From topology to metric: modal logic and quantification in metric spaces

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ABSTRACT. We propose a framework for comparing the expressive power and computational behaviour of modal logics designed for reasoning about qualitative aspects of metric spaces. Within this framework we can compare such well-known logics as **S4** (for the topology induced by the metric), **wK4** (for the derivation operator of the topology), variants of conditional logic, as well as logics of comparative similarity. One of the main problems for the new family of logics is to delimit the borders between 'decidable' and 'undecidable.' As a first step in this direction, we consider the modal logic with the operator 'closer to a set τ_0 than to a set τ_1' interpreted in metric spaces. This logic contains **S4** with the universal modality and corresponds to a very natural language within our framework. We prove that over arbitrary metric spaces this logic is ExpTime-complete. Recall that over $\mathbb{R}$, $\mathbb{Q}$, and $\mathbb{Z}$, as well as their finite subspaces, this logic is undecidable.

Keywords: metric, topology, comparative similarity, conditional logic.

1 Introduction

Various 'modal-like' propositional logics have been introduced for reasoning about qualitative aspects of metric spaces. Recall, for example, the modal logic **S4** whose diamond and box are interpreted as the closure and interior operators of the induced topology [10], the modal logic **wK4** whose box is interpreted as the derivation operator on the topological space (see, e.g., [10, 2]), the extensions of these logics with the universal modality, conditional logic with a binary operator comparing distances between points [8], and the logics of comparative similarity [12]. In all of these cases, the truth conditions for the modal operators correspond to certain simple quantifier patterns of first-order logic:

- the interior $\mathbb{I}X$ of a set X is the set

$$\big\{w \mid \exists a \in \mathbb{R}^{>0}\, \forall v\, (d(w,v) < a \to v \in X)\big\},$$

- the universal box $\Box$ is defined by

$$\Box X \;=\; \big\{w \mid \forall a \in \mathbb{R}^{>0}\, \forall v\, (d(w,v) < a \to v \in X)\big\},$$

- the derived set ∂X of X is

$$\big\{w \mid \forall a \in \mathbb{R}^{>0}\, \exists v\, (0 < d(w,v) < a \wedge v \in X)\big\},$$

- the 'closer operator' $X \leftleftarrows Y$ for 'X is closer than Y' in the language $\mathcal{CSL}$ of comparative similarity or distances [12] is defined by

$$\big\{w \mid \exists a \in \mathbb{R}^{>0}(\exists v \in X(d(w,v) < a) \wedge \neg\exists v \in Y(d(w,v) < a))\big\},$$

- the interpretation of some variants of the conditional implication operator $X > Y$ [8] can be given by the formula (see, e.g., [1, 11])

$$(X \leftleftarrows (X \wedge \neg Y)) \vee \Box\neg X.$$

Of course, in the modal languages the quantifier patterns above are only implicit and therefore 'forgotten.' This probably explains why the relation between the first-order logic and different modal logics for metric spaces has received very little attention from the modal logic community; see [6] for a brief review of the available results.

In this paper, we first make the quantifier patterns above explicit by introducing 'modal' operators of the form $\exists^{<x}$, $\exists^{>x}$, $\exists^{=x}$, $\exists^{\leq x}_{>0}$ (and their duals $\forall^{<x}$, $\forall^{>x}$, etc.), where the variable x ranges over the *positive real numbers* and can be bound by the quantifiers $\forall x$ and $\exists x$. Intuitively, if x is assigned a value $a \in \mathbb{R}^{>0}$, then $\exists^{<x}X$ denotes the set of all points that are located at distance $< a$ from at least one point in X. In this language the intended meaning of the operators considered above can be represented in a clear and concise manner:

$$\mathbb{I}\tau \;=\; \exists x \forall^{<x}\tau, \tag{1.1}$$

$$\Box\tau \;=\; \forall x \forall^{<x}\tau, \tag{1.2}$$

$$\partial\tau \;=\; \forall x \exists^{\leq x}_{>0}\tau, \tag{1.3}$$

$$\tau_1 \leftleftarrows \tau_2 = \exists x(\exists^{<x}\tau_1 \wedge \neg\exists^{<x}\tau_2), \tag{1.4}$$

$$\tau_1 > \tau_2 \;=\; \exists x\left(\exists^{<x}\tau_1 \wedge \neg\exists^{<x}(\tau_1 \wedge \neg\tau_2)\right) \vee \Box\neg\tau_1. \tag{1.5}$$

We observe that the resulting modal language, called $\mathcal{QMS}$ (for qualitative metric system), has the same expressive power as the two-variable fragment of a certain two-sorted first-order language for metric spaces, and thereby obtain a first insight into the relation between first-order and (qualitative) modal languages for metric spaces. As follows from (1.1)–(1.3), the logics $\mathbf{wK4}, \mathbf{S4}$ and $\mathbf{S4}_u$ (that is, $\mathbf{S4}$ enriched with the universal modalities) give rise to $\mathcal{CTL}$-like fragments of $\mathcal{QMS}$, with modal operators corresponding to the quantifier pattern 'a quantifier over the reals followed by a quantifier over the space.' These observations motivate the following general research programme:

> *Classify and investigate fragments of $\mathcal{QMS}$ with respect to their expressive power and computational behaviour.*

In this paper, we contribute to this programme by investigating the $\mathcal{CTL}^+$-like extension of $\mathbf{S4}_u$ which allows quantifier patterns of the following form: a quantifier over the reals followed by a Boolean combination of quantifiers $\exists^{<x}$ over the space. A typical example is the closer operator $\leftleftarrows$ of $\mathcal{CSL}$.

First we show that this language has indeed the same expressive power over metric spaces as the modal logic $\mathcal{CSL}$ with sole operator $\leftleftarrows$ (although it might be exponentially more succinct). The computational properties of $\mathcal{CSL}$ over certain classes of metric spaces have already been investigated. We know from [12] that the satisfiability problem for $\mathcal{CSL}$ is ExpTime-complete over metric spaces with the so-called *min-condition*:

$$d(X,Y) \;=\; \inf\{d(x,y) \mid x \in X, y \in Y\} \;=\; \min\{d(x,y) \mid x \in X, y \in Y\},$$

for all sets X and Y. Moreover, in this case the logic enjoys the finite model property. On the other hand, it has been shown in [12] that over (arbitrary or arbitrary finite subspaces of) $\mathbb{R}$, $\mathbb{Q}$, and $\mathbb{Z}$ the logic $\mathcal{CSL}$ can simulate arbitrary Diophantine equations, and so is undecidable. The major open and technically challenging problem has been to investigate the behaviour of $\mathcal{CSL}$ over arbitrary metric spaces, where already very simple formulas like $\neg(p \leftleftarrows \neg p) \sqcap \neg(\neg p \leftleftarrows p)$ require infinite converging sequences. Here we present a solution to this problem by proving that over arbitrary metric spaces $\mathcal{CSL}$ is still ExpTime-complete.

$\mathcal{CSL}$ is closely related to certain conditional logics. In conditional logic, the min-condition above is often called the *limit assumption*, and spaces are not required to be symmetric. It has been shown in [12] that over possibly non-symmetric metric spaces with the min-condition, the closer operator $\leftleftarrows$ has the same expressive power as the conditional implication $>$. The resulting conditional logic is known as the logic of frames satisfying the normality, reflexivity, strict centering, uniformity and connectedness conditions [4]. In this paper, we do not consider distance spaces that are more general than metric spaces. However, the reader can easily modify the decidability proof given below for $\mathcal{CSL}$ over metric spaces in order to prove the decidability of $\mathcal{CSL}$ over distance spaces without symmetry and/or the triangle inequality.

2 The logic $\mathcal{QMS}$

In the examples above, we needed only one variable x over distances. In general, however, it is useful to have countably many *distance variables* $\{x_1, x_2, \dots\}$ and, in order to represent *constraints* on relations between distances, an additional set Σ of formulas over these variables. As the distance variables range over $\mathbb{R}^{>0}$ we can take, for example,

- the set Σ_0 of inequalities $x_i < x_j$,
- the set Σ_1 of linear rational equalities $a_1x_1 + \cdots + a_nx_n = a_{n+1}$,
- the set Σ_2 of linear rational inequalities $a_1x_1 + \cdots + a_nx_n \leq a_{n+1}$.

Suppose now that we have such a set Σ of constraints. Let $\{p_1, p_2, \dots\}$ be a countably infinite set of *set variables*. The $\mathcal{QMS}[\Sigma]$-*terms* are defined inductively as follows, where $\varkappa \in \Sigma$:

$$\begin{aligned}\tau \;::=\; p_i \;\mid\; \varkappa \;\mid\; \neg\tau \;\mid\; \tau_1 \sqcap \tau_2 \;\mid\; \\ \exists x_i \tau \;\mid\; \exists^{<x_i}\tau \;\mid\; \exists^{=x_i}\tau \;\mid\; \exists^{>x_i}\tau \;\mid\; \exists^{\leq x_i}_{>x_j}\tau \;\mid\; \exists^{\leq x_i}_{>0}\tau.\end{aligned}$$

The intended *metric models* for this language are structures of the form

$$\mathfrak{I} \;=\; \big(\mathfrak{D}, p_1^{\mathfrak{I}}, p_2^{\mathfrak{I}}, \dots\big) \tag{2.1}$$

where $\mathfrak{D} = (\Delta, d)$ is a *metric space* and the $p_i^{\mathfrak{I}}$ are subsets of Δ.

