Completeness and incompleteness in first-order modal logic: an overview

Valentin Shehtman

In this talk we describe the present nontrivial situation in semantics for first-order modal and superintuitionistic logics. The same topic was considered in our talk of 1990, cf. [11]; nowadays the field has become more developed, but perhaps less clear.

We consider normal monomodal logics extending **QK**, the pure quantified version of **K** and also the logics with equality. A logic is a set of formulas in the full first-order language with a single modal connective $\Box$ and without function symbols and constants; this set should be closed under Substitution, Modus Ponens, Generalization, and Necessitation and it should contain the standard axioms of classical logic (with equality — if necessary) and the axioms of **K**.

We also consider superintuitionistic logics; such a logic is a set of formulas in the same first-order language without $\Box$, containing the axioms of first-order intuitionistic calculus (in an appropriate form) and closed under Substitution, Modus Ponens, and Generalization.

QL denotes the pure quantified extension of a propositional logic **L**.

The main problem is to find a reasonable balance between syntax and semantics. Kripke semantics gives a plausible solution in the field of propositional modal logic (with some exceptions). But the whole situation in the first-order case can be briefly characterized as a big gap: simply axiomatizable logics may have a complicated semantical description, and the other way round — "relatively simple" classes of frames may correspond to non-axiomatizable logics.

Here are some issues to be discussed.

1. In standard Kripke semantics there are incompleteness results for large families of logics. For example, every logic **QL**, for any $\mathbf{L} \supseteq \mathbf{S4}$ such that $\mathbf{S5} \not\subseteq \mathbf{L}$ and $\mathbf{L} \not\subseteq \mathbf{S4.3}$ is incomplete [5]. Another family consists of the logics **QL** for any intermediate propositional logic **L** of finite depth ≥ 2, cf. [8].

2. Unlike the propositional case, elementarity and Kripke-completeness are not closely related; moreover, within a certain class of first-order modal formulas every elementary formula axiomatizes an incomplete logic [2].

Advances in Modal Logic, Volume 6.

3. Sahlqvist's completeness theorem does not transfer to Kripke semantics of first-order modal logics; a simple example of an incomplete logic with Sahlqvist axioms is $\mathbf{QK} + \Diamond\top + \Box\Box p \rightarrow \Box p$, cf. [4]. Moreover, Kripke-complete logics are usually not canonical (in a natural sense); so completeness proofs in Kripke semantics are rather nontrivial, cf. [3], [13].

4. Unlike the propositional case, one should not expect axiomatizability for logics of non-elementary classes of frames, cf. general results in [15], [14]. Typical examples — the superintuitionistic logic of all frames based on an arbitrary Nötherian poset of unrestricted height is Π^1_1-hard; the superintuitionistic logic of a class of Kripke frames based on a class of finite posets is recursively axiomatizable iff this logic is characterized by a single finite frame.

5. The hyperdoctrinal semantics, where all first-order modal logics are complete [12], is rather complicated, so its real applicability is questionable. Recent results on completeness with respect to general Kripke-type frames presented at this conference [7] are of different flavour — this semantics is "first-order" in Thomason's sense, i.e. valuations do not range over all subsets.

6. There exists a Kripke-type ("second order") semantics, for which an analogue of Sahlqvist Theorem can be proved [10]. This is "simplicial" (or "metaframe") semantics introduced in [16]. However the convenience of simplicial frames is dubious. Functor semantics (introduced by S. Ghilardi) seems more efficient, but here a general completeness theorem is known only for superintuitionistic logics of the form $\mathbf{QL}$, with $\mathbf{L}$ canonical [6]. Generally speaking, simplicial frames are not representable of by set-valued functors, and we do not know any sufficient condition for representability. For modal logics functor semantics is weaker than simplicial [16], but we do not know if this is the case for superintuitionistic logics.

7. Algebraic semantics with equality also seems a good candidate to deal with – note that it is adequate for all propositional logics. Nevertheless we do not know if every first-order modal logic is algebraically complete.

