Topological Modal Logics with Difference Modality

Andrey Kudinov

ABSTRACT. We consider propositional modal logic with two modal operators $\Box$ and $[\neq]$. In topological semantics $\Box$ is interpreted as an interior operator and $[\neq]$ as difference. We show that some important topological properties are expressible in this language. In addition, we present a few logics and proofs of f.m.p. and of completeness theorems.

1 Introduction.

This paper deals with the topological semantics of modal logic. The study of topological semantics of modal logic was started in 1944 by McKinsey and Tarski [12]. Recently, this topic has been attracting more attention partly due to applications in AI (cf. [2] and [8]). Reading the modal box as an interior operator one can easily show that logic of all topological spaces is **S4**. In addition, McKinsey and Tarski proved that **S4** is also the complete logic of the reals, Cantor space and indeed of any metric separable space without isolated points (for a new proof of this fact see [1]). Therefore, all these spaces are modally equivalent, hence many natural properties of topological spaces such as connectedness, density-in-itself and T_1 are undefinable. For more information on spatial logics and spatial reasoning see [1, 9, 15, 16].

There are two ways of enriching the definability of a language: to change semantics or to extend the language. According to the first way, Esakia in [6] and Shehtman in [16] considered the derivational logic (more recent paper on this [3]). According to the other way, we can add the universal modality. In this new language we can express connectedness (cf. [15]).

In this paper, however, we add the difference modality (or modality of inequality) $[\neq]$, interpreted as "true everywhere except here". Using the difference modality was suggested by several people independently (in [10] for one). This modality and its interpretation in Kripke frames were studied more deeply in [14]. It has been shown that the difference modality greatly increases the expressive power of a language (cf. [11, 14]). The expressive power of this language in topological spaces has been studied by Gabelaia in [9]; the author presented axioms that defines T_1 and T_0 spaces. Added to the topological modal logic, the difference modality allows us to express topological properties that were unreachable before. The topological properties mentioned in the end of the first paragraph become definable. The universal modality is expressible as well in the following way: $[\forall]A = [\neq]A \wedge A$.

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Here we also introduce three logics: **S4D**, **S4DS**, and **S4DT₁S**. We prove their f.m.p. and the following completeness theorems: **S4D** is complete with respect to all topological spaces (Theorem 31), **S4DS** is complete with respect to all dense-in-themselves topological spaces (Theorem 33), and **S4DT₁S** is complete with respect to any zero-dimensional dense-in-itself metric space (Theorem 36).

2 Definitions and basic notions.

Let us introduce some notations the reader will meet in this paper. Assume that B is a set, $R, R' \subseteq B \times B$ are relations on B, then

$$\begin{array}{l} R\restriction_A = R \cap (A \times A), \text{ for any } A \subseteq B; \\ Id_B = \{(x,x) \mid x \in B\}; \\ R^+ = R \cup Id_B; \\ R \circ R' = \{(x,z) \mid \exists y\ (xRy\ \&\ yR'z)\}; \\ R^1 = R,\ R^n = R^{n-1} \circ R; \\ R^* = \bigcup_{n=1}^{\infty} R^n. \end{array}$$

In this paper, we study propositional modal logics with two modal operators, $\Box$ and $[\neq]$. A formula is defined as follows:

$$\phi ::= p \mid \bot \mid \phi \to \phi \mid \Box\phi \mid [\neq]\phi.$$

The standard classic logic operators $(\vee, \wedge, \neg, \top, \equiv)$ are expressed in terms of $\to$ and $\bot$. The dual modal operators $\Diamond, \langle\neq\rangle$ are defined in the usual way as $\Diamond A = \neg\Box\neg A$, $\langle\neq\rangle A = \neg[\neq]\neg A$ respectively. $[\forall]A$ stands for $[\neq]A \wedge A$.

DEFINITION 1. A *bimodal logic* (or *a logic,* for short) is a set of modal formulas closed under Substitution $\left(\frac{A(p_i)}{A(B)}\right)$, Modus Ponens $\left(\frac{A,\ A\to B}{B}\right)$ and two Generalization rules $\left(\frac{A}{\Box A},\ \frac{A}{[\neq]A}\right)$; containing all classical tautologies and the following axioms

$$\begin{array}{l} \Box(p \to q) \to (\Box p \to \Box q), \\ [\neq](p \to q) \to ([\neq]p \to [\neq]q). \end{array}$$

$\mathbf{K_2}$ denotes *the minimal bimodal logic.*

Let L be a logic and let Γ be a set of formulas, then $L + \Gamma$ denotes the minimal logic containing L and Γ. If $\Gamma = \{A\}$, then we write $L + A$ rather then $L + \{A\}$.

In this paper, however, we consider a few additional axioms:

$$\begin{array}{ll} (B_D) & p \to [\neq]\langle\neq\rangle p \\ (4_D^-) & (p \wedge [\neq]p) \to [\neq][\neq]p \\ (T_\Box) & \Box p \to p \\ (4_\Box) & \Box p \to \Box\Box p \\ (D_\Box) & [\forall]p \to \Box p \\ (AT_1) & [\neq]p \to [\neq]\Box p \\ (DS) & [\neq]p \to \Diamond p \end{array}$$

The first two axioms are for $[\neq]$ and they are from the paper by de Rijke [14]. These axioms correspond to some basic properties of inequality: symmetry and pseudo-transitivity[1] respectively.