To interpret $\mathcal{QMS}[\Sigma]$-terms in metric models, we require *assignments* $\mathfrak{a}$ of *positive* real numbers $\mathfrak{a}(x_i) \in \mathbb{R}^{>0}$ to the distance variables x_i.[1] Given such an assignment $\mathfrak{a}$, we define the *extension* $p_i^{\mathfrak{I},\mathfrak{a}} \subseteq \Delta$ of a set variable p_i to be $p_i^{\mathfrak{I}}$. The *extension* $\varkappa^{\mathfrak{I},\mathfrak{a}} \in \{\emptyset, \Delta\}$ of $\varkappa \in \Sigma$ is defined by setting $\varkappa^{\mathfrak{I},\mathfrak{a}} = \Delta$ iff $(\mathbb{R}^{>0}, \mathfrak{a}) \models \varkappa$, and $\varkappa^{\mathfrak{I},\mathfrak{a}} = \emptyset$, otherwise.[2] The inductive definition of the *extension* $\tau^{\mathfrak{I},\mathfrak{a}} \subseteq \Delta$ of a $\mathcal{QMS}[\Sigma]$-term τ is now as usual for the Booleans and as follows for the remaining operators:

$$\begin{aligned}
(\exists^{=x_i}\tau)^{\mathfrak{I},\mathfrak{a}} &= (\exists^{=\mathfrak{a}(x_i)}\tau)^{\mathfrak{I}}, \qquad & (\exists^{<x_i}\tau)^{\mathfrak{I},\mathfrak{a}} &= (\exists^{<\mathfrak{a}(x_i)}\tau)^{\mathfrak{I}},\\
(\exists^{>x_i}\tau)^{\mathfrak{I},\mathfrak{a}} &= (\exists^{>\mathfrak{a}(x_i)}\tau)^{\mathfrak{I}}, \qquad & (\exists^{<x_i}_{>x_j}\tau)^{\mathfrak{I},\mathfrak{a}} &= (\exists^{<\mathfrak{a}(x_i)}_{>\mathfrak{a}(x_j)}\tau)^{\mathfrak{I}},\\
(\exists x_i\, \tau)^{\mathfrak{I},\mathfrak{a}} &= \bigcup\{\tau^{\mathfrak{I},\mathfrak{b}} \mid \mathfrak{b}(x_j) = \mathfrak{a}(x_j),\ \text{for } x_j \neq x_i\},
\end{aligned}$$

where, for $a, b \in \mathbb{R}^{>0}$,

$$\begin{aligned}
(\exists^{=a}\tau)^{\mathfrak{I}} &= \{x \in \Delta \mid \exists y\ (d(x,y) = a\ \wedge\ y \in \tau^{\mathfrak{I}})\},\\
(\exists^{<a}\tau)^{\mathfrak{I}} &= \{x \in \Delta \mid \exists y\ (d(x,y) < a\ \wedge\ y \in \tau^{\mathfrak{I}})\},\\
(\exists^{>a}\tau)^{\mathfrak{I}} &= \{x \in \Delta \mid \exists y\ (d(x,y) > a\ \wedge\ y \in \tau^{\mathfrak{I}})\},\\
(\exists^{<b}_{>a}\tau)^{\mathfrak{I}} &= \{x \in \Delta \mid \exists y\ (a < d(x,y) < b\ \wedge\ y \in \tau^{\mathfrak{I}})\}.
\end{aligned}$$

EXAMPLE 1. The sublanguage of $\mathcal{QMS}[\Sigma_2]$ with expressions of the form $\exists\vec{x}\ (\varkappa \wedge \tau)$, where $\varkappa$ is a conjunction of linear rational inequalities and τ is a $\mathcal{QMS}[\emptyset]$-term containing only the operators $\exists^{<x_i}$ and $\exists^{\leq x_i}$ and, additionally, quantifiers $\exists x_i$ only directly in front of $\exists^{<x_i}$ (as in the interior operator) has been investigated in [13]. In particular, it was proved that the satisfiability problem for this language is decidable.

To put this new language into a more familiar context, consider the following *two-sorted* first-order language $\mathcal{FM}[\Sigma]$. Its terms of sort $\mathbb{R}^{\geq 0}$ are the individual variables $x_1, x_2, \dots$, and terms of sort *object* are the individual variables $w_1, w_2, \dots$. The signature of $\mathcal{FM}$ also contains a countably infinite set $\{P_1, P_2, \dots\}$ of unary predicates, binary predicates $<$ and $=$, and a binary function symbol d. The $\mathcal{FM}[\Sigma]$-*formulas* φ are defined inductively as follows, where $\varkappa \in \Sigma$:

$$\begin{aligned}
\varphi \ ::=\ P_j(w_i) \ \mid\ \varkappa \ \mid\ d(w_i,w_j) < x_k \ \mid \\
x_k = 0 \ \mid\ \neg\varphi \ \mid\ \varphi_1 \wedge \varphi_2 \ \mid\ \exists x_i\varphi \ \mid\ \exists w_i\varphi.
\end{aligned}$$

[1] We quantify over *positive* real numbers rather than *non-negative* ones in order to obtain short and transparent definitions of standard topological operators; see (1.1). The expressiveness of the language does not depend on this assumption.

[2] It is straightforward to give a more conventional truth-definition for formulas in Σ by extending the language $\mathcal{QMS}$ with formulas and not regarding the members of Σ as terms. The semantics given here is a bit more concise.

$\mathcal{FM}[\Sigma]$ is interpreted in metric models $\mathfrak{I}$ of the form (2.1) with the help of two assignments $(\mathfrak{a}, \mathfrak{o})$, where $\mathfrak{a}$ assigns a non-negative real number to each x_i, while $\mathfrak{o}$ assigns an object from Δ to each w_i. The *satisfaction relation* $(\mathfrak{I}, \mathfrak{a}, \mathfrak{o}) \models \varphi$ is defined in the obvious way. Denote by $\mathcal{FM}_2[\Sigma]$ the fragment of $\mathcal{FM}[\Sigma]$ with only two variables of sort *object*.

The following expressive completeness result can be proved by an almost straightforward extension of the proof of Theorem 2.2 from [7] to quantifiers over distances. The succinctness result can be proved using the example and technique from [3]; see also [9].

THEOREM 2. *Let $\Sigma \supseteq \Sigma_0$. Then the language $\mathcal{QMS}[\Sigma]$ is expressively complete for the language $\mathcal{FM}_2[\Sigma]$ over metric models. More precisely, for each $\mathcal{QMS}[\Sigma]$-term τ, one can construct an $\mathcal{FM}_2[\Sigma]$-formula φ with a single free variable of sort object such that, for all metric models $\mathfrak{I}$ with assignments $\mathfrak{a}$ and all $o \in \Delta$,*

$$o \in \tau^{\mathfrak{I},\mathfrak{a}} \quad \textit{iff} \quad (\mathfrak{I}, \mathfrak{a}) \models \varphi[o], \tag{2.2}$$

and conversely, for each $\mathcal{FM}_2[\Sigma]$-formula φ with exactly one free variable of sort object, there exists a $\mathcal{QMS}[\Sigma]$-term τ such that (2.2) *holds for all metric models $\mathfrak{I}$ with assignments $\mathfrak{a}$ and all $o \in \Delta$.*

$\mathcal{FM}_2[\Sigma]$ is, however, exponentially more succinct than $\mathcal{QMS}[\Sigma]$.

To classify and investigate $\mathcal{QMS}[\Sigma]$ and its various fragments can be regarded as an interesting and challenging research programme, with possible applications for reasoning about distances and similarity in various application domains. At this early stage, however, there are more open problems than answers. Here we mention just some of them (see also [6]). First, does the expressive completeness result above hold for the language $\mathcal{QMS}[\Sigma]$ with $\Sigma = \emptyset$? So far, the proof requires inequalities $x_i < x_j$ to be available in both languages. Second, the fragments of $\mathcal{QMS}[\Sigma]$ discussed above contain only the distance operators $\exists^{<x_i}$. Proofs of decidability results for those fragments often employ a certain *tree model property* (formulated in terms of tree metric spaces) as well as a technique that is close to standard unravelling (alias bisimulation). Is it possible to define a natural notion of bisimulation on metric spaces which could explain the 'good' behaviour of those fragments of $\mathcal{QMS}[\Sigma]$? Is there a natural characterisation of the fragment of $\mathcal{FM}[\Sigma]$ which is invariant under such bisimulations? Finally, an interesting problem is to find 'maximal' decidable fragments of $\mathcal{QMS}[\Sigma]$. It is not difficult to see using the technique of [7] that satisfiability of $\mathcal{QMS}[\Sigma_1]$-terms is undecidable. But we conjecture that much weaker fragments are undecidable already.

The contribution of this paper to the research programme above is an analysis of the computational behaviour and the expressive power of the fragment of $\mathcal{QMS}[\Sigma]$ known as the *logic of comparative similarity* [12].

3 The logic $\mathcal{CSL}$

Let us consider the $\mathcal{CTL}^+$-like fragment of $\mathcal{QMS}[\emptyset]$ where the role of branch quantifiers is played by $\exists x$ and $\forall x$, while instead of temporal operators we have the quantifiers $\exists^{<x}$ and $\forall^{<x}$. Thus, in this fragment we allow the quantifiers $\exists x$ to be applied to Boolean combinations of terms of the form $\exists^{<x}\tau$ and atoms p_i. More precisely, take a variable x and define the set $\mathcal{CLV}_{open}$ of *open terms* σ and the set $\mathcal{CLV}$ of *terms* τ by induction as follows:

$$\begin{array}{rcl} \tau & ::= & p_i \quad | \quad \neg\tau \quad | \quad \tau_1 \sqcap \tau_2 \quad | \quad \exists x\sigma, \\ \sigma & ::= & \tau \quad | \quad \neg\sigma \quad | \quad \sigma_1 \sqcap \sigma_2 \quad | \quad \exists^{<x}\tau. \end{array}$$

Another language we consider in this section is called $\mathcal{CSL}$ (which stands for the *logic of comparative similarity*) [12]. Its terms are defined by

$$\tau \quad ::= \quad p_i \quad | \quad \neg\tau \quad | \quad \tau_1 \sqcap \tau_2 \quad | \quad \tau_1 \leftleftarrows \tau_2,$$

where $\leftleftarrows$, the *closer operator*, is interpreted in a metric model $\mathfrak{I}$ of the form (2.1) as follows (this is equivalent to the definition of $\leftleftarrows$ in the introduction):

$$(\tau_1 \leftleftarrows \tau_2)^{\mathfrak{I}} \quad = \quad \{x \in \Delta \mid d(x, \tau_1^{\mathfrak{I}}) < d(x, \tau_2^{\mathfrak{I}})\}.$$

Here the *distance* $d(x, Y)$ *from a point* $x \in \Delta$ *to a subset* $Y \subseteq \Delta$ is defined as usual:

$$d(x, Y) \;=\; \begin{cases} \inf\{d(x, y) \mid y \in Y\}, & \text{if } Y \neq \emptyset \\ \infty, & \text{if } Y = \emptyset. \end{cases}$$

Two terms τ_1 and τ_2 are said to be *equivalent*, $\tau_1 \equiv \tau_2$ in symbols, if $\tau_1^{\mathfrak{I}} = \tau_2^{\mathfrak{I}}$ for every metric model $\mathfrak{I}$. It is not hard to see that

$$\tau_1 \leftleftarrows \tau_2 \;\equiv\; \exists x\,(\exists^{<x}\tau_1 \sqcap \neg\exists^{<x}\tau_2).$$

Thus, $\mathcal{CSL}$ can be regarded as a sublanguage of $\mathcal{CLV}$. Despite its apparent simplicity, this language turns out to be quite expressive. In particular, $\top \leftleftarrows \neg\tau$ is interpreted as the interior of τ in the topological space induced by the metric, and $\top \leftrightarrows \tau$ as the closure of τ, where $\top$ is the whole space $(p_1 \sqcup \neg p_1)$ and

$$\tau_1 \leftrightarrows \tau_2 \;=\; \neg(\tau_1 \leftleftarrows \tau_2) \sqcap \neg(\tau_2 \leftleftarrows \tau_1)$$

is the set of points located at the same distance from τ_1 and τ_2. The universal modalities can be also expressed via $\leftleftarrows$:

$$\exists\tau \;\equiv\; (\tau \leftleftarrows \bot) \qquad \text{and} \qquad \forall\tau \;\equiv\; \neg(\neg\tau \leftleftarrows \bot)$$

(here $\bot$ stands for $p_1 \sqcap \neg p_1$). Thus, the logic $\mathcal{CSL}$ contains full $\mathbf{S4}_u$. We now show that actually $\mathcal{CSL}$ is as expressive as full $\mathcal{CLV}$.