8. Modal and superintuitionistic logics are not so closely related as in the propositional case. It seems unclear, whether every superintuitionistic logic is a fragment of some modal logic extending $\mathbf{QS4}$. For example, this problem arises for simplicially complete logics that are not Kripke-complete, because intuitionistically sound simplicial frames may be not modally sound [16].

9. Very little is known about topological semantics, which is between (standard) Kripke and algebraic. For some recent completeness results

see [1]. The logics of familiar spaces (like the real line) with the constant domain are still unknown.

10. In many cases for simple, but Kripke-incomplete logics, nothing is known about completeness even in simplicial semantics – e.g. for $\mathbf{QS4.1}$ or $\mathbf{QH} + \neg\neg\exists x P(x) \rightarrow \exists x \neg\neg P(x)$. In other cases, when simplicial semantics is adequate, it is unclear, what happens in more convenient semantics (like functor or topological) – a typical example is $\mathbf{QS4.2} + \forall x \Box P(x) \rightarrow \Box \forall x P(x)$.

BIBLIOGRAPHY

[1] Arló-Costa, H., Pacuit, E.: *First-order classical modal logic: applications in logics of knowledge and probability.* In: Van der Meyden, R., ed. Proceedings of TARK X (2005), July 10-12, 2005, Singapore. `www.tark.org/proceedings/tark_jul10_05/proceedings.html`

[2] Astretsov, S.: *Elementarity and incompleteness for predicate modal logics with constant domains.* International Conference "Computer Science applications of modal logic", September 5-9, 2005, Abstracts. Poncelet Math. Laboratory, 2005, Moscow. `www.mccme.ru/poncelet/`

[3] Corsi, G., Ghilardi, S.: *Directed frames.* Archive for math. logic, **29** (1989), 53–67.

[4] Gasquet, O.: *A new incompleteness result in Kripke semantics.* Fundamenta Informatica, **24** (1995), 407–415.

[5] Ghilardi, S.: *Incompleteness results in Kripke semantics.* J. Symbolic Logic, **56** (1991), 517–538.

[6] Ghilardi, S.: *Quantified extensions of canonical propositional intermediate logics.* Studia Logica, **51** (1992), 195–214.

[7] Goldblatt, R., Mares, E.: *A general semantics for quantified modal logic* (2006), see this Volume.

[8] Ono, H.: *Incompleteness of semantics for intermediate predicate logics. I. Kripke's semantics.* Proc. Jap. Acad., **49:** (1973), 711–713,

[9] Ono, H.: *Model extension theorem and Craig's interpolation theorem for intermediate predicate logics.* Reports in Math. Logic, No. 15 (1983), 41–58.

[10] Shehtman, V.: *Sahlqvist-type completeness theorem in modal predicate logic.* In: 10th International Congress in Logic, Methodology and Philosophy of Science, Florence, 1995, Abstracts, 220.

[11] Shehtman, V., Skvortsov, D.: *Semantics of non-classical first order predicate logics.* In Petkov, P., ed., Proceedings of Summer School and Conference on Mathematical Logic, Sept. 13–23 in Chaika, Bulgaria, 105–116. Plenum Press, 1990.

[12] Shirasu, H.: *Hyperdoctrinal semantics for non-classical predicate logics.* PhD thesis, Japan Advanced Institute of Science and Technology, 1997.

[13] Skvortsov, D.: *On the predicate logic of linear Kripke frames and some of its extensions.* Studia Logica, **81** (2005), 261–282.

[14] Skvortsov, D.: *The superintuitionistic predicate logic of finite Kripke frames is not recursively axiomatizable.* J. Symbolic Logic, **70** (2005), 451–459.

[15] Skvortsov, D.: *On non-axiomatizability of superintuitionistic predicate logics of some classes of well-founded and dually well-founded Kripke frames.* Journal of Logic and Computation (2006), to appear.

[16] Skvortsov, D., Shehtman, V.: *Maximal Kripke-type semantics for modal and superintuitionistic predicate logics.* Annals of Pure and Applied Logic, **63** (1993), 69–101.

Valentin Shehtman

Institute for Information Transmission Problems
Russian Academy of Sciences

B.Karetny 19, Moscow, Russia, 127994
shehtman@netscape.net