# Regarding Overlaps in 'Topologic'

Bernhard Heinemann

ABSTRACT. We study a particular extension of Moss and Parikh's logic of knowledge and effort with regard to covering properties of spaces of sets. As is known, one can only specify properties of points close to the actual one by means of formulas of that logic, and this may be viewed as a deficiency. In this paper, we introduce a new modality describing overlaps, which is intended to rectify that shortcoming. By cooperating with knowledge the overlap operator in fact allows us to access remote points as well. This could be interesting in connection with a modal treatment of compactness. The outcome of the paper is a soundness and completeness theorem for the logic of set spaces including the overlap operator, the decidability of this logic, some interesting connections with the usual logic of knowledge, and a hybrid version of the logic of overlapping which appears in a sense more natural than previous hybridizations of the initial system.

## 1 Introduction

The starting point to this paper is Moss and Parikh's approach to reasoning about knowledge of some agent; see [3]. The bi-modal system invented there facilitates a fairly abstract description of procedures gaining knowledge. As a by-product, a certain *topological component* of knowledge was revealed. In fact, since knowledge is represented by the collection of all *knowledge states* of the agent, knowledge acquisition appears as a *shrinking procedure* regarding this space of sets. In this way, some particular concepts from topology like *closeness* or *neighbourhood* turn up naturally in connection with knowledge.

Moss and Parikh called one of their system $\mathsf{topologic}$, and we adopt this naming here. We recall some features of the language underlying $\mathsf{topologic}$. As it has just been indicated, formulas may contain two one-place operators: a modality $\mathsf{K}$ representing knowledge of the agent and another one, $\Box$, representing (computational) effort. The domains for evaluating formulas are triples $(X, \mathcal{O}, V)$ which we call *set spaces,* consisting of a non-empty set $X$ of states, a set $\mathcal{O}$ of subsets of $X$ representing the knowledge states of the agent, and a valuation $V$ determining the states where the atomic propositions are true. The operator $\mathsf{K}$ then quantifies across any knowledge state $U \in \mathcal{O}$, whereas $\Box$ quantifies 'downward' across $\mathcal{O}$. That is, more knowledge, i.e., a closer proximity to states of 'complete' knowledge, can be achieved by *decreasing* (with respect to the set inclusion relation), and just this is modeled by $\Box$.

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Some classes of set spaces including the ordinary topological ones, could be characterized by means of topologic subsequently; cf [5, 6, 10, 20]. However, more expressive power is needed to specify less basic properties of scenarios in which topology plays a part.

To this end, both a *temporal* and a *hybrid* version of topologic were introduced in [11] and [12], respectively, and further developed later on. Hybrid topologic, in particular, turned out to be quite a useful system; cf, eg, [15].

In this paper we nevertheless go 'back to the roots'. That is, we do neither change the original operators of topologic (as in the temporal setting) nor go beyond the purely modal framework (as it was done by using hybrid means of expression). Instead, we add a further 'spatial' modality, called the *overlap operator,* to the basic system and investigate the accompanying modal logic. (This is at least true up to the last but one section, where we hybridize that logic though.)

We now motivate the introduction of the new modality. Due to the semantics of topologic, formulas can only speak about properties of points close to the actual one; in other words, by evaluating a formula we cannot 'leave' the neighbourhood in question (see Definition 2 below). The overlap operator, however, will allow us to do exactly this. Thus we will have tools to hand for quantifying not only 'downward' but also 'diagonally' across $\mathcal{O}$. This may be viewed as a very first step towards a modal treatment of *coverings* of spaces. (Unfortunately, we have to confine ourselves to this first step here. The long-term goal is a treatment of *compactness,* actually, but we are still far away from this.)

It should be remarked that the investigation into overlapping is new for topologic, but not for modal logic in general. Overlapping is a very natural spatial concept and occurs, therefore, in many papers on modal logics of space. Standing in for all of them we mention two papers from different areas of application, [18] and [19].