THEOREM 3. *For every $\mathcal{CLV}$-term τ, there is a $\mathcal{CSL}$-term τ^* with $\tau \equiv \tau^*$.*

Proof. Observe first that, for all $\mathcal{CLV}$-terms $\tau_1, \dots, \tau_n$ and ρ, we have

$$\prod_{i=1}^{n} \exists x(\exists^{<x}\tau_i \sqcap \neg\exists^{<x}\rho) \;\equiv\; \exists x(\prod_{i=1}^{n} \exists^{<x}\tau_i \sqcap \neg\exists^{<x}\rho).$$

(For a proof of this observation use the fact that distances are taken from the *linearly ordered* set $\mathbb{R}^{>0}$.)

Now the proof proceeds by induction on the construction of τ. The term τ is a Boolean combination of terms of the form $\exists x\sigma$ and atoms p_i. As both languages have the Boolean operators and the set variables p_i, it is sufficient to define the translation of a term of the form $\exists x\sigma$. We may assume that

$$\exists x\sigma \;=\; \exists x \bigsqcup_{i=1}^{n} \prod_{j=1}^{m_i} \rho_i^j,$$

where the ρ_i^j are negated or non-negated terms of the form $\exists^{<x}\tau$, p_i, or $\exists x\sigma'$. Clearly

$$\exists x\sigma \;\equiv\; \bigsqcup_{i=1}^{n} \exists x \prod_{j=1}^{m_i} \rho_i^j.$$

As $\neg\exists^{<x}\tau_1 \sqcap \neg\exists^{<x}\tau_2 \equiv \neg\exists^{<x}(\tau_1 \sqcup \tau_2)$, we obtain

$$\exists x\sigma \;\equiv\; \bigsqcup_{i=1}^{n} \exists x \prod_{j=1}^{k_i} (\exists^{<x}\tau_{ij} \sqcap \neg\exists^{<x}\tau_i \sqcap \beta_{ij}),$$

where the β_{ij} are negated or non-negated terms of the form $\exists x\sigma'$ or p_i. By the observation above, we then have

$$\exists x\sigma \;\equiv\; \bigsqcup_{i=1}^{n} \prod_{j=1}^{k_i} \big(\exists x(\exists^{<x}\tau_{ij} \sqcap \neg\exists^{<x}\tau_i) \sqcap \beta_{ij}\big).$$

By the induction hypotheses, we have translations τ_{ij}^*, τ_i^* and β_{ij}^* in $\mathcal{CSL}$. Then the translation we need can be obtained by taking

$$(\exists x\sigma)^* \;=\; \bigsqcup_{i=1}^{n} \prod_{j=1}^{k_i} \big((\tau_{ij}^* \leftleftarrows \tau_i^*) \sqcap \beta_{ij}^*\big),$$

which proves the theorem. ■

Observe that the translation from $\mathcal{CLV}$ to $\mathcal{CSL}$ above introduces an exponential blow-up. The question whether $\mathcal{CLV}$ is indeed exponentially more succinct than $\mathcal{CSL}$ remains open.

4 Decidability of $\mathcal{CSL}$

A typical decidability proof for a modal (temporal, dynamic, etc.) logic L proceeds as follows. Given a formula φ, we take a proper 'closure' $\mathsf{cl}\,\varphi$ of the set $\mathsf{sub}\,\varphi$ of subformulas of φ, introduce a syntactical notion of a 'type' approximating those subsets of $\mathsf{cl}\,\varphi$ that can be realised in models for L, and then show how to construct an L-model for a given type t with $\varphi \in t$ by providing a 'witness type' for each $\Diamond\psi \in t$, that is, a type t' such that $\psi \in t'$ and $\chi \in t'$, for every $\Box\chi \in t$. This general scheme can be applied to $\mathcal{CSL}$ as well. As usual, however, the devil (or God?) is in the details.

Let us figure out first what a $\mathcal{CSL}$-type is. Throughout this section we assume that we are given a $\mathcal{CSL}$-term τ. Denote by $\mathsf{sub}\,\tau$ the set of subterms of τ. As we need to *compare* distances between types containing certain subterms of τ, we introduce the set

$$\mathsf{com}\,\tau \;=\; \{\bot, \top\} \,\cup\, \{\varphi \mid \varphi \leftleftarrows \psi \in \mathsf{sub}\,\tau \;\text{ or }\; \psi \leftleftarrows \varphi \in \mathsf{sub}\,\tau, \;\text{ for some } \psi\}.$$

Finally, we define $\mathsf{cl}\,\tau \supseteq \mathsf{sub}\,\tau$, the *closure* of $\mathsf{sub}\,\tau$, to be the smallest set of terms with the following properties:

- $\mathsf{cl}\,\tau$ is closed under single negations, and
- $\mathsf{cl}\,\tau$ contains $\varphi \leftleftarrows \psi$, for every $\varphi, \psi \in \mathsf{com}\,\tau$.

Clearly, the size of $\mathsf{cl}\,\tau$ is polynomial in the size $|\tau|$ of τ.

Suppose now that we have a metric model $\mathfrak{I}$ of the form (2.1) and a point $x \in \Delta^{\mathfrak{I}}$. Then the τ-*type* of x in $\mathfrak{I}$ is the set

$$t^{\mathfrak{I}}(x) \;=\; \{\varphi \in \mathsf{cl}\,\tau \mid x \in \varphi^{\mathfrak{I}}\}.$$

Clearly, this set is Boolean closed. Moreover, the model $\mathfrak{I}$ determines a natural linear quasi-order $\leq_{t^{\mathfrak{I}}(x)}$ on $\mathsf{com}\,\tau$: for all $\varphi, \psi \in \mathsf{com}\,\tau$, we have

$$\varphi \leq_{t^{\mathfrak{I}}(x)} \psi \quad \text{iff} \quad d(x, \varphi^{\mathfrak{I}}) \leq d(x, \psi^{\mathfrak{I}}) \quad \text{iff} \quad \neg(\psi \leftleftarrows \varphi) \in t^{\mathfrak{I}}(x).$$

Observe that

- φ is a $\leq_{t^{\mathfrak{I}}(x)}$-minimal element iff $d(x, \varphi^{\mathfrak{I}}) = 0$, i.e., $x \in \mathbb{C}\varphi^{\mathfrak{I}}$, and
- φ is a $\leq_{t^{\mathfrak{I}}(x)}$-maximal element iff $d(x, \varphi^{\mathfrak{I}}) = \infty$, i.e., $\varphi^{\mathfrak{I}} = \emptyset$.

This suggests the following syntactical approximation of the 'real' τ-types.

A subset t of $\mathsf{cl}\,\tau$ is said to be *Boolean closed* if $\top \in t$ and the following conditions are satisfied: (a) $\varphi \in t$ iff $\neg\varphi \notin t$, for all $\neg\varphi \in \mathsf{cl}\,\tau$, and (b) $\varphi \sqcap \psi \in t$ iff $\varphi \in t \wedge \psi \in t$, for all $\varphi \sqcap \psi \in \mathsf{cl}\,\tau$. With every Boolean closed $t \subseteq \mathsf{cl}\,\tau$ we associate the following binary relation $\leq_t$ on $\mathsf{com}\,\tau$:

$$\leq_t \;\;=\;\; \{(\varphi, \psi) \mid \neg(\psi \leftleftarrows \varphi) \in t\}.$$

Now, a τ-*type* (or simply a *type* if τ is understood) is a Boolean closed subset t of $\mathsf{cl}\,\tau$ such that

- $\leq_t$ is a linear quasi-order on $\mathsf{com}\,\tau$,
- all terms in $t \cap \mathsf{com}\,\tau$ are $\leq_t$-minimal elements,
- $\bot$ is a $\leq_t$-maximal element.

Denote by $<_t$ and $\simeq_t$ the strict linear order and the equivalence relation induced by $\leq_t$, respectively. It is easy to see that

$$<_t \;=\; \{(\varphi, \psi) \mid (\varphi \leftleftarrows \psi) \in t\}.$$

Let $\min t$ denote the set of $\leq_t$-minimal elements. It should be clear that there are at most exponentially ($2^{O(|\tau|^2)}$) many τ-types.

Recall that $\varphi \leftrightarrows \psi \;=\; \neg(\varphi \leftleftarrows \psi) \sqcap \neg(\psi \leftleftarrows \varphi)$. Clearly, $\top \leftrightarrows \varphi$ is actually equivalent to $\neg(\top \leftleftarrows \varphi)$, while $\bot \leftrightarrows \varphi$ is equivalent to $\neg(\varphi \leftleftarrows \bot)$.

Before we proceed to our next notion, let us consider an example explaining an essential difference between $\mathcal{CSL}$ and standard modal logics.

EXAMPLE 4. To satisfy the term $\neg p \sqcap (q \leftleftarrows \bot) \sqcap (p \leftleftarrows q) \sqcap (p \leftrightarrows \top)$

- we need, by the first conjunct, a point x from $\neg p$;
- by the second conjunct, we need a 'witness' y for $q \leftleftarrows \bot$, that is, a point y that belongs to q;
- by the third conjunct, we need a witness z for $p \leftleftarrows q$, that is, z belongs to p, neither x nor z are in q, and x should be closer to z than to y;
- finally, the fourth conjunct says that x must be 'infinitely close' to p, that is, we need an *infinite* sequence $\{z_i \mid i \in \omega\}$ of points from p converging to x. Note that, by the third conjunct, only a finite number of the z_i can be in q.