The next two axioms are axioms for **S4**. These axioms have well-known correspondence to the properties of topological interior operator: $\mathbf{I}Y \subseteq Y$ and $\mathbf{I}Y \subseteq \mathbf{II}Y$ respectively (where Y is an arbitrary set). We denote the interior and the closure operators by $\mathbf{I}$ and $\mathbf{C}$ respectively.

Axiom $(D_\Box)$ is needed to connect $\Box$ and $[\neq]$ and to make sure that $[\forall]$ is the universal modality.

The meaning of the next two axioms will be explained later.

In this paper we study the following three logics:

$$\begin{aligned}
&\mathbf{S4D} = \mathbf{K_2} + \{B_D, 4_D^-, D_\Box, T_\Box, 4_\Box\},\\
&\mathbf{S4DS} = \mathbf{S4D} + DS,\\
&\mathbf{S4DT_1S} = \mathbf{S4DS} + AT_1.
\end{aligned}$$

3 Topological models.

Let us define topological models.

DEFINITION 2. A *topological model* is a pair $(\mathfrak{X}, \theta)$, where $\mathfrak{X}$ is a topological space and θ is a function assigning to each proposition letter p a subset $\theta(p)$ of $\mathfrak{X}$. The function θ is called *a valuation*.

DEFINITION 3. The truth of a formula at a point of a topological model is defined by induction:

$$\begin{array}{rlll}
\text{(i)} & \mathfrak{X},\theta,x \models p & \text{iff} & x \in \theta(p)\\
\text{(ii)} & \mathfrak{X},\theta,x \not\models \bot & & \\
\text{(iii)} & \mathfrak{X},\theta,x \models \phi \to \psi & \text{iff} & \mathfrak{X},\theta,x \not\models \phi \textit{ or } \mathfrak{X},\theta,x \models \psi\\
\text{(iv)} & \mathfrak{X},\theta,x \models \Box\phi & \text{iff} & \text{there is a neighborhood } U \text{ of } x \text{ such}\\
 & & & \text{that for any } y \in U \quad \mathfrak{X},\theta,y \models \phi\\
\text{(v)} & \mathfrak{X},\theta,x \models [\neq]\phi & \text{iff} & \mathfrak{X},\theta,y \models \phi \text{ for any } y \neq x
\end{array}$$

If U is a subset of $\mathfrak{X}$, then $\mathfrak{X},\theta,U \models A$ denotes that $\mathfrak{X},\theta,x \models A$ for any $x \in U$. A formula A is called *valid* in a topological space $\mathfrak{X}$ (notation: $\mathfrak{X} \models A$), if it is true at any point under any valuation. Also in notation $\mathfrak{X},\theta,x \models A$ we will omit the space and/or the valuation, if it is clear what space and/or valuation we consider.

DEFINITION 4. The *D-logic* of a class of topological spaces $\mathcal{T}$ (in notation $L_D(\mathcal{T})$) is the set of all formulas that are valid in all topological spaces from $\mathcal{T}$.

Let us describe the classes of topological spaces axiomatized by (AT_1), (DS).

DEFINITION 5. A T_1-*space* is a topological space such that all its one-element subsets are closed.

[1] In this paper relation R is pseudo-transitive iff R^+ is transitive. In some papers this property is called weakly transitive (cf. [6, 3])

As we mentioned in introduction there is an axiom that defines T_1 spaces in [9], but it has a little bit different form. And due to the next lemma they are equivalent on topological spaces.

LEMMA 6. *Let $\mathfrak{X}$ be a topological space then $\mathfrak{X} \models AT_1$ iff $\mathfrak{X}$ is a T_1-space.*

Proof. ($\Rightarrow$) Ad absurdum. Suppose there exists $x \in \mathfrak{X}$ such that $\{x\}$ is not closed. Hence $\mathfrak{X} - \{x\} \neq \mathbf{I}(\mathfrak{X} - \{x\})$. Let $U = \mathfrak{X} - \{x\}$. There exists

(3.1) $y \in U - \mathbf{I}U$

We take a valuation θ in $\mathfrak{X}$ such that $\theta(p) = U$. Then $x \models [\neq]p$ but $x \models AT_1$; hence $x \models [\neq]\Box p$. Since $y \neq x$, we have $y \models \Box p$, which means that y together with some its neighborhood is in U. This contradicts to (3.1).

($\Leftarrow$) Assume that $\mathfrak{X}$ is a T_1-space. Let $\mathfrak{X}, \theta, x \models [\neq]p$ then $\theta(p) \supseteq \mathfrak{X} - \{x\}$. We need to prove that $x \models [\neq]\Box p$. It means that for all $y \in \mathfrak{X} - \{x\}$ $y \models \Box p$. Take any $y \in \mathfrak{X} - \{x\}$. Since $\mathfrak{X} - \{x\}$ is open, there exists an open $U \ni y$ and $U \subseteq \mathfrak{X} - \{x\}$. So $U \models p$, then $y \models \Box p$, hence $x \models [\neq]\Box p$.

■

DEFINITION 7. Let $\mathfrak{X}$ be a topological space. A point $x \in \mathfrak{X}$ is called isolated, if $\{x\}$ is open. $\mathfrak{X}$ is called *dense-in-itself*, if it has no isolated points.

LEMMA 8. *Let $\mathfrak{X}$ be a topological space then $\mathfrak{X} \models DS$ iff $\mathfrak{X}$ is dense-in-itself.*

Proof. ($\Rightarrow$) Ad absurdum. Assume that $\mathfrak{X}$ is not dense-in-itself and $x \in \mathfrak{X}$ is isolated.

Let us take a valuation θ in $\mathfrak{X}$ such that $\theta(p) = \mathfrak{X} - \{x\}$; then $x \models [\neq]p$. Since $\{x\}$ is open and $x \models \neg p$, it follows that $x \models \Box\neg p$ or equivalently, $x \models \neg\Diamond p$. This contradicts to the axiom (DS).

($\Leftarrow$) Assume that $\mathfrak{X}$ is dense-in-itself and $(\mathfrak{X}, \theta), x \models [\neq]p$; then there are two cases:
(i) $\theta(p) = \mathfrak{X}$, in this case it is obvious that $(\mathfrak{X}, \theta), x \models \Diamond p$;
(ii) $\theta(p) = \mathfrak{X} - \{x\}$, then $(\mathfrak{X}, \theta), x \models \Diamond p$ since $x \in \mathbf{C}(\mathfrak{X} - \{x\}) = \mathfrak{X}$. ■

4 Kripke frames and models.

Kripke frames and models are well-known basic notions of modal logic (cf. [4] and [5]).

DEFINITION 9. A *Kripke frame* is a tuple $F = (W, R_1, \ldots R_n)$ such that

(i) W is a non-empty set,
(ii) R_i for $i = 1 \ldots n$ are binary relations on W.

In this paper however, we consider Kripke frames with one or two relations only. The first is denoted as R and the second (if it is present) — as R_D.

DEFINITION 10. A *Kripke model* is a pair $\mathcal{M} = (F, \theta)$, where F is a frame and θ is a valuation (a function from the set of all proposition letters to the set of all subsets of W).

$\mathcal{M}, x \models A$ denotes that formula A is true in model $\mathcal{M}$ at point x; $\mathcal{M} \models A$ denotes that A is true at all points of model $\mathcal{M}$; $F \models A$ denotes that $(F, \theta), x \models A$ for all valuations θ and all points $x \in W$; $F, x \models A$ denotes that $(F, \theta), x \models A$ for all valuations θ. For a subset $U \subseteq W$ $\mathcal{M}, U \models A$ denotes that for any $x \in U$ $(\mathcal{M}, x \models A)$.

DEFINITION 11. The *logic* of a class of frames $\mathcal{F}$ (in notation $L(\mathcal{F})$) is the set of all formulas that are valid in all frames from $\mathcal{T}$. For a single frame F, $L(F)$ stands for $L(\{F\})$.

DEFINITION 12. A frame F is called *a* Λ*-frame* for a modal logic Λ, if $\Lambda \subseteq L(F)$.

DEFINITION 13. A *p-morphism* from a Kripke frame $F = (W, R, R_D)$ onto a Kripke frame $F' = (W', R', R'_D)$ is a map $f : W \to W'$ satisfying the following conditions:

1. f is surjective;
2. $\forall x \forall y (xRy \Rightarrow f(x)R'f(y))$ and the same for R_D and R'_D;
3. $\forall x \forall z (f(x)R'z \Rightarrow \exists y (xRy \& f(y) = z))$ and the same for R_D and R'_D.

In notation: $f : F \twoheadrightarrow F'$.

DEFINITION 14. By cone F^x we will understand the frame

$$(W^x, R\restriction_{W^x}, R_D\restriction_{W^x}),$$

where $W^x = (R \cup R_D)^+(x)$. If for some x $F = F^x$ then F called *rooted*.

The following two lemmas are well-known (cf. [4] and [5]).

LEMMA 15. *Let* $F = (W, \ldots)$ *be a Kripke frame, then*

$$L(F) = \bigcap \{L(F^x \mid x \in W\}.$$

LEMMA 16. *(p-morphism Lemma)* $f : F \twoheadrightarrow F'$ *implies* $L(F) \subseteq L(F')$.

In this paper we consider only **S4D**-frames. The axioms $B_D, 4^-_D, D_\Box, T_\Box, 4_\Box$ put constraints on relations R and R_D. So from now on we assume that all Kripke frames satisfy the following conditions:

- R is reflexive (axiom $T_\Box$) and transitive ($4_\Box$),
- R_D is symmetric (B_D)
- R_D is pseudo-transitive (4^-_D),
- $R \subseteq R_D \cup Id_W$ $(D_\Box)$.

Note that we can further assume that $R_D \cup Id = W \times W$, because according to Lemma 15 we can consider only generated subframes.

Now let us see what formulas AT_1 and DS mean in a Kripke frame.

Let $F = (W, R, R_D)$ be a **S4D**-frame, then $Top(F) = Top(W, R)$ denotes the topological space on the set W with the topology $\{R(V) \mid V \subseteq W\}$. For formulas with the difference modality, validity in F and $Top(F)$ may not be equivalent. This is because R_D may not be the real inequality relation.

DEFINITION 17. Let R be a transitive reflexive relation on W. Then $x \in W$ is called *R-minimal* (respectively R-maximal), if for any y, yRx (respectively xRy) implies $x = y$.

DEFINITION 18. Let $F = (W, R, R_D)$ be an **S4D**-frame; we say that F is a *T_1-frame* (or has the *T_1–property*), if all R_D-irreflexive points are R-minimal.

LEMMA 19. *Let $F = (W, R, R_D)$ be* **S4D***-frame. Then $F \models AT_1$ iff F is a T_1-frame.*

Proof. ($\Rightarrow$) Suppose $F \models AT_1$ and there exists an R-non-minimal and R_D-irreflexive point in F. To be more specific, let x and y be two different points such that $\neg xR_Dx$ and yRx. Take a valuation θ such that $\theta(p) = W - \{x\}$. Then $x \models [\neq]p$ and $x \models \neg p$, thus $y \models \Diamond\neg p$. Since $x \neq y$ and yRx, we have xR_Dy, $x \models \langle\neq\rangle\Diamond\neg p$. Hence $x \models \neg[\neq]\Box p$. This contradicts $x \models AT_1$.

($\Leftarrow$) Assume that F is a T_1-frame and for some valuation for F we have $x \models [\neq]p$. Let us show that $x \models [\neq]\Box p$. As we mentioned above, generated subframes preserve validity, so we can assume that $F = F^x$ hence $R_D(x) \cup \{x\} = W$. There are two possibilities:

1) xR_Dx. Then $y \models p$ for any $y \in W$, hence for all $y \in W$ we have $y \models \Box p$; so $x \models [\neq]\Box p$.

2) $\neg xR_Dx$. Then $y \models p$ for every $y \neq x$. By assumption, $y \neq x$, yRz implies $z \neq x$, hence $z \models p$. So for any $y \neq x$ $y \models \Box p$; hence $x \models [\neq]\Box p$. ■

DEFINITION 20. Let $F = (W, R, R_D)$ be an **S4D**-frame; we say that F is a *DS-frame*, if every R_D-irreflexive point has an R-successor (called just a *successor* further on).

LEMMA 21. *Let $F = (W, R, R_D)$ be an* **S4D***-frame. Then $F \models DS$ iff F is a DS-frame.*

Proof. ($\Rightarrow$) Suppose $F \models DS$ and there exists an R_D-irreflexive point x without successors. We take a valuation θ such that $\theta(p) = W - \{x\}$; then $x \models [\neq]p$ but $x \not\models \Diamond p$. This contradicts $F \models DS$.

($\Leftarrow$) Suppose that every R_D-irreflexive point in F has a successor. Let us prove that for any $x \in W$ $x \models DS$. Suppose $(F, \theta), x \models [\neq]p$, then there are two cases: (i) x is R_D-reflexive; then $\theta(p) = W$, and so $x \models \Diamond p$ since R is reflexive; (ii) x is R_D-irreflexive, then $\theta(p) \supseteq W - \{x\}$, and by our assumption, there exists $y \neq x$ such that xRy; hence $y \models p$ and $x \models \Diamond p$. ■

5 Kripke completeness and finite model property.

All our axioms are Sahlqvist formulas. So we easily obtain Kripke completeness for logics **S4D**, **S4DS**, $\mathbf{S4DT_1S}$.

Following the common way of proving f.m.p. we use filtration (cf. [5] and [4]).

DEFINITION 22. Let $M = (F, \theta)$ be a Kripke model, where $F = (W, R, R_D)$ is a Kripke frame and Ψ is a set of formulas closed under subformulas. Let $\approx_\Psi$ be the equivalence relation on the elements of W defined as follows:

$$w \approx_\Psi v \text{ iff for all } \phi \text{ in } \Psi\text{:}(M, w \models \psi \text{ iff } M, v \models \phi).$$

By $[w]$ we denote the equivalence class of w. Suppose $M' = (F', \theta')$ and $F' = (W', R', R'_D)$ such that

1. $W' = W_\Psi = \{[w] \mid w \in W\}$.
2. If wRv then $[w]R'[v]$ (and similarly for R_D),
3. If $[w]R'[v]$ then for all $\Box\phi \in \Psi$; $M, w \models \Box\phi$ only if $M, v \models \phi$ (and similarly for R_D and $[\neq]$).
4. $\theta'(p) = \{[w] \mid M, w \models p\}$, for all atomic symbols p in Ψ.

Then M' is called a filtration of M through Ψ.

LEMMA 23. (Filtration Lemma) *Let M' be a filtration of M through Ψ, then for any $x \in M_1$ and for any $\psi \in \Psi$*

$$M, w \models \psi \iff M', [w] \models \psi.$$

LEMMA 24. *Let F_1 be an* **S4D***-frame, $M_1 = (F_1, \theta_1)$ a model, Ψ a finite set of formulas closed under subformulas. Then there exists a filtration M_2 of M_1 through Ψ, such that $M_2 = (F_2, \theta_2)$ and F_2 is an* **S4D***-frame.*

Proof. Let $M' = (W_\Psi, R', R'_D, \theta')$ be the minimal filtration of M. The minimal filtration is well-known (cf. [5] or [12]). Briefly, $[x]R'[y]$ iff there exist $x' \in [x]$ and $y' \in [y]$ such that $x'R_1y'$, and the same for R'_D and R_{D1}.

Let R_2 be the transitive closure of R':

$$R_2 = R'^* = \bigcup_{n \geq 1} R'^n;$$

and let R_{D2} be the pseudo-transitive closure of R'_D:

$$R_{D2} = R'^*_D - (Id - R'_D).$$

Note that the only difference between the pseudo-transitive and the transitive closure is that the irreflexive points remain irreflexive.

One can easily see that the reflexivity of R' is inherited by R_2, and the reflexivity of R' follows from the reflexivity of R_1. In the same way the symmetry of R_{D1} implies the symmetry of R_{D2}. The transitivity of R_2 and the pseudo-transitivity of R_{D2} are provided by construction. Next, we can easily show that $R' \subseteq R'_D \cup Id$; hence $R_2 \subseteq R_{D2} \cup Id$ holds.

To complete the proof, we have to show that the relations R_2 and R_{D2} satisfy the definition of filtration. Since Filtration Lemma for the minimal filtration and its transitive closure in transitive logics are well-known (cf. [4]), we will only check R_{D2}.

1. For arbitrary $w, v \in W_1$ assume $vR_{D1}w$, let us prove that $[v]R_{D2}[w]$. If $[v] \neq [w]$, the proof is the same as for the transitive closure. So assume $[w] = [v]$; then $[v]R'_D[w]$, and so $[v]R_{D2}[w]$.

2. Assume $[v]R_{D2}[w]$, let us prove that for all $[\neq]\psi \in \Psi$; $M_1, v \models [\neq]\psi$ only if $M_1, w \models \psi$. If $[v] \neq [w]$, then the proof is the same as for the transitive closure. If $[w] = [v]$, then from $[v]R_{D2}[v]$ follows $[v]R'_D[v]$. But R'_D was already filtration.

So we obtain a filtration that reduces M_1 to a finite model over an **S4D**-frame. ■

THEOREM 25. *Let L be one of the logics:* **S4D**, **S4DS**, $\mathbf{S4DT_1S}$. *Then L has the finite model property.*

Proof. Assume that A is a formula such that A is not in L. Hence, A is refuted in some generated submodel $M_1 = (W_1, R_1, R_{D1}, \theta)$ of the canonical model of logic L. Note that since M_1 is a generated submodel, $R_{D1} \cup Id_{W_1}$ is the universal relation.

Let Ψ be the set of all subformulas of formula A. By Lemma 24 there exists model $M_2 = (F_2, \theta_2)$ such that F_2 is a **S4D**-frame and M_2 is a filtration of M_1 through Ψ.

Since M_2 is a filtration, A is refuted in M_2. So it remains to prove (if needed) the T_1–property and the DS-property for F_2.

Let us prove that axiom AT_1 is valid in frame F_2. By Lemma 19, it is sufficient to prove that for any η such that $\neg\eta R_{D2}\eta$ there does not exist ψ such that $\psi \neq \eta$ and $\psi R_2 \eta$.

Assume the contrary, i.e. there exists a point $\psi \neq \eta$ such that $\psi R_2 \eta$. Then consider their inverse images: $[x] = \eta$ & $[y] = \psi$. By construction of R_2 we obtain

$$y \approx_\Psi y_0 R_1 z_0 \approx_\Psi y_1 \ldots y_k R_1 z_k \approx_\Psi x$$

Since $y \not\approx_\Psi x$, we can take maximal l such that $y_l \not\approx_\Psi z_l$. By transitivity of $\approx_\Psi$ we conclude that $z_l \approx_\Psi x$ and $[z_l] = [x] = \eta$. Assume that $z_l \neq x$, since $R_{D1} \cup Id$ is the universal relation $z_l R_{D1} x$, hence $[z_l]R_{D2}[x]$; which contradicts $\neg\eta R_{D2}\eta$.

So $y_l \neq x$ & $y_l R_1 x (= z_l)$, at the same time reflexivity is preserved under filtration; hence $\neg x R_{D1} x$. So we came to a contradiction, because the generated subframe $F_1 = (W_1, R_1, R_{D1})$ of the canonical model has the T_1–property.

Now let $[x]$ be an R_{D2}-irreflexive point. Hence, x is also an R_{D1}-irreflexive point, so for some y, xR_1y and $xR_{D1}y$, then $[x]R_2[y]$ and $[x]R_{D2}[y]$, so $[y] \neq [x]$; hence $[x]$ is not maximal. ■

6 Topological completeness.

Let us define analogue of p-morphism for maps from topological space onto finite **S4D**-frame.

DEFINITION 26. Let $\mathfrak{X}$ be a topological space and let $F = (W, R, R_D)$ be a finite Kripke frame. A function $f : \mathfrak{X} \to F$ is called a *cd-p-morphism*, if it is surjective and satisfies the following two conditions

(6.1) $\mathbf{C}f^{-1}(w) = f^{-1}(R^{-1}(w))$,

(6.2) $R_D^{-1}(f^{-1}(w)) = f^{-1}(R_D^{-1}(w))$,

where $R_D =$ " $\neq$ " in $\mathfrak{X}$ (in particular $R_D^{-1}(\{x\}) = \mathfrak{X} - \{x\}$). In notation $f : \mathfrak{X} \overset{cd}{\twoheadrightarrow} F$.

