\beta!$ and $\neg(\beta!)$ are both false. This characterization of ? and ! reveals an odd behavior of $\neg$: when taken along with ?-marked formulas it is a paracomplete negation, but along with !-marked formulas it behaves like a paraconsistent negation. As a consequence of this, it is expected that many classical logical principles (in our case the excluded middle principle, the law of non-contradiction, the *reductio ad absurdum* principle, contraposition law, and others) shall not hold in paranormal modal logic. Regarding non-modal formulas, we can make $\neg$ behave exactly like classical negation, for our initial considerations do not require this sort of formulas to behave non-classically. Due to all this, it seems fair to classify our negation as a *paranormal modality-dependent negation*.

Despite these general considerations, there are many gaps to be filled. The crucial one seems to be how we will conceive a Kripkean semantic structure able to produce a model which satisfies both $\alpha?$ and $\neg(\alpha?)$, or that is such that neither $\alpha!$ nor $\neg(\alpha!)$ are satisfied. In order to avoid arbitrariness and give an initial answer to (3), we will try to limit the technical choices we have in the construction of paranormal modal logic by taking guidance of the intuitions contained in a possible application, to be presented in the next Section. After all, to have an intuitive justification was one of the conditions that according to Jaskowski a logic of contradictory systems should satisfy [13].

[6] In our informal use of the notions of possible world and satisfaction in this section and in the next one we will drop the reference to the notion of an accessibility and act as if the relation between the worlds were universal.

3 Why Paranormal Modal Logic? The Skeptical and Credulous Approaches to induction

The first answer we would be inclined to give to the question that begins the title of this section is that maybe answering (1) would be by itself a worthy task. After all, it was this sort of theoretical curiosity that led to the development of non-euclidean geometries and da Costa's calculi. In both cases, the applications came after. However, as we saw, there is a higher theoretical purpose involved in the creation of a paranormal modal logic. As we have said, the existence of such a logic might improve our understanding about the relations that exist between modal logic and paraconsistent and paracomplete logics. That this is so will be shown in the sections that will follow. Moreover, from the side of modal logic our work might be of some relevance. We have seen before different ways to base modal logic in some non-standard logic [9, 11, 14]. At this time we provide a minimal "traditional" modal logic based on a paraconsistent and paracomplete logic, that is to say, a paraconsistent and paracomplete modal logic which is as close as possible to traditional normal modal logic.

Nevertheless, and despite this, we do have an application which justifies the sort of modal logic presented here and which, we hope, provides an answer to question (3). This application is strongly connected with the main motivation for the construction of the systems of which paranormal modal logic is, say, a refined version [5, 17, 18]: the problem of dealing with contradictions that are sure to arise from the use of defeasible rules of inference [18]. As we have said, much of what we will set in this section will guide important choices to be made in the development of our logic, including our reading of the modal operators ! and ?.

The problem of inductive inconsistencies, extensively studied in both artificial intelligence and philosophy of science[7], strongly suggests the existence of at least two different approaches to the formal analysis of induction, which we shall call the *skeptical* and the *credulous* approaches to induction. They can be explained as follows. Consider a situation where some of our inductive rules of inference conflict with each other. In this case, there will be contradictory conclusions. One way to be able to keep reasoning classically about them is to keep all the conclusions we may obtain from our rules of inference without getting into a contradiction in different sets of formulas. We will call the deductive closure of each one of these sets an *inductive extension*. A skeptical approach to induction then is a position that rejects all sorts of contradictions and accepts as authentic inductive conclusions only those formulas that are in the intersection of all inductive extensions. It is a sort of rigid approach which requires quite a lot to accept something as a valid inductive conclusion. On the other hand, a credulous approach would take contradictions into consideration and accept the union of all inductive extensions, corresponding then to a tolerant approach which

[7]For some connections between the analysis of inductive inferences in artificial intelligence and philosophy of science see [21].

quite easily gives a formula the status of an authentic conclusion[8].

Now suppose that we decide, as it is common in philosophy of science, to distinguish inductive conclusions from ordinary formulas using some probability-like term like the term "plausibility." In this case we will have two different ways to appraise the plausibility of statements. We will say that α is plausible according to a skeptical approach iff α is true in all inductive extensions, and that α is plausible according to a credulous approach iff α is true in at least one extension. Here we see that the skeptical and credulous approaches work as evaluation functions which assess in different ways the truthfulness of plausible statements.

As one might have concluded, this picture can be represented with the aid of a modal framework. Let ! correspond to the skeptical plausibility and ? correspond to the credulous plausibility so that $\alpha!$ means that α is plausible according to the skeptical approach and $\alpha?$ that α is plausible according to the credulous approach. Let also each inductive extension correspond to a possible world, which we shall call from now on *plausible world*. In this case, we will have a quite familiar picture where $\alpha!$ is true iff α is true in all plausible worlds, and $\alpha?$ is true iff α is true in at least one plausible world. Given this, one might be wondering what is special about all that: traditional modal logic seems to be able to deal quite easily with the situation. Unfortunately this is not so, and the reason it is not so resides in the fact that having two dual ways to evaluate plausible formulas, a rigid and a tolerant one, makes negation to behave in a way that traditional logic cannot deal with.

According to the meaning we have set above, $\neg(\alpha!)$ and $\neg(\alpha?)$ will mean something like (I) "α is implausible according to a skeptical position" and (II) "α is implausible according to a credulous position," respectively. Note however that due to the qualification on the implausibility, which can be either skeptical or credulous, there is an ambiguity in the reading of these two sentences. This can be shown with the help of brackets. The first one, for example, can either mean that (i) it is not the case that [α is plausible according to a skeptical position] or that (ii) [it is not the case that α is plausible] according to a skeptical position. Clearly enough, the negation involved in (i) corresponds to the classical negation present in traditional modal logic: since "α is plausible according to a skeptical position" is true iff α is true in all worlds, (i) will be true if α is false in at least one world. That is not however the case with (ii). There, something very different is being said: it says that the whole sentence "it is not the case that α is plausible" is true according to a skeptical position. Now, how will this be evaluated in our plausible world framework?

If we stick to the idea behind the notion of a skeptical approach explained above, to evaluate "α is not plausible" according to a skeptical position means that we will be very strict in the matter of accepting "α is not

[8]The reader familiar with nonmonotonic literature should have already noticed the similarity of concepts. To a discussion of these two approaches from the standpoint of artificial intelligence see [22].

plausible" as true. In other words, we will require the maximum we can to give "α is not plausible" the value true or, going the other way round and taking true as 1 and false as 0, we will try the best we can to minimize the truth-value of "α is not plausible." I cannot think of anything else that could satisfy this except the situation where α is false in *all* plausible worlds. This would fairly express the idea behind [α being not plausible] according to a rigid, skeptical approach. Therefore, (ii) is true iff α is false in all plausible worlds.