The subsequent technical part of this paper is organized as follows. In the next section we integrate the overlap operator into the language of topologic. We give some examples of valid formulas of the extended language there, too, and we touch on the question of expressive power. Afterwards, in Section 3, we present an axiomatization of the new logic and prove its soundness and completeness with respect to the class of all set spaces. We also show in this section that the set of all theorems of the logic is decidable. In Section 4, we take a small step further towards coverings of set spaces by considering the reflexive and transitive closure of the relation which results from joining overlapping with moving inside sets. We prove (only) Kripke completeness and decidability in this case, and we comment on an interesting connection with the common logic of knowledge. (See the textbooks [4] or [17] for an exhaustive treatment regarding this.) We then discuss a hybrid extension of the logic of overlapping. This system once again demonstrates the power of names in the context of topologic. Moreover, it is also more natural (in a sense that will be made precise later on) than other approaches to this topic in the past (cf [13, 14]). Finally, we give a brief summary of the paper and point to further aspects and future research.

Apart from the appendix, which relies on several concepts and results from [3] and [14], respectively, the present paper is, taken by and large, self-contained.

## 2 Incorporating the overlap operator

In this section, the language of $\mathsf{topologic}$ is extended by a unary operator, $\mathsf{O}$, describing overlaps of the actual neighbourhood situation. We first define the new syntax and semantics, and give some comments related to expressiveness afterwards.

Let PROP $= \{A, B, \dots\}$ be a denumerable set of symbols called *proposition letters*. We define the set WFF of *well-formed formulas over* PROP by the rule

$$\alpha \ ::= \ A \mid \neg\alpha \mid \alpha \wedge \beta \mid \mathsf{K}\alpha \mid \Box\alpha \mid \mathsf{O}\alpha.$$

The operators $\mathsf{K}$ and $\Box$ represent *knowledge* and *effort,* respectively, as it is common for $\mathsf{topologic}$. $\mathsf{O}$ is called the *overlap operator*. The missing boolean connectives $\top, \bot, \vee, \rightarrow, \leftrightarrow$ are treated as abbreviations, as needed. The duals of $\mathsf{K}$, $\Box$ and $\mathsf{O}$ are denoted $\mathsf{L}$, $\Diamond$ and $\mathsf{P}$, respectively.

We now turn to semantics. First, we define the domains for interpreting formulas. For a given set $X$, we let $\mathcal{P}(X)$ denote the powerset of $X$.

DEFINITION 1 (Set frames and spaces).

1. Let $X \neq \emptyset$ be a set and $\mathcal{O} \subseteq \mathcal{P}(X)$ a set of subsets of $X$. Then $\mathcal{F} := (X, \mathcal{O})$ is called a *set frame*.

2. Let $\mathcal{F} = (X, \mathcal{O})$ be a set frame. The *set* $\mathcal{N}_{\mathcal{F}}$ *of neighbourhood situations* of $\mathcal{F}$ is defined by

$$\mathcal{N}_{\mathcal{F}} := \{(x, U) \mid x \in U \text{ and } U \in \mathcal{O}\}.$$

   (Mostly, neighbourhood situations are written without brackets later on.)

3. Let $\mathcal{F}$ be as above. A mapping $V :$ PROP $\longrightarrow \mathcal{P}(X)$ is called an *$\mathcal{F}$–valuation.*

4. A *set space* is a triple $\mathcal{M} := (X, \mathcal{O}, V)$, where $\mathcal{F} := (X, \mathcal{O})$ is a set frame and $V$ an $\mathcal{F}$–valuation; $\mathcal{M}$ is then called *based on* $\mathcal{F}$.

The relation of satisfaction between neighbourhood situations and formulas, is now defined with regard to set spaces.

DEFINITION 2 (Satisfaction and validity). Let $\mathcal{M} = (X, \mathcal{O}, V)$ be a set space.