Note that since f is surjective, (6.2) is equivalent to the following: if w is R_D-irreflexive then $f^{-1}(w)$ is one-element.

LEMMA 27. *If F is a finite Kripke frame, $\mathfrak{X}$ is a topological space and $f : \mathfrak{X} \overset{cd}{\twoheadrightarrow} F$ then $L_D(\mathfrak{X}) \subseteq L(F)$.*

Proof. Note that f is a *cd-p*-morphism and $\mathbf{C}$ distributes over finite[2] unions. So for $U \subseteq W$ we have

$$\begin{aligned} f^{-1}(R^{-1}(U)) &= f^{-1}(\bigcup_{w\in U} R^{-1}(w)) = \bigcup_{w\in U} f^{-1}(R^{-1}(w)) \overset{(6.1)}{=} \\ &= \bigcup_{w\in U} \mathbf{C}f^{-1}(w) = \mathbf{C}f^{-1}(U) \end{aligned} \tag{6.3}$$

In other terms, f is an interior map between topological spaces $\mathfrak{X}$ and $Top(F)$.

Similarly

$$\begin{aligned} f^{-1}(R_D^{-1}(U)) &= f^{-1}(\bigcup_{w\in U} R_D^{-1}(w)) = \bigcup_{w\in U} f^{-1}(R_D^{-1}(w)) \overset{(6.2)}{=} \\ &= \bigcup_{w\in U} R_D^{-1} f^{-1}(w) = R_D^{-1} f^{-1}(U). \end{aligned} \tag{6.4}$$

Now let θ be an arbitrary valuation on the frame F. Take a valuation Θ on $\mathfrak{X}$ such that $\Theta(p) = f^{-1}(\theta(p))$. Then a standard inductive argument shows that for any formula ϕ

(6.5) $\Theta(\phi) = f^{-1}(\theta(\phi))$,

[2] $\mathbf{C}$ does not distribute over infinite unions so finiteness of F is essential.

where $\theta(\phi) = \{v \,|(F,\theta), v \models \phi\}$ and $\Theta(\phi) = \{x \,|(\mathfrak{X},\Theta), x \models \phi\}$.

For this proof we rewrite all formulas using $\Diamond$ and $\langle\neq\rangle$ (rather then $\Box$ or $[\neq]$).

There are only two nontrivial cases:

i) $\phi \equiv \Diamond\psi$. Then

$$f^{-1}(\theta(\Diamond\psi)) = f^{-1}(R^{-1}(\theta(\psi))) \overset{(6.3)}{=} \mathbf{C}f^{-1}(\theta(\psi)) \overset{induction}{=} \mathbf{C}\Theta(\psi) = \Theta(\Diamond\psi).$$

ii) $\phi \equiv \langle\neq\rangle\psi$. Then

$$\begin{aligned} f^{-1}(\theta(\langle\neq\rangle\psi)) = f^{-1}(R_D^{-1}(\theta(\psi))) \overset{(6.4)}{=} R_D^{-1}f^{-1}(\theta(\psi)) \overset{induction}{=} & \ R_D^{-1}\Theta(\psi) \\ = & \ \Theta(\langle\neq\rangle\psi). \end{aligned}$$

Now if $\phi \notin \mathbf{L}(F)$, there exists a valuation θ such that $\theta(\phi) \neq W$. By (6.5) $\Theta(\phi) = f^{-1}(\theta(\phi))$, and so $\Theta(\phi) \neq \mathfrak{X}$ since f is subjective. Thus $\phi \notin \mathbf{L}(\mathfrak{X})$. ■

The following proposition uses ideas from [14, 6]

PROPOSITION 28. *Let $F = (W, R, R_D)$ be a* **S4D**-*Kripke frame, $R_D \cup Id_W = W \times W$. There exists* **S4D**-*Kripke frame $F' = (W', R', R'_D)$, such that $F' \twoheadrightarrow F$ and $x'R'_Dy'$ iff $x' \neq y'$.*

Proof. Let us put $W^0 = \{x \in W | xR_Dx\}$ and $W^\times = W - W^0$. Then

(6.6) $W' = W^\times \cup W^0 \times \{0, 1\}$

Let us define the function $f : F' \to F$ such that

$$f(x') = \begin{cases} x, & \text{if } x' = x \in W^\times; \\ x, & \text{if } x' = (x, i); \end{cases}$$

and the relation R':

$$x'R'y' \iff f(x')Rf(y')$$

Let us prove that f is a p-morphism.

1. Obviously f is surjective.

2. Assume that $x'R'y'$ then by definition of R' $f(x')Rf(y')$. Assume that $x'R'_Dy'$ (or $x' \neq y'$), $f(x') = x$ and $f(y') = y$. If $x \neq y$ then xR_Dy. If $x = y$ then $x' = (x, 0)$ and $y' = (x, 1)$ (or vice verse); using (6.6) we conclude that $xR_Dx = y$.

3. Assume that $f(x')Ry$. If $y \in W^0$ then $y' = (y, 0)$ or $y' = y$ otherwise. Easy to see that $f(y') = y$ and $x'R'y'$. Assume that $f(x')R_Dy$. Case when $f(x') \neq y$ is obvious so let $f(x') = y$. It means that $y \in W^0$ and $x' = (y, i)$. So we put $y' = (y, (i + 1) \bmod 2)$ and this will do.

■

COROLLARY 29. *Let $\mathcal{C}$ be the class* **S4D***-frames of the form* $F = (W, R, \neq)$ *then* **S4D** *is complete with respect to* $\mathcal{C}$.

It is easy to show that for any **S4D**-frame $F = (W, R, \neq)$

$$Top(F) \xrightarrow{cd} F$$

but we can prove a stronger statement:

LEMMA 30. *Let* (F, θ) *be a Kripke model then for any formula* A *and* $x \in W$

$$F, \theta, x \models A \iff Top(F), \theta, x \models A$$

Proof. By induction on the complexity of A. The only case that is not trivial or classical is when $A = [\neq]B$.

$$F, \theta, x \models [\neq]B \iff \forall y\, (y \neq x \Rightarrow F, \theta, y \models B)$$

but by induction it holds iff

$$\forall y\, (y \neq x \Rightarrow Top(F), \theta, y \models B) \iff Top(F), \theta, x \models [\neq]B$$

■

THEOREM 31. **S4D** *is the D-logic of all topological spaces.*

Proof. Let A be a formula that is not in **S4D**. Then by Corollary 29 there exists a Kripke frame $F = (W, R, \neq)$ such that $F \not\models A$. By Lemma 30 we obtain $Top(F) \not\models A$. ■

PROPOSITION 32. *Let* $F = (W, R, \neq)$ *be a DS-frame, then* $Top(F)$ *is a dense-in-itself topological space.*

Proof. In $Top(F)$ the least open neighborhood of point x is $R(x)$. Since F is a DS-frame, $R(x) - \{x\} \neq \varnothing$; hence $Top(F)$ is dense-in-itself. ■

THEOREM 33. **S4DS** *is logic of all dense-in-itself topological spaces.*

Proof. From Theorem 25 we know that **S4DS** is complete with respect to all finite DS-frames. Now we can apply Proposition 32 and Lemma 30. ■

If a logic contains the axiom (AT_1) then we cannot use the above methods. Indeed if $F = (W, R, \neq)$ and $F \models AT_1$, then $R = Id_W$. The logic of such frames will be the logic of isolated points. So we need to find more sophisticated ways.

Recall a few definitions.

DEFINITION 34. A non-empty topological space $\mathfrak{X}$ is called *zero-dimensional* if clopen sets constitute its open base.

DEFINITION 35. A pair (X, ρ) called metric space if X is a set and ρ is a function from $X \times X$ onto $\mathbb{R}$, such that $\rho(x,y) \geq 0$, $\rho(x,y) = 0$ iff $x = y$, $\rho(x,y) = \rho(y,x)$, and $\rho(x,y) + \rho(y,z) \geq \rho(x,z)$.

On metric space can be defined natural topology based on open balls: $\{y \mid \rho(x,y) < r\}$.

THEOREM 36. $\mathbf{S4DT_1S}$ *is compete with respect to any zero-dimensional dense-in-itself metric space.*

Proof. Let $\mathfrak{X}$ be a zero-dimensional dense-in-itself metric space and ρ be the distance in it; $O(x,r)$ denotes the open ball $\{y \in \mathfrak{X} | \rho(x,y) < r\}$.

We know from Theorem 25 that $\mathbf{S4DT_1S}$ is complete with respect to all finite DS-T_1-frames. If we prove that for an arbitrary finite DS-T_1-frame $F = (W, R, R_D)$,

$$\mathfrak{X} \xrightarrow{cd} F$$

then we prove the theorem.

We use induction on the size of F. Consider three cases.

Case I. $W = R(w_0)$, $R_D = W \times W$ for some w_0. Since $\mathbf{S4}$ is complete with respect to $\mathfrak{X}$(cf. [1]) and $F^- = (W, R)$ is an $\mathbf{S4}$-frame, there exists a continuous function $f : \mathfrak{X} \to Top(F^-)$. It is easy to check that $f : \mathfrak{X} \xrightarrow{cd} F$.

Case II. $W = R(w_0)$, $R_D = W \times W - (w_0, w_0)$. Since W is finite let us enumerate all points in W starting with w_0: $W = \{w_0, w_1, w_2, \ldots w_n\}$. Any generated subframe F^{w_i} for $i > 0$ satisfies case I.

Take an arbitrary point x_0 and clopen sets $Y_0, Y_1, \ldots$ such that

$$\{x_0\} \subset \ldots \subset Y_n \subset \ldots \subset Y_1 \subset Y_0 = \mathfrak{X}$$

and

$$Y_n \subseteq O(x_0, \frac{1}{n})$$

for every $n > 0$. We can do it because $\mathfrak{X}$ is zero-dimensional (cf. [13]).