Similarly for the credulous version of (ii): (ii') [it is not the case that α is plausible] according to a credulous position. (ii') involves being very tolerant in the matter of giving "α is not plausible" the value true or, going the other way round, it involves trying to maximize the truth-value of "α is not plausible." This of course means that being false in at least one world will suffice for us to attribute true to the sentence "α is not plausible." Therefore, (ii') is true iff α is false in *at least one* plausible world.

If one agrees on this analysis[9] as well as on the idea that the notion of implausibility is to be analyzed, represented or described in terms of the notion of plausibility, he will have to conclude that the ambiguity involved in (I) and (II) should (or at least might) be understood as primordially related to the negation, in such a way that a formal system like ours will have to have at least two negations, one for each reading of (I) and (II). In either case, both negations will evaluate $(\neg\alpha)!$ and $(\neg\alpha)?$ in the usual way, for there is no ambiguity involved here. $(\neg\alpha)!$ and $(\neg\alpha)?$ will be true, respectively, iff α is false in all plausible worlds and iff α is false in at least one plausible world. But this is exactly the evaluation, we have agreed above, that should be given to $\neg(\alpha!)$ and $\neg(\alpha?)$ in order to account for our second reading. Therefore, concerning our non-standard negation, we will have the strange relations expressed by the formulas $\neg(\alpha!)\rightarrow(\neg\alpha)!$ and $(\neg\alpha)?\rightarrow\neg(\alpha?)$.

The second important thing about this negation is that, since $\alpha!$ is true iff α is true in all worlds and $\neg(\alpha!)$ is true iff α is false in all worlds, it might happen that neither $\alpha!$ nor $\neg(\alpha!)$ are true, that is to say, a behavior exactly like the one a truly paracomplete negation should have and that initially motivated our theoretical investigation. Similarly, $\alpha?$ is true iff α is true in at least one world. Since $\neg(\alpha?)$ is true iff α is false in at least one world, then we will have that a model might satisfy both $\alpha?$ and $\neg(\alpha?)$, making $\neg$ correspond to what we have called a truly paraconsistent negation.

In this way, it seems that if we really stick to the meaning given to the skeptical and credulous positions, we will have to admit that in order to formalize a logic of skeptical and credulous plausibility inside a possible world framework we will have to develop something like a paranormal modal logic. From now on, we will make reference to this application to elucidate some key formal principles of our system.

[9] For the skeptic readers, a much more detailed argumentation is given in [22].

4 Language and Calculus

In this section we present the simplest paranormal modal system corresponding in traditional modal logic to system K. We call it $K_?$.

DEFINITION 1. Let $\Im$ be a propositional language. The paranormal modal language $\Im_?$ is defined as follows:

(i) If p is a propositional symbol of $\Im$, then p$\in \Im_?$;
(ii) If $\alpha, \beta \in \Im_?$, then $\neg\alpha, \alpha \to \beta, \alpha \wedge \beta, \alpha \vee \beta \in \Im_?$;
(iii) If $\alpha \in \Im_?$, then $\alpha!$, $\alpha?$ $\in \Im_?$;
(iv) Nothing else belongs to $\Im_?$.

We call the set of all propositional symbols P. The logical symbols $\to$, $\wedge$ and $\vee$ are interpreted according to their usual meaning. $\neg$ will represent what we have named paranormal modality-dependent negation: regarding the elements of $\Im$, $\neg$ behaves classically, but in connection with !-marked and ?-marked formulae it behaves paracompletely and paraconsistently, respectively.

DEFINITION 2. Let $\alpha, \beta \in \Im_?$ be any formulae of $\Im_?$. We define the derived symbols $\leftrightarrow$,$\perp$ and $\sim$ as follows:

(i) $\alpha \leftrightarrow \beta =_{def} (\alpha \to \beta) \wedge (\beta \to \alpha)$;
(ii) $\perp =_{def}$ p $\wedge\neg$p, where p$\in$P is an arbitrary propositional symbol;
(iii) $\sim \alpha =_{def} \alpha \to \perp$.

The symbols $\leftrightarrow$ and $\perp$ are taken in accordance with their usual meaning. $\sim$ is a derived symbol intent to represent the classical negation. This is the negation we shall use to formalize the first reading of implausible sentences we saw in the last section.

DEFINITION 3. Let $\alpha \in \Im_?$ be a formula. We say that α is ?-*free* iff ? does not occur in α and that α is !-*free* iff ! does not occur in α. If α is both ?-free and !-free we call it a ?!-free formula.

DEFINITION 4. The axiomatic of paranormal modal logic $K_?$ is as follows:

Positive Classical Axioms
P1: $\alpha \to (\beta \to \alpha)$
P2: $(\alpha \to (\beta \to \varphi)) \to ((\alpha \to \beta) \to (\alpha \to \varphi))$
P3: $\alpha \wedge \beta \to \alpha$
P4: $\alpha \wedge \beta \to \beta$
P5: $\alpha \to (\beta \to \alpha \wedge \beta)$
P6: $\alpha \to \alpha \vee \beta$
P7: $\beta \to \alpha \vee \beta$
P8: $(\alpha \to \beta) \to ((\varphi \to \beta) \to (\alpha \vee \varphi \to \beta))$
P9: $((\alpha \to \beta) \to \alpha) \to \alpha$

Paranormal Classical Axioms
A1: $(\alpha \to \beta) \to ((\alpha \to \neg\beta) \to \neg\alpha)$, wherein β is ?-free and α is !-free
A2: $\neg\alpha \to (\alpha \to \beta)$, wherein α is ?-free
A3: $\alpha \vee \neg\alpha$, wherein α is !-free

Non-Positive Additional Classical Axioms
N1: $\neg(\alpha \to \beta) \leftrightarrow (\alpha \wedge \neg\beta)$

N2: $\neg(\alpha \wedge \beta) \leftrightarrow (\neg\alpha \vee \neg\beta)$
N3: $\neg(\alpha \vee \beta) \leftrightarrow (\neg\alpha \wedge \neg\beta)$
N4: $\neg\neg\alpha \leftrightarrow \alpha$
Paranormal Modal Axioms
K1: $\alpha? \leftrightarrow \sim ((\sim \alpha)!)$
K2: $(\neg\alpha)! \leftrightarrow \neg(\alpha!)$
K3: $(\neg\alpha)? \leftrightarrow \neg(\alpha?)$
Modal Axioms
$K_?$: $(\alpha \to \beta)! \to (\alpha! \to \beta!)$
Rules of Inference
MP: $\alpha, \alpha \to \beta/\beta$
$N_?$: $\alpha/\alpha!$

The calculus of paranormal modal logic is a modal extension of positive classical logic (axioms P1-P9) in such a way as to consider a modality-dependent paranormal negation. The schemas of formula A1-A3 correspond to the negative axioms of classical logic restricted in such a way as to consider the paraconsistent and paracomplete behaviors of ? and !, respectively. Trivially enough, since these restrictions apply only to non-modal formulas, the non modal fragment of paranormal modal logic behaves exactly like classical logic. Axiom schemas N1-N4 are meant to restore the deductive power of paranormal modal logic weakened by the restrictions imposed on axioms A1-A3. The paranormal modal axioms K1-K3 set the basic properties of the modal operators ! and ?. K1 states that in connection with classical negation $\sim$, ? and ! are the dual operators of each other. K2 and K3 state, respectively, that the skeptical plausibility of $\neg\alpha$ is equivalent to the skeptical implausibility of α, and that the credulous plausibility of $\neg\alpha$ is equivalent to the credulous implausibility of α. $K_?$ is the paranormal equivalent to normal modal logic's axiom K and, finally, MP is *modus ponens* and $N_?$ is the paranormal rule of necessitation.

Due to K1, one might think that ? or ! could be introduced as definitions and that either K2 or K3 could be obtained as theorem. Unfortunately this is not so, the reason being the non-standard behavior of $\to$, which shall become clear when we present the semantics in the next section.

According to the best known contemporary taxonomy of paraconsistent logic [19], our logic belongs to the group of what has been labeled 'positive-plus' logics, that is, logics which, laying the blame for trivialization on the classical theory of negation, begin with a positive fragment of classical logic, and then add to it a weakened account of negation (which in our case is done by the restrictions imposed on axioms A1-A3.) We believe this style of attaining paraconsistency (and paracompleteness) is extremely suitable for our purposes in that it leaves explicit from an axiomatic point of view where paranormal modal logic effectively differs from normal modal logic. This difference can be seen very clearly if we consider that a fair axiomatic for K can be obtained from the above one by excluding P9, the non-positive additional classical axioms and the paranormal modal axioms, and removing the qualifications imposed on A1-A3[10]. This is also valid for all other normal modal systems, for in order to obtain, for example, $T_?$, $S4_?$ and $S5_?$ from $K_?$, we proceed in exactly the same way as we do to obtain T, S4 and S5 from K: add ($T_?$) $\alpha! \rightarrow \alpha$ to $K_?$ and you have system $T_?$; add ($4_?$) $\alpha! \rightarrow \alpha!!$ to $T_?$ and you have system $S4_?$; add ($B_?$) $\alpha \rightarrow \alpha?!$ to $S4_?$ and you have $S5_?$, and so on and so forth.

Letting A,B$\subseteq \Im_?$ be two sets of formulas, A representing the set of global premises and B the set of local ones, $\alpha \in \Im_?$ a formula and $L_?$ a paranormal modal logic, we define α's being $L_?$-deducted from A and B (in symbols: $A + B \vdash_{L?} \alpha$) in the standard way[11]. If $B = \emptyset$ we write $A \vdash_{L?} \alpha$ and if $A = B = \emptyset$, we write $\vdash_{L?} \alpha$, in which case we say that α is a $L_?$-theorem. In the case of $K_?$ for example, we say that α is $K_?$-deducted from A and B (in symbols: $A + B \vdash_{K?} \alpha$) and that α is a $K_?$-theorem (in symbols: $\vdash_{k?} \alpha$).

Theorems 5 and 6 below show some $K_?$-theorems which might give a flavor of the non-standard aspect of the calculus of paranormal modal logic.

THEOREM 5. *Some formulae of $\Im_?$ that satisfy one of the following schemas of formula are not $K_?$-theorems:*

$(\alpha \rightarrow \beta) \rightarrow (\alpha \rightarrow \neg\beta) \rightarrow \neg\alpha$; $\neg\alpha \rightarrow (\alpha \rightarrow \beta)$; $\neg\alpha \vee \alpha$;
$(\alpha \rightarrow \beta) \rightarrow (\neg\beta \rightarrow \neg\alpha)$; $(\neg\beta \rightarrow \neg\alpha) \rightarrow (\alpha \rightarrow \beta)$; $\neg(\alpha \wedge \neg\alpha)$;
$\neg\alpha \vee \beta \rightarrow (\alpha \rightarrow \beta)$; $(\alpha \rightarrow \beta) \rightarrow \neg\alpha \vee \beta$.

THEOREM 6. *Some formulae of $\Im_?$ that satisfy one of the following schemas of formula are not $K_?$-theorems.*

$\alpha? \rightarrow \neg((\neg\alpha)!)$; $\neg((\neg\alpha)!) \rightarrow \alpha?$; $\neg(\alpha!) \rightarrow (\neg\alpha)?$; $(\neg\alpha)? \rightarrow \neg(\alpha!)$;
$\alpha! \rightarrow \neg((\neg\alpha)?)$; $\neg((\neg\alpha)?) \rightarrow (\alpha!)$; $\neg(\alpha?) \rightarrow (\neg\alpha)!$; $(\neg\alpha)! \rightarrow \neg(\alpha?)$;

5 Semantics

In this section we present the semantics of paranormal modal logic. Again we tried to use a formal structure that resembles as most as possible traditional modal logic. The two definitions that follow define a semantic structure that is identical to the one we use to evaluate the truth-value of formulas in normal modal logic. The difference will be the function that will make use of this structure to recursively attribute true or false to all sorts of formulas, to be defined in Definition 9

DEFINITION 7. A frame F is a duple $\langle W, R\rangle$ where W is a non-empty set of entities called *(plausible) words* and R is a binary relation on W called *accessibility relation.*

DEFINITION 8. A model M is a triple $\langle W, R, v\rangle$ where $F = \langle W, R\rangle$ is a frame and v is function mapping pairs composed by elements of P and

[10]If we wish, we could only remove the qualifications on A1-A3 as well as K2 and K3 that we would still have an axiomatic for K, only a redundant one.

[11]From an axiomatic point of view, the difference between global and local premises is that the rule of necessitation can be applied only to those formulas derived exclusively from the set of global premises. For a textbook-style presentation of modal logic that uses global and local premises in the definition of the relations of deductibility and logical consequence see [10].

elements of W to truth-values 0 and 1. We say that the model M is *based on* F and that $w \in W$ is a world of M.