Thus, we require witnesses of two sorts: (i) those that are at some finite distance from a given point x, and (ii) those that represent infinite sequences converging to x (that the points of such a sequence can always be chosen to be of the same type follows from Lemma 5 below).

Two important facts should also be observed in connection with the example above. First, the concrete values of the distances $d(x, y)$ and $d(x, z)$ are of no importance at all; what really matters is that they should satisfy the inequalities $d(x, y) > d(x, z) > 0$. At the same time, the value $\lim_{i\to\infty} d(x, z_i)$ must be zero. The logic of comparative similarity *cannot* speak of any particular distance except 0.

The second important fact is that if a term requires some witness at a positive distance then a single witness—rather than an infinite sequence of witnesses as in (ii) above—is always enough.

LEMMA 5. *Let $\mathfrak{I}$ be a metric model.*

(1) *Suppose $x \in \Delta^{\mathfrak{I}}$ and $\varphi^{\mathfrak{I}} \neq \emptyset$ for some $\varphi \in \mathsf{com}\,\tau$. Then there is a type t with $\varphi \in t$ such that $d(x, \varphi^{\mathfrak{I}}) = d(x, t^{\mathfrak{I}})$, where $t^{\mathfrak{I}} = \{y \in \Delta^{\mathfrak{I}} \mid t^{\mathfrak{I}}(y) = t\}$. Moreover, the pair of types $(t^{\mathfrak{I}}(x), t)$ satisfies the following conditions:*

- *if $\top <_{t^{\mathfrak{I}}(x)} \varphi$ then $\varphi <_{t^{\mathfrak{I}}(x)} \psi$ implies $\varphi <_t \psi$, for all $\psi \in \mathsf{com}\,\tau$,*
- *if $\varphi \simeq_{t^{\mathfrak{I}}(x)} \top$ then $\chi <_{t^{\mathfrak{I}}(x)} \psi$ implies $\chi <_t \psi$, for all $\psi, \chi \in \mathsf{com}\,\tau$.*

(2) *For all $x, y \in \Delta^{\mathfrak{I}}$ and $\psi \in \mathsf{com}\,\tau$, we have $\psi <_{t^{\mathfrak{I}}(x)} \bot$ iff $\psi <_{t^{\mathfrak{I}}(y)} \bot$.*

Proof. To show (2), it is enough to observe that $\psi <_{t^{\mathfrak{I}}(x)} \bot$ iff $\psi^{\mathfrak{I}} \neq \emptyset$ iff $\psi <_{t^{\mathfrak{I}}(y)} \bot$, for all $x, y \in \Delta^{\mathfrak{I}}$ and $\psi \in \mathsf{com}\,\tau$.

Let us show (1) for the case $\varphi \simeq_{t^{\mathfrak{I}}(x)} \top$. Since $d(x, \varphi^{\mathfrak{I}}) = 0$, either $x \in \varphi^{\mathfrak{I}}$ or there is an infinite sequence $\{z_i \in \varphi^{\mathfrak{I}} \mid i \in \omega\}$ converging to x. In the former case we set $t = t^{\mathfrak{I}}(x)$. In the latter one, since the number of types is finite, there is an infinite subsequence $\{z_{i_j}\}$ of $\{z_i\}$ whose points are of the same type, and so we can set $t = t^{\mathfrak{I}}(z_{i_j})$.

Let $\chi <_{t^{\mathfrak{I}}(x)} \psi$ for some $\psi, \chi \in \mathsf{com}\,\tau$. Then $\varepsilon = d(x, \psi^{\mathfrak{I}}) - d(x, \chi^{\mathfrak{I}}) > 0$. Choose some $y \in t^{\mathfrak{I}}$ with $d(x, y) < \varepsilon/2$. By the triangle inequality we then have $d(y, \psi^{\mathfrak{I}}) - d(y, \chi^{\mathfrak{I}}) \geq \varepsilon - 2\,d(x, y) > 0$, which means $\chi <_t \psi$.

The case of $\top <_{t^{\mathfrak{I}}(x)} \varphi$ is considered analogously. ■

We now define a notion of a φ-link using which we can provide witnesses for terms from a given type. Let s, t be types and $\varphi \in \mathsf{com}\,\tau$. Two cases are possible:

- Suppose that $\top <_s \varphi <_s \bot$. Then we say that the pair (s, t) is a φ-*link* (of types) if $\varphi \in t$ and, for all $\psi \in \mathsf{com}\,\tau$, we have

$$\psi <_s \bot \;\leftrightarrow\; \psi <_t \bot \quad \text{and} \quad \varphi <_s \psi \;\rightarrow\; \varphi <_t \psi$$

  (note that $\varphi <_t \psi$ is equivalent here to $\top <_t \psi$, since $\varphi \in t$).

- Suppose that $\varphi \simeq_s \top$. Then we say that (s, t) is a φ-*link* (of types) if $\varphi \in t$ and, for all $\psi, \chi \in \mathsf{com}\,\tau$, we have

$$\psi <_s \bot \;\leftrightarrow\; \psi <_t \bot \quad \text{and} \quad \chi <_s \psi \;\rightarrow\; \chi <_t \psi$$

  (note that the second implication is equivalent to $<_s \subseteq <_t$, and we have $\min t \subseteq \min s$).

In the latter case we will also say that (s, t) is a *short link*, while in the former the link will be called *long*. Clearly, a link (s, t) is short iff $<_s \subseteq <_t$.

Unfortunately, the notion of a link above does not take into account a possible interaction of two (or more) short links. To be more specific, consider the following situation. Suppose that t_0 is a type and $\varphi \notin t_0$, for some $\varphi \in \mathsf{com}\,\tau$ such that $\varphi \in \min t_0$ (i.e., $\varphi \simeq_{t_0} \top$). Then we need a short φ-link (t_0, t_1). Assume further that $\psi \notin t_1$, for some $\psi \in \mathsf{com}\,\tau$ with $\psi \in \min t_0$. This means that we also need a (long or short) ψ-link (t_1, t_2). But then, according to Lemma 6 below, (t_0, t_2) must be a short ψ-link, which by no means follows from the definition of a link.

LEMMA 6. *Let $\mathfrak{I}$ be a model, $x \in \Delta^{\mathfrak{I}}$, and let $\varphi, \psi \in \mathsf{com}\,\tau$ be such that $d(x, \varphi^{\mathfrak{I}}) = d(x, \psi^{\mathfrak{I}}) = 0$. Let s, t be types with $\varphi \in s$, $\psi \in t$, and let $S \subseteq s^{\mathfrak{I}}$ be such that $d(x, S) = d(x, \varphi^{\mathfrak{I}}) = 0$ and $d(y, t^{\mathfrak{I}}) = d(y, \psi^{\mathfrak{I}})$, for all $y \in S$. Then $d(x, t^{\mathfrak{I}}) = d(x, \psi^{\mathfrak{I}}) = 0$.*

Proof. Take an arbitrary $\varepsilon > 0$. As $d(x, S) = 0$, there is $y \in S$ with $d(x, y) < \varepsilon/2$. Then $d(y, t^{\mathfrak{I}}) = d(y, \psi^{\mathfrak{I}}) \leq d(y, x) + d(x, \psi^{\mathfrak{I}}) < \varepsilon/2$, and so $d(x, t^{\mathfrak{I}}) \leq d(x, y) + d(y, \psi^{\mathfrak{I}}) < \varepsilon$, i.e., $d(x, t^{\mathfrak{I}}) = 0$, as $\varepsilon > 0$ is arbitrary. ■

Thus we should be careful when constructing sequences of links starting with a short one, in particular, we should remember some previous links in the sequence. Let us consider possible scenarios when we start with a short link (t_0, t_1).

1. Suppose that $<_{t_0} = <_{t_1}$ and we need a φ-link (t_1, t_2) for some $\varphi \in \mathsf{com}\,\tau$. In this case the types t_0 and t_1 contain precisely the same terms of the form $\chi_1 \leftleftarrows \chi_2$ and can only differ in Boolean terms. It follows that (t_1, t_2) is a φ-link iff (t_0, t_2) is a φ-link. This means that the choice of t_2 does not depend on the link (t_0, t_1).

2. Suppose that $<_{t_0} \subsetneq <_{t_1}$ and we need a φ-link (t_1, t_2) for some $\varphi \in \mathsf{com}\,\tau$. As we have $\min t_1 \subseteq \min t_0$, three cases are possible.

2.1: $\varphi \in \min t_1$. Then for any φ-link (t_1, t_2) we have $<_{t_0} \subset <_{t_1} \subseteq <_{t_2}$, and so (t_0, t_2) is also a short φ-link. Thus, no additional requirement should be imposed on (t_1, t_2).

2.2: $\varphi \in \min t_0 \setminus \min t_1$. In this case, when choosing a (long) φ-link (t_1, t_2), we must also ensure that (t_0, t_2) is a short φ-link.

2.3: $\varphi \notin \min t_0$, and so $\varphi \notin \min t_1$. In this case (t_0, t_1) does not have any influence on subsequent links at all.

3. Suppose that $<_{t_0} \subsetneq <_{t_1}$ and (t_1, t_2) is a φ-link, for $\varphi \in \min t_0 \setminus \min t_1$ (as in **2.2**), and so (t_0, t_2) is a short φ-link with $<_{t_0} \subseteq <_{t_2}$. Suppose also that we are looking for a ψ-link (t_2, t_3). As (t_1, t_2) is a long link, t_1 has no influence on the choice of t_3. However, (t_0, t_2) should be taken into account. We again have three cases.

3.1: $\psi \in \min t_2$. Then for any ψ-link (t_2, t_3) the pair (t_0, t_3) will automatically be a ψ-link.

3.2: $\varphi \in \min t_0 \setminus \min t_2$. Then, when choosing a long φ-link (t_2, t_3), we must also ensure that (t_0, t_3) is a short ψ-link.

3.3: $\varphi \notin \min t_0$. In this case no additional requirement is needed.

This analysis suggests the following definitions. A sequence $\mathbf{t} = (t_0, \dots, t_n)$ of τ-types is called a *block* if we have the inclusions $<_{t_0} \subset \cdots \subset <_{t_{n-1}} \subseteq <_{t_n}$ (which means that all pairs $(t_0, t_1), \dots, (t_{n-1}, t_n)$ are short links). We call t_n the *type* of $\mathbf{t}$, while $(t_0, \dots, t_{n-1})$ is understood as its 'history' or 'heredity.' We say that $\mathbf{t}$ is *realised* in a model $\mathfrak{I}$ of the form (2.1) if there exist subsets $X_0 \subseteq t_0^{\mathfrak{I}}, \dots, X_n \subseteq t_n^{\mathfrak{I}}$ such that $d(x_i, X_{i+1}) = 0$ for all $i < n$ and $x_i \in X_i$.

It is easy to see that the size of $\mathsf{com}\,\tau$, and so the length of any block, is bounded by $|\tau|$. Therefore, the number of different blocks is at most exponential in $|\tau|$.

Now, for $\varphi \in \mathsf{com}\,\tau$, we introduce a notion of a *φ-link of blocks*, which specialises the notion of a φ-link of types. Let $\mathbf{s}$ and $\mathbf{t}$ be blocks with $\mathbf{s} = (s_0, \dots, s_m)$. Consider four cases.

- Suppose that $\varphi \notin \min s_0$. Then $(\mathbf{s}, \mathbf{t})$ is called a *φ-link* (of blocks) if $\mathbf{t} = (t)$ and (s_m, t) is a φ-link of types. In this case the long link (s_m, t) allows us to 'forget' everything that happened before t.
- Suppose that $\varphi \in \min s_{n-1} \setminus \min s_n$, for some $n \leq m$. Then $(\mathbf{s}, \mathbf{t})$ is a *φ-link* (of blocks) if $\mathbf{t} = (s_0, \dots, s_{n-1}, t)$ and (s_m, t) is a φ-link of types. In this case (s_n, t) is a long link, while (s_{n-1}, t) is a short one, and so s_{n-1} and its 'heredity' should be kept.
- Suppose that $\varphi \in \min s_m \setminus s_m$ and $<_{s_{m-1}} = <_{s_m}$. Then $(\mathbf{s}, \mathbf{t})$ is a *φ-link* (of blocks) if $\mathbf{t} = (s_0, \dots, s_{m-1}, t)$ and (s_m, t) is a φ-link of types. In this case s_{m-1} and s_m carry the same information on 'heredity' of t, so we can drop s_m.
- Suppose that $\varphi \in \min s_m \setminus s_m$ and $<_{s_{m-1}} \subset <_{s_m}$. Then $(\mathbf{s}, \mathbf{t})$ is a *φ-link* (of blocks) if $\mathbf{t} = (s_0, \dots, s_m, t)$ and (s_m, t) is a φ-link of types.

Let D be a set of blocks and T the set of all types occurring in blocks from D. We call D a *diagram* if the following conditions hold:

$$\text{there exists } (t) \in D \text{ with } \tau \in t, \tag{4.1}$$

$$\text{for all } s, t \in T \text{ and } \varphi \in \mathsf{com}\,\tau, \text{ we have } \varphi <_s \bot \text{ iff } \varphi <_t \bot, \tag{4.2}$$

$$\begin{aligned}&\text{for all } \mathbf{s} = (s_0, \dots, s_n) \in D \text{ and } \varphi <_{s_n} \bot,\ \varphi \notin s_n, \text{ there exists}\\ &\mathbf{t} \in D \text{ such that } (\mathbf{s}, \mathbf{t}) \text{ is a } \varphi\text{-link.}\end{aligned} \tag{4.3}$$

The rather abstract notions of block, link and diagram for the given term τ will become more transparent from the proofs of Lemmas 7 and 10.

LEMMA 7. *Let $\mathfrak{I}$ be a metric model where $\tau^{\mathfrak{I}} \neq \emptyset$, and let D be the set of blocks realised in $\mathfrak{I}$. Then D is a diagram.*

Before we turn to the proof of Lemma 7, we need to establish the following technical property.

LEMMA 8. *Suppose that a block $\mathbf{s} = (s_0, \dots, s_m)$ is realised in $\mathfrak{I}$, and $X_0 \subseteq s_0^{\mathfrak{I}}, \dots, X_m \subseteq s_m^{\mathfrak{I}}$ are such that $d(x_i, X_{i+1}) = 0$ for all $x_i \in X_i$ and $i < m$. Suppose also that $\varphi <_{s_m} \bot$ and $\varphi \notin s_m$. Then there exist a type t and subsets $Y_0 \subseteq X_0, \dots, Y_m \subseteq X_m$ with the following properties:*

$$d(y, t^{\mathfrak{I}}) = d(y, \varphi^{\mathfrak{I}}), \quad \text{for all } y \in Y_m,$$

$$d(y, Y_{l+1}) = 0, \quad \text{for all } y \in Y_l \text{ and } l < m.$$

Proof. Choose an arbitrary $x_\lambda \in X_0$, where λ denotes the empty sequence. As $d(x_i, X_{i+1}) = 0$ for all $x_i \in X_i$ and $i < m$, we can choose $x_\alpha \in X_l$, for all $\alpha \in \omega^l$ and $l \leq m$, so that

$$x_\alpha = \lim_{i \to \infty} x_{(\alpha, i)}.$$

Now, for every $\alpha \in \omega^m$, there exists a type t_α such that $\varphi \in t_\alpha$ and $d(x_\alpha, \varphi^{\mathfrak{I}}) = d(x_\alpha, t_\alpha^{\mathfrak{I}})$. Since the number of types is finite, we obtain a partition $\omega^m = A_0 \cup \dots \cup A_n$, where $t_\alpha = t_{\alpha'}$, for all $\alpha, \alpha' \in A_r$, $r \leq n$.

Say that a subset $A \subseteq \omega^m$ is *essential* if we have

$$(\exists^\infty a_0) \dots (\exists^\infty a_{m-1})\ (a_0, \dots, a_{m-1}) \in A, \tag{4.4}$$

where $\exists^\infty$ means 'there exist infinitely many.'

CLAIM 9. A_r is an essential subset of ω^m for some $r \leq n$.

Proof. We proceed by induction on m. Clearly, for $m = 1$ an essential subset is simply an infinite subset. So in this case Claim 9 is trivial.

Suppose that $m > 1$ and Claim 9 holds for $m-1$. For $A \subseteq \omega^m$ and $a \in \omega$, let $A|_a$ denote $\{\alpha \in \omega^{m-1} \mid (a, \alpha) \in A\}$. Then, for every $a \in \omega$, we have $\omega^{m-1} = A_0|_a \cup \dots \cup A_n|_a$. Hence there exists $r(a) \leq n$ such that $A_{r(a)}|_a$ is essential in ω^{m-1}. As $r(a)$ has only a finite number of possible values, there exists $r < m$ such that $A_r|_a$ is essential in ω^{m-1} for infinitely many $a \in \omega$. This means that A_r is essential in ω^m. ■

So let $A = A_r$ be an essential subset of ω^m for some $r \in \omega^m$. For every $l \leq m$, let

$$A_{(l)} \;=\; \{\alpha \in \omega^l \mid (\alpha, \beta) \in A \text{ for some } \beta \in \omega^{m-l}\}$$

(in particular, $A_{(m)} = X$ and $A_{(0)} = \{\lambda\}$). Then the following property is a straightforward consequence of the definition of essential sets:

for all $l < m$ and $\alpha \in A_{(l)}$, there are infinitely many $a \in \omega$ such that $(\alpha, a) \in A_{(l+1)}$. (4.5)

It remains to put $Y_l = \{x_\alpha \mid \alpha \in A_{(l)}\}$, for all $l \leq m$. ■

We are now in a position to prove Lemma 7.

Proof. Clearly, D satisfies (4.1) and (4.2). Let us prove (4.3). Suppose that a block $\mathbf{s} = (s_0, \dots, s_m)$ is realised in $\mathfrak{I}$ and $X_0 \subseteq s_0^{\mathfrak{I}}, \dots, X_m \subseteq s_m^{\mathfrak{I}}$ are such that $d(x_i, X_{i+1}) = 0$ for all $i < m$ and all $x_i \in X_i$. Let $\varphi <_{s_m} \perp$, $\varphi \notin s_m$. By Lemma 8, there exist a type t and subsets $Y_0 \subseteq X_0, \dots, Y_m \subseteq X_m$ with the following properties:

$$d(y, t^{\mathfrak{I}}) = d(y, \varphi^{\mathfrak{I}}), \quad \text{for all } y \in Y_m,$$
$$d(y, Y_{l+1}) = 0, \quad \text{for all } y \in Y_l \text{ and } l < m.$$

Note that the latter property implies that $d(y, Y_{l'}) = 0$, for all $y \in Y_l$ and $l < l' \leq m$. Four cases are now possible.

Case 1: $\varphi \notin \min s_0$. Then the block $\mathbf{t} = (t)$ is realised in $\mathfrak{I}$, because $t^{\mathfrak{I}} \neq \emptyset$, and $(\mathbf{s}, \mathbf{t})$ is a φ-link by construction.

Case 2: $\varphi \in \min s_{n-1} \setminus \min s_n$ for some $n \leq m$. Let us show that $\mathbf{t} = (s_0, \dots, s_{n-1}, t)$ is a block realised in $\mathfrak{I}$. Take any $u \in Y_{n-1}$ and $v \in Y_m$. Then $d(u, t^{\mathfrak{I}}) \leq d(u, v) + d(v, t^{\mathfrak{I}}) = d(u, v) + d(v, \varphi^{\mathfrak{I}}) \leq d(u, v) + d(v, u) + d(u, \varphi^{\mathfrak{I}}) = 2\, d(u, v)$, and so $d(y, t^{\mathfrak{I}}) = 0$. We obtain $<_{s_{n-1}} \subseteq <_t$, since $u \in Y_{n-1} \subseteq s_{n-1}^{\mathfrak{I}}$. Thus $\mathbf{t}$ is a block. By considering the sets $Y_0, \dots, Y_{n-1}, t^{\mathfrak{I}}$, we see that $\mathbf{t}$ is realised in $\mathfrak{I}$. Finally, $(\mathbf{s}, \mathbf{t})$ is a φ-link by construction.

Case 3: $\varphi \in \min s_m \setminus s_m$ and $<_{s_{m-1}} = <_{s_m}$. Similarly to the previous case we obtain that $d(u, t^{\mathfrak{I}}) = 0$ for all $u \in Y_{m-1}$ and therefore $<_{s_{m-1}} \subseteq <_t$. Thus, $\mathbf{t} = (s_0 \dots, s_{m-1}, t)$ is a block, $\mathbf{t}$ is realised in $\mathfrak{I}$ (consider the sets $Y_0, \dots, Y_{m-1}, t^{\mathfrak{I}}$), and $(\mathbf{s}, \mathbf{t})$ is a φ-link.

Case 4: $\varphi \in \min s_m \setminus s_m$ and $<_{s_{m-1}} \subset <_{s_m}$. Then $d(u, t^{\mathfrak{I}}) = d(u, \varphi^{\mathfrak{I}}) = 0$ for all $u \in Y_m$. Therefore $\mathbf{t} = (s_0 \dots, s_m, t)$ is a block, $\mathbf{t}$ is realised in $\mathfrak{I}$ (consider the sets $Y_0, \dots, Y_m, t^{\mathfrak{I}}$), and $(\mathbf{s}, \mathbf{t})$ is a φ-link. ■

LEMMA 10. *Let D be a diagram. Then there exists a model $\mathfrak{I}$ with $\tau^{\mathfrak{I}} \neq \emptyset$.*

Proof. Our first goal is to 'unravel' D into a certain tree that will serve as the underlying set of the model we need.

Let T be the set of all types from blocks in D. Let $\varphi_0, \dots, \varphi_{k-1}$ be all different members of the set $\{\varphi \in \mathsf{com}\,\tau \mid \varphi <_t \bot \text{ for all } t \in T\}$. We are going to unravel D into a tree $\Delta \subseteq (\{0, \dots, k-1\} \times \omega)^*$ together with three labelling functions $tp : \Delta \to T$, $bl : \Delta \to D$ and $hr : \Delta \to \Delta^*$ the intended meaning of which is as follows.

For all $\alpha \in \Delta$, $bl(\alpha)$ is some block in D of the type $tp(\alpha)$, and $tp(\alpha)$ should be the type of α in Δ after we turn Δ into a proper metric model. And if α is a child of β, i.e., $\alpha = \beta(i, j)$, for some $i < k$ and $j \in \omega$, then $(bl(\beta), bl(\alpha))$ should be a φ_i-link of blocks; in particular, we should have that $(tp(\alpha), tp(\beta))$ is a φ_i-link of types, $\varphi_i \in tp(\beta)$. Therefore, for $\alpha \in \Delta$, we set that $\alpha(i, 0)$ belongs to Δ iff φ_i does not belong to $tp(\alpha)$. Moreover, nodes $\alpha(i, j)$ with $j > 0$ are included into Δ iff φ_i belongs to $\min tp(\alpha)$, i.e., α should be a limit point of all the $\alpha(i, j)$. Finally, if $bl(\alpha) = (t_0, \dots, t_n)$, then $hr(\alpha) = (\alpha_0, \dots, \alpha_{n-1})$, where α_m is the node 'responsible' for the presence of t_m in $bl(\alpha)$.

We proceed by induction. First we choose some $(t_*) \in D$ with $\tau \in t_*$, and set

$$\lambda \in \Delta, \quad tp(\lambda) = t_*, \quad bl(\lambda) = (t_*), \quad hr(\lambda) = \lambda$$

(recall that λ denotes an empty sequence). Suppose now that $\alpha \in \Delta$ is constructed and $tp(\alpha), bl(\alpha), hr(\alpha)$ are defined, say,

$$tp(\alpha) = s_m, \quad bl(\alpha) = (s_0, \dots, s_m), \quad hr(\alpha) = (\alpha_0, \dots, \alpha_{m-1})$$

(i.e., $hr(\alpha) = \lambda$ when $m = 0$). Consider an arbitrary $i < k$. According to (4.3), there exists some $\mathbf{t} = (t_0, \dots, t_n)$ in D such that $(bl(\alpha), \mathbf{t})$ is a φ_i-link (note that $n \leq m + 1$). Three cases are now possible.

$\varphi_i \in tp(\alpha)$. Then we set $\alpha(i, j) \notin \Delta$ for all $j \in \omega$.

$\varphi_i \notin \min tp(\alpha)$. Set $\alpha(i, 0) \in \Delta$ and $\alpha(i, j) \notin \Delta$ for all $j > 0$.

$\varphi_i \in \min tp(\alpha) \setminus tp(\alpha)$. Then we set $\alpha(i, j) \in \Delta$, for all $j \in \omega$.

Now, for all j with $\alpha(i, j) \in \Delta$ we define:

$$tp(\alpha(i, j)) = t_n, \quad bl(\alpha(i, j)) = \mathbf{t}, \quad hr(\alpha(i, j)) = (\alpha_0, \dots, \alpha_{n-1}),$$

where α_m stands for α, if $n = m + 1$. Clearly, we have the following:

LEMMA 11. *Let* $\alpha \in \Delta$ *and* $hr(\alpha) = (\alpha_0, \dots, \alpha_{n-1})$, $bl(\alpha) = (t_0, \dots, t_n)$. *Then* $hr(\alpha_m) = (\alpha_0, \dots, \alpha_{m-1})$ *and* $bl(\alpha_m) = (t_0, \dots, t_m)$, *for all* $m < n$.

The next step is to convert Δ into a metric space $\mathfrak{D} = (\Delta, d^{\mathfrak{D}})$, i.e., to define a distance function $d^{\mathfrak{D}}$ on Δ. Recall that, by the construction of Δ, if $\varphi \in \mathsf{com}\,\tau$ and $\varphi \in tp(\beta)$ for some $\beta \in \Delta$, then every $\alpha \in \Delta$ with $\varphi \notin tp(\alpha)$ has a child β' with $\varphi \in tp(\beta')$. The main idea behind the construction of $d^{\mathfrak{D}}$ is to ensure that such a β' can always be chosen so that $d^{\mathfrak{D}}(\alpha, \beta') \le d^{\mathfrak{D}}(\alpha, \beta)$. For this purpose we introduce a number of numerical parameters that will be defined by simultaneous induction on $\alpha \in \Delta$. These parameters are:

- The distance $d^{\alpha} = d^{\mathfrak{D}}(\alpha', \alpha)$, where α' is the parent of α.
- A sequence of numbers $c(\alpha)$ of the same length as $bl(\alpha)$. The distances d^{β}, for all children β of α, will form several slots within the interval $(0, 1)$, and $c(\alpha)$ stores some information on the boundaries of these slots.
- We use the following notation, for all $i < k$:

$$d_\alpha(i) = \begin{cases} 0, & \text{if } \varphi_i \in \min tp(\alpha), \\ d^{\alpha(i,0)}, & \text{if } \varphi_i \notin \min tp(\alpha). \end{cases}$$

- A 'sufficiently small' number $\varepsilon(\alpha)$ which is defined as follows. Suppose $c(\alpha) = (c_0, \dots, c_n)$. Then

$$\begin{aligned} \varepsilon(\alpha) \;=\; \min\big(&\{d_\alpha(i) - d_\alpha(j) \mid i, j < k,\ d_\alpha(i) > d_\alpha(j)\} \cup \\ &\{c_m - d_\alpha(i) \mid m \le n,\ i < k,\ c_m > d_\alpha(i)\}\big). \end{aligned}$$

  Roughly speaking, $\varepsilon(\alpha)$ measures the space available for 'splitting' the values $d_\alpha(i) = d_\alpha(j)$ with $i \neq j$.

We now list the principal conditions (4.6)–(4.11) that determine the choice of distances:

1) *For all* $\gamma \in \Delta$ *and* $i, j < k$,

$$d_\gamma(i) < d_\gamma(j) \;\leftrightarrow\; \varphi_i <_{tp(\gamma)} \varphi_j. \tag{4.6}$$

2) *Let* $\gamma \in \Delta$ *be such that* $hr(\gamma) = \lambda$, $bl(\gamma) = (t)$, $c(\gamma) = (c)$. *Then, for all* $i < k$, $j \in \omega$,

$$2c/3 \le d_\gamma(i) < c, \quad \textit{if } \; \varphi_i \notin \min t, \tag{4.7}$$

$$0 < d^{\gamma(i,j)} \le \varepsilon(\gamma)/2, \quad \textit{if } \; \varphi_i \in \min t \setminus t. \tag{4.8}$$

3) *Let* $\gamma \in \Delta$ *be such that* $hr(\gamma) = (\gamma_0, \dots, \gamma_{n-1})$, $bl(\gamma) = (t_0, \dots, t_n)$, $c(\gamma) = (c_0, \dots, c_n)$, *where* $n > 0$. *Then, for all* $i < k$, $j \in \omega$,

$$d_{\gamma_{n-1}}(i) \le d_\gamma(i) < d_{\gamma_{n-1}}(i) + c_n/3, \quad \textit{if } \; \varphi_i \notin \min t_{n-1}, \tag{4.9}$$

$$2c_n/3 \le d_\gamma(i) < c_n, \quad \textit{if } \; \varphi_i \in \min t_{n-1} \setminus \min t_n, \tag{4.10}$$

$$0 < d^{\gamma(i,j)} \le \varepsilon(\gamma)/2, \quad \textit{if } \; \varphi_i \in \min t_n \setminus t_n, \tag{4.11}$$

And in the process of construction we will prove that the following property is satisfied as well:

LEMMA 12. *Let* $\gamma \in \Delta$ *and* $hr(\gamma) = (\gamma_0, \dots, \gamma_{n-1})$, $bl(\gamma) = (t_0, \dots, t_n)$, $c(\gamma) = (c_0, \dots, c_n)$. *Then, for all* $m < n$, *we have*

$$c_{m+1} \leq \varepsilon(\gamma_m)/2, \quad c_{m+1} \leq c_m/2, \quad c(\gamma_m) = (c_0, \dots, c_m). \tag{4.12}$$

Let us now turn to the construction. First, let $c(\lambda) = (1)$ and $d^\lambda = 2/3$ (the latter is defined simply for convenience). Suppose now that d^α and $c(\alpha) = (c_0, \dots, c_n)$ are defined for some $\alpha \in \Delta$, condition (4.12) is satisfied for $\gamma = \alpha$, and conditions (4.6)–(4.12) are satisfied if γ is any ancestor of α. Let $hr(\alpha) = (\alpha_0, \dots, \alpha_{n-1})$ and $bl(\alpha) = (t_0, \dots, t_n)$. Two cases are possible.

Case 1: $n = 0$, i.e., $hr(\alpha) = \lambda$, $bl(\alpha) = (t_0)$, and $c(\alpha) = (c_0)$. Then, for all $i < k$ with $\varphi_i \notin \min t_0$, we can choose values $d_\alpha(i) = d^{\alpha(i,0)}$ in such a way that (4.6) and (4.7) are satisfied for $\gamma = \alpha$. Now $\varepsilon(\alpha)$ is defined, and we set $d^{\alpha(i,j)} = \varepsilon(\alpha)/(2j+2)$, for all $i < k$, $j \in \omega$ with $\varphi_i \in \min t_0 \setminus t_0$. Thus, (4.8) is satisfied as well for $\gamma = \alpha$, while (4.9)–(4.11) do not apply to this case. We further set

$$\begin{aligned} c(\alpha(i,0)) &= (c_0/2), \quad \text{if } \ \varphi_i \notin \min t_0, \\ c(\alpha(i,j)) &= (c_0, d^{\alpha(i,j)}), \quad \text{if } \ \varphi_i \in \min t_0 \setminus t_0, \end{aligned}$$

for all $i < k$, $j \in \omega$. This makes (4.12) satisfied on the children of α (recall that $d^{\alpha(i,j)} \leq \varepsilon(\alpha)/2$ and $\varepsilon(\alpha) \leq c_n$ by definition).

Case 2: $n > 0$, i.e., $hr(\alpha)$ is a nonempty sequence. Since $(t_0, \dots, t_n)$ is a block, we have $<_{t_{n-1}} \subseteq <_{t_n}$. And in view of (4.12) we have $c_n < d_{\alpha_{n-1}}(i)$ for all $i < k$ with $\varphi_i \notin \min t_{n-1}$. Therefore, for all $i < k$ with $\varphi_i \notin \min t_n$, we can choose values $d_\alpha(i) = d^{\alpha(i,0)}$ satisfying (4.6) and (4.9)–(4.10). Now $\varepsilon(\alpha)$ is defined, and we set $d^{\alpha(i,j)} = \varepsilon(\alpha)/(2j+2)$, for all $i < k$, $j \in \omega$ with $\varphi_i \in \min t_n \setminus t_n$. Thus, (4.11) is satisfied for $\gamma = \alpha$, while (4.7)–(4.8) do not apply to this case.

Further, consider any $i < k$ with $\varphi_i \notin t_n$. We then have several possibilities. First, let $\varphi_i \notin \min t_0$. Then $hr(\alpha(i,0)) = \lambda$, and we set $c(\alpha(i,0)) = (c_0/2)$. Clearly, (4.12) holds for $\gamma = \alpha(i,0)$.

Let $\varphi_i \in \min t_{m-1} \setminus \min t_m$ for some $1 \leq m \leq n$. Then $hr(\alpha(i,0)) = (\alpha_0, \dots, \alpha_{m-1})$, and we set $c(\alpha(i,0)) = (c_0, \dots, c_{m-1}, c_m/2)$. Now (4.12) holds for $\gamma = \alpha(i,0)$ in view of the induction hypothesis.

Let $\varphi_i \in \min t_n$ and $<_{t_{n-1}} = <_{t_n}$. Then $hr(\alpha(i,j)) = (\alpha_0, \dots, \alpha_{n-1})$, and we set $c(\alpha(i,j)) = (c_0, \dots, c_{n-1}, d^{\alpha(i,j)})$, for all $j \in \omega$. Again, (4.12) holds for $\gamma = \alpha(i,j)$ by the induction hypothesis.

Let finally $\varphi_i \in \min t_n$ and $<_{t_{n-1}} \subset <_{t_n}$. Then $hr(\alpha(i,j)) = (\alpha_0, \dots, \alpha_n)$, and we set $c(\alpha(i,j)) = (c_0, \dots, c_n, d^{\alpha(i,j)})$, for all $j \in \omega$. Recall that $d^{\alpha(i,j)} \leq \varepsilon(\alpha)/2$ and $\varepsilon(\alpha) \leq c_n$ by definition. Therefore (4.12) holds for $\gamma = \alpha(i,j)$ by the induction hypothesis.

Thus we define all the distances $d^\beta = d^{\mathfrak{D}}(\alpha, \beta)$, where α is a parent of β. Then we extend $d^{\mathfrak{D}}$ to the entire Δ by setting

$d^{\mathfrak{D}}(\alpha, \alpha) = 0$, for all $\alpha \in \Delta$,

$d^{\mathfrak{D}}(\beta, \alpha) = d^{\mathfrak{D}}(\alpha, \beta)$, if α is a parent of β,

$d^{\mathfrak{D}}(\alpha, \beta) = d^{\mathfrak{D}}(\alpha, \alpha_1) + \dots + d^{\mathfrak{D}}(\alpha_n, \beta)$, if $\alpha, \alpha_1, \dots, \alpha_n, \beta$ is the shortest path from α to β.

This distance function satisfies the following properties:

LEMMA 13. *Let $\alpha \in \Delta$ and $hr(\alpha) = (\alpha_0, \dots, \alpha_{n-1})$, $bl(\alpha) = (t_0, \dots, t_n)$, $c(\alpha) = (c_0, \dots, c_n)$.*

1) *Let $m < n$. Then, for all $i < k$ with $\varphi_i \notin \min t_m$, we have*

$$0 \;\le\; d_\alpha(i) - d_{\alpha_m}(i) \;<\; (c_{m+1} + \dots + c_n)/3 \;<\; 2c_{m+1}/3. \tag{4.13}$$

2) *For all $i < k$ and $1 \le m \le n$, we have*

$$\begin{aligned} &2c_0/3 \le d_\alpha(i) < c_0, \quad \text{if } \varphi_i \notin \min t_0, \\ &2c_m/3 \le d_\alpha(i) < c_m, \quad \text{if } \varphi_i \in \min t_{m-1} \setminus \min t_m, \end{aligned} \tag{4.14}$$

Proof. Let us prove (4.13) first. Note that, by (4.12), we have

$$c_{m+1} + \dots + c_n \;\le\; (1 + 1/2 + \dots + 1/2^{n-m-1})\, c_{m+1} \;<\; 2c_{m+1},$$

for any $m < n$. This proves the right-hand side inequality in (4.13). We then proceed by induction on $n - m$.

For $m = n-1$, (4.13) follows directly from (4.9). Let now $m \le n-2$ and $\varphi_i \notin t_m$, for some $i < k$. By Lemma 11 and (4.12), we have $hr(\alpha_{n-1}) = (\alpha_0, \dots, \alpha_{n-2})$, $bl(\alpha_{n-1}) = (t_0, \dots, t_{n-1})$ and $c(\alpha_{n-1}) = (c_0, \dots, c_{n-1})$. Therefore, by the induction hypothesis, we have

$$0 \;\le\; d_{\alpha_{n-1}}(i) - d_{\alpha_m}(i) \;<\; (c_{m+1} + \dots + c_{n-1})/3.$$

Combining this with (4.9) we obtain the required inequalities.

We now prove (4.14). Let $1 \le m \le n$ and $\varphi_i \in \min t_{m-1} \setminus \min t_m$, or $m = 0$ and $\varphi_i \notin \min t_0$. By Lemma 11 and (4.12) we have $hr(\alpha_m) = (\alpha_0, \dots, \alpha_{m-1})$, $bl(\alpha_m) = (t_0, \dots, t_m)$ and $c(\alpha_m) = (c_0, \dots, c_m)$. Therefore, by (4.10), we have $2c_m/3 \le d_{\alpha_m}(i) < c_m$, and moreover $d_{\alpha_m}(i) \le c_m - \varepsilon(\alpha_m)$ by the definition of $\varepsilon(\alpha_m)$. But then, by applying (4.13) and (4.12), we obtain $2c_m/3 \le d_\alpha(i) < c_m - \varepsilon(\alpha_m) + 2c_{m+1}/3 < c_m$. ■

LEMMA 14. *Let α be a parent of β in Δ. Then, for all $i < k$, we have:*

$$|d_\alpha(i) - d_\beta(i)| \;\le\; d^\beta.$$

Proof. Suppose that $hr(\alpha) = (\alpha_0, \dots, \alpha_{n-1})$, $bl(\alpha) = (t_0, \dots, t_n)$, $c(\alpha) = (c_0, \dots, c_n)$, and $\beta = \alpha(j, l)$, $t = tp(\beta)$. Then (t_n, t) is a φ_j-link. Let $i < k$.

First, assume $\varphi_i \in \min t$. Then $d_\beta(i) = 0$, $\varphi_j \not<_t \varphi_i$, and so $\varphi_j \not<_{t_n} \varphi_i$. By (4.6), we obtain $d_\alpha(i) \le d_\alpha(j)$, which implies $0 \le d_\alpha(i) - d_\beta(i) \le d^\beta$, since $d_\alpha(j) \le d^\beta$.

Therefore we further assume that $\varphi_i \notin \min t$. Four cases are possible.

Case 1: $\varphi_j \notin \min t_0$. Then $\beta = \alpha(j,0)$, $c(\beta) = (c_0/2)$, and $d^\beta = d_\alpha(j) \in [2c_0/3, c_0)$, $d_\alpha(i) < c_0$, $d_\beta(i) \in [c_0/3, c_0/2)$. Hence we have $|d_\alpha(i) - d_\beta(i)| \leq c_0 - c_0/3 \leq d^\beta$.

Case 2: $\varphi_j \in \min t_{m-1} \setminus \min t_m$, for $1 \leq m \leq n$. Then $\beta = \alpha(j,0)$, $d^\beta = d_\alpha(j) \in [2c_m/3, c_m)$, and $hr(\beta) = (\alpha_0, \ldots, \alpha_{m-1})$, $bl(\beta) = (t_0, \ldots, t_{m-1}, t)$ $c(\beta) = (c_0, \ldots, c_{m-1}, c_m/2)$. Suppose first that $\varphi_i \in \min t_{m-1} \setminus t$. Then $d_\alpha(i) < c_m$ and $d_\beta(i) \in [c_m/3, c_m/2)$, i.e., $|d_\alpha(i) - d_\beta(i)| \leq c_m - c_m/3 \leq d^\beta$.

Suppose now that $\varphi_i \notin \min t_{m-1}$. Then $0 \leq d_\beta(i) - d_{\alpha_{m-1}}(i) < c_m/6$ by (4.9), and $0 \leq d_\alpha(i) - d_{\alpha_{m-1}}(i) < 2c_m/3$ by (4.13). Therefore, we have $|d_\alpha(i) - d_\beta(i)| \leq 2c_m/3 \leq d^\beta$.

Case 3: $\varphi_j \in \min t_n$ and $<_{t_{n-1}} = <_{t_n}$. Then $hr(\beta) = (\alpha_0, \ldots, \alpha_{n-1})$ and $c(\beta) = (c_0, \ldots, c_{n-1}, d^\beta)$. Suppose that $i \in \min t_n$. Then $d_\alpha(i) = 0$ and $d_\beta(i) < d^\beta$ by the construction. Hence $|d_\beta(i) - d_\alpha(i)| < d^\beta$.

Suppose now that $i \notin \min t_n$. Then we have $i \notin \min t_{n-1}$ and $d_\alpha(i), d_\beta(i) \in [d_{\alpha_{n-1}}(i), d_{\alpha_{n-1}}(i) + d^\beta/3)$ by (4.9). Hence $|d_\beta(i) - d_\alpha(i)| < d^\beta/3$.

Case 4: $\varphi_j \in \min t_{n-1}$ and $<_{t_{n-1}} = <_{t_n}$. Similar to the previous one. ■

LEMMA 15. *Let $\alpha, \beta \in \Delta$ and $\varphi_i \in tp(\beta)$, for some $i < k$. Then*

$$d_\alpha(i) \ \leq \ d^{\mathfrak{D}}(\alpha, \beta).$$

Proof. First, note that $d_\beta(i) = 0$. Let $\alpha_0, \ldots, \alpha_n$ be the shortest path between α and β (i.e., $\alpha_0 = \alpha$, $\alpha_n = \beta$). We proceed by induction on n.

Let $n = 0$, i.e., $\alpha = \beta$. Then $d_\alpha(i) = d_\beta(i) = 0 = d^{\mathfrak{D}}(\alpha, \beta)$.

Let now $n \geq 1$. We have $d^{\mathfrak{D}}(\alpha, \beta) = d^{\mathfrak{D}}(\alpha, \alpha_1) + d^{\mathfrak{D}}(\alpha_1, \beta)$ by the definition of d^β. Then $|d_\alpha(i) - d_{\alpha_1}(i)| \leq d^{\mathfrak{D}}(\alpha, \alpha_1)$ by Lemma 14, and $|d_{\alpha_1}(i) - d_\beta(i)| \leq d^{\mathfrak{D}}(\alpha_1, \beta)$ by the induction hypothesis. Thus, we obtain $d_\alpha(i) = |d_\alpha(i) - d_\beta(i)| \leq d^{\mathfrak{D}}(\alpha, \beta)$, as required. ■

Define now a model $\mathfrak{I} = (\mathfrak{D}, p_1^{\mathfrak{I}}, p_2^{\mathfrak{I}}, \ldots)$ by the following rule, for every atomic term p_i:

$$p_i^{\mathfrak{I}} \ = \ \{\alpha \in \Delta \mid p_i \in tp(\alpha)\}.$$

LEMMA 16. *For every $\alpha \in \Delta$ and $\varphi \in \mathsf{cl}\,\tau$, we have*

$$\alpha \in \varphi^{\mathfrak{I}} \quad \textit{iff} \quad \varphi \in tp(\alpha). \tag{4.15}$$

Proof. We proceed by induction on the construction of $\varphi \in \mathsf{cl}\,\tau$. If φ is an atomic term, then (4.15) holds by the definition of $\mathfrak{I}$. If $\varphi = \neg\psi_0$ or $\varphi = \psi_0 \sqcap \psi_1$, then (4.15) follows easily from the induction hypothesis.

So let now $\varphi = \psi_0 \leftleftarrows \psi_1$. Recall that D is the initial diagram, and T is the set of types occurring in blocks from D. Suppose that $\psi_0 \notin \{\varphi_0, \ldots, \varphi_{k-1}\}$. Then $\psi_0 \simeq_t \bot$, for all $t \in T$. Hence, by the definition of a type, $\psi_0 \notin t$ and $\psi_0 \leftleftarrows \psi_1 \notin t$, for all $t \in T$. On the one hand, we obtain by the induction hypothesis that $\psi_0^{\mathfrak{I}} = \emptyset$, and so $(\psi_0 \leftleftarrows \psi_1)^{\mathfrak{I}} = \emptyset$. On the other hand, we see that $\{\alpha \in \Delta \mid \psi_0 \leftleftarrows \psi_1 \in tp(\alpha)\} = \emptyset$. Thus, (4.15) is satisfied in this case.

We therefore assume from now on that $\psi_0 \in \{\varphi_0, \dots, \varphi_{k-1}\}$. Then by the construction of Δ, we have, for every $\alpha \in \Delta$, that either $\psi_0 \in tp(\alpha)$, or $\psi_0 \in tp(\beta)$ for some child β of α. Hence $\psi_0^{\mathfrak{I}} = \{\alpha \in \Delta \mid \psi_0 \in tp(\alpha)\} \neq \emptyset$ by the induction hypothesis. Suppose now that $\psi_1 \notin \{\varphi_0, \dots, \varphi_{k-1}\}$. Then $\psi_1^{\mathfrak{I}} = \emptyset$ similarly to the above, and $\psi_1 \simeq_t \bot$, hence $\psi_0 \leftleftarrows \psi_1 \in t$, for all $t \in T$. We obtain that $\psi_0 \leftleftarrows \psi_1^{\mathfrak{I}} = \Delta$, and $\{\alpha \in \Delta \mid \psi_0 \leftleftarrows \psi_1 \in tp(\alpha)\} = \Delta$, i.e., (4.15) is satisfied in this case as well.

Suppose finally that $\psi_1 \in \{\varphi_0, \dots, \varphi_{k-1}\}$, let $\psi_0 = \varphi_0$ and $\psi_1 = \varphi_1$ for concreteness. Then $\varphi_i^{\mathfrak{I}} = \{\alpha \in \Delta \mid \varphi_i \in tp(\alpha)\} \neq \emptyset$, $i = 0, 1$, similarly to the above. Therefore, by applying Lemma 15 and (4.6), we obtain for all $\alpha \in \Delta$: $\quad \alpha \in (\varphi_0 \leftleftarrows \varphi_1)^{\mathfrak{I}} \;\leftrightarrow\; d_\alpha(0) < d_\alpha(1) \;\leftrightarrow\; \varphi_0 \leftleftarrows \varphi_1 \in tp(\alpha)$. ■

Thus, by Lemma 16, $\lambda \in \tau^{\mathfrak{I}}$ which completes the proof of Lemma 10. ■

We can now prove the main result of this paper.

THEOREM 17. *The satisfiability problem for $\mathcal{CSL}$-terms in metric models is* ExpTime*-complete.*

Proof. The ExpTime-hardness of this problem was shown in [12]. Let us prove the upper bound.

Given a term τ, let B be the set of all blocks for τ. As property (4.3) is clearly preserved under set unions, B contains the largest subset D' satisfying (4.3). It is not hard to see that D' can be constructed using the following elimination procedure (see, e.g., [5]).

Step 0: set $D_0 = B$.

Step $n+1$: suppose that D_n is constructed. For each $\mathbf{s} \in D_n$ and each $\varphi \in \mathsf{com}\,\tau$ such that $\varphi <_s \bot$ and $\varphi \notin s$, where s is the type of $\mathbf{s}$, we check whether there is a φ-link $(\mathbf{s}, \mathbf{t})$ for some $\mathbf{t} \in D_n$. If this is not the case then we set $D_{n+1} = D_n \setminus \mathbf{s}$ and go to the next step. Otherwise, we set $D' = D_n$.

As $|B| = 2^{O(|\tau|^2)}$, it should be clear that D' can be constructed in exponential time in $|\tau|$.

Suppose now that (4.1) does not hold for D'. Then obviously no diagram for τ exists, and so τ is not satisfiable by Lemma 7. So assume that $(t) \in D'$, $\tau \in t$ and consider the set

$$D \;=\; \{\mathbf{s}_n \mid ((t), \mathbf{s}_0), \dots, (\mathbf{s}_{n-1}, \mathbf{s}_n) \text{ are } \varphi\text{-links, for some } \mathbf{s}_0, \dots, \mathbf{s}_n \in D'\}.$$

Then D still satisfies (4.1) and (4.3), and by the definition of a link it satisfies (4.2) as well. Thus, D is a diagram and τ is satisfiable by Lemma 10.

It follows that τ is satisfiable iff D' satisfies (4.1), with the latter being verifiable in exponential time. ■

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BIBLIOGRAPHY

[1] J. P. Delgrande. Preliminary considerations on the modelling of belief change operators by metric spaces. In *Proceedings of NMR*, pages 118–125, 2004.

[2] L. Esakia. Intuitionistic logic and modality via topology. *Annals of Pure and Applied Logic*, 127:155–170, 2004.

[3] K. Etessami, M. Vardi, and T. Wilke. First-order logic with two variables and unary temporal logic. In *Proceedings of the 12th IEEE Symp. Logic in Computer Science*, pages 228–235, 1997.

[4] N. Friedman and J. Halpern. On the complexity of conditional logics. In *Proceedings of KR'94*, pages 202–213, 1994.

[5] D. Harel, D. Kozen, and J. Tiuryn. *Dynamic Logic*. The MIT Press, 2000.

[6] A. Kurucz, F. Wolter, and M. Zakharyaschev. Modal logics for metric spaces: Open problems. In S. Artemov, H. Barringer, A. d'Avila Garcez, L. Lamb, and J. Woods, editors, *We Will Show Them! Essays in Honour of Dov Gabbay, Volume Two*, pages 193–208. College Publications, 2005.

[7] O. Kutz, H. Sturm, N.-Y. Suzuki, F. Wolter, and M. Zakharyaschev. Logics of metric spaces. *ACM Transactions on Computational Logic*, 4:260–294, 2003.

[8] D. Lewis. *Counterfactuals*. Blackwell, Oxford, 1973.

[9] C. Lutz, U. Sattler, and F. Wolter. Description logics and the two-variable fragment. In D. L. McGuiness, P. F. Pater-Schneider, C. Goble, and R. Möller, editors, *Proceedings of the 2001 International Workshop in Description Logics (DL-2001)*, pages 66–75, Stanford, California, USA, 2001.

[10] J.C.C. McKinsey and A. Tarski. The algebra of topology. *Annals of Mathematics*, 45:141–191, 1944.

[11] K. Schlechta. *Coherent Systems*. Elsevier, 2004.

[12] M. Sheremet, D. Tishkovsky, F. Wolter, and M. Zakharyaschev. Comparative similarity, tree automata, and Diophantine equations. In G. Sutcliffe and A. Voronkov, editors, *LPAR*, volume 3835 of *Lecture Notes in Computer Science*, pages 651–665. Springer, 2005.

[13] F. Wolter and M. Zakharyaschev. A logic for metric and topology. *Journal of Symbolic Logic*, 70:795–828, 2005.

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