Since $Y_n \subseteq O(x_0, \frac{1}{n})$, it follows that

$$\bigcap_n Y_n = \{x_0\}$$

and further we obtain

$$\mathfrak{X} - \{x_0\} = \mathfrak{X} - \bigcap_n Y_n = \bigsqcup_n \mathfrak{X}_n,$$

where $\mathfrak{X}_n = Y_n - Y_{n+1}$. Each set $\mathfrak{X}_n$ is open, metric, dense-in-itself, and zero-dimensional.

For any open neighborhood U of x_0 there exists n such that $O(x_0, \frac{1}{n}) \subset U$, it follows that $Y_n \subset U$, hence for all $i \geq n$ $\mathfrak{X}_i \subset U$.

So, by induction, for any $j > 0$ there exists

$$f_j : \mathfrak{X}_j \xrightarrow{cd} F^{w_k}, \text{ where } (k-1) \equiv j \pmod{n}$$

Now consider

$$f(x) = \begin{cases} w_0, & \text{if } x = x_0; \\ f_j(x), & \text{if } x \in \mathfrak{X}_j. \end{cases}$$

Let us prove that $f : \mathfrak{X} \overset{cd}{\twoheadrightarrow} F$.

First, we note that f is surjective.

Second, we check (6.1). Assume that $y \in \mathfrak{X}_j$ then:

$$y \in \mathbf{C}f^{-1}(w) \Longrightarrow y \in \mathbf{C}f_j^{-1}(w) = f_j^{-1}(R^{-1}(w)) \subseteq f^{-1}(R^{-1}(w));$$

and the other way around:

$$y \in f^{-1}(R^{-1}(w)) \Longrightarrow f_j^{-1}(R^{-1}(w)) = y \in \mathbf{C}f_j^{-1}(w) \subseteq \mathbf{C}f^{-1}(w).$$

Now assume that $y = x_0$. For any $w \in W$, $w_0 \in R^{-1}(w)$; hence

$$x_0 \in f^{-1}(R^{-1}(w)).$$

On the other hand, for some i, $w = w_i$ and for any open neighborhood of x_0, there exists m such that $(i-1) \equiv m \pmod n$ and $U \supset \mathfrak{X}_m \supset f_m^{-1}(w_i)$. In other words, x_0 is a limit point for $f^{-1}(w_i)$, hence $x_0 \in \mathbf{C}f^{-1}(w_i)$.

Third, we check (6.2). Since $f^{-1}(w_0)$ is a one-element set, (6.2) holds.

Case III. Everything else. Let us take all R-minimal R-clusters of F and from each one of them we choose an arbitrary point. So we get the following set: $\{v_1, v_2, \ldots, v_k\}$. Standard unravelling arguments show that

$$F' = F^{v_1} \sqcup F^{v_2} \sqcup \ldots \sqcup F^{v_n} \twoheadrightarrow F$$

So we need to show that $\mathfrak{X} \overset{cd}{\twoheadrightarrow} F'$.

Since F is a $\mathbf{S4DT_1S}$-frame, each F^{v_i} satisfies case I or case II.

Since $\mathfrak{X}$ is zero-dimensional, we can present $\mathfrak{X}$ as the disjunctive union of clopen subsets:

$$\mathfrak{X} = \mathfrak{X}_1 \sqcup \ldots \sqcup \mathfrak{X}_{k-1} \sqcup \mathfrak{X}_k.$$

By induction we have

$$f_1 : \mathfrak{X}_1 \overset{cd}{\twoheadrightarrow} F^{v_1},$$

$$\ldots$$

$$f_{k-1} : \mathfrak{X}_{k-1} \overset{cd}{\twoheadrightarrow} F^{v_{k-1}},$$

$$f_k : \mathfrak{X}_k \overset{cd}{\twoheadrightarrow} F^{v_k}$$

It is easy to show that $f = f_1 \sqcup \ldots \sqcup f_k$ (if $x \in \mathfrak{X}_i$ then $f(x) = f_i(x)$) is a cd-p-morphism. ■

The immediate and obvious corollary of this theorem is that $\mathbf{S4DT_1S}$ is complete with respect to all dense-in-themselves T_1 spaces.

7 Conclusions and open problems.

The language with difference modality shows much more expressive power than the basic topological language, and even more than the basic language with universal modality. We can express density-in-itself, T_1, and connectedness in it. Moreover the axiom

$$(AE_1) \qquad [\neq]p \wedge \neg p \wedge \Box(p \rightarrow \Box q \vee \Box\neg q) \rightarrow \Box(p \rightarrow q) \vee \Box(p \rightarrow \neg q)$$

distinguishes $\mathbb{R}$ from $\mathbb{R}^2$ (cf. [7]). It was proved that logic $\mathbf{S4DT_1S}$ + (AE_1) + "connectedness" is complete with respect to $\mathbb{R}^n$, $n \geq 2$ (the full proof is to be published). We still do not know the D-logic of $\mathbb{R}$ and whether $\mathbf{S4D} + (AT_1)$ is complete with respect to all T_1 spaces.

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Andrey Kudinov

Department of Mathematical Logic and Theory of Algorithms
Faculty of Mechanics and Mathematics Moscow State University
Leninskie gory, Moscow, 119992 RUSSIA

kudinov.andrey@gmail.com