DEFINITION 9. The *max-min modal valuations* are functions Ω and $\mho$ which, given a model $M = \langle W, R, v\rangle$ and a world $w \in W$, map formulas of $\Im_?$ to truth-values 0 and 1. Ω and $\mho$ are defined as follows:

(i) $\Omega_{M,w}(p) = \mho_{M,w}(p) = 1$ iff $v_w(p) = 1$;
(ii) $\Omega_{M,w}(\neg\alpha) = 1$ iff $\mho_{M,w}(\alpha) = 0$;
(iii) $\mho_{M,w}(\neg\alpha) = 1$ iff $\Omega_{M,w}(\alpha) = 0$;
(iv) $\Omega_{M,w}(\alpha \to \beta) = 1$ iff $\Omega_{M,w}(\alpha) = 0$ or $\Omega_{M,w}(\beta) = 1$;
(v) $\mho_{M,w}(\alpha \to \beta) = 1$ iff $\Omega_{M,w}(\alpha) = 0$ or $\mho_{M,w}(\beta) = 1$;
(vi) $\Omega_{M,w}(\alpha \wedge \beta) = 1$ iff $\Omega_{M,w}(\alpha) = 1$ and $\Omega_{M,w}(\beta) = 1$;
(vii) $\mho_{M,w}(\alpha \wedge \beta) = 1$ iff $\mho_{M,w}(\alpha) = 1$ and $\mho_{M,w}(\beta) = 1$;
(viii) $\Omega_{M,w}(\alpha \vee \beta) = 1$ iff $\Omega_{M,w}(\alpha) = 1$ or $\Omega_{M,w}(\beta) = 1$;
(ix) $\mho_{M,w}(\alpha \vee \beta) = 1$ iff $\mho_{M,w}(\alpha) = 1$ or $\mho_{M,w}(\beta) = 1$;
(x) $\Omega_{M,w}(\alpha?) = 1$ iff, for some $w' \in W$ such that wRw', $\Omega_{M,w'}(\alpha) = 1$;
(xi) $\mho_{M,w}(\alpha?) = 1$ iff, for all $w' \in W$ such that wRw', $\mho_{M,w'}(\alpha) = 1$;
(xii) $\Omega_{M,w}(\alpha!) = 1$ iff, for all $w' \in W$ such that wRw', $\Omega_{M,w'}(\alpha) = 1$;
(xiii) $\mho_{M,w}(\alpha!) = 1$ iff, for some $w' \in W$ such that wRw', $\mho_{M,w'}(\alpha) = 1$;.

Ω and $\mho$ are functions that, depending on the modal operator attached to the formula, evaluate it either skeptically or credulously (if the formulas is a non-modal one, the evaluation is, we can say, neutral.) They represent therefore, depending on the formula being evaluated, our skeptical and credulous approaches to induction. In what follows we shall try to elucidate some key aspects of Ω and $\mho$[12].

Our first concern will be how, through (x)-(xiii), modal formulas are evaluated. While Ω, which is the function we shall use in the definition of logical consequence, gives the intended meaning of ! and ?, $\mho$ does the opposite and evaluates !-formulas credulously and ?-formulas skeptically. The need for this "anomalous" interpretation has to do with the way Ω and $\mho$ deal with the negation, which is materialized in items (ii) and (iii). The rationale behind these rules can be seen through a simple interpretation of the meaning of functions Ω and $\mho$. Consider $\neg(\alpha?)$. Since Ω will evaluate it credulously, $\mho$ evaluates $\alpha?$ skeptically. We can read $\Omega_{M,w}(\neg(\alpha?)) = 1$ as "according to a credulous position that tries to maximize the truthfulness of formulas, 'not [α is plausible]' is true." This can be said to be equivalent to "the task of maximizing the truthfulness of 'not [α is plausible]' was successful," which in its turn is the same as "the task of maximizing the falsehood of 'α is plausible' was successful." Now consider $\mho_{M,w}(\alpha?) = 0$. It can be read as "according to a position that tries to minimize the truthfulness of formulas, 'α is plausible' is not true," which in its turn is equivalent to "the task of minimizing the truth of 'α is plausible' failed," which is pretty the same as "the task of maximizing the falsehood of 'α

[12]It should be said that a full explanation of the intricacies of this unusual way to evaluate modal formulas would need much more space. For a full account of this and other aspects of paranormal modal logic, including a first-order formulation, see [22].

is plausible' was successful." That is why $\Omega_{M,w}(\neg\alpha) = 1$ iff $\mho_{M,w}(\alpha) = 0$. The same can be done to show that $\mho_{M,w}(\neg\alpha) = 1$ iff $\Omega_{M,w}(\alpha) = 0$. Thus we see why to evaluate $\neg(\alpha?)$ credulously and $\neg(\alpha!)$ skeptically we need to evaluate $\alpha?$ skeptically and $\alpha!$ credulously.

Our final comment concerns item (iv), which is asymmetric with respect to (v). It can be shown that making use of the simple idea of skeptical and credulous evaluation as described here, the correct form of (iv) would be $\Omega_{M,w}(\alpha \rightarrow \beta) = 1$ iff $\mho_{M,w}(\alpha) = 0$ or $\Omega_{M,w}(\beta) = 1$. From a purely conceptual point of view, $\alpha \rightarrow \beta$ is true according to a maximal position, for example, iff α is false according to a minimal position or β is true according to a maximal one. However, if we follow that, we would not be able to preserve *modus ponens*, since our consequence relation is defined over Ω[13] .

Alike to the axiomatic presented in the last section, there is a strong parallel between paranormal modal logic's semantics and normal modal logic's one. As we said, the notion of semantic model is exactly the same; what changes is the way to use the model to evaluate the truth-value of formulas. Roughly speaking, we just have, in addition to the usual way to evaluate modal formulas (which is, except for the asymmetry of (iv), captured by Ω), another function intended to reverse the position taken by Ω. This additional function is essential for the proper treatment of negation in (ii) and (iii), which is the key of the non-standard behavior of ! and ?. Aside that, other paranormal modal systems are obtained in the standard way by restricting the accessibility relation.

DEFINITION 10. Let $M = \langle W, R, v\rangle$ be a model, $w \in W$ a world of M and $\alpha \in \Im_?$ a formula: α *is satisfied by* M at w (in symbols: $M, w \Vdash \alpha$) iff $\Omega_{M,w}(\alpha) = 1$; α *is satisfied by* M (in symbols: $M \Vdash \alpha$) iff, for all $w' \in W$, $M, w' \Vdash \alpha$.

From the two notions above we define, in the customary way, the notion of logical consequence. Let $A, B \subseteq \Im_?$ be two sets of formulae, A representing the global premises and B the local ones, and $\alpha \in \Im_?$ a formula. If α is a logical consequence of A and B (all classes of models being considered) we say that α is a $K_?$-logical consequence of A (in symbols: $A + B \models_{k?} \alpha$). In the case $A = B = \emptyset$, we say that α is $K_?$-valid (in symbols: $\models_{k?} \alpha$).

THEOREM 11. *$K_?$ is sound and complete (in symbols: Let $A, B \subseteq \Im_?$ be two sets of formulas and $\alpha \in \Im_?$ a formula. $A + B \vdash_{K?} \alpha$ iff $A + B \models_{K?} \alpha$).*

6 Normal and Paranormal Modal Logic: K and $K_?$

In this section we show the results which, from a formal point of view, more strongly relate paranormal modal logic with normal modal logic. Let $\Im_\Diamond$ be the language of propositional normal modal logic,$\Diamond$ being taken either as a primitive or a derived symbol, and $\vdash_K$ and $\models_K$ the relations of deductibility and logical consequence of modal logic K.

[13] Although acknowledging some *ad hoc* flavor in this maneuver, we do not think this sole point represents a threat to our use of the skeptical and credulous approaches to induction to provide an intuition for our logic.

DEFINITION 12. We define the function Φ: $\Im_\Diamond \to \Im_?$ as follows:
(i) $\Phi(p) = p$, where $p \in P$;
(ii) $\Phi(\neg\alpha) = \sim \Phi(\alpha)$;
(iii) $\Phi(\alpha \oplus \beta) = \Phi(\alpha) \oplus \Phi(\beta)$, where $\oplus \in \{\wedge, \vee, \to\}$;
(iv) $\Phi(\Diamond\alpha) = \Phi(\alpha)?$;
(v) $\Phi(\Box\alpha) = \Phi(\alpha)!$.

THEOREM 13. *Let* $\alpha \in \Im_?$. *If* $\vdash_K \alpha$, *then* $\vdash_{K?} \Phi(\alpha)$.

The meaning of theorem 13 is that, when taken in conjunction with $\sim$, paranormal modalities ! and ? are indistinguishable from normal modalities $\Box$ and $\Diamond$. This of course implies that the full expressive power of normal modal logic is contained in paranormal modal logic. Theorems 17-20 below show the stronger result that the inverse is also true.

DEFINITION 14. We define the functions Π and $\amalg$ of the form: $\Im_? \to \Im_\Diamond$ as follows:
(i) $\Pi(p) = \amalg(p) = p$;
(ii) $\Pi(\alpha?) = \Diamond\Pi(\alpha)$;
(iii) $\amalg(\alpha?) = \Box \amalg (\alpha)$;
(iv) $\Pi(\alpha!) = \Box\Pi(\alpha)$;
(v) $\amalg(\alpha!) = \Diamond \amalg (\alpha)$;
(vi) $\Pi(\neg\alpha) = \neg \amalg (\alpha)$;
(vii) $\amalg(\neg\alpha) = \neg\Pi(\alpha)$;
(viii) $\Pi(\alpha \oplus \beta) = \Pi(\alpha) \oplus \Pi(\beta)$, where $\oplus \in \{\wedge, \vee, \to\}$;
(ix) $\amalg(\alpha \oplus \beta) = \amalg(\alpha) \oplus \amalg(\beta)$, where $\oplus \in \{\wedge, \vee\}$;
(x) $\amalg(\alpha \to \beta) = \Pi(\alpha) \to \amalg(\beta)$.

DEFINITION 15. We define the functions Δ and ∇ of the form: $\Im_\Diamond \to \Im_?$ as follows:
(i) $\Delta(p) = \nabla(p) = p$;
(ii) $\Delta(\Diamond\alpha) = \Delta(\alpha)?$;
(iii) $\nabla(\Diamond\alpha) = \nabla(\alpha)!$;
(iv) $\Delta(\Box\alpha) = \Delta(\alpha)!$;
(v) $\nabla(\Box\alpha) = \nabla(\alpha)?$;
(vi) $\Delta(\neg\alpha) = \neg\nabla(\alpha)$;
(vii) $\nabla(\neg\alpha) = \neg\Delta(\alpha)$;
(viii) $\Delta(\alpha \oplus \beta) = \Delta(\alpha) \oplus \Delta(\beta)$, where $\oplus \in \{\wedge, \vee, \to\}$;
(ix) $\nabla(\alpha \oplus \beta) = \nabla(\alpha) \oplus \nabla(\beta)$, where $\oplus \in \{\wedge, \vee\}$;
(x) $\nabla(\alpha \to \beta) = \Delta(\alpha) \to \nabla(\beta)$.

DEFINITION 16. Let $A \subseteq \Im_?$ and $B \subseteq \Im_\Diamond$.
(i) $\Pi(A) = \{\Pi(\alpha) | \alpha \in A\}$;
(ii) $\amalg(A) = \{\amalg(\alpha) | \alpha \in A\}$;
(iii) $\Delta(B) = \{\Delta(\alpha) | \alpha \in B\}$;
(iv) $\nabla(B) = \{\nabla(\alpha) | \alpha \in B\}$.

THEOREM 17. *Let* $A, B \subseteq \Im_\Diamond$ *and* $\alpha \in \Im_\Diamond$. $A + B \vdash_K \alpha$ *iff* $\Delta(A) + \Delta(B) \vdash_{K?} \Delta(\alpha)$.

THEOREM 18. *Let* $A, B \subseteq \Im_?$ *and* $\alpha \in \Im_?$. $A + B \vdash_{K?} \alpha$ *iff* $\Pi(A) + \Pi(B) \vdash_K \Pi(\alpha)$.

THEOREM 19. *Let* $A, B \subseteq \Im_\diamond$ *and* $\alpha \in \Im_\diamond$. $A + B \models_K \alpha$ *iff* $\Delta(A) + \Delta(B) \models_{K?} \Delta(\alpha)$.

THEOREM 20. *Let* $A, B \subseteq \Im_?$ *and* $\alpha \in \Im_?$. $A + B \models_{K?} \alpha$ *iff* $\Pi(A) + \Pi(B) \models_K \Pi(\alpha)$.

Definitions 14 and 15 formalize a translation that comes naturally when we look at the semantics of normal and paranormal modal logics. Using them, theorems 17-20 state that, according to this translation, both normal and paranormal logics can be fully embedded inside each other. The implications of this are obvious. Since formulas resulting from the application of Δ can be seen as abbreviations inside $\Im_\diamond$, it might be said that there is a formal paraconsistent and paracomplete inferential relation (in addition to a conceptual one) based on a truly paranormal modality-dependent negation inside normal modal logic. This we think strengths the thesis we have mentioned in Section 1 about normal modal logic being paranormal.

This result might give room for a sort of objection that questions the whole worthiness of our endeavor: if all the expressive and inferential power of paranormal modal logic is already contained in normal modal logic, what is the point of developing and studying it? Well, taking this objection seriously might allow us to refer to theorems 18 and 19 and claim that since all the expressive and inferential power of normal modal logic is contained in paranormal modal logic, why not to question instead the supremacy of normal modal logic? After all, given the above stated equivalence between the two systems, the only real reason for this supremacy would be the historical one that one logic was discovered, created, or whatever, before the other. Of course things are not so simple. In fact, the whole situation is very alike to one that might be created from the fact that it is possible to translate S5 into the fragment of monadic classical first-order logic with only one variable, and vice-versa [24].

I particularly prefer to stick to the formal apparatus available to us and think of two different but really strongly connected formal systems; so strongly connected that might be taken as different aspects of the same thing. From this perspective, I can only think of those funny toys we often see on cartoons which at one time look like a car and at other time look like a completely different object, say, an airplane. Despite the toy being, at one specific time, from the point of view of the child who plays with it at that time, only a car, all the stuffs needed for the airplane are already there, inside the car.

7 Conclusion

In this paper we presented what we have called paranormal modal logic with two main purposes in mind. The first, if the reader allows me to take the inverse order, was to formalize a logic of plausibility based on what we have called the skeptical and the credulous approaches to induction. The conceptual framework provided by this theory of induction was the main intuitive justification we used to rationalize some (somehow controversial) conceptual features of our system. Second, we intended to throw some light upon the relations that hold between modal logic and paraconsistent and paracomplete logic. We think we have succeeded in this task in that the simple fact of having a sound and complete paraconsistent and paracomplete modal logic which, from the formal point of view, is as close as possible to traditional modal logic already gives important hints about what in modal logic can be said to be connected to paraconsistency and paracompleteness and consequently might help in the development of a paranormal modal logic and what is incompatible and need to be given up in such endeavor. Also, and surely more surprising, we showed that paranormal and normal modal logics can from a representational and inferential point of view be fully embedded in each other.

Perhaps one might object to the above conclusions that we should have done a more general work, perhaps from a meta-level, instead of taking two specific modal systems. Well, despite having based our analysis on two systems, K and $K_?$, the conclusions we drew are trivially applicable to the two classes of logical systems definable upon K and $K_?$. Second, despite being "just one modal logic," we believe normal modal logic K and normal logics in general are very distinguished representatives of the class of modal logics, in that conclusions of the sort we got here about them are surely relevant for the whole field of modal logic.

BIBLIOGRAPHY

[1] Batens, Diderik (2002), On the Remarkable Correspondence between Paraconsistent Logics, Modal Logics and Ambiguity Logics. In Paraconsistency: The Logical Way to the Inconsistency, eds. W. A. Carnielli, M. E. Coniglio e I. D' Ottaviano: 445-454, New York: Marcel Dekker.

[2] Béziau, J. Y. (199), The future of paraconsistent logic. Logical Studies 2: 1-23.

[3] Béziau, J. Y. (2002), S5 is a paraconsistent logic and so is first-order classical logic. Logical Studies, 9:301-309.

[4] Béziau, J. Y. (2005), Paraconsistent logic from a modal viewpoint. Journal of Applied Logic, 3(1):7-14.

[5] Buchsbaum, Arthur R. and T. Pequeno (1993), Uma Família de Lógicas Paraconsistentes e/ou Paracompletas com Semânticas Recursivas. Coleçâo Documentos, Série Lógica e Teoria da Ciência 14, Universidade de São Paulo.

[6] da Costa, N. C. A. (1963), Calculs propositionnels pour les systèmes formels inconsistants. Comtes Rendus de l'Académie des Sciences de Paris 257: 3790-3793.

[7] da Costa, N. C. A. (1974), On the Theory of Inconsistent Formal Systems. Notre Dame Journal of Formal Logic 15.

[8] Dosen, K. (1986), Negation as a modal operator. Reports on Mathematical Logic 20:15-28.

[9] Fagin, Ronald and Y. H. Joseph and M. Y. Vardi (1996), A Nonstandard Approach to the Logical Omniscience Problem. Artificial Intelligence, 79: 203–240

[10] Fitting, Melvin (1993), Basic Modal Logic. In Handbook of Logic in Artificial Intelligence and Logic Programming, Vol. 1, Logical Foundations, eds. D. Gabbay and D. Hogger and J. Robinson, Oxford: Oxford University Press.

[11] Hintikka, J. (1975), Impossible possible worlds vindicated. Journal of Philosophical Logic 4: 475-484.

[12] Hughes, G. E and M. J. Cresswell (1996), A New Introduction to Modal Logic. New York: Routledge.
[13] Jaśkowski, S. (1948), A propositional calculus for inconsistent deductive systems (in Polish). Studia Societatis Scientiarum Torunensis, Sectio A, 5:57-77. Translated into English in Studia Logica, 24:143-157, 1967.
[14] Kripke, S. (1965), Semantical analysis of modal logic II: non-normal modal propositional calculi. In The theory of models, Vol. 9, eds. J. W. Addison, L. Henkin and A. Tarski: 206-220, Amsterdam: North-Holland.
[15] Loparíc, Andréa and N. da Costa (1984), Paraconsistency, Paracompletenes and Valuations. Logique et Analyse 106: 119-131.
[16] Marcos, J. (2005), Nearly every normal modal logic is paranormal. Logique et Analyse 48: 279-300.
[17] Martins, A. T. C and M. Pequeno and T. Pequeno (2002), A Multiple Worlds Semantics to a Paraconsistent Nonmonotonic Logic. In Paraconsistency: The Logical Way to the Inconsistency, eds. W. A. Carnielli, M. E. Coniglio e I. D' Ottaviano: 187-211, New York: Marcel Dekker.
[18] Pequeno, Tarcisio H. C. and Arthur R. Buchsbaum (1991), The Logic of Epistemic Inconsistency. In Principles of Knowledge Representation and Reasoning: Proceedings of Second International Conference, San Mateo: Morgan Kaufman.
[19] Priest, G. and R. Routley and J. Norman (eds.) Paraconsistent Logic: Essays on the Inconsistent. München: Philosophia Verlag, 1989.
[20] Rescher, Nicholas and Robert Brandom (1980), The Logic of Inconsistency: A study in nonstandard possible-world semantics and ontology. Basil Blackwell.
[21] Silvestre, Ricardo and T. Pequeno (2005), A Logic of Inductive Implication or AI Meets Philosophy of Science II. In Proceedings of the 18th Conference of the Canadian Society for the Computational Studies of Intelligence, (LNAI 3501), eds. B. Kégl and G. Lapalme (eds.): 232-243, Berlin-Heidelberg: Springer-Verlag.
[22] Silvestre, Ricardo (2005), Induction and Plausibility: A Formal Approach from the Standpoint of Artificial Intelligence. Ph.D. dissertation, University of Montreal, Montreal.
[23] Vakarelov, D. (1989), Consistency, completeness and negation. In Paraconsistent Logic: Essays on the inconsistent, eds. G. Priest, R. Sylvan and J. Norman, Philosophia Verlag: 328-363.
[24] Wajsberg, M (1933), Ein erweiteter Klassenkalkül. Monatshefte für Mathematik und Physik 40: 113-126.

Appendix: Sketch of the Proofs of Theorems

We will here just sketch the proofs of theorems 11, 17, 18, 19 and 20. In the spirit of the main purpose of the paper, we will base the proof of theorem 11 on theorems 17-20. Therefore the proof of theorems 17-20 shall come first. For a full account of the proofs see [22]. In order to prove theorems 17-20 some key lemmas are needed. Let P1,..., P11, K, NP, MP and N be the axiomatic for K in which the axioms P1, ..., P8 are as in definition 4, P9, P10 and P11 are the non-qualified axioms corresponding to A1, A2 and A3, respectively, K is $\Box(\alpha \rightarrow \beta) \rightarrow (\Box\alpha \rightarrow \Box\beta)$, NP is $\Diamond\alpha \leftrightarrow \neg\Box\alpha$, N is $\alpha/\Box\alpha$ and MP is *modus ponens*. Let also Ψ be the evaluation function of normal modal logic (one which functions like Ω) and $\Vdash_\Psi$ (and $\Vdash_\Omega$) the respective satisfaction relation.

LEMMA 21. *Let* $\alpha \in \Im_?$ *and* $\beta \in \Im_\Diamond$. $\Delta(\Pi(\alpha)) = \alpha$ *and* $\Pi(\Delta(\beta)) = \beta$. *Let* $A \subseteq \Im_?$ *and* $B \subseteq L_\Diamond$. $\Pi(\Delta(A)) = A$ *and* $\Pi(\Delta(B)) = B$.

LEMMA 22. *Let* $A, B \subseteq \Im_?$ *and* $\alpha \in \Im_?$. *If* $A + B \vdash_{K?} \alpha$ *then* $\Pi(A) + \Pi(B) \vdash_K \Pi(\alpha)$.

LEMMA 23. *Let* $A, B \subseteq \Im_\diamond$ *and* $\alpha \in \Im_\diamond$. *If* $A + B \vdash_K \alpha$ *then* $\Delta(A) + \Delta(B) \vdash_{K?} \Delta(\alpha)$.

LEMMA 24. *Let* $\alpha \in \Im_\diamond$ *be a formula,* $M = \langle W, R, v\rangle$ *a model and* $w \in W$ *a world of* M. $M, w \Vdash_\Psi \alpha$ *iff* $M, w \Vdash_\Omega \Delta(\alpha)$ *or, equivalently,* $\Psi_{M,w}(\alpha) = 1$ *iff* $\Omega_{M,w}(\Delta(\alpha)) = 1$.

LEMMA 25. *Let* $\alpha \in \Im_?$ *a formula,* M *be a model and* w *a world of* M. $M, w \Vdash_{\Omega?} \alpha$ *iff* $M, w \Vdash_\Psi \Pi(\alpha)$ *or, equivalently,* $\Omega_{M,w}(\alpha) = 1$ *iff* $\Psi_{M,w}(\Pi(\alpha)) = 1$.

Lemma 21 follows from the fact that $\Delta$and Π are the inverse functions of each other. Proof of lemmas 22 and 23 are rather long. The key idea however is simple. In the case of lemma 23, we do the proof by induction on the size of the K-derivation of α from A and B. The base of induction is trivial for the case where $\alpha \in A \cup B$. In the case α is an axiom of K, we have to analyze the possibility of α's being an instance of each one of K's axiom schemas. For each one of these possibilities, we show that there is a $K_?$-derivation of $\Delta(\alpha)$. Regarding K's classical axioms, the cases where α is an instance of P1-P8 are trivial. To show that if α is an instance of P9, P10, P11 then there is a $K_?$-derivation of $\Delta(\alpha)$ requires much more labour (and space.) In order to deal with P9, for example, we are considering $\alpha \equiv (\varphi \to \beta) \to ((\varphi \to \neg\beta) \to \varphi)$. $\Delta(\alpha)$ then is $(\Delta\varphi \to \Delta\beta) \to ((\Delta\varphi \to \neg\nabla\beta) \to \neg\nabla\varphi)$. We then prove that there exists a $K_?$-derivation of $(\Delta\varphi \to \Delta\beta) \to ((\Delta\varphi \to \neg\nabla\beta) \to \neg\nabla\varphi)$. In order to do that, we have to do another proof by induction, now on the size of φ. K and NP are quite easy. In the step of induction we consider K-derivations of α from A and B of size $n > 1$ and suppose that for K-derivations of sizes smaller than n the result holds. That is to say, if $A + B \vdash_K \varphi$ and the size of the derivation of φ from A and B is smaller than n, then $\Delta(A) + \Delta(B) \vdash_{K?} \Delta(\varphi)$. Then we just prove that with this supposition, either considering MP or N the result will hold.

The proof of lemma 22 follows the same idea, just that at this time we have to consider $K_?$-derivations and Π and Π.

The proof of lemma 24 is done by induction on the size of α. The basic case where $\alpha \equiv p$ is trivial. In the hypothesis of induction we take an arbitrary formula α and suppose that the result holds for all formulae φ of size $m < n$, where n is α's size. We then prove that, if this is the case, the result also holds for α. This is done by considering all possible forms α may have. The only situation which poses some difficulty is the case where $\alpha \equiv \neg\varphi$. For all others, the proof is trivial. In this difficult case, $\Delta\alpha \equiv \Delta(\neg\varphi) \equiv \neg\nabla\varphi$. We have then to prove that $\Psi_{M,w}(\neg\varphi) = 1$ iff $\Omega_{M,w}(\neg\nabla\varphi) = 1$, which is done by induction on the size of φ. For the hypothesis of induction (the basis is trivial), we suppose that the result holds for formulae of size smaller than φ's size. We then show that, if this supposition holds, independently of the form of φ, the general result that $\Psi_{M,w}(\neg\varphi) = 1$ iff $\Omega_{M,w}(\neg\nabla\varphi) = 1$ also holds. As usual, we consider all forms φ may have. The proof of lemma 25 is almost identical to lemma 24's. All we have to do is to properly erase the occurrences of Δ and consider function Π along with Ψ.

With these lemmas at hand, we can proceed to the proofs of theorems 17-20.

Theorem 17. Let $A, B \subseteq \Im_\diamond$ and $\alpha \in \Im_\diamond$. $A + B \vdash_K \alpha$ iff $\Delta(A) + \Delta(B) \vdash_{K?} \Delta(\alpha)$.

