Downward-directed transitive frames with universal relations

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abstract. In this paper we identify modal logics of some bimodal Kripke frames corresponding to geometrical structures. Each of these frames is a set of 'geometrical' objects with some natural accessibility relation plus the universal relation. For these logics we present finite axiom systems and prove completeness.

We also show that all these logics have the finite model property and are PSPACE-complete. To prove this, we show that under certain restrictions, adding the universal modality preserves 'good' properties of a monomodal logic.

Keywords: universal modality, relativistic temporal logics, regions, completeness, finite model property

1 Introduction

The subject of this paper is in the field of spatial and temporal modal logics. We focus mainly on relativistic time, and also consider some interval and regional structures.

Our basic temporal relations are *causal* ($\preceq$) and *chronological* ($\prec$) accessibility. Modal axiomatizations of real space and its domains ordered by $\preceq$ can be found in [3], [14]. Analogous results on $\prec$ were obtained in [12], [10]. In [15] it was noted that relativistic logics can be interpreted in interval or regional semantics. This approach was further developed in [13]. Thus at present we have quite a few monomodal logics axiomatizing relativistic spacetime, interval or regional structures; all these logics have the finite model property (for short, FMP) and they are known to be PSPACE-complete. But the expressive power of these systems is rather weak. On the other hand, richer spatial structures may have undecidable or even non recursively enumerable logics (cf. [6], [8], [13]). This brings up the following standard question: how to improve the expressivity preserving the FMP and complexity?

Consider the following property of relativistic time: any two points are accessible from another point, in other words, any two points have common past. This property (*downward-directedness*) holds in real space $\mathbb{R}^n$, but it may fail in its subsets, in particular, in the upper half-space $\mathbb{R}^{n-1} \times \{t \mid t > 0, t \in \mathbb{R}\}$. However, the monomodal logic of the upper half-space (ordered by $\prec$ or $\preceq$) is equal to the logic of the whole space. Using the universal modality, we can express downward-directedness with the modal formula $A^{\downarrow} = \exists p \wedge \exists q \to \exists(\Diamond p \wedge \Diamond q)$.

In this paper we axiomatize modal logics of some geometrical downward-directed frames expanded with universal relations, and show that the FMP and PSPACE-decidability are preserved.

The problem of enriching a modal language with the universal modality was first systematically investigated in [5]. Given a normal monomodal logic L, we consider the logic LU - the fusion of L and **S5** plus the *containment* axiom $\Diamond p \to \exists p$. Among others, it was proved in [5] that LU inherits strong Kripke completeness and compactness from L. However, some properties of L can be lost: [17] gives an example of a monomodal logic L with the FMP such that LU lacks the FMP; in [7], it was shown that there exists a decidable monomodal logic L such that LU is undecidable.

Nevertheless, we prove that adding the universal modality together with the axiom $A^{\downarrow}$ preserves properties of transitive logics such as Kripke-completeness, the finite model property, decidability within a certain polynomially closed complexity class.

2 Basic notions

In this paper we consider normal monomodal and bimodal propositional logics.

Let $FM(\Diamond)$ be the set of all formulas constructed using a countable set of propositional variables $PV = \{p_1, p_2, \dots\}$, propositional constant $\bot$ (*false*), and connectives $\to$, $\Diamond$. To obtain $FM(\Diamond, \exists)$, we enrich the language with the unary modal operator $\exists$. We define $\Box$, $\forall$, $\neg$, $\vee$, $\wedge$, $\top$ in the usual way, in particular, $\Box A := \neg\Diamond\neg A$, $\forall A := \neg\exists\neg A$. Also let $\Diamond^+ A := \Diamond A \vee A$, $\Box^+ A := \Box A \wedge A$.

For a monomodal (bimodal) logic L and a formula $A \in FM(\Diamond)$ ($A \in FM(\Diamond, \exists)$), $\mathrm{L} + A$ denotes the smallest monomodal (bimodal) logic containing $\mathrm{L} \cup \{A\}$. The notation $\mathrm{L} \vdash A$ means $A \in \mathrm{L}$.

$Sub(A)$ denotes the set of all subformulas of A; for $\odot \in \{\Diamond, \exists\}$, $Sub_{\odot}(A) := \{\odot A \mid \odot A \in Sub(A)\}$.

A *(Kripke) frame* is a tuple $(W, R_1, \dots, R_n)$, where $W \neq \varnothing$, $R_i \subseteq W \times W$. In this paper we always assume that R_1 is *transitive*. We consider only monomodal or bimodal frames, i.e., $n \leq 2$.

A *(Kripke) model* $\mathfrak{M}$ over a frame $\mathfrak{F}$ is a pair $(\mathfrak{F}, \theta)$, where $\theta : PV \to 2^W$, 2^W denotes the power set of W. The truth of a formula in a model is defined in the standard way. In particular, for a model $\mathfrak{M}$ over a frame (W, R_1, R_2) and a formula $\exists A \in FM(\Diamond, \exists)$, we put for all $x \in W$

$$\mathfrak{M}, x \vDash \exists A \quad \text{iff} \quad \mathfrak{M}, y \vDash A \text{ for some } y \in R_2(x)$$

The notations $x \in \mathfrak{M}$, $x \in \mathfrak{F}$ mean $x \in W$. The notation $\mathfrak{M} \vDash A$ means that $\mathfrak{M}, x \vDash A$ for all $x \in \mathfrak{M}$. If for any model $\mathfrak{M}$ over $\mathfrak{F}$ we have $\mathfrak{M} \vDash A$ then we say that A is *valid* in $\mathfrak{F}$, $\mathfrak{F} \vDash A$ in symbols. For a class $\mathcal{F}$ of frames,

$\mathbf{L}(\mathcal{F})$ denotes the *modal logic determined by* $\mathcal{F}$, i.e., the set of all formulas that are valid in all frames from $\mathcal{F}$. For a single frame $\mathfrak{F}$, $\mathbf{L}(\mathfrak{F})$ abbreviates $\mathbf{L}(\{\mathfrak{F}\})$.

For a logic L, if $\mathbf{L}(\mathfrak{F}) \supseteq$ L, then we say that $\mathfrak{F}$ is an L-*frame*. A formula A is *satisfiable in a model* $\mathfrak{M}$ if for some $x \in \mathfrak{M}$ we have $\mathfrak{M}, x \models A$; A is *satisfiable at a point x in a frame* $\mathfrak{F}$ if $\mathfrak{M}, x \models A$ for some model $\mathfrak{M}$ over $\mathfrak{F}$. A is *satisfiable in a frame* $\mathfrak{F}$ if A is satisfiable at some $x \in \mathfrak{F}$. For a class $\mathcal{F}$ of frames, A is $\mathcal{F}$-*satisfiable* if A is satisfiable in some $\mathfrak{F} \in \mathcal{F}$. A is L-*satisfiable* if A is satisfiable in some L-frame.

Consider a monomodal frame (W, R). For $x \in W$ let $R(x) := \{y \mid xRy\}$. Id_W denotes the equality relation on W, and R^r denotes the reflexive closure of R, i.e., $R^r := R \cup Id_W$.

Consider some first-order properties of a relation R:

seriality:	$\forall x \exists y\ xRy$;
Church–Rosser property:	$\forall x \forall y_1 \forall y_2 \exists z (xRy_1 \wedge xRy_2 \to y_1Rz \wedge y_2Rz)$;
McKinsey property:	$\forall x \exists y \in R(x)\ R(y) = \{y\}$;
irreflexive McKinsey property:	$\forall x \exists y \in R^r(x)\ R(y) = \varnothing$;
2-density:	$\forall x \forall y_1 \forall y_2 \exists z (xRy_1 \wedge xRy_2 \to xRz \wedge zRy_1 \wedge zRy_2)$.

For a monomodal frame (W, R), we have the following correspondence between modal axioms and first-order properties (recall that we consider only transitive frames):

$A4 := \Diamond\Diamond p \to \Diamond p$	transitivity;
$AT := p \to \Diamond p$	reflexivity;
$AD := \Diamond\top$	seriality;
$A1 := \Box\Diamond p \to \Diamond\Box p$	McKinsey property;
$A2 := \Diamond\Box p \to \Box\Diamond p$	Church − Rosser property.

Put $A1^- := \Diamond^+\Box\bot$, $Ad_2 := \Diamond p_1 \wedge \Diamond p_2 \to \Diamond(\Diamond p_1 \wedge \Diamond p_2)$.

The following proposition is straightforward:

PROPOSITION 1. *For a frame* $\mathfrak{F} = (W, R)$,

- $\mathfrak{F} \models A1^-$ *iff* $\mathfrak{F}$ *satisfies irreflexive McKinsey property;*
- $\mathfrak{F} \models Ad_2$ *iff* $\mathfrak{F}$ *is 2-dense iff for any* $n \geq 1$, $\mathfrak{F}$ *satisfies*

$$\forall x \forall y_1 \ldots \forall y_n (\{y_1, \ldots, y_n\} \subseteq R(x) \to \exists z \in R(x)\ \{y_1, \ldots, y_n\} \subseteq R(z)).$$

As usual, $\mathbf{K}$ denotes the smallest normal monomodal logic. Let

$$\mathbf{K4} := \mathbf{K} + A4, \qquad \mathbf{S4} := \mathbf{K4} + AT,$$
$$\mathbf{Cr} := \mathbf{K4} + Ad_2 + AD, \qquad \mathbf{CrB} := \mathbf{K4} + Ad_2 + A1^-.$$

For a logic L let $\mathrm{L}.1 := \mathrm{L} + A1,\ \mathrm{L}.2 := \mathrm{L} + A2$.

Consider a monomodal frame $\mathfrak{F} = (W, R)$. For $V \subseteq W$, let $R|V := R \cap (V \times V)$. For $x \in \mathfrak{F}$ let $\mathfrak{F}\langle x\rangle := (R^r(x), R|R^r(x))$. If for some $x \in \mathfrak{F}$ we have $\mathfrak{F} = \mathfrak{F}\langle x\rangle$, we say that $\mathfrak{F}$ is *rooted* (or *a cone*), and x is a *root* of $\mathfrak{F}$. We say that $x \in \mathfrak{F}$ is *minimal* in $\mathfrak{F}$, if yRx implies xRy for any $y \in \mathfrak{F}$. Let $\mathfrak{F}^u$ denote the frame with the additional *universal* relation: $\mathfrak{F}^u := (W, R, W \times W)$. For a model $\mathfrak{M} = (\mathfrak{F}, \theta)$ we put $\mathfrak{M}^u := (\mathfrak{F}^u, \theta)$. For a class $\mathcal{F}$ of monomodal frames we put $\mathcal{F}^u := \{\mathfrak{F}^u \mid \mathfrak{F} \in \mathcal{F}\}$.

Consider frames $\mathfrak{F} = (W, R_1, \ldots, R_n)$ and $\mathfrak{G} = (V, S_1, \ldots, S_n)$. Recall that a surjective map $f : W \to V$ is a *p-morphism* from $\mathfrak{F}$ onto $\mathfrak{G}$ (in notation, $f : \mathfrak{F} \twoheadrightarrow \mathfrak{G}$), if for any $x \in W$, $1 \le i \le n$, we have $f(R_i(x)) = S_i(f(x))$. The notation $\mathfrak{F} \twoheadrightarrow \mathfrak{G}$ means that there exists a p-morphism from $\mathfrak{F}$ onto $\mathfrak{G}$. Recall that $\mathfrak{F} \twoheadrightarrow \mathfrak{G}$ implies $\mathbf{L}(\mathfrak{F}) \subseteq \mathbf{L}(\mathfrak{G})$. For monomodal frames $\mathfrak{F}$ and $\mathfrak{G}$ the following two facts are trivial: if $\mathfrak{F}$ and $\mathfrak{G}$ are isomorphic then $\mathfrak{F}^u$ and $\mathfrak{G}^u$ are isomorphic; if $f : \mathfrak{F} \twoheadrightarrow \mathfrak{G}$ then $f : \mathfrak{F}^u \twoheadrightarrow \mathfrak{G}^u$, and so $\mathbf{L}(\mathfrak{F}^u) \subseteq \mathbf{L}(\mathfrak{G}^u)$.

The following syntactical introduction of the universal modality is due to [5]. For a monomodal logic L, let LU denote the smallest normal bimodal logic containing L and the formulas

$$A4_\exists = \exists\exists p \to \exists p, \quad AT_\exists = p \to \exists p, \quad AB_\exists = p \to \forall\exists p;$$
$$A_\subseteq = \Diamond p \to \exists p.$$

It follows that (W, R_1, R_2) is LU-frame iff $\mathfrak{F} = (W, R_1)$ is L-frame, R_2 is an equivalence relation on W, and $R_1 \subseteq R_2$.

3 Translation

Given a monomodal rooted frame $\mathfrak{F}$, it is possible to show that the satisfiability in $\mathfrak{F}^u$ can be reduced to the satisfiability in $\mathfrak{F}$. For this purpose we use the following construction, proposed in [7].

Consider a formula $A \in FM(\Diamond, \exists)$. In this section we assume that $PV(A) \subseteq \{p_1, \ldots, p_m\}$, $Sub(A) = \{A_1, \ldots, A_n\}$, and $Sub_\exists(A) = \{\exists A_{i_1}, \ldots, \exists A_{i_l}\}$.

Fix some variables $q_1, \ldots, q_l \notin PV(A)$. For any $B \in Sub(A)$ we define the formula $[B]$ as follows:

$$[p] := p,\ [\bot] := \bot,\ [B_1 \to B_2] := [B_1] \to [B_2],\ [\Diamond B] := \Diamond[B],\ [\exists A_{i_j}] := q_j.$$

LEMMA 2. *[7] Given a formula $A \in \mathrm{FM}(\Diamond, \exists)$ and a model $\mathfrak{M} = ((W, R), \theta)$ such that:*

(3.1) *if $\mathfrak{M}^u \vDash \exists A_{i_j}$ then $\theta(q_j) = W$, otherwise $\theta(q_j) = \varnothing$.*

Then for any $y \in W$ and any $B \in Sub(A)$, we have

$$\mathfrak{M}^u, y \vDash B \iff \mathfrak{M}, y \vDash [B]$$

Proof. By induction on the length of the formula. We consider only the case when $B = \exists A_{i_j}$: $\mathfrak{M}^u, y \vDash \exists A_{i_j}$ iff for some $z \in W$ $\mathfrak{M}^u, z \vDash A_{i_j}$ iff $\theta(q_j) = W$ iff $\mathfrak{M}, y \vDash [\exists A_{i_j}]$ (recall that $[\exists A_{i_j}] = q_j$). ∎

We put

$$A_\exists := \bigwedge_{j=1}^{l} \big((\exists[A_{i_j}] \to \forall q_j) \wedge (\neg\exists[A_{i_j}] \to \forall\neg q_j)\big) \wedge [A]$$

(as usual, we put $\bigwedge_{i\in\varnothing} C_i := \top$)

LEMMA 3. *[7] A is satisfiable in $\mathfrak{F}^u$ iff $A_\exists$ is satisfiable in $\mathfrak{F}^u$.*

Proof. ($\Rightarrow$) For some $\mathfrak{M}_0 = (\mathfrak{F}, \theta_0)$, $x \in W$, we have $\mathfrak{M}_0^u, x \vDash A$. Since $q_j \notin PV(A)$, $1 \le j \le l$, there exists a model $\mathfrak{M} = (\mathfrak{F}, \theta)$ such that $\theta(p_i) = \theta_0(p_i)$, $1 \le i \le m$, and $\mathfrak{M}$ satisfies (3.1). Then $\mathfrak{M}^u, x \vDash A$, and by Lemma 2 $\mathfrak{M}^u, x \vDash A_\exists$.
($\Leftarrow$) For some $\mathfrak{M} = (\mathfrak{F}, \theta)$, $x \in W$, we have $\mathfrak{M}^u, x \vDash A_\exists$. By induction on the length of the formula one can see that for any $y \in \mathfrak{F}$ and any $B \in Sub(A)$, we have: $\mathfrak{M}^u, y \vDash B$ iff $\mathfrak{M}, y \vDash [B]$. Let us consider the case when $B = \exists A_{i_j}$:
$\mathfrak{M}^u, y \vDash \exists A_{i_j}$ iff for some z $\mathfrak{M}^u, z \vDash A_{i_j}$ iff for some z $\mathfrak{M}, z \vDash [A_{i_j}]$ iff $\mathfrak{M}^u, y \vDash \exists[A_{i_j}]$ iff $\mathfrak{M}, y \vDash q_j$ (note that $\mathfrak{M}^u \vDash \forall q_j$ or $\mathfrak{M}^u \vDash \forall\neg q_j$).
Since $\mathfrak{M}, x \vDash [A]$, we have $\mathfrak{M}^u, x \vDash A$. ∎

Let us slightly modify the considered translation:

$$A_\Diamond := \bigwedge_{j=1}^{l} \big((\Diamond^+[A_{i_j}] \to \Box^+ q_j) \wedge (\neg\Diamond^+[A_{i_j}] \to \Box^+\neg q_j)\big) \wedge \Diamond^+[A]$$

Note that $A_\Diamond \in FM(\Diamond)$.

Consider a rooted frame $\mathfrak{F} = \mathfrak{F}\langle x\rangle$. Then for any model $\mathfrak{M}$ over $\mathfrak{F}$ and $B \in FM(\Diamond)$ we have: $\mathfrak{M}^u \vDash \exists B$ iff $\mathfrak{M}, x \vDash \Diamond^+ B$. Thus we obtain the following reformulation of Lemma 3:

LEMMA 4. *Let $\mathfrak{F}$ be a cone, and let x be a root of $\mathfrak{F}$. Then for any $A \in$ FM$(\Diamond, \exists)$, A is satisfiable in $\mathfrak{F}^u$ iff $A_\Diamond$ is satisfiable at x in $\mathfrak{F}$.*

4 Downward-directed frames

In this section we prove transfer theorems for logics of downward-directed transitive frames.

A monomodal frame $\mathfrak{F} = (W, R)$ is *downward-directed* iff $\mathfrak{F}$ satisfies

$$\forall x \forall y \exists z (zRx \wedge zRy).$$

(W, R) is *weakly downward-directed* iff (W, R^r) is downward-directed.

Consider the following axioms:

$$A^\Downarrow := \exists p \wedge \exists q \to \exists(\Diamond^+ p \wedge \Diamond^+ q)$$
$$A^\downarrow := \exists p \wedge \exists q \to \exists(\Diamond p \wedge \Diamond q)$$

Straightforward arguments show

PROPOSITION 5. *For a frame* $\mathfrak{F} = (W, R)$,

- $\mathfrak{F}^u \models A^{\Downarrow}$ *iff* $\mathfrak{F}$ *is weakly downward-directed;*
- $\mathfrak{F}^u \models A^{\downarrow}$ *iff* $\mathfrak{F}$ *is downward-directed.*

Clearly, every rooted frame is weakly downward-directed, so $\mathfrak{F} = \mathfrak{F}\langle x \rangle$ implies $\mathfrak{F}^u \models A^{\Downarrow}$. Moreover, if x is reflexive then $\mathfrak{F}$ is downward-directed and $\mathfrak{F} \models A^{\downarrow}$. On the other hand, let x be minimal in $\mathfrak{F}$. Then $\mathfrak{F}^u \models A^{\Downarrow}$ implies $\mathfrak{F} = \mathfrak{F}\langle x \rangle$; if $\mathfrak{F}^u \models A^{\downarrow}$ then we also have that x is reflexive.

For a monomodal logic L, we put

$$\mathrm{LU}^{\downarrow} := \mathrm{LU} + A^{\downarrow},\ \mathrm{LU}^{\Downarrow} := \mathrm{LU} + A^{\Downarrow}.$$

For a bimodal logic M, let $\mathfrak{F}^{\mathrm{M}} = (W^{\mathrm{M}}, R^{\mathrm{M}}, U^{\mathrm{M}})$ denote its canonical frame, $\mathfrak{M}^{\mathrm{M}}$ denote its canonical model (the notion of *canonicity* is defined in the standard way, see e.g. [1]). Note that if $\mathrm{M} \supseteq \mathbf{K}\mathrm{U}$ then $R^{\mathrm{M}} \subseteq U^{\mathrm{M}}$ and U^{M} is an equivalence relation on W^{M} [5]. One can see that $A^{\downarrow},\ A^{\Downarrow}$ are Sahlqvist formulas, so $A^{\downarrow},\ A^{\Downarrow}$ are canonical.

By induction on n it is not hard to check

PROPOSITION 6. *For* $n \geq 1$,

- $\mathbf{K4}\mathrm{U}^{\Downarrow} \vdash \exists p_1 \wedge \ldots \wedge \exists p_n \rightarrow \exists(\Diamond^+ p_1 \wedge \ldots \wedge \Diamond^+ p_n)$.
- $\mathbf{K4}\mathrm{U}^{\downarrow} \vdash \exists p_1 \wedge \ldots \wedge \exists p_n \rightarrow \exists(\Diamond p_1 \wedge \ldots \wedge \Diamond p_n)$.

The following lemma shows that the canonical frame for a logic $\mathrm{M} \supseteq \mathbf{K4}\mathrm{U}^{\Downarrow}$ is a disjoint union of cones with universal relations:

LEMMA 7. *Consider a logic* $\mathrm{M} \supseteq \mathbf{K4}\mathrm{U}$. *Let* W *be an equivalence class modulo* U^{M}, $R := R^{\mathrm{M}}|W$, $\mathfrak{F} := (W, R)$. *Then*

- *if* $\mathrm{M} \vdash A^{\Downarrow}$ *then* $\mathfrak{F}$ *has a root;*
- *if* $\mathrm{M} \vdash A^{\downarrow}$ *then* $\mathfrak{F}$ *has a reflexive root.*

Proof. Let $\Psi = \{\Diamond B \mid B \in y \text{ for some } y \in W\}$. If Ψ is M-consistent then by the Lindenbaum lemma there exists $x \in W^{\mathrm{M}}$ such that $x \supseteq \Psi$. Then by the construction, $W = R(x)$.

If $\mathrm{M} \vdash A^{\downarrow}$ then any finite subset of Ψ is M-consistent (see Proposition 6), so Ψ is M-consistent.

Suppose that $\mathrm{M} \vdash A^{\Downarrow}$ and Ψ is M-inconsistent. Then $\Phi := \{\Diamond B_1, \ldots, \Diamond B_n\}$ is M-inconsistent for some $\Diamond B_1, \ldots, \Diamond B_n \in \Psi$. By Proposition 6, there exists $x \in W$ such that $x \ni \bigwedge\limits_{1 \leq i \leq n} \Diamond^+ B_i$. Then for any $x' \in W$, $x' R x$ implies $x' \supset \Phi$. It follows that x is minimal in $\mathfrak{F}$. Since $\mathfrak{F} \models A^{\Downarrow}$, $\mathfrak{F} = \mathfrak{F}\langle x \rangle$. ■

For a formula A, let $A^0 := \neg A$, $A^1 := A$.

THEOREM 8. *Let $\mathcal{F}$ be a class of rooted frames closed under taking cones. Then* $\mathbf{L}(\mathcal{F}^u) = \mathbf{L}(\mathcal{F})\mathrm{U}^{\Downarrow}$.

Proof. Put $\mathrm{L} := \mathbf{L}(\mathcal{F})$, $\mathrm{M} := \mathrm{LU}^{\Downarrow}$. Since $\mathcal{F}^u$ consists of rooted LU-frames, we have $\mathbf{L}(\mathcal{F}^u) \supseteq \mathrm{M}$.

To prove the converse inclusion, consider an M-consistent formula A. Then $A \in y_0$ for some $y_0 \in W^{\mathrm{M}}$. Let $Sub_E(A) = \{B_1, \ldots, B_l\}$. Put $B := A \wedge \bigwedge\limits_{1 \le j \le l} B_j^{v_j}$, where $v_1, \ldots, v_l$ are defined as follows: if $B_j \in y_0$ then $v_j := 1$, otherwise $v_j := 0$. Thus $B \in y_0$.

Note that for any M-consistent formula C, $q \notin PV(C)$, $v \in \{0,1\}$, we have $C \wedge \forall q^v$ is M-consistent. Indeed, suppose that $C \wedge \forall q^v$ is M-inconsistent, i.e., $\mathrm{M} \vdash \forall q^v \to \neg C$. Replacing q with $\bot$ or $\top$, we obtain $\mathrm{M} \vdash \forall \top \to \neg C$. Trivially, $\mathrm{M} \vdash \forall \top$, so $\mathrm{M} \vdash \neg C$: a contradiction.

Let $q_1, \ldots, q_l \notin PV(B)$. It follows that $B \wedge \bigwedge\limits_{1 \le j \le l} \forall q_j^{v_j}$ is M-consistent, so for some $y \in W^{\mathrm{M}}$ we have $\{B, \forall q_1^{v_1}, \ldots, \forall q_l^{v_l}\} \subseteq y$.

Let $W := U^{\mathrm{M}}(y)$, and let $\mathfrak{M}^u = ((W, R, U), \theta)$ be the restriction $\mathfrak{M}^{\mathrm{M}}|W$, i.e., $R := R^{\mathrm{M}}|W$, $U := U^{\mathrm{M}}|W = W \times W$, and for any $p \in PV$, $\theta(p) := \{x \in W \mid p \in x\}$. As well as in $\mathfrak{M}^{\mathrm{M}}$, for any $z \in W$ and any $C \in FM(\Diamond, \exists)$ we have: $C \in x$ iff $\mathfrak{M}^u, x \vDash C$.

Put $\mathfrak{F} := (W, R)$. By Lemma 2, we have $A_\exists \in y$. By Lemma 7, $\mathfrak{F} = \mathfrak{F}\langle x \rangle$ for some $x \in W$. Then $A_\Diamond \in x$, thus $A_\Diamond$ is L-consistent.

Then $A_\Diamond$ is satisfiable at some $z \in \mathfrak{G}$, $\mathfrak{G} \in \mathcal{F}$, thus $A_\Diamond$ is satisfiable at the root of $\mathfrak{H} = \mathfrak{G}\langle z \rangle$, and by Lemma 4, A is satisfiable in $\mathfrak{H}^u$. Since $\mathcal{F}$ closed under taking cones, we have $\mathfrak{H} \in \mathcal{F}$ and so $\mathfrak{H}^u \in \mathcal{F}^u$. Thus A is $\mathcal{F}^u$-satisfiable.

It follows that $\mathbf{L}(\mathcal{F}^u) \subseteq \mathrm{M}$. ■

COROLLARY 9. *For a monomodal transitive Kripke-complete logic* L*,*

- $\mathrm{LU}^{\Downarrow}$ *is Kripke-complete;*
- *if* L *has the FMP, then* $\mathrm{LU}^{\Downarrow}$ *has the FMP;*
- *for any* $A \in \mathrm{FM}(\Diamond, \exists)$*,* A *is* $\mathrm{LU}^{\Downarrow}$*-satisfiable iff* $A_\Diamond$ *is* L*-satisfiable;*
- L *and* $\mathrm{LU}^{\Downarrow}$ *are polynomially equivalent.*

Proof. Suppose $\mathrm{L} = \mathbf{L}(\mathcal{G})$. Put $\mathcal{F} = \{\mathfrak{G}\langle x \rangle \mid \mathfrak{G} \in \mathcal{G},\ x \in \mathfrak{G}\}$. Then $\mathrm{L} = \mathbf{L}(\mathcal{F})$. By Theorem 8, $\mathrm{LU}^{\Downarrow} = \mathbf{L}(\mathcal{F}^u)$.

By Lemma 4, for any formula $A \in FM(\Diamond, \exists)$ we have:

$$A \text{ is } \mathcal{F}^u\text{-satisfiable iff } A_\Diamond \text{ is } \mathcal{F}\text{-satisfiable.}$$

One can readily check that $A_\Diamond$ can be computed in time polynomial in the length of A, so $\mathrm{LU}^{\Downarrow}$ is polynomially reduced to L. Trivially, for any $B \in FM(\Diamond)$ we have: B is L-satisfiable iff B is $\mathrm{LU}^{\Downarrow}$-satisfiable, so L and $\mathrm{LU}^{\Downarrow}$ are polynomially equivalent. ■

It is well-known that the logics **S4**, **S4.2**, **S4.1** have the FMP and are PSPACE-complete (see e.g. [2]). Clearly, if $\mathrm{L} \vdash AT$ then $\mathrm{LU}^{\downarrow} = \mathrm{LU}^{\Downarrow}$. Therefore we have

COROLLARY 10. *The logics* $\mathbf{S4}\mathrm{U}^{\downarrow}$, $\mathbf{S4.2}\mathrm{U}^{\downarrow}$, $\mathbf{S4.1}\mathrm{U}^{\downarrow}$ *have the FMP and are PSPACE-complete.*

Thus adding the universal modality together with the axiom $A^{\Downarrow}$ preserves Kripke completeness, the FMP, and the complexity of a transitive monomodal logic L. However, the situation with the axiom $A^{\downarrow}$ is more delicate: the following example shows that $\mathrm{LU}^{\downarrow}$ may lack the FMP.

Consider the logic $\mathbf{GL} := \mathbf{K} + \Box(\Box p \to p) \to \Box p$. This logic is Kripke-complete, transitive and has the FMP: **GL** is complete with respect to the class of all finite strictly ordered trees (see e.g. [2]). The logic $\mathbf{GL}\mathrm{U}^{\downarrow}$ is consistent and has frames, for instance $\mathbf{L}((\mathbb{N}, >)^u) \supseteq \mathbf{GL}\mathrm{U}^{\downarrow}$. But it is not hard to see that this logic has no finite frames. (To simplify this example one can replace $A^{\downarrow}$ with the axiom $\exists p \to \exists \Diamond p$, expressing seriality of R^{-1}.)

Nevertheless, under some additional assumptions, $\mathrm{LU}^{\downarrow}$ inherits the above mentioned properties of L.

For a monomodal frame $\mathfrak{F}$, let $\widetilde{\mathfrak{F}} := \mathfrak{C}_1 + \mathfrak{F}$, where $\mathfrak{C}_1$ is a reflexive singleton, + denotes the ordinal sum of frames.

For a formula $A \in FM(\Diamond, \exists)$, put

$$A_{\Diamond}^{refl} := A_{\Diamond} \wedge \bigwedge_{\Diamond B \in Sub(A_{\Diamond})} (B \to \Diamond B)$$

LEMMA 11. *Suppose* $A_{\Diamond}^{refl}$ *is satisfiable at* y *in* $\mathfrak{F}$, $\mathfrak{G} := \widetilde{\mathfrak{F}\langle y\rangle}$. *Then* A *is satisfiable in* $\mathfrak{G}^u$.

Proof. For some model $\mathfrak{M} = (\mathfrak{F}, \eta)$, we have $\mathfrak{M}, y \vDash A_{\Diamond}^{refl}$. Let y_0 denote the root of $\mathfrak{G}$. Put $\mathfrak{N} := (\mathfrak{G}, \theta)$, where $\theta^{-1}(y_0) := \eta^{-1}(y)$, and $\theta^{-1}(z) := \eta^{-1}(z)$ for any $z \in \mathfrak{F}\langle y\rangle$.

By induction on the length of the formula one can readily check that $\mathfrak{N}, y_0 \vDash A_{\Diamond}$. By Lemma 4, A is satisfiable in $\mathfrak{G}^u$. ■

THEOREM 12. *Let* $\mathcal{F}$ *be a class of frames such that:*
any $\mathfrak{F} \in \mathcal{F}$ *has a reflexive root;*
$\mathfrak{F} \in \mathcal{F}$, $y \in \mathfrak{F}$ *implies* $\widetilde{\mathfrak{F}\langle y\rangle} \in \mathcal{F}$.
Then $\mathbf{L}(\mathcal{F}^u) = \mathbf{L}(\mathcal{F})\mathrm{U}^{\downarrow}$.

Proof. Put $\mathrm{L} := \mathbf{L}(\mathcal{F})$, $\mathrm{M} := \mathrm{LU}^{\downarrow}$. Given an M-consistent formula A, we have to show that A is $\mathcal{F}^u$-satisfiable.

Similarly to the proof of Theorem 8, for some reflexive $x \in W^{\mathrm{M}}$ we have $A_{\Diamond} \in x$. So $A_{\Diamond}^{refl} \in x$, then $A_{\Diamond}^{refl}$ is L-consistent. Thus $A_{\Diamond}^{refl}$ is satisfiable at some $y \in \mathfrak{F}$, $\mathfrak{F} \in \mathcal{F}$. Let $\mathfrak{G} := \widetilde{\mathfrak{F}\langle y\rangle}$. By Lemma 11, A is satisfiable in $\mathfrak{G}^u$. Since $\mathfrak{G} \in \mathcal{F}$, A is $\mathcal{F}^u$-satisfiable.

It follows that $\mathbf{L}(\mathcal{F}^u) \subseteq \mathrm{M}$. Clearly, $\mathbf{L}(\mathcal{F}^u) \supseteq \mathrm{M}$, thus $\mathbf{L}(\mathcal{F}^u) = \mathrm{M}$. ■

COROLLARY 13. *Let* L *be a monomodal transitive Kripke-complete logic such that for any cone* $\mathfrak{F}$, *if* $\mathbf{L}(\mathfrak{F}) \supseteq \mathrm{L}$ *then* $\mathbf{L}(\widetilde{\mathfrak{F}}) \supseteq \mathrm{L}$. *Then*

- $\mathrm{LU}^{\downarrow}$ *is Kripke-complete;*
- *if* L *has the FMP, then* $\mathrm{LU}^{\downarrow}$ *has the FMP;*
- *for any* $A \in \mathrm{FM}(\Diamond, \exists)$, A *is* $\mathrm{LU}^{\downarrow}$*-satisfiable iff* $A_{\Diamond}^{refl}$ *is* L*-satisfiable;*
- L *and* $\mathrm{LU}^{\downarrow}$ *are polynomially equivalent.*

Proof. Let $\mathcal{G}$ be the class of all (finite) L-frames, and let $\mathcal{F}$ be the class of all (finite) cones from $\mathcal{G}$ with reflexive root. Then $\mathrm{L} \subseteq \mathbf{L}(\mathcal{F})$.

Assume that $A \in FM(\Diamond)$ is L-satisfiable. $\mathrm{L} = \mathbf{L}(\mathcal{G})$ implies that A is satisfiable at some $z \in \mathfrak{F}$, $\mathfrak{F} \in \mathcal{G}$. Trivially, A is satisfiable in $\widetilde{\mathfrak{F}\langle z \rangle} \in \mathcal{F}$. Thus $\mathrm{L} = \mathbf{L}(\mathcal{F})$. Since $\mathcal{F}$ satisfies the conditions of Theorem 12, $\mathrm{LU}^{\downarrow} = \mathbf{L}(\mathcal{F}^u)$.

By Lemmas 4, 11, for any formula $A \in FM(\Diamond, \exists)$ we have:

$$A \text{ is } \mathcal{F}^u\text{-satisfiable iff } A_{\Diamond}^{refl} \text{ is } \mathcal{F}\text{-satisfiable.}$$

To complete the proof, note that the length of $A_{\Diamond}^{refl}$ is polynomial in the length of A. ■

In [12], it was shown that the logics **Cr**, **Cr.2** have the FMP. The method proposed in [12] was used in [10] to prove the FMP of **CrB**. The complexity of 2-dense logics was studied in [11], where PSPACE-completeness of **Cr**, **Cr.2** was proved. A slight modification of this proof yields the PSPACE-completeness of **CrB**. By Proposition 1, if $\mathrm{L} \in \{\mathbf{Cr}, \mathbf{Cr.2}, \mathbf{CrB}\}$ and $\mathfrak{F}$ is L-frame then $\widetilde{\mathfrak{F}}$ is L-frame. Therefore we have

COROLLARY 14. *The logics* $\mathbf{Cr}\mathrm{U}^{\downarrow}$, $\mathbf{Cr.2}\mathrm{U}^{\downarrow}$, $\mathbf{CrB}\mathrm{U}^{\downarrow}$ *have the FMP and are PSPACE-complete.*

5 Intervals, regions, and Minkowski spacetime

In this section we quote some results on modal axiomatization of relativistic spacetime and related interval and regional structures (for a detailed survey of this topic, see [13]).

5.1 Causal and chronological modalities

Let us recall the definition of *causal accessibility* $\preceq$ and *chronological accessibility* $\prec$ in Minkowski spacetime. For $(x_1, \dots, x_n)$, $(y_1, \dots, y_n) \in \mathbb{R}^n$, $n \geq 2$, we put:

$$(x_1, \dots, x_n) \preceq (y_1, \dots, y_n) \Leftrightarrow \sum_{i=1}^{n-1} (y_i - x_i)^2 \leq (x_n - y_n)^2 \ \&\ x_n \leq y_n,$$

$$(x_1, \dots, x_n) \prec (y_1, \dots, y_n) \Leftrightarrow \sum_{i=1}^{n-1} (y_i - x_i)^2 < (x_n - y_n)^2 \ \&\ x_n < y_n.$$

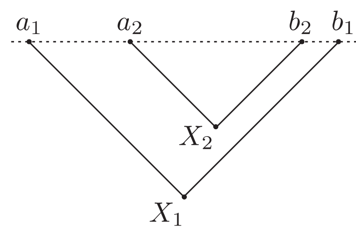


Figure 1.

For $D \subseteq \mathbb{R}^2$, $R \in \{\prec, \preceq\}$ let (D, R) abbreviate $(D, R|D)$. Put

$$\mathbb{R}^n_< := \{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid x_n < 0\}, \quad \mathbb{R}^n_\leq := \{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid x_n \leq 0\}.$$

The first results on modal axiomatization of relativistic relations are due to Goldblatt [3] and Shehtman [14], where modal logics of real space and its domains ordered by $\preceq$ were described. Analogous results on the relation $\prec$ were recently obtained in [12],[10].

THEOREM 15. *Let $n \geq 2$.*
$\mathbf{L}(\mathbb{R}^n, \preceq) = \mathbf{S4.2}$*,* $\mathbf{L}(\mathbb{R}^n_<, \preceq) = \mathbf{S4}$ *[3];* $\mathbf{L}(\mathbb{R}^n_\leq, \preceq) = \mathbf{S4.1}$ *[14];*
$\mathbf{L}(\mathbb{R}^n, \prec) = \mathbf{Cr.2}$*,* $\mathbf{L}(\mathbb{R}^n_<, \prec) = \mathbf{Cr}$ *[12];* $\mathbf{L}(\mathbb{R}^n_\leq, \prec) = \mathbf{CrB}$ *[10].*

5.2 Intervals and regions

Consider the sets of strict and non-strict intervals $\mathcal{I}$, $\mathcal{I}^*$ on the real line and the relations $\sqsupseteq$, $\sqsupset$:

$$\begin{aligned}
&\mathcal{I} := \{[a,b] \mid a,b \in \mathbb{R}, a < b\}, \quad \mathcal{I}^* := \{[a,b] \mid a,b \in \mathbb{R}, a \leq b\};\\
&[a_1, b_1] \sqsupseteq [a_2, b_2] := a_1 \leq a_2 \text{ and } b_2 \leq b_1,\\
&[a_1, b_1] \sqsupset [a_2, b_2] := a_1 < a_2 \text{ and } b_2 < b_1.
\end{aligned}$$

In [15], it was noted that there exists a simple isomorphism between the frames $(\mathcal{I}^*, \sqsupseteq, \sqsupset)$ and $(\mathbb{R}^2_\leq, \preceq, \prec)$, Fig 1.

PROPOSITION 16. *[15]*
$(\mathcal{I}, \sqsupseteq, \sqsupset) \cong (\mathbb{R}^2_<, \preceq, \prec)$*,* $(\mathcal{I}^*, \sqsupseteq, \sqsupset) \cong (\mathbb{R}^2_\leq, \preceq, \prec)$*.*

Thus the following fact is an immediate consequence of Theorem 15:

COROLLARY 17.
$\mathbf{L}(\mathcal{I}, \sqsupseteq) = \mathbf{S4}$*,* $\mathbf{L}(\mathcal{I}^*, \sqsupseteq) = \mathbf{S4.1}$*;*
$\mathbf{L}(\mathcal{I}, \sqsupset) = \mathbf{Cr}$*;* $\mathbf{L}(\mathcal{I}^*, \sqsupset) = \mathbf{CrB}$*.*

We consider the following sets as *regions*:

- *balls* in $\mathbb{R}^n$:

  $$\mathcal{B}_n := \{B(X, r) \mid X \in \mathbb{R}^n, r > 0\}, \quad \mathcal{B}^*_n := \{B(X, r) \mid X \in \mathbb{R}^n, r \geq 0\},$$

  where $B(X, r)$ is the closed ball with center X of radius r;

- *bricks* in $\mathbb{R}^n$:
$$\mathcal{R}_n := \{ \prod_{1\leq i\leq n} [a_i, b_i] \mid [a_1, b_1], \ldots, [a_n, b_n] \in \mathcal{I}\},$$
$$\mathcal{R}^*_n := \{ \prod_{1\leq i\leq n} [a_i, b_i] \mid [a_1, b_1], \ldots, [a_n, b_n] \in \mathcal{I}^*\};$$

- $\mathcal{CN}_n$ (respectively, $\mathcal{CV}_n$) is the set of all non-empty compact regular[1] sets with connected (respectively, convex) interior in $\mathbb{R}^n$. $\mathcal{CN}^*_n$ (respectively, $\mathcal{CV}^*_n$) is obtained from $\mathcal{CN}_n$ (respectively, $\mathcal{CV}_n$) by adding all singletons.

For a set $U \subseteq \mathbb{R}^n$, $\mathbf{I}U$ denotes its interior. Put

$$U \Supset V := \mathbf{I}U \supseteq V \quad (U \text{ is a } \textit{non-tangential proper part} \text{ of } V).$$

Note that if $W \in \{\mathcal{B}_1, \mathcal{CN}_1, \mathcal{CV}_1, \mathcal{R}_1\}$, then $(\mathcal{I}, \sqsupseteq, \sqsupset) = (W, \supseteq, \Supset)$ and $(\mathcal{I}^*, \sqsupseteq, \sqsupset) = (W^*, \supseteq, \Supset)$. Moreover, the following holds:

LEMMA 18. *[13] For $n \geq 1$, $W \in \{\mathcal{B}_n, \mathcal{CN}_n, \mathcal{CV}_n, \mathcal{R}_n\}$,*
$(W, \supseteq, \Supset) \twoheadrightarrow (\mathcal{I}, \sqsupseteq, \sqsupset)$, $(W^*, \supseteq, \Supset) \twoheadrightarrow (\mathcal{I}^*, \sqsupseteq, \sqsupset)$.

THEOREM 19. *[13] Let $W \in \{\mathcal{B}_n, \mathcal{CN}_n, \mathcal{CV}_n, \mathcal{R}_n\}$, $n \geq 1$. Then*
$\mathbf{L}(W, \supseteq) = \mathbf{S4}$, $\mathbf{L}(W^*, \supseteq) = \mathbf{S4.1}$;
$\mathbf{L}(W, \Supset) = \mathbf{Cr}$, $\mathbf{L}(W^*, \Supset) = \mathbf{CrB}$.

6 Main completeness results

Let $X, Y \in \mathbb{R}^n$. One can see that there exists a point $Z \in \mathbb{R}^n_<$ such that $Z \prec X$, $Z \prec Y$ (and therefore $Z \preceq X$, $Z \preceq Y$), and we have the following

LEMMA 20. *Let $\mathfrak{F} := (W, R)$, where $R \in \{\preceq, \prec\}$, $W \in \{\mathbb{R}^n, \mathbb{R}^n_\leq, \mathbb{R}^n_<\}$, $n \geq 2$. Then $\mathfrak{F}$ is downward-directed, and thus $\mathbf{L}(\mathfrak{F}^u) \supseteq \mathbf{L}(\mathfrak{F})\mathrm{U}^\downarrow$*

The following technical lemmas are needed for the sequel.

LEMMA 21. *Consider a frame $\mathfrak{F} = (W, R)$ and an infinite sequence of sets $W_0 \subseteq W_1 \subseteq \ldots$ such that $\bigcup_i W_i = W$. Put $\mathfrak{F}_i = (W_i, R|W_i)$ and suppose that for all $i \geq 0$, there exists a p-morphism $p_i : \mathfrak{F}_{i+1} \twoheadrightarrow \mathfrak{F}_i$, $p_i|W_i = Id_{W_i}$. Then $\mathfrak{F} \twoheadrightarrow \mathfrak{F}_0$.*

Proof. Put $f_i := p_i \cdot \ldots \cdot p_0$. Then $f_i : \mathfrak{F}_i \twoheadrightarrow \mathfrak{F}_0$, $i \geq 0$. One can see that $f := \bigcup_i f_i$ is the required p-morphism. ■

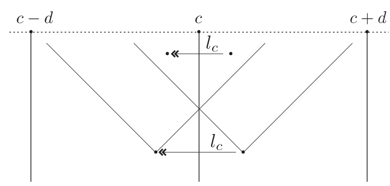
LEMMA 22. *For $R \in \{\preceq, \prec\}$ we have:*
$(\mathbb{R}^2_<, R) \twoheadrightarrow ([-1, 1] \times \mathbb{R}_<, R)$, $(\mathbb{R}^2_\leq, R) \twoheadrightarrow ([-1, 1] \times \mathbb{R}_\leq, R)$.

Proof. For any $c \in \mathbb{R}$, we define the maps r_c, l_c as follows. For $x, t \in \mathbb{R}$, we put

$$r_c(x,t) := \begin{cases} (2c - x, t) & x \leq c \\ (x, t) & x > c \end{cases} \qquad l_c(x,t) := \begin{cases} (x, t) & x \leq c \\ (2c - x, t) & x > c \end{cases}$$

[1] Recall that *regular closed sets* are the closures of open sets

(a)



(b)

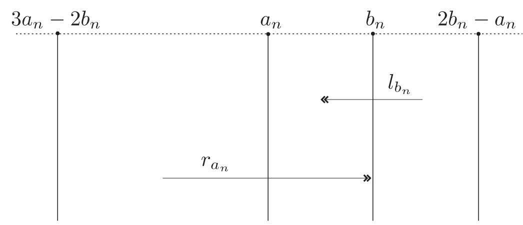


Figure 2.

It is not difficult to check that

$$r_c : ([c-d, c+d] \times \mathbb{R}_<, R) \twoheadrightarrow ([c, c+d] \times \mathbb{R}_<, R),$$

$$l_c : ([c-d, c+d] \times \mathbb{R}_<, R) \twoheadrightarrow ([c-d, c] \times \mathbb{R}_<, R),$$

for any $c, d \in \mathbb{R},\ d \geq 0$ (see Fig 2,a).

Consider a sequence of segments $[a_0, b_0], [a_1, b_1], \ldots,$ where $a_0 := -1$, $b_0 := 1$, $a_{n+1} := 3a_n - 2b_n$, $b_{n+1} := 2b_n - a_n$. Using maps l_{a_n}, r_{b_n}, we obtain

$$([a_{n+1}, b_{n+1}] \times \mathbb{R}_<, R) \twoheadrightarrow ([a_n, b_n] \times \mathbb{R}_<, R),$$

Fig 2,b. Clearly, a_n tends to $-\infty$, b_n tends to $+\infty$,[2] and therefore $\mathbb{R}^2_< = \bigcup\limits_i ([a_i, b_i] \times \mathbb{R}_<)$. By Lemma 21, $(\mathbb{R}^2_<, R) \twoheadrightarrow ([a_0, b_0] \times \mathbb{R}_<, R)$.

In complete analogy, $(\mathbb{R}^2_\leq, R) \twoheadrightarrow ([-1, 1] \times \mathbb{R}_\leq, R)$. ■

For a relation $R \subseteq W \times W$, let $R^{\bowtie} := W \times W - (R \cup R^{-1} \cup Id_W)$.

LEMMA 23. *Consider a 2-dense frame $\mathfrak{F} = (W, R)$, $x \in W$, and suppose that the following holds:*

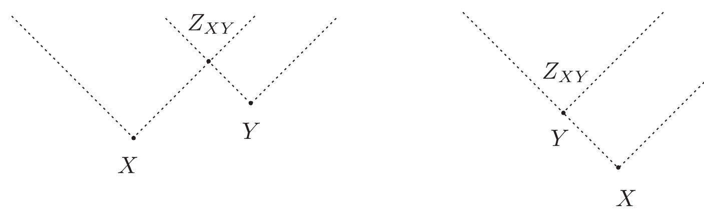
$$\forall y \in R^{\bowtie}(x)\ \exists z_y\ (R(z_y) = R(x) \cap R(y)).$$

Then for any $A \in \mathrm{FM}(\Diamond, \exists)$ we have: if $A^{\mathfrak{r}efl}_{\Diamond}$ is satisfiable at x in $\mathfrak{F}$ then A is satisfiable in $\mathfrak{F}^u$.

Proof. Suppose that for some $\mathfrak{N} = (\mathfrak{F}\langle x \rangle, \eta)$ we have $\mathfrak{N}, x \vDash A^{\mathfrak{r}efl}_{\Diamond}$.

Let $W_1 := R^r(x)$, $W_2 := R^{-1}(x) - W_1$, $W_3 := W - (W_1 \cup W_2) = R^{\bowtie}(x)$.

[2]By direct calculation, $a_n = \frac{1-4^{n+1}}{3}$, $b_n = \frac{2 \cdot 4^n + 1}{3}$



(a) (b)

Figure 3.

For any $y \in W$ we define $y' \in W_1$ as follows. If $y \in W_1$, we put $y' := y$; if $y \in W_2$, we put $y' := x$. To define y' when $y \in W_3$, consider the set of formulas

$$\Psi_y := \{\Diamond B \in Sub(A_\Diamond) \mid \exists u \in R(z_y)\ \mathfrak{N}, u \vDash B\}.$$

Suppose $\Psi_y = \{\Diamond B_1, \ldots, \Diamond B_l\}$. Then $\mathfrak{N}, u_i \vDash B_i$ for some points $u_1, \ldots, u_l \in R(z_y)$. Since $\mathfrak{F}$ is 2-dense, by Proposition 1 we have $R(v) \supseteq \{u_1, \ldots, u_l\}$ for some $v \in R(z_y)$. We put $y' := v$.

Consider a model $\mathfrak{M} = (\mathfrak{F}, \theta)$ such that $\theta^{-1}(y) = \eta^{-1}(y')$ for any $y \in W$. We claim that for any $C \in Sub(A_\Diamond)$ and any $y \in W$,

(6.1) $\mathfrak{M}, y \vDash C \iff \mathfrak{N}, y' \vDash C$

The proof is by induction on the length of the formula. Consider the only non-trivial case $C = \Diamond B$, $y \notin W_1$. Note that $R(y') \subseteq R(y)$, so $\mathfrak{N}, y' \vDash \Diamond B$ implies $\mathfrak{M}, y \vDash \Diamond B$ (by induction hypothesis). Conversely, assume that $\mathfrak{M}, y \vDash \Diamond B$. Thus $\mathfrak{M}, v \vDash B$ for some $v \in R(y)$, and by induction hypothesis $\mathfrak{N}, v' \vDash B$. There may be two options:

- $y \in W_2$. Then $y' = x$. Since $v' \in R^r(x)$, we have $\mathfrak{N}, x \vDash \Diamond^+ B$. Due to the definition of $A_\Diamond^{refl}$, $\mathfrak{N}, x \vDash B \to \Diamond B$. Thus $\mathfrak{N}, x \vDash \Diamond B$.

- $y \in W_3$. Then $v \in W_1 - \{x\}$ or $v \in W_3$ (because $yR^{\bowtie}x$). In the former case $v' = v \in R(y) \cap R(x)$, so $v' \in R(z_y)$. If $v \in W_3$ then $R(z_v) \subseteq R(z_y)$, and since $v' \in R(z_v)$, $v' \in R(z_y)$. So, in either case, we have $v' \in R(z_y)$. Thus $\Diamond B \in \Psi_y$. Therefore for some $u \in R(y')$ we have $\mathfrak{N}, u \vDash B$, so $\mathfrak{N}, y' \vDash \Diamond B$.

Since $\mathfrak{N}, x \vDash A_\Diamond$, and due to (6.1), we obtain that $A_\exists$ is satisfiable in $\mathfrak{M}^u$. By Lemma 3, A is satisfiable in $\mathfrak{F}^u$. ■

Observe that for any $X, Y \in \mathbb{R}^2$, there exists a unique point Z_{XY} such that $\prec(X) \cap \prec(Y) = \prec(Z_{XY})$, Fig. 3.

Now we are ready to prove the following key

LEMMA 24. *Let* $\mathfrak{F} := (W, R)$, *where* $R \in \{\preceq, \prec\}$, $W \in \{\mathbb{R}^2, \mathbb{R}^2_\leq, \mathbb{R}^2_<\}$. *Then* $\mathbf{L}(\mathfrak{F}^u) = \mathbf{L}(\mathfrak{F})\mathrm{U}^\downarrow$, *namely*

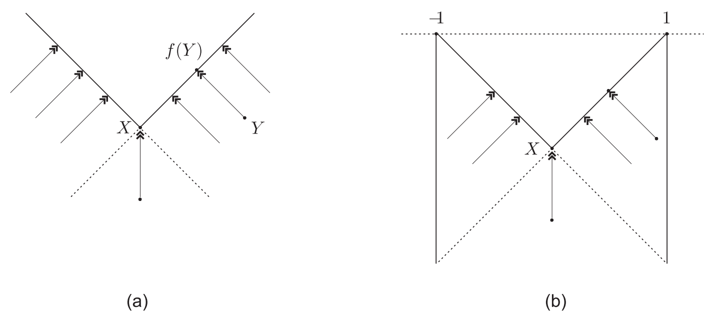


Figure 4.

$\mathbf{L}((\mathbb{R}^2,\preceq)^u) = \mathbf{S4.2}\mathrm{U}^{\downarrow}$, $\mathbf{L}((\mathbb{R}^2_<,\preceq)^u) = \mathbf{S4}\mathrm{U}^{\downarrow}$, $\mathbf{L}((\mathbb{R}^2_\leq,\preceq)^u) = \mathbf{S4.1}\mathrm{U}^{\downarrow}$;
$\mathbf{L}((\mathbb{R}^2,\prec)^u) = \mathbf{Cr.2}\mathrm{U}^{\downarrow}$, $\mathbf{L}((\mathbb{R}^2_<,\prec)^u) = \mathbf{Cr}\mathrm{U}^{\downarrow}$, $\mathbf{L}((\mathbb{R}^2_<,\prec)^u) = \mathbf{CrB}\mathrm{U}^{\downarrow}$.

Proof. First, consider the frames ordered by $\preceq$.

Let $\mathfrak{F}$ be a cone in $(\mathbb{R}^2,\preceq)$, and let $\mathfrak{G}$ be a cone in $(\mathbb{R}^2_<,\preceq)$. Then $\mathbf{L}(\mathfrak{F}) = \mathbf{S4.2}$, $\mathbf{L}(\mathfrak{G}) = \mathbf{S4}$ [3],[14]. Put

$$\mathcal{F} := \{\mathfrak{F}\langle X\rangle \mid X \in \mathfrak{F}\},\quad \mathcal{G} := \{\mathfrak{G}\langle X\rangle \mid X \in \mathfrak{G}\}.$$

By Theorem 8 we have: $\mathbf{L}(\mathcal{F}^u) = \mathbf{L}(\mathcal{F})\mathrm{U}^{\Downarrow}$, $\mathbf{L}(\mathcal{G}^u) = \mathbf{L}(\mathcal{G})\mathrm{U}^{\Downarrow}$. All frames in $\mathcal{F}$ (in $\mathcal{G}$) are isomorphic, thus all frames in $\mathcal{F}^u$ (in $\mathcal{G}^u$) are isomorphic. Thus

$$\mathbf{L}(\mathfrak{F}^u) = \mathbf{L}(\mathcal{F}^u) = \mathbf{L}(\mathcal{F})\mathrm{U}^{\Downarrow} = \mathbf{L}(\mathfrak{F})\mathrm{U}^{\Downarrow}.$$

Similarly, $\mathbf{L}(\mathfrak{G}^u) = \mathbf{L}(\mathfrak{G})\mathrm{U}^{\Downarrow}$. Since $\mathfrak{F}$ and $\mathfrak{G}$ are reflexive, we obtain $\mathbf{L}(\mathfrak{F}^u) = \mathbf{S4.2}\mathrm{U}^{\downarrow}$, $\mathbf{L}(\mathfrak{G}^u) = \mathbf{S4}\mathrm{U}^{\downarrow}$.

Let $\mathfrak{H}$ be a cone in $(\mathbb{R}^2_\leq,\preceq)$ whose root belongs to $\mathbb{R}^2_<$. By Theorem 8, we obtain $\mathbf{L}(\{\mathfrak{H}^u,\mathfrak{C}_1^u\}) = \mathbf{L}(\mathfrak{H})\mathrm{U}^{\Downarrow}$, where $\mathfrak{C}_1$ is a reflexive singleton. Trivially, $\mathbf{L}(\mathfrak{H}^u) \subseteq \mathbf{L}(\mathfrak{C}_1^u)$, so $\mathbf{L}(\mathfrak{H}^u) = \mathbf{L}(\mathfrak{H})\mathrm{U}^{\Downarrow}$. Since $\mathbf{L}(\mathfrak{H}) = \mathbf{S4.1}$ [14], we get $\mathbf{L}(\mathfrak{H}^u) = \mathbf{S4.1}\mathrm{U}^{\downarrow}$.

Without loss of generality we may assume that the frames $\mathfrak{F}$, $\mathfrak{G}$, $\mathfrak{H}$ have the same root $X = (0,-1)$. Let us define the map f as follows: for any $Y \in \mathbb{R}^2$, put

$$f(Y) := \begin{cases} Y & X \preceq Y \\ X & Y \preceq X \\ Z_{XY} & \text{otherwise} \end{cases}$$

It is not difficult to see that $f : (\mathbb{R}^2,\preceq) \twoheadrightarrow \mathfrak{F}$ (Fig. 4,a), so $\mathbf{L}((\mathbb{R}^2,\preceq)^u) \subseteq \mathbf{S4.2}\mathrm{U}^{\downarrow}$. We also have $f : ([-1,1]\times\mathbb{R}_<,\preceq) \twoheadrightarrow \mathfrak{G}$, $f : ([-1,1]\times\mathbb{R}_\leq,\preceq) \twoheadrightarrow \mathfrak{H}$ (Fig. 4,b), and by Lemma 22, $(\mathbb{R}^2_<,\preceq) \twoheadrightarrow \mathfrak{G}$ and $(\mathbb{R}^2_\leq,\preceq) \twoheadrightarrow \mathfrak{H}$. Thus $\mathbf{L}((\mathbb{R}^2_<,\preceq)^u) \subseteq \mathbf{S4}\mathrm{U}^{\downarrow}$, $\mathbf{L}((\mathbb{R}^2_\leq,\preceq)^u) \subseteq \mathbf{S4.1}\mathrm{U}^{\downarrow}$,

The converse inclusions hold by Lemma 20.

Now consider the frames ordered by $\prec$.

Suppose that $A \in FM(\Diamond, \exists)$ is $\mathbf{Cr.2}\mathrm{U}^{\downarrow}$-satisfiable. By Corollary 13, $A_{\Diamond}^{refl}$ is $\mathbf{Cr.2}$-satisfiable. Since $\mathbf{Cr.2} = \mathbf{L}(\mathbb{R}^2, \prec)$, $A_{\Diamond}^{refl}$ is satisfiable in $(\mathbb{R}^2, \prec)$. By Lemma 23, A is satisfiable in $(\mathbb{R}^2, \prec)^u$. Thus $\mathbf{L}((\mathbb{R}^2, \prec)^u) \subseteq \mathbf{Cr.2}\mathrm{U}^{\downarrow}$, and by Lemma 20, $\mathbf{L}((\mathbb{R}^2, \prec)^u) = \mathbf{Cr.2}\mathrm{U}^{\downarrow}$.

Analogously, if A is $\mathbf{Cr}\mathrm{U}^{\downarrow}$-satisfiable then $A_{\Diamond}^{refl}$ is satisfiable at some X in $(\mathbb{R}^2_<, \prec)$, and without loss of generality we may assume that $X = (0, -1)$. By Lemmas 20, 22, 23, we obtain $\mathbf{L}((\mathbb{R}^2_<, \prec)^u) = \mathbf{Cr}\mathrm{U}^{\downarrow}$.

Finally, consider a $\mathbf{CrB}\mathrm{U}^{\downarrow}$-satisfiable formula A. Then $A_{\Diamond}^{refl}$ is satisfiable in $(\mathbb{R}^2_\leq, \prec)$. It is easy to see that if $A_{\Diamond}^{refl}$ is satisfiable at some non-serial point then $A_{\Diamond}^{refl}$ is satisfiable at some $X \in \mathbb{R}^2_<$. Similarly to the previous cases, $\mathbf{L}((\mathbb{R}^2_\leq, \prec)^u) = \mathbf{CrB}\mathrm{U}^{\downarrow}$. ■

For $X = (x_1, \ldots, x_n) \in \mathbb{R}^n$, $n \geq 2$, put $p_n(X) := (x_1, x_n)$. Then $p_n : (\mathbb{R}^n, \preceq) \twoheadrightarrow (\mathbb{R}^2, \preceq)$, $p_n : (\mathbb{R}^n, \prec) \twoheadrightarrow (\mathbb{R}^2, \prec)$, [3]. So by Lemmas 20, 24, we have the following

THEOREM 25. *For* $n \geq 2$,
$\mathbf{L}((\mathbb{R}^n, \preceq)^u) = \mathbf{S4.2}\mathrm{U}^{\downarrow}$*;*
$\mathbf{L}((\mathbb{R}^n, \prec)^u) = \mathbf{Cr.2}\mathrm{U}^{\downarrow}$.

Lemma 24 together with Proposition 16 yields

PROPOSITION 26.
$\mathbf{L}((\mathcal{I}, \sqsupseteq)^u) = \mathbf{S4}\mathrm{U}^{\downarrow}$, $\mathbf{L}((\mathcal{I}^*, \sqsupseteq)^u) = \mathbf{S4.1}\mathrm{U}^{\downarrow}$*;*
$\mathbf{L}((\mathcal{I}, \sqsupset)^u) = \mathbf{Cr}\mathrm{U}^{\downarrow}$, $\mathbf{L}((\mathcal{I}^*, \sqsupset)^u) = \mathbf{CrB}\mathrm{U}^{\downarrow}$.

Let $W \in \{\mathcal{B}_n, \mathcal{C}N_n, \mathcal{C}V_n, \mathcal{R}_n\}$, $R \in \{\supseteq, \Supset\}$, $n \geq 1$. It is easy to see that $\mathfrak{F} := (W, R)$ is downward-directed, so $\mathbf{L}(\mathfrak{F}^u) \supseteq \mathbf{L}(\mathfrak{F})\mathrm{U}^{\downarrow}$; by Lemma 18, Proposition 26, and by Theorem 19, $\mathbf{L}(\mathfrak{F}^u) \subseteq \mathbf{L}(\mathfrak{F})\mathrm{U}^{\downarrow}$. Thus we obtain

THEOREM 27. *Let* $W \in \{\mathcal{B}_n, \mathcal{C}N_n, \mathcal{C}V_n, \mathcal{R}_n\}$, $n \geq 1$. *Then*
$\mathbf{L}((W, \supseteq)^u) = \mathbf{S4}\mathrm{U}^{\downarrow}$, $\mathbf{L}((W^*, \supseteq)^u) = \mathbf{S4.1}\mathrm{U}^{\downarrow}$*;*
$\mathbf{L}((W, \Supset)^u) = \mathbf{Cr}\mathrm{U}^{\downarrow}$, $\mathbf{L}((W^*, \Supset)^u) = \mathbf{CrB}\mathrm{U}^{\downarrow}$.

7 Acknowledgements

The author is grateful to Prof. Valentin Shehtman for his help and also to the anonymous referees for their useful comments.

The work on this paper was supported by Poncelet Laboratory (UMI 2615 of CNRS and Independent University of Moscow) and by RFBR-NWO grant 047.011.2004.040.

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