Complexity of intuitionistic and Visser's basic and formal logics in finitely many variables

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ABSTRACT. The main result of the paper is that the decision problem for the variable free fragment of **BPL**, one-variable fragment of **FPL**, and two-variable fragment of **Int** is PSPACE-complete. Some relevant questions are also discussed.

Keywords: complexity, decision problem, propositional logic, intuitionistic logic, Visser's formal logic, non-classical logic, complexity function.

1 Introduction

It is well known that practically all standard propositional logics are decidable but the decision problem for them is usually very hard: even for classical propositional logic **Cl** only exponential time algorithms deciding it are known, and the decision (satisfiability) problem for **Cl** is NP-complete [10]. Complexity problems for non-classical (more exactly, superintuitionistic) logics were first explicitly mentioned by A. V. Kuznetsov in 1975, see [18]. One of the questions raised by A. V. Kuznetsov was the problem of polynomial approximability of intuitionistic propositional logic **Int** and some of its extensions. A. V. Kuznetsov showed [19] that if this problem is solved positively for **Int**, then **Int** and **Cl** are polynomially equivalent. As follows from [19], it is the same situation with some other non-classical propositional logics.

In 1977 R. E. Ladner proved that the decision problem for **K**, **T**, **S4** is PSPACE-complete [20]. In 1979 R. Statman proved that the decision problem for **Int** is also PSPACE-complete [33]. So, a positive solution to Kuznetsov's question would imply that NP = PSPACE.

Later Ladner's and Statman's ideas were applied by other authors to prove PSPACE-hardness for very wide classes of logics: for example, see [1, 4, 9, 11, 40].

Because the decision problem even for 'simple' logics — such as **K**, **T**, **K4**, **S4**, **S5**, **Int**, etc. — is quite hard, it is natural to consider some restrictions for these logics and the decision problem for corresponding fragments. We shall consider the situation when the propositional language contains only finitely many variables. The main reason for this restriction is that in applications we need only finitely many variables, and it is even sufficient to have a fixed number of variables.

Apparently, the question about complexity of non-classical propositional logics in finitely many variables arose immediately after Ladner's and Statman's results were published (both R. E. Ladner and R. Statman used infinitely many propositional variables in their proofs). As an open problem[1] it was put by A. V. Chagrov in 1980s at several conferences and published later in [9, Problem 18.4].

For any propositional logic L and any natural number n, by $L(n)$ we denote the least fragment of L containing all formulas constructed from variables $p_1, \ldots, p_n$. By $L(0)$ we denote the variable free fragment of L.

The question about complexity of the decision problem for $L(n)$ is quite well studied in the case of standard modal logics. So, E. Spaan [32] in fact proved PSPACE-hardness for $\mathbf{K}(1)$; later J. Y. Halpern [15] presented the same result for $\mathbf{K}(1)$, $\mathbf{T}(1)$, $\mathbf{S4}(1)$ (some inaccuracies in [15] in the case of $\mathbf{S4}$ were corrected in [8]). In [8] PSPACE-hardness and NP-hardness is proved for n-variable fragments of wide classes of normal modal logics; in particular, it is proved that the decision problem for $\mathbf{K}(0)$, $\mathbf{K4}(0)$, $\mathbf{S4}(1)$, $\mathbf{GL}(1)$, $\mathbf{Grz}(1)$ is PSPACE-complete (for $\mathbf{GL}(1)$ see also [36]).

REMARK 1. Of course, from [8] one can easily obtain similar results for non-normal modal logics too. In fact in [8] it is proved that *all sets of formulas* (in particular, all non-normal modal logics) between $\mathbf{K}(0)$ and $\mathbf{K4}(0)$, between $\mathbf{K}(1)$ and $\mathbf{GL}(1)$, between $\mathbf{K}(1)$ and $\mathbf{Grz}(1)$ have PSPACE-hard decision problem; the same is true for NP-hardness results presented in [8]. One can also extend results of [8] to some non-normal logics incomparable with $\mathbf{K}$. For example, using the fact that for any formula φ,

$$\varphi \in \mathbf{S4} \quad \Longleftrightarrow \quad \Box\Box\top \to \varphi \in \mathbf{S3},$$

we obtain PSPACE-hardness (and hence, PSPACE-completeness) of the decision problem for $\mathbf{S3}(1)$.

In this paper I present some related results for non-modal logics, which have an intuitionistic-like Kripke semantics. First of all I mean intuitionistic propositional logic **Int**, Visser's basic propositional logic **BPL**, and Visser's formal propositional logic[2] **FPL**.

Intuitionistic logic is well-known and quite well studied in both the propositional and predicate cases; because I cannot say the same about Visser's basic and formal logics, I give some references and some reasons for interest in these logics.

First of all, the reader can see for these logics the paper [38] of A. Visser, where **BPL** and **FPL** are defined (for predicate case of **BPL** see [25]). The

[1]More exactly, as the conjecture that **Int** in finitely many variables is decidable in polynomial time.

[2]The logic **FPL** has at least two names: *formal propositional logic* and *fixed point logic*. We have an 'interesting' coincidence: these names have the same abbreviation, i. e., **FPL**.

main logic in [38] is **FPL**, which is obtained by interpreting implication as formal provability (in Peano Arithmetic); **BPL** is used in [38] as a 'technical' logic.

Some advanced results about **BPL** and **FPL** the reader can find, for example, in [5, 6, 35, 39]. I note only that in [35], in particular, **BPL** is described as a Hilbert-type calculus (A. Visser presented it in [38] by a set of rule schemes) and that some 'undecidability' results for **FPL** were presented at the conference 'Computer Science Applications of Modal Logic' [7]. As for complexity results for these logics, I give several references below (see Remarks 2 and 3).

Both **BPL** and **FPL** are connected with the modal logics **K4** and **GL** respectively (note that **K4** may be also considered as a 'technical' logic: we are interested mainly in its extensions such as **GL**, **S4**, **Grz**, etc.) via the following Gödel translation T:

$$\begin{array}{rcl} \mathsf{T}(\bot) & = & \Box\bot; \\ \mathsf{T}(p) & = & \Box p, \text{ where } p \text{ is a propositional variable;} \\ \mathsf{T}(\varphi\wedge\psi) & = & \mathsf{T}(\varphi)\wedge\mathsf{T}(\psi); \\ \mathsf{T}(\varphi\vee\psi) & = & \mathsf{T}(\varphi)\vee\mathsf{T}(\psi); \\ \mathsf{T}(\varphi\to\psi) & = & \Box(\mathsf{T}(\varphi)\to\mathsf{T}(\psi)). \end{array}$$

Exactly, as it follows from [38], for every formula φ,

$$\begin{array}{rcl} \varphi\in\mathbf{BPL} & \Longleftrightarrow & \mathsf{T}(\varphi)\in\mathbf{K4}; \\ \varphi\in\mathbf{FPL} & \Longleftrightarrow & \mathsf{T}(\varphi)\in\mathbf{GL}. \end{array}$$

It is also well known [9] that for every formula φ,

$$\varphi\in\mathbf{Int} \quad \Longleftrightarrow \quad \mathsf{T}(\varphi)\in\mathbf{S4},$$

so, we may mean **BPL** and **FPL** as 'superintuitionistic' fragments of **K4** and **GL** respectively. Roughly speaking, in particular, we may mean the implication in **Int**, **BPL**, and **FPL** as the strong implication in **S4**, **K4**, and **GL**, correspondingly (but instead of variables and $\bot$ we use 'boxed' variables and 'boxed' $\bot$).

Thus, **Int**, **BPL**, and **FPL** are 'naturally' connected with well-known modal logics; moreover, for these modal logics we know about their complexity in finitely many variables [8]. At the same time there are some 'technical' problems[3] to (easily) extend the proofs in [8] for the cases of **Int**, **BPL**, and **FPL**: the connectives in these logics are not classical and we do not have 'pure' modalities in the language. There is also an observation of another kind: the decision problem for $\mathbf{S4}(1)$ is PSPACE-complete while the one for $\mathbf{Int}(1)$ is in P: cf. [21]. The last observation gives us a hope that the situation with the complexity of $\mathbf{Int}(n)$, $\mathbf{BPL}(n)$, and $\mathbf{FPL}(n)$ differs from the one for corresponding modal logics.

[3]The main problem is with negation. But I omit any details here: after all I am going to give a proof.

In this paper I am going to clarify the situation: unfortunately (or maybe, fortunately!) the decision problem for **Int**(2), **BPL**(0), and **FPL**(1) is PSPACE-complete. See Section 3, Theorem 4. This theorem has some corollaries about the complexity of wide classes of logics and their positive fragments in finitely many variables (including some extensions of **Int**); the corollaries are presented in Section 4. As an application of the construction in Theorem 4 we obtain exponential lower bounds on the complexity function for all PSPACE-hard fragments of logics considered in this paper; for details see Section 5.

REMARK 2. Most results in this article are presented in other papers — see [26, 27, 28, 29] — but they are in Russian (the exception [29] does not contain any proof), so, as I think, actually they are not known in modal logic society, and I am thankful to AiML for that I can present these results here.

2 Definitions

Our definitions for propositional language and Kripke semantics for it mainly correspond to [9] and [38]; for complexity problems the reader can consult [13, 14, 16, 22]. So, the reader can omit this section.

We shall consider formulas constructed from propositional variables $p_1, p_2, p_3, \ldots$ and constant $\bot$ with $\wedge$, $\vee$, $\rightarrow$, and brackets. We define $\neg$ and $\top$ as the usual abbreviations: $\neg\varphi = \varphi \rightarrow \bot$, $\top = \neg\bot$. We shall use the following standard conventions on representation of formulas: we assume $\neg$ to connect formulas more strongly than $\wedge$ and $\vee$, which in turn are stronger than $\rightarrow$, and omit those brackets that can be recovered according to this priority of the connectives.

Below we shall mainly consider positive formulas. A formula φ is called *positive* if φ is constructed from propositional variables with $\wedge$, $\vee$, and $\rightarrow$. For a logic (a set of formulas) L denote by L^+ the positive fragment of L.

Let us define intuitionistic-like Kripke semantics for this language (the difference from the usual definitions, as in, for example, [9], is in that some worlds in Kripke frames can be irreflexive).

An *intuitionistic Kripke frame* is a pair $\mathfrak{F} = \langle W, R\rangle$ consisting of a non-empty set W and a relation R on W, which is transitive and antisymmetric. The elements of W are called the *worlds* of the frame $\mathfrak{F}$, and the relation R is called the *accessibility relation*. If for some $w, w' \in W$ the relation wRw' holds, we say that w' *is accessible from* w.

A *valuation* in an intuitionistic frame $\mathfrak{F} = \langle W, R\rangle$ is a map v associating with each propositional variable p some subset $v(p)$ of W such that for every $w \in v(p)$ and every $w' \in W$, wRw' implies $w' \in v(p)$.

An *intuitionistic Kripke model* is a pair $\mathfrak{M} = \langle \mathfrak{F}, v\rangle$, where $\mathfrak{F}$ is an intuitionistic frame and v is a valuation in $\mathfrak{F}$.

Let $\mathfrak{M} = \langle \mathfrak{F}, v\rangle$ be a model and w be a world in the frame $\mathfrak{F} = \langle W, R\rangle$. By induction on the construction of formula φ we define a relation $(\mathfrak{M}, w) \models \varphi$:

$$\begin{array}{lcl}
(\mathfrak{M}, w) \not\models \bot; & & \\
(\mathfrak{M}, w) \models p & \iff & w \in v(p), \text{ where } p \text{ is a propositional variable;} \\
(\mathfrak{M}, w) \models \psi' \wedge \psi'' & \iff & (\mathfrak{M}, w) \models \psi' \text{ and } (\mathfrak{M}, w) \models \psi''; \\
(\mathfrak{M}, w) \models \psi' \vee \psi'' & \iff & (\mathfrak{M}, w) \models \psi' \text{ or } (\mathfrak{M}, w) \models \psi''; \\
(\mathfrak{M}, w) \models \psi' \to \psi'' & \iff & \text{for every } w' \in W \text{ such that } wRw', \\
 & & (\mathfrak{M}, w') \not\models \psi' \text{ or } (\mathfrak{M}, w') \models \psi''.
\end{array}$$

From the definition it follows that $(\mathfrak{M}, w) \models \top$ and

$$(\mathfrak{M}, w) \models \neg\psi \iff \text{for every } w' \in W \text{ such that } wRw', \ (\mathfrak{M}, w') \not\models \psi,$$

for every $w \in W$.

If $(\mathfrak{M}, w) \models \varphi$ holds, we say that φ *is true at the world* w *in* $\mathfrak{M}$; otherwise that φ *is refuted at* w. We say that φ *is true in a model* $\mathfrak{M} = \langle \mathfrak{F}, v\rangle$ defined on a frame $\mathfrak{F} = \langle W, R\rangle$ if $(\mathfrak{M}, w) \models \varphi$, for every $w \in W$; if φ is true in $\mathfrak{M}$, we write $\mathfrak{M} \models \varphi$. We say that φ *is true in a frame* $\mathfrak{F} = \langle W, R\rangle$ if φ is true in every model based on $\mathfrak{F}$; if φ is true in $\mathfrak{F}$, we write $\mathfrak{F} \models \varphi$. We say that φ *is true at the world* w *in frame* $\mathfrak{F}$ if $(\mathfrak{M}, w) \models \varphi$, for every model $\mathfrak{M}$ defined on $\mathfrak{F}$; if φ is true at the world w in frame $\mathfrak{F}$, we write $(\mathfrak{F}, w) \models \varphi$.

We define propositional intuitionistic logic **Int** to be the set of formulas that are true in every reflexive frame, Visser's basic propositional logic **BPL** to be the set of formulas that are true in every frame, and Visser's formal propositional logic **FPL** to be the set of formulas that are true in every irreflexive and finite frame.

Let us recall the definitions of the complexity classes P, NP, and PSPACE. We consider only algorithmic recognition problems, i. e., problems that can be formulated as questions '$x \in X$?', for some set X (as an example, the decision problem for a logic L looks as '$\varphi \in L$?'). The complexity classes P, NP, and PSPACE are the classes of problems that can be solved respectively by polynomial time deterministic algorithms, by polynomial time nondeterministic algorithms, and by polynomial space nondeterministic[4] algorithms.

We are mainly interested in PSPACE-hard and PSPACE-complete problems. A problem '$x \in X$?' is called PSPACE-*hard* if any problem '$y \in Y$?' in PSPACE is polynomially reducible to '$x \in X$?', i. e., if there exists a polynomial time function (algorithm) f such that for every y,

$$y \in Y \iff f(y) \in X.$$

A problem '$x \in X$?' is called PSPACE-*complete* if it belongs to PSPACE and is PSPACE-hard.

[4]We can replace 'nondeterministic' with 'deterministic' here; the equivalence between the definitions follows from [30].

3 Main Result

Recall (see the Introduction) that **Int**, **BPL**, and **FPL** are connected with **S4**, **K4**, and **GL**, correspondingly, via the Gödel translation T. It is known [8] that the decision problem for **K4**(0), **GL**(1), **S4**(1) is PSPACE-complete. It is also known that the decision problem for **GL**(0) is in P, and hence the one for **FPL**(0) is also in P. Finally, the decision problem for **Int**(1) is also in P: cf. [21]. Thus, there is a reason to consider the question of the complexity of **Int**(n) for $n \geq 2$, **BPL**(n) for $n \geq 0$, and **FPL**(n) for $n \geq 1$.

In 1985 A. V. Chagrov [4] proved PSPACE-hardness for many logics, in particular, for **Int**, **BPL**, and **FPL** in the language with implication only (for PSPACE-hardness of **BPL** and **FPL** see also [2]). Below we shall use more weak facts, which follow from [4]: the decision problem for $\mathbf{Int}^+$, $\mathbf{BPL}^+$, and $\mathbf{FPL}^+$ is PSPACE-complete.

REMARK 3. In fact, below we can use the facts that **Int**, **BPL**, and **FPL** have PSPACE-complete decision problem for their implicative fragments, moreover there is a reason to do so: one connective in the language means for some of our proofs only one case in the induction step. But on the one hand, [4] is in Russian and unfortunately it is difficult to access even in Russia. On the other hand, PSPACE-completeness of $\mathbf{Int}^+$, $\mathbf{BPL}^+$, $\mathbf{FPL}^+$ is sufficient for the aim of this article, and there is no problem to find corresponding proofs in English: for $\mathbf{Int}^+$ it is given in [9, 33] and, which is important, one can easily modify the proof in [9] for the cases of $\mathbf{BPL}^+$ and $\mathbf{FPL}^+$ (it is sufficient to replace the last occurrences of q with $(q \to q) \to q$ in both formulas on page 560 in [9]); or see [2] (to obtain PSPACE-hardness for $\mathbf{BPL}^+$ and $\mathbf{FPL}^+$ one can easily eliminate $\bot$ in the corresponding proofs for **BPL** and **FPL** in [2]).

Our main aim is in to prove the following statement.

THEOREM 4. *The decision problem for* $\mathbf{Int}^+(2)$*,* $\mathbf{BPL}(0)$*, and* $\mathbf{FPL}(1)$ *is* PSPACE*-complete.*

PROOF. The fact that this problem is in PSPACE follows from the PSPACE-completeness of the decision problem for **Int**, **BPL**, and **FPL** (in the language with infinitely many variables). So, it is sufficient to prove PSPACE-hardness of this problem.

The corresponding proof for **BPL**(0) and **FPL**(1) is similar to the one for the logic **K4** in [8]. There are some differences for the case of $\mathbf{Int}^+(2)$ but the general idea is the same. Below I give a proof for $\mathbf{Int}^+(2)$ (it is more difficult in a 'technical' sense than the ones for **BPL**(0) and **FPL**(1)) and a detailed sketch of proof for the cases of **BPL**(0) and **FPL**(1).

We start with $\mathbf{Int}^+(2)$. Let p and q be two propositional variables. We define the following positive formulas constructed from p and q:

$$\begin{array}{rclrcl}
D_1 & = & p \to q; \quad D_2 \ = \ q \to p; & A_1^1 & = & A_1^0 \wedge A_2^0 \to B_1^0 \vee B_2^0; \\
D_3 & = & D_1 \wedge D_2 \to p \vee q; & A_2^1 & = & A_1^0 \wedge B_1^0 \to A_2^0 \vee B_2^0; \\
A_1^0 & = & D_2 \to D_1 \vee D_3; & A_3^1 & = & A_1^0 \wedge B_2^0 \to A_2^0 \vee B_1^0; \\
A_2^0 & = & D_3 \to D_1 \vee D_2; & B_1^1 & = & A_2^0 \wedge B_1^0 \to A_1^0 \vee B_2^0; \\
B_1^0 & = & D_1 \to D_2 \vee D_3; & B_2^1 & = & A_2^0 \wedge B_2^0 \to A_1^0 \vee B_1^0; \\
B_2^0 & = & A_1^0 \wedge A_2^0 \wedge B_1^0 \to D_1 \vee D_2 \vee D_3; & B_3^1 & = & B_1^0 \wedge B_2^0 \to A_1^0 \vee A_2^0.
\end{array}$$

To define other formulas we need, let us fix the following order $\prec$ on the set $\{(i,j) : i,j \geq 2\}$:

$$(i,j) \prec (i',j') \iff \text{either } i+j < i'+j', \text{ or } i+j = i'+j' \text{ and } i < i'.$$

Clearly, this order is strict and linear. So, we can enumerate all such pairs of natural numbers by means of some function, say $g(x)$, accordingly to this order, i. e., $g(1) = (2,2)$, $g(2) = (2,3)$, $g(3) = (3,2)$, $g(4) = (3,3)$, etc.

Suppose the formulas $A_1^k, \ldots, A_m^k, B_1^k, \ldots, B_m^k$ are already defined, for some $k \geq 1$. Let $i,j \in \{2,\ldots,m\}$ and let $(i,j) = g(s)$; then we put

$$A_s^{k+1} \ = \ A_1^k \to B_1^k \vee A_i^k \vee B_j^k; \qquad B_s^{k+1} \ = \ B_1^k \to A_1^k \vee A_i^k \vee B_j^k.$$

We call A_i^k and B_i^k formulas of level k. By N_k denote the number of formulas A_i^k of level k. Clearly, N_k is also the number of formulas B_i^k of level k. It is easy to see that

$$N_0 = 2, \ N_1 = 3, \ N_{k+2} = (N_{k+1} - 1)^2.$$

For a formula A we denote by $|A|$ the length of A. Let us fix some number l such that

$$l > \max_{1 \leq i \leq 2} \{|A_i^0| + |B_i^0|\};$$

for example, we can take $l = 200$. It is easy to show that $|A_i^k| < 5^k \cdot l$ and $|B_i^k| < 5^k \cdot l$ (by induction on k).

It is not hard to see that there exists k_0 such that $5^k \cdot l < N_k$, for any $k \geq k_0$; one can check that as k_0 we can take any number greater than or equal to 6. Let $k_0 = 6$.

We construct a Kripke model refuting all formulas A_i^k and B_i^k such that, for every formula A_i^k and B_i^k, there exists a unique maximal world, at which this formula is refuted. Let

$$W \ = \ \{c_0, d_1, d_2, d_3\} \cup \{a_i^k, b_i^k : k \geq 0, \ 1 \leq i \leq N_k\}.$$

We call a_i^k and b_j^k worlds of level k. To define the accessibility relation R

on W, let

$$\begin{array}{rcl} R_0 &=& \{\langle d_1, c_0\rangle, \langle d_2, c_0\rangle, \langle d_3, c_0\rangle, \langle a_1^0, d_1\rangle, \langle a_2^0, d_1\rangle, \langle b_2^0, d_1\rangle,\\ && \langle a_2^0, d_2\rangle, \langle b_1^0, d_2\rangle, \langle b_2^0, d_2\rangle, \langle a_1^0, d_3\rangle, \langle b_1^0, d_3\rangle, \langle b_2^0, d_3\rangle\};\\ R_1^a &=& \{\langle a_1^1, b_1^0\rangle, \langle a_1^1, b_2^0\rangle, \langle a_2^1, a_2^0\rangle, \langle a_2^1, b_2^0\rangle, \langle a_3^1, a_2^0\rangle, \langle a_3^1, b_1^0\rangle\};\\ R_1^b &=& \{\langle b_1^1, a_1^0\rangle, \langle b_1^1, b_2^0\rangle, \langle b_2^1, a_1^0\rangle, \langle b_2^1, b_1^0\rangle, \langle b_3^1, a_1^0\rangle, \langle b_3^1, a_2^0\rangle\};\\ R_1 &=& R_1^a \cup R_1^b, \end{array}$$

and let, for every $k \geq 1$,

$$\begin{array}{rcl} R_{k+1}^a &=& \{\langle a_m^{k+1}, b_1^k\rangle, \langle a_m^{k+1}, a_i^k\rangle, \langle a_m^{k+1}, b_j^k\rangle : A_m^{k+1} = A_1^k \to B_1^k \vee A_i^k \vee B_j^k\};\\ R_{k+1}^b &=& \{\langle b_m^{k+1}, a_1^k\rangle, \langle b_m^{k+1}, a_i^k\rangle, \langle b_m^{k+1}, b_j^k\rangle : B_m^{k+1} = B_1^k \to A_1^k \vee A_i^k \vee B_j^k\};\\ R_{k+1} &=& R_{k+1}^a \cup R_{k+1}^b. \end{array}$$

We put

$$R' = \bigcup_{k=1}^{\infty} R_k$$

and take as R the reflexive and transitive closure of R'. Let $\mathfrak{F} = \langle W, R\rangle$ and $\mathfrak{M} = \langle \mathfrak{F}, v\rangle$, where v is defined in the following way:

$$\begin{array}{rcl} (\mathfrak{M}, w) \models p &\iff& w = c_0 \text{ or } w = d_1;\\ (\mathfrak{M}, w) \models q &\iff& w = c_0 \text{ or } w = d_2. \end{array}$$

The model $\mathfrak{M}$ is depicted on Fig. 1.

LEMMA 5. *Let w be a world of $\mathfrak{M}$. Then*

$$\begin{array}{rcl} (\mathfrak{M}, w) \not\models A_m^k &\iff& wRa_m^k;\\ (\mathfrak{M}, w) \not\models B_m^k &\iff& wRb_m^k. \end{array}$$

PROOF proceeds by easy induction on k. To prove the basis of induction it is useful to check that for every world w in $\mathfrak{M}$ and for every $i \in \{1, 2, 3\}$,

$$(\mathfrak{M}, w) \not\models D_i \iff wRd_i.$$

❑

Let φ be a positive intuitionistic formula and let $p_1, \ldots, p_n$ be all variables of φ. We denote by k the least natural number such that $|\varphi| < 5^k \cdot l$. Note that

$$N_{k+6} \ > \ 5^{k+6} \cdot l \ > \ 5^6 \cdot |\varphi| \ > \ |\varphi| \ > \ n,$$

and so the following formulas are well-defined for every $i \in \{1, \ldots, n\}$:

$$\alpha_i \ = \ A_i^{k+6} \vee B_i^{k+6}.$$

Let φ_α be the result of substituting the formulas $\alpha_1, \ldots, \alpha_n$ respectively for $p_1, \ldots, p_n$ in φ.

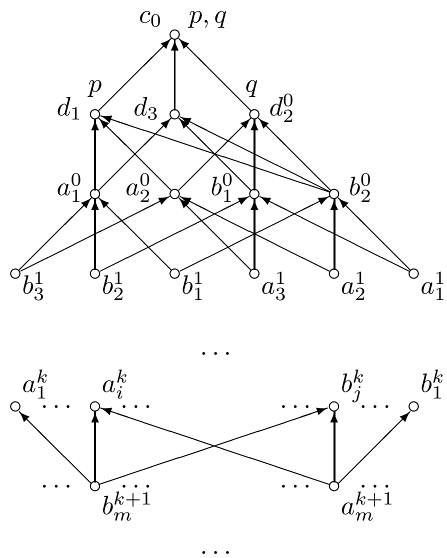

Figure 1. Model $\mathfrak{M}$

LEMMA 6. *There exists a number m such that for any φ, $|\varphi_\alpha| < m \cdot |\varphi|^2$.*

PROOF. From the fact that k is the least natural number satisfying the condition $|\varphi| < 5^k \cdot l$ it follows that $5^{k-1} \cdot l \leq |\varphi|$, and we obtain

$$5^{k+6} \cdot l \ \leq \ 5^7 \cdot |\varphi|.$$

Since $|A_i^{k+6}| < 5^{k+6} \cdot l$ and $|B_i^{k+6}| < 5^{k+6} \cdot l$, we have

$$|\alpha_i| \ < \ 2 \cdot 5^{k+6} \cdot l \ \leq \ 2 \cdot 5^7 \cdot |\varphi|.$$

Hence

$$|\varphi_\alpha| \ \leq \ |\varphi| \cdot \max_{1 \leq i \leq n} |\alpha_i| \ < \ 2 \cdot 5^7 \cdot |\varphi|^2,$$

i.e., we can take $m = 2 \cdot 5^7$. ❑

Using Lemma 6 one can show that the formula φ_α is computed from φ by a polynomial time algorithm.

LEMMA 7. *For every positive intuitionistic formula φ,*

$$\varphi \in \mathbf{Int} \quad \Longleftrightarrow \quad \varphi_\alpha \in \mathbf{Int}.$$

PROOF. If $\varphi \in \mathbf{Int}$, then $\varphi_\alpha \in \mathbf{Int}$ because **Int** is closed under Substitution.

Let φ be a positive intuitionistic formula such that $\varphi \notin \mathbf{Int}$. Then there exists a model $\mathfrak{M}_\varphi = \langle \mathfrak{F}_\varphi, v_\varphi \rangle$ defined on reflexive, transitive, and antisymmetric frame $\mathfrak{F}_\varphi = \langle W_\varphi, R_\varphi \rangle$ such that $(\mathfrak{M}_\varphi, w_0) \not\models \varphi$, for some $w_0 \in W_\varphi$.

Let us construct an intuitionistic model refuting φ_α. Without a loss of generality we may assume that $W \cap W_\varphi$ is empty. We put $W^* = W \cup W_\varphi$. Let

$$\begin{array}{rcl} R' & = & \{\langle w, a_i^{k+6}\rangle, \langle w, b_i^{k+6}\rangle : w \in W_\varphi,\, (\mathfrak{M}_\varphi, w) \not\models p_i,\, 1 \le i \le n\} \cup \\ & & \cup\, \{\langle w, a_{n+1}^{k+6}\rangle, \langle w, b_{n+1}^{k+6}\rangle : w \in W_\varphi\}. \end{array}$$

Let R^* be the reflexive and transitive closure of the relation $R \cup R_\varphi \cup R'$ and let $\mathfrak{F}^* = \langle W^*, R^*\rangle$. We define the model $\mathfrak{M}^* = \langle \mathfrak{F}^*, v^*\rangle$ on $\mathfrak{F}^*$ by putting for every $w \in W^*$

$$\begin{array}{lcl} (\mathfrak{M}^*, w) \models p & \iff & w = c_0 \text{ or } w = d_1; \\ (\mathfrak{M}^*, w) \models q & \iff & w = c_0 \text{ or } w = d_2. \end{array}$$

For every subformula ψ of the formula φ, let ψ_α be the result of substituting the formulas $\alpha_1, \dots, \alpha_n$ respectively for $p_1, \dots, p_n$ in ψ. By induction on the construction of ψ let us show that for every $w \in W_\varphi$,

$$(\mathfrak{M}^*, w) \models \psi_\alpha \iff (\mathfrak{M}_\varphi, w) \models \psi.$$

Let $\psi = p_m$. If $(\mathfrak{M}_\varphi, w) \not\models p_m$, then a_m^{k+6} and b_m^{k+6} are accessible from w in $\mathfrak{M}^*$. Using Lemma 5, we see that $(\mathfrak{M}^*, a_m^{k+6}) \not\models A_m^{k+6}$ and $(\mathfrak{M}^*, b_m^{k+6}) \not\models B_m^{k+6}$, hence $(\mathfrak{M}^*, w) \not\models \alpha_m$, i. e., $(\mathfrak{M}^*, w) \not\models \psi_\alpha$.

Suppose $(\mathfrak{M}^*, w) \not\models \alpha_m$, for some world $w \in W_\varphi$. Let $g(m) = (i, j)$, i. e., $A_m^{k+6} = A_1^{k+5} \to B_1^{k+5} \vee A_i^{k+5} \vee B_j^{k+5}$, $B_m^{k+6} = B_1^{k+5} \to A_1^{k+5} \vee A_i^{k+5} \vee B_j^{k+5}$. Then there exist worlds w' and w'' in $\mathfrak{M}^*$ such that wR^*w', wR^*w'', and

$$\begin{array}{lll} (\mathfrak{M}^*, w') \models A_1^{k+5}, & (\mathfrak{M}^*, w') \not\models B_1^{k+5}, & (\mathfrak{M}^*, w') \not\models A_i^{k+5} \vee B_j^{k+5}; \\ (\mathfrak{M}^*, w'') \not\models A_1^{k+5}, & (\mathfrak{M}^*, w'') \models B_1^{k+5}, & (\mathfrak{M}^*, w'') \not\models A_i^{k+5} \vee B_j^{k+5}. \end{array}$$

Note that $w', w'' \in W$. Indeed, for every world $u \in W_\varphi$, we have $uR^*a_{n+1}^{k+6}$ and $uR^*b_{n+1}^{k+6}$. Since $(\mathfrak{M}^*, a_{n+1}^{k+6}) \not\models B_1^{k+5}$ and $(\mathfrak{M}^*, b_{n+1}^{k+6}) \not\models A_1^{k+5}$, we obtain $(\mathfrak{M}^*, u) \not\models A_1^{k+5}$, $(\mathfrak{M}^*, u) \not\models B_1^{k+5}$. But $(\mathfrak{M}^*, w') \models A_1^{k+5}$, $(\mathfrak{M}^*, w'') \models B_1^{k+5}$, thus $w', w'' \notin W_\varphi$, i. e., $w', w'' \in W$.

Since w' and w'' are R^*-accessible from a world of W_φ, then their levels are not greater than $k+6$. Besides, $(\mathfrak{M}^*, w') \not\models A_m^{k+6}$ and $(\mathfrak{M}^*, w'') \not\models B_m^{k+6}$, therefore the levels of w' and w'' are not less than $k+6$. So, w' and w'' are worlds of level $k+6$. But the unique world of level $k+6$ refuting A_m^{k+6} is a_m^{k+6} and the unique world of level $k+6$ refuting B_m^{k+6} is b_m^{k+6}, hence $w' = a_m^{k+6}$ and $w'' = b_m^{k+6}$.

Thus $wR^*a_m^{k+6}$. This means that there exists $u \in W_\varphi$ such that $wR_\varphi u$ and $uR'a_m^{k+6}$. But if $uR'a_m^{k+6}$, then, by the definition of $\mathfrak{F}^*$, we have both $uR'b_m^{k+6}$ and $(\mathfrak{M}_\varphi, u) \not\models p_m$. Since $wR_\varphi u$, we obtain $(\mathfrak{M}_\varphi, w) \not\models p_m$.

Let ψ' and ψ'' be subformulas of φ such that for every $w \in W_\varphi$,

$$\begin{array}{lcl} (\mathfrak{M}^*, w) \models \psi'_\alpha & \iff & (\mathfrak{M}_\varphi, w) \models \psi'; \\ (\mathfrak{M}^*, w) \models \psi''_\alpha & \iff & (\mathfrak{M}_\varphi, w) \models \psi'', \end{array}$$

and let $\psi = \psi' \to \psi''$. If $(\mathfrak{M}_\varphi, w) \not\models \psi$, then there is a world w' in $\mathfrak{M}_\varphi$ accessible from w such that $(\mathfrak{M}_\varphi, w') \models \psi'$ and $(\mathfrak{M}_\varphi, w') \not\models \psi''$; but then $(\mathfrak{M}^*, w') \models \psi'_\alpha$ and $(\mathfrak{M}^*, w') \not\models \psi''_\alpha$, hence $(\mathfrak{M}^*, w) \not\models \psi_\alpha$.

Suppose for some $w \in W_\varphi$, $(\mathfrak{M}^*, w) \not\models \psi_\alpha$. Then there is $w' \in W^*$ such that wR^*w', $(\mathfrak{M}^*, w') \models \psi'_\alpha$, and $(\mathfrak{M}^*, w') \not\models \psi''_\alpha$. Note that the formulas $\alpha_1, \ldots, \alpha_n$ are true in every world $u \in W$ of level not greater than $k+6$. Hence in all such worlds the formulas constructed from $\alpha_1, \ldots, \alpha_n$ using $\to$, $\wedge$, and $\vee$, must be true; in particular, ψ''_α must be true. From $(\mathfrak{M}^*, w') \not\models \psi''_\alpha$ it follows that $w' \notin W$, i. e., $w' \in W_\varphi$. By the inductive hypothesis, we obtain $(\mathfrak{M}_\varphi, w') \models \psi'$ and $(\mathfrak{M}_\varphi, w') \not\models \psi''$, hence $(\mathfrak{M}_\varphi, w) \not\models \psi$.

The cases $\psi = \psi' \wedge \psi''$ and $\psi = \psi' \vee \psi''$ are trivial.

As a result we obtain $(\mathfrak{M}^*, w_0) \not\models \varphi_\alpha$, and hence $\varphi_\alpha \notin$ **Int**. ❑

As a corollary we immediately obtain that the decision problem for $\mathbf{Int}^+(2)$ is PSPACE-complete.

The case with Visser's logics **BPL** and **FPL** is simpler. For the proof of PSPACE-hardness for **BPL**(0) and **FPL**(1) we define 'modality' $\Box$ as the following abbreviation: $\Box\psi = \top \to \psi$. Let also $\Box^1\psi = \Box\psi$ and for any $n \geq 1$, let $\Box^{n+1}\psi = \Box\Box^n\psi$.

Let us define formulas β_n and γ_n, for $n \geq 1$:

$$\begin{array}{rcl} \beta_n & = & (\Box^{n+2}\bot \to \Box^{n+1}\bot) \to (\Box^{n+1}\bot \to \Box^n\bot) \vee \Box^{n+2}\bot; \\ \gamma_n & = & \Box^2 p \to (\Box^{n+1}\bot \to \Box^n\bot \vee p). \end{array}$$

Let φ be a positive intuitionistic formula and let $p_1, \ldots, p_n$ be all variables of φ. We denote by φ_β the result of substituting the formulas $\beta_1, \ldots, \beta_n$ respectively for $p_1, \ldots, p_n$ in φ and by φ_γ the result of substituting the formulas $\gamma_1, \ldots, \gamma_n$ respectively for $p_1, \ldots, p_n$ in φ.

Clearly, φ_β and φ_γ are computed from φ by some polynomial time algorithms. Hence, to prove PSPACE-hardness for **BPL**(0) it is sufficient to prove the following statement.

LEMMA 8. *For every positive formula φ,*

$$\varphi \in \mathbf{BPL} \iff \varphi_\beta \in \mathbf{BPL}.$$

PROOF. If $\varphi \in$ **BPL**, then $\varphi_\beta \in$ **BPL** because **BPL** is closed under Substitution.

Let $\varphi \notin$ **BPL**. We show that $\varphi_\alpha \notin$ **BPL**. There exists a model $\mathfrak{M} = \langle \mathfrak{F}, v\rangle$ on some transitive and antisymmetric Kripke frame $\mathfrak{F} = \langle W, R\rangle$ such that $(\mathfrak{M}, w_0) \not\models \varphi$, for some $w_0 \in W$.

Let $\mathfrak{F}_k = \langle W_k, R_k\rangle$ be the Kripke frame, depicted on Fig. 2 (b_k is a reflexive world, other are irreflexive, R_k is transitive), i. e.,

$$\begin{array}{rcl} W_k & = & \{b_k, a^k_1, a^k_2, \ldots, a^k_{k+2}\}; \\ wR_kw' & \iff & \text{either } w = w' = b_k, \\ & & \text{or } w = a^k_{k+2},\ w' = b_k, \\ & & \text{or } w = a^k_i,\ w' = a^k_j,\ i > j. \end{array}$$

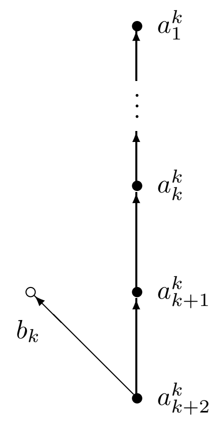

Figure 2. Frame $\mathfrak{F}_k$

Frame $\mathfrak{F}_k$ has the following property: if we add to it as a root some world w, then we obtain a frame $\mathfrak{F}'_k$ such that for every $m \geq 1$

$$(\mathfrak{F}'_k, w) \not\models \beta_m \iff k = m.$$

Using this observation it is easy to construct a frame of **BPL** refuting φ_β. Let

$$W^* = W \cup \bigcup_{k=1}^{n+1} W_k,$$

let R^* be the transitive closure of the relation

$$R \cup \bigcup_{k=1}^{n+1} R_k \cup \{\langle w, a_{k+2}^k \rangle : w \in W, (\mathfrak{M}, w) \not\models p_k\} \cup \{\langle w, a_{n+3}^{n+1} \rangle : w \in W\},$$

and let $\mathfrak{F}^* = \langle W^*, R^* \rangle$. In other worlds, for every variable p_k of φ and every world w in $\mathfrak{F}$ such that $(\mathfrak{M}, w) \not\models p_k$ we make $\mathfrak{F}_k$ to be accessible from w; and also for every world w in $\mathfrak{F}$ we make $\mathfrak{F}_{n+1}$ to be accessible from w.

For every subformula ψ of formula φ, let ψ_β be the result of substituting the formulas $\beta_1, \ldots, \beta_n$ respectively for $p_1, \ldots, p_n$ in ψ. By induction on construction of ψ one can show that for every $w \in W$

$$(\mathfrak{F}^*, w) \models \psi_\beta \iff (\mathfrak{M}, w) \models \psi.$$

Let $\psi = p_k$. If $(\mathfrak{M}, w) \not\models p_k$, then $wR^*a_{k+2}^k$, hence $(\mathfrak{F}^*, w) \not\models \beta_k$, i. e., $(\mathfrak{F}^*, w) \not\models \psi_\beta$.

Suppose $(\mathfrak{F}^*, w) \not\models \psi_\beta$, i. e., $(\mathfrak{F}^*, w) \not\models \beta_k$. Then there exists $w' \in W^*$ such that wR^*w',

$$(\mathfrak{F}^*, w') \models \Box^{k+2}\bot \to \Box^{k+1}\bot, \quad (\mathfrak{F}^*, w') \not\models (\Box^{k+1}\bot \to \Box^k\bot) \vee \Box^{k+2}\bot.$$

We claim $w' \notin W$. Indeed, suppose $w' \in W$. Then, by the definition of $\mathfrak{F}^*$, we have $w' R^* a_{n+3}^{n+1}$ (i. e., $\mathfrak{F}_{n+1}$ is accessible from w'). It is easy to see that $(\mathfrak{F}_{n+1}, a_{n+3}^{n+1}) \not\models \Box^{k+2}\bot \to \Box^{k+1}\bot$, hence $(\mathfrak{F}^*, a_{n+3}^{n+1}) \not\models \Box^{k+2}\bot \to \Box^{k+1}\bot$, and so $(\mathfrak{F}^*, w') \not\models \Box^{k+2}\bot \to \Box^{k+1}\bot$. We have a contradiction, therefore $w' \notin W$.

Since $w' \notin W$, we have $w' \in W_m$, for some $m \in \{1, \dots, n+1\}$. One can easily check that for any $s \geq 1$,

$$(\mathfrak{F}_s, u) \not\models \beta_k \quad \Longleftrightarrow \quad s = k \text{ and } u = a_{k+2}^k.$$

For our case it means that $w' = a_{k+2}^k$. By the construction of $\mathfrak{F}^*$, we see that $(\mathfrak{M}, w) \not\models p_k$.

The cases $\psi = \psi' \wedge \psi''$ and $\psi = \psi' \vee \psi''$ are trivial. The proof for the case $\psi = \psi' \to \psi''$ proceeds in the same way as in Lemma 7.

So, we obtain $(\mathfrak{F}^*, w_0) \not\models \varphi_\beta$, and hence $\varphi_\beta \notin \mathbf{BPL}$. ❑

So, the decision problem for **BPL**(0) is PSPACE-complete. To prove PSPACE-hardness for **FPL**(1) it is sufficient to prove

LEMMA 9. *For every positive formula φ,*

$$\varphi \in \mathbf{FPL} \quad \Longleftrightarrow \quad \varphi_\gamma \in \mathbf{FPL}.$$

PROOF. The proof proceeds in the same way as for Lemma 8. Instead of frames $\mathfrak{F}_k$ one may take models $\mathfrak{M}_k$: the worlds in $\mathfrak{M}_k$ are numbers $1, \dots, k+2$, the accessibility relation is the relation $>$, and the valuation for p is defined so that p is true in the worlds $1, \dots, k$ and not true in $k+1$ and $k+2$. ❑

So, the decision problem for **FPL**(1) is PSPACE-complete, and Theorem 4 is proved. ❑

4 Corollaries

Of course, from Theorem 4 we immediately obtain

COROLLARY 10. *The decision problem for* **Int**(2) *is* PSPACE-*complete.*

From Theorem 4 one can obtain corollaries about complexity of the decision problem for n-variable fragments of wide classes of logics. I give some of them here.

First of all, with the help of the translation T we obtain some results in [8].

COROLLARY 11. [8]. *The decision problem for* **K4**(0)*,* **GL**(1)*,* **S4**(2) *is* PSPACE-*complete.*

REMARK 12. The logic **S4** has PSPACE-complete decision problem in the language with one variable: see [8].

Let us consider the logic of the weak law of the excluded middle **KC** (for details the reader can consult [9]), which is defined by adding to **Int** the corresponding axiom: $\mathbf{KC} = \mathbf{Int} + \neg p \vee \neg\neg p$.

The decision problem for $\mathbf{KC}^+$ is PSPACE-complete. The proof of this fact is similar to the one for $\mathbf{Int}^+$ [33]; in particular, to prove PSPACE-hardness of $\mathbf{KC}^+$ it is sufficient to consider the same formulas.

V. A. Jankov [17] proved that, for every superintuitionistic logic L (i. e., an extension of $\mathbf{Int}$),

$$L^+ = \mathbf{Int}^+ \quad \Longleftrightarrow \quad L \subseteq \mathbf{KC}.$$

Hence from Theorem 4 we obtain the following statements.

COROLLARY 13. *The decision problem for* $\mathbf{KC}(2)$ *is* PSPACE-*complete.*

COROLLARY 14. *Let* L *be a logic such that* $\mathbf{Int} \subseteq L \subseteq \mathbf{KC}$*. Then the decision problem for* $L(2)$ *is* PSPACE-*hard.*

Corollary 14 has its own corollaries.

COROLLARY 15. *Let* L *be a logic such that* $L \subset \mathbf{Cl}$ *and it can be obtained by adding to* $\mathbf{Int}$ *a one-variable axiom. Then the decision problem for* $L(2)$ *is* PSPACE-*hard.*

PROOF immediately follows from [21] and Corollary 14. Indeed, using the Nishimura lattice [21], one can see that the greatest consistent extension of $\mathbf{Int}$ obtained by adding to it a one-variable axiom is $\mathbf{KC}$. ❑

For the next corollary we need a definition: a formula φ is called *implication free* if φ is constructed from propositional variables with $\wedge$, $\vee$ and $\neg$.

COROLLARY 16. *Let* L *be a logic such that* $L \subset \mathbf{Cl}$ *and it can be obtained by adding to* $\mathbf{Int}$ *an implication free axiom. Then the decision problem for* $L(2)$ *is* PSPACE-*hard.*

PROOF. It is not hard to see that in this case $\mathbf{Int} + \varphi \subseteq \mathbf{KC}$ (see also [3]), and so the decision problem for the two-variable fragment of L is PSPACE-hard by Corollary 14. ❑

We consider two special logics: the Kreisel–Putnam logic $\mathbf{KP}$ and the Medvedev logic $\mathbf{ML}$. The logic $\mathbf{KP}$ is defined by adding to $\mathbf{Int}$ the Kreisel–Putnam formula:

$$\mathbf{KP} \;=\; \mathbf{Int} + (\neg p \to q \vee r) \to (\neg p \to q) \vee (\neg p \to r).$$

The logic $\mathbf{ML}$ is an extension of $\mathbf{KP}$. It can be defined semantically in the following way, see [9]. Let $n > 0$ and let W_n be the family of non-empty subsets of a set with n elements. Let wR_nw' mean $w' \subseteq w$, for every $w, w' \in W_n$. The pair $\mathfrak{M}_n = \langle W_n, R_n \rangle$ is clearly a Kripke frame. Then

$$\mathbf{ML} \;=\; \{\varphi : \mathfrak{F}_n \models \varphi, \text{ for any } n > 0\}.$$

These two logics are rather complex: it is known that $\mathbf{KP}$ is decidable [12] but it is not known if the decision problem for $\mathbf{KP}$ belongs to PSPACE. The decidability of $\mathbf{ML}$ is still an open problem.

Since $\mathbf{ML}^+ = \mathbf{Int}^+$ [37], from [17] we obtain $\mathbf{ML} \subseteq \mathbf{KC}$, and hence we proved

COROLLARY 17. *The decision problem for logics* $\mathbf{KP}(2)$ *and* $\mathbf{ML}(2)$ *is* PSPACE-*hard.*

Note, that to prove PSPACE-hardness of **BPL**, **FPL**, **Int**, **KC** (in full language) we can use the same (positive) formulas (see [4]; one can also construct such formulas as a slight modification of the ones in [33] or [9], see Remark 3). More exactly, there exists a polynomial time algorithm f such that for every quantified Boolean formula φ,

$$\varphi \text{ is satisfiable} \iff f(\varphi) \notin L,$$

where $L \in \{\mathbf{BPL}, \mathbf{FPL}, \mathbf{Int}, \mathbf{KC}\}$, $f(\varphi)$ is a positive formula (for complexity of satisfiability problem for quantified Boolean formulas see [34]).

Using this observation we obtain the following statements.

COROLLARY 18. *Let L be a logic such that* $\mathbf{BPL} \subseteq L \subseteq \mathbf{KC}$. *Then the decision problem for* $L^+(2)$ *is* PSPACE-*hard.*

PROOF. Let $\varphi^* = f(\varphi)$, where f is the function mentioned above, φ is a quantified Boolean formula. If φ is not satisfiable, then $\varphi^* \in \mathbf{BPL}$, hence $\varphi^*_\alpha \in \mathbf{BPL}$ (because **BPL** is closed under Substitution), so $\varphi^*_\alpha \in L^+(2)$. If φ is satisfiable, then $\varphi^* \notin \mathbf{Int}$, hence, by Lemma 7, we have $\varphi^*_\alpha \notin \mathbf{Int}$, wherefore, by [17], $\varphi^*_\alpha \notin \mathbf{KC}$, and so $\varphi^*_\alpha \notin L^+(2)$. ❑

COROLLARY 19. *Let L be a logic such that* $\mathbf{BPL} \subseteq L \subseteq \mathbf{FPL}$. *Then the decision problem for* $L(1)$ *and* $L^+(2)$ *is* PSPACE-*hard.*

PROOF. It is easy to understand that the decision problem for $L(1)$ is PSPACE-hard. To prove PSPACE-hardness for $L^+(2)$ we may simply redefine the 'modality' $\Box$: $\Box\psi = (q \to q) \to \psi$, here q is a propositional variable different from p. All semantic properties of this 'new' $\Box$ are the same as of the 'old' one, hence if in the definition of γ_n we use the 'new' $\Box$ instead of the 'old' $\Box$ and q instead of all other occurrences of $\bot$, then we obtain a positive version of φ_γ, and so, PSPACE-hardness for $L^+(2)$. ❑

Using the same argumentation and formulas β_n it is easy to show that the decision problem for $\mathbf{BPL}^+(1)$ is PSPACE-complete too.

COROLLARY 20. *The decision problem for the logic* $\mathbf{BPL}^+(1)$ *is* PSPACE-*complete.*

Note also that $\mathbf{BPL}^+(0)$ is trivial; in particular, it is decidable in polynomial time.

5 Complexity function

Recall that the complexity function $f_L(n)$ for a logic (a set of formulas) L is defined in the following way (see [9]):

$$f_L(n) = \max_{\substack{|\varphi| \le n \\ \varphi \notin L}} \min_{\substack{\mathfrak{F} \models L \\ \mathfrak{F} \not\models \varphi}} |\mathfrak{F}|,$$

where $|\mathfrak{F}|$ is the cardinality of the frame $\mathfrak{F}$ and $|\varphi|$ is the length of the formula φ.

The complexity function for a logic L is a way to estimate the complexity of the decision problem for L. For example, it can provide us with bounds for some computational resources (for example, time resources) required for algorithms deciding L by searching for countermodels for tested formulas.

Moreover, the study of the complexity function for **Int** and for some other logics may be helpful in the search for an answer to Kuznetsov's question presented in [19] (see also [18]): is the problem '$\varphi \in$ **Int**?' polynomially reducible to the problem '$\varphi \in$ **Cl**?'? So, in fact A. V. Kuznetsov [19] showed that if the complexity function for **Int** is polynomial, then **Int** is polynomially equivalent to **Cl**. Note that **Int** here may be replaced with other logics — **BPL**, **FPL**, **K**, **T**, **K4**, etc. — which have PSPACE-hard decision problem and 'good' semantics. Note also that a positive answer to such question means that NP = PSPACE.

At the end of 1970s and later M. V. Zakharyaschev established for a lot of standard non-classical logics (in particular including **Int** and **KC**) that the complexity function for these logics is exponential, see [9, 23, 24], i. e., for some positive numbers k, m, r, s,

$$2^{k \cdot n^m} \leqslant f_L(n) \leqslant 2^{r \cdot n^s}.$$

So, the complexity function for these logics actually cannot be bounded by any polynomial function.

Zakharyaschev's results are in close connection with Ladner's and Statman's ones: using Zakharyaschev's formulas it is not hard to prove PSPACE-hardness of **Int**, **K**, **T**, **S4**, etc., and from Ladner's and Statman's constructions it is not hard to obtain that the complexity function for the corresponding logics is exponential.

In general we can observe a certain connection between the complexity function for a logic L and the complexity of L with respect to complexity classes such as NP and PSPACE. For example, known logics with NP-complete decision problem — **Cl**, **S5**, **S4.3**, etc. — have polynomial complexity function, known logics with PSPACE-complete decision problem — **Int**, **BPL**, **FPL**, **K**, **T**, etc. — have exponential complexity function; note also that the complexity function for every logic between **KP** and **ML** has the lower bound of order $2^{2^{k \cdot n}}$, for some $k > 0$ (for details see [9]).

All these results are obtained for logics with infinitely many variables in the language (I do not take into account our — with A. V. Chagrov —

article [8]), and it is natural to solve the problem about bounds of the complexity function for PSPACE-hard n-variable fragments of **Int**, **KC**, **BPL**, etc. Of course, it will be surprising if the complexity function for some of these fragments is polynomial, and it is really not the case: it has exponential lower bound in every case of the logics considered here. To prove it, it is sufficient to take corresponding Zakharyaschev's (or Statman's, or Chagrov's) formulas and substitute α_n's, or β_n's, or γ_n's in accordance with what logic we consider.

So, we have the following corollaries from the construction of Theorem 4.

COROLLARY 21. *Let a logic L be such that $\mathbf{BPL}(1) \subseteq L(1) \subseteq \mathbf{FPL}(1)$. Then the lower bound of the complexity function for $L(1)$ is exponential.*

COROLLARY 22. *Let a logic L be such that $\mathbf{BPL}^+(2) \subseteq L(2) \subseteq \mathbf{FPL}^+(2)$, or $\mathbf{BPL}^+(2) \subseteq L(2) \subseteq \mathbf{Int}^+(2)$. Then the lower bound of the complexity function for $L(2)$ is exponential.*

Taking into account results in [9] concerning the complexity function, in some cases we can specify the result.

COROLLARY 23. *The complexity function for the logics $\mathbf{BPL}(0)$, $\mathbf{FPL}(1)$, $\mathbf{Int}(2)$, $\mathbf{KC}(2)$, $\mathbf{BPL}^+(1)$, $\mathbf{FPL}^+(2)$, $\mathbf{Int}^+(2)$ is exponential.*

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