Proof. By lemma 23, if $A+B \vdash_K \alpha$ then $\Delta(A)+\Delta(B) \vdash_{K?} \Delta(\alpha)$. Suppose that $\Delta(A) + \Delta(B) \vdash_{K?} \Delta(\alpha)$ but $A + B \nvdash_K \alpha$. If $A + B \nvdash_K \alpha$, by lemma 21, we have that $\Pi(\Delta(A)) + \Pi(\Delta(B)) \nvdash_K \Pi(\Delta(\alpha))$. By lemma 22, we have then that $\Delta(A) + \Delta(B) \nvdash_{K?} \Delta(\alpha)$, which is a contradiction. Therefore, if $\Delta(A) + \Delta(B) \vdash_{K?} \Delta\alpha)$ then $A + B \vdash_K \alpha$. ■

Theorem 18. Let $A, B \subseteq \Im_?$ and $\alpha \in \Im_?$. $A + B \vdash_{K?} \alpha$ iff $\Pi(A) + \Pi(B) \vdash_K \Pi(\alpha)$.

Proof. By lemma 22, if $A+B \vdash_{K?} \alpha$ then $\Pi(A)+\Pi(B) \vdash_K \Pi(\alpha)$. Suppose that $\Pi(A) + \Pi(B) \vdash_K \Pi(\alpha)$ but $A + B \nvdash_{K?} \alpha$. If $A + B \nvdash_{K?} \alpha$, by lemma 21, we have that $\Delta(\Pi(A)) + \Delta(\Pi(B)) \nvdash_{K?} \Delta(\Pi(\alpha))$. By lemma 23, we have then that $\Pi(A) + \Pi(B) \nvdash_K \Pi(\alpha)$, which is a contradiction. Therefore, if $\Pi(A) + \Pi(B) \vdash_K \Pi(\alpha)$ then $A + B \vdash_{K?} \alpha$. ■

Theorem 19. Let $A, B \subseteq \Im_\diamond$ and $\alpha \in \Im_\diamond$. $A + B \models_K \alpha$ iff $\Delta(A) + \Delta(B) \models_{K?} \Delta(\alpha)$.

Proof. Suppose that $A + B \models_K \alpha$ but $\Delta(A) + \Delta(B) \not\models_{K?} \Delta(\alpha)$. If $\Delta(A) + \Delta(B) \not\models_{K?} \Delta(\alpha)$, then there is a model M and a world w of M such that $M \Vdash_\Omega \Delta(\phi)$, for all $\Delta(\phi) \in \Delta(A), M, w \Vdash_\Omega \Delta(\lambda)$, for all $\Delta(\lambda) \in \Delta(B)$, and $M, w \nVdash_\Omega \Delta(\alpha)$. But if $M \Vdash_\Omega \Delta(\phi)$ for all $\Delta(\phi) \in \Delta(A), M, w \Vdash_\Omega \Delta(\lambda)$ for all $\Delta(\lambda) \in \Delta(B)$ and $M, w \nVdash_\Omega \Delta(\alpha)$, by lemma 24 we have that $M \Vdash_{\Psi?} \phi$ for all $\phi \in A, M, w \Vdash_\Psi \lambda$ for all $\lambda \in B$ and $M, w \nVdash_\Psi \alpha$. Consequently, $A+B \not\models_K \alpha$, which is a contradiction. Therefore, if $A + B \models_K \alpha, \Delta(A) + \Delta(B) \models_{K?} \Delta(\alpha)$. Suppose now that $\Delta(A) + \Delta(B) \models_{K?} \Delta(\alpha)$ but $A + B \not\models_K \alpha$. If $A+B \not\models_K \alpha$ then there is a model M and a world w of M such that $M \Vdash_\Psi \phi$ for all $\phi \in A, M, w \Vdash_\Psi \lambda$ for all $\lambda \in B$ and $M, w \nVdash_\Psi \alpha$. But if $M \Vdash_\Psi \phi$ for all $\phi \in A, M, w \Vdash_\Psi \lambda$ for all $\lambda \in B$ and $M, w \nVdash_\Psi \alpha$, then by lemma 24 $M \Vdash_\Omega \Delta(\phi)$ for all $\Delta(\phi) \in \Delta(A), M, w \Vdash_\Omega \Delta(\lambda)$ for all $\Delta(\lambda) \in \Delta(B)$ and $M, w \nVdash_\Omega \Delta(\alpha)$. Consequently, $\Delta(A) + \Delta(B) \not\models_{K?} \Delta(\alpha)$, which is a contradiction. Therefore, if $\Delta(A)+\Delta(B) \models_{K?} \Delta(\alpha)$ then $A+B \models_K \alpha$. ■

Theorem 20. Let $A, B \subseteq \Im_?$ and $\alpha \in \Im_?$. $A + B \models_{K?} \alpha$ iff $\Pi(A) + \Pi(B) \models_K \Pi(\alpha)$.

Proof. The proof of this theorem follows the same idea of theorem 19's. We have just to properly erase the occurrences of Δ and consider function Π along with Ψ as well as to use lemma 25 instead of lemma 24. ■

Theorem 11. $K_?$ is sound and complete (in symbols: Let $A, B \subseteq \Im_?$ be two sets of formulae and $\alpha \in \Im_?$ a formula. $A + B \vdash_{K?} \alpha$ iff $A + B \models_{K?} \alpha$).

Proof. Let us first prove the left-right direction (soundness): for any $A, B \subseteq \Im_?$ and $\alpha \in \Im_?$, if $A + B \vdash_{K?} \alpha$ then $A + B \models_{K?} \alpha$. Suppose that $A + B \vdash_{K?} \alpha$ and $A + B \not\models_{K?} \alpha$. If $A + B \not\models_{K?} \alpha$, by theorem 20 we have that $\Pi(A) + \Pi(B) \not\models_K \Pi(\alpha)$. By the soundness theorem of normal modal logic K[14], we have $\Pi(A) + \Pi(B) \nvdash_K \Pi(\alpha)$. From that, along with theorem 18, we have that $A + B \nvdash_{K?} \alpha$, which is a contradiction. Therefore, if $A + B \vdash_{K?} \alpha$ then $A + B \models_{K?} \alpha$. The right-left direction (completeness) is proved through the same reference to normal modal logic K. Suppose that $A + B \models_{K?} \alpha$ and $A + B \nvdash_{K?} \alpha$. If $A + B \nvdash_{K?} \alpha$, by theorem 18 we have that $\Pi(A) + \Pi(B) \nvdash_K \Pi(\alpha)$. By the completeness theorem of normal modal logic K, we have then that $\Pi(A) + \Pi(B) \not\models_K \Pi(\alpha)$. From that, along with theorem 20, we have that $A + B \not\models_{K?} \alpha$, which is a contradiction. Therefore, if $A + B \models_{K?} \alpha$ then $A + B \vdash_{K?} \alpha$. ■

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[14]For the proof of soundness and completeness of normal modal logic K with local and global premises see [10].