1. Let $x, U$ be a neighbourhood situation of $\mathcal{F} = (X, \mathcal{O})$. Then

$$\begin{array}{lcl} x, U \models_{\mathcal{M}} A & :\Longleftrightarrow & x \in V(A) \\ x, U \models_{\mathcal{M}} \neg\alpha & :\Longleftrightarrow & x, U \not\models_{\mathcal{M}} \alpha \\ x, U \models_{\mathcal{M}} \alpha \wedge \beta & :\Longleftrightarrow & x, U \models_{\mathcal{M}} \alpha \text{ and } x, U \models_{\mathcal{M}} \beta \\ x, U \models_{\mathcal{M}} \mathsf{K}\alpha & :\Longleftrightarrow & \forall\, y \in U : y, U \models_{\mathcal{M}} \alpha \\ x, U \models_{\mathcal{M}} \Box\alpha & :\Longleftrightarrow & \forall\, U' \in \mathcal{O} : (x \in U' \subseteq U \Rightarrow x, U' \models_{\mathcal{M}} \alpha) \\ x, U \models_{\mathcal{M}} \mathsf{O}\alpha & :\Longleftrightarrow & \forall\, U' \in \mathcal{O} : (x \in U' \Rightarrow x, U' \models_{\mathcal{M}} \alpha), \end{array}$$

   where $A \in$ PROP and $\alpha, \beta \in$ WFF. In case $x, U \models_{\mathcal{M}} \alpha$ is true we say that $\alpha$ *holds in* $\mathcal{M}$ *at* the neighbourhood situation $x, U$.

2. A formula $\alpha$ is called *valid in* $\mathcal{M}$ (written '$\mathcal{M} \models \alpha$'), iff it holds in $\mathcal{M}$ at every neighbourhood situation of the frame $\mathcal{M}$ is based on.

The following comments on this definition may be appropriate. The meaning of proposition letters is independent of neighbourhoods by definition, thus 'stable' with respect to $\Box$ and $\mathsf{O}$. This fact is reflected by a special axiom below. – Intuitively, shrinking is a special case of overlapping. This is captured by the exact definition as well. – Note that the semantics of the overlap operator 'dismisses' the set component of the actual neighbourhood situation. Because of that, this operator turns out to be a very good-natured modality; see below.[1] – The general case of overlapping can be obtained by an iterated application of $\mathsf{O}$ and $\mathsf{K}$; see Section 4.

Now, we look for the basic set space validities. The formula schema

$$\mathsf{K}\Box\alpha \rightarrow \Box\mathsf{K}\alpha$$

is the typical validity of topologic. This schema was called the *Cross Axiom* in the paper [3] and plays a key role in the completeness and the decidability proof for that logic. The Cross Axiom describes the interaction between knowledge and effort. It is now natural to ask whether knowledge and overlapping cooperate in a similar manner. The reader can easily convince himself or herself that the answer to this question is 'no'. On the other hand, the effort and the overlap operator are obviously related as displayed by item 1 of the following proposition. Besides that we state that $\mathsf{O}$ satisfies all the S5 laws.

PROPOSITION 3. *Let* $\mathcal{M}$ *be any set space and* $\alpha \in$ WFF *a formula. Then*

*1.* $\mathcal{M} \models \mathsf{O}\alpha \rightarrow \Box\alpha$,

*2.* $\mathcal{M} \models \mathsf{O}\alpha \rightarrow \alpha \wedge \mathsf{O}\mathsf{O}\alpha$,

*3.* $\mathcal{M} \models \alpha \rightarrow \mathsf{O}\mathsf{P}\alpha$.

[1] One of the anonymous referees pointed to the kinship between the overlap operator and the *bi-persistent sentences* of topologic (cf [3], Sec. 3.1). In fact, every formula $\mathsf{O}\alpha$ proves to be bi-persistent. However, the overlap operator is not definable in topologic.

The proof of Proposition 3 is straightforward from Definition 2 and, therefore, omitted.

Finally in this section, we ask for interesting properties that can be expressed with the aid of the new language. We give a simple, but instructive example.

DEFINITION 4. Let $\mathcal{F} = (X, \mathcal{O})$ be a set frame. $\mathcal{F}$ is called *directed,* iff for all $x \in X$ and any $U, U' \in \mathcal{O}$ that overlap at $x$ there exists some $\tilde{U} \in \mathcal{O}$ such that $x \in \tilde{U} \subseteq U \cap U'$.

Directed frames were studied in [20], where it was shown among other things that the class of all directed frames is not finitely axiomatizable in the language of $\mathsf{topologic}$. (As to a treatment of this class by means of hybrid logic and comments on the arising differences, cf [15].) We now show that the new language is strong enough to capture directedness.

PROPOSITION 5. *Let $\mathcal{F} = (X, \mathcal{O})$ be a set frame. Then, $\mathcal{F}$ is directed iff for all set spaces $\mathcal{M}$ based on $\mathcal{F}$ the formula schema*

$$\mathsf{P}\Box\alpha \to \Diamond\alpha$$

*is valid in $\mathcal{M}$.*

**Proof.** We only prove the sufficiency of the condition. To this end, suppose that $\mathcal{F}$ is not directed. Then there are $x \in X$ and $U, U' \in \mathcal{O}$ such that $x \in U \cap U'$, but for all $\tilde{U} \in \mathcal{O}$ containing $x$ we have that $\tilde{U} \not\subseteq U$ or $\tilde{U} \not\subseteq U'$. Now, take a proposition letter $A \in \mathrm{PROP}$ and any $\mathcal{F}$–valuation $V$ satisfying $V(A) = U'$. Let $\mathcal{M}$ be the resulting set space and $\alpha := \mathsf{K}A$. Then $x, U' \models_{\mathcal{M}} \Box\alpha$, hence $x, U \models_{\mathcal{M}} \mathsf{P}\Box\alpha$. But $x, U \not\models_{\mathcal{M}} \Diamond\alpha$ as every $\tilde{U} \in \mathcal{O}$ such that $x \in \tilde{U} \subseteq U$ contains a point $y \notin U'$ which, therefore, falsifies $A$. Thus $\mathcal{M} \not\models \mathsf{P}\Box\alpha \to \Diamond\alpha$ for some formula $\alpha$. ■

Note that the modal correspondent to weak directedness, $\Diamond\Box\alpha \to \Box\Diamond\alpha$, can be derived from $\mathsf{P}\Box\alpha \to \Diamond\alpha$ by using the S5 axioms for $\mathsf{O}$.

We will briefly come back to the topic just touched on in the penultimate section of this paper.

## 3 Completeness and decidability

Our starting point to this section is the system of axioms for $\mathsf{topologic}$ from [3]. We then add the schemata which the new modality is involved in. We prove the soundness and completeness of the resulting list with respect to the class of all set spaces. Later on in this section, we also show that the new logic is decidable.

One schema from the just mentioned axiomatization of $\mathsf{topologic}$ is missing in the following block since it is subsumed by the later Axiom 9.

1. All instances of propositional tautologies.

2. $\mathsf{K}(\alpha \to \beta) \to (\mathsf{K}\alpha \to \mathsf{K}\beta)$

3. $\mathsf{K}\alpha \rightarrow (\alpha \wedge \mathsf{KK}\alpha)$

4. $\mathsf{L}\alpha \rightarrow \mathsf{KL}\alpha$

5. $\Box(\alpha \rightarrow \beta) \rightarrow (\Box\alpha \rightarrow \Box\beta)$

6. $\Box\alpha \rightarrow \alpha \wedge (\Box\Box\alpha)$

7. $\mathsf{K}\Box\alpha \rightarrow \Box\mathsf{K}\alpha$,

where $A \in$ PROP and $\alpha, \beta \in$ WFF. In this way, it is expressed that for every Kripke model validating these axioms

- the accessibility relation $\xrightarrow{\mathsf{K}}$ belonging to the knowledge operator is an equivalence,
- the accessibility relation $\xrightarrow{\Box}$ belonging to the effort operator is reflexive and transitive, and
- the relations for knowledge and effort commute as described by the Cross Axiom (schema 7 above).

The next group of axioms concerns the overlap operator and its interaction with the effort operator $\Box$; cf Proposition 3.

8. $\mathsf{O}(\alpha \rightarrow \beta) \rightarrow (\mathsf{O}\alpha \rightarrow \mathsf{O}\beta)$

9. $(A \rightarrow \mathsf{O}A) \wedge (\mathsf{P}A \rightarrow A)$

10. $\mathsf{O}\alpha \rightarrow \mathsf{OO}\alpha$

11. $\alpha \rightarrow \mathsf{OP}\alpha$

12. $\mathsf{O}\alpha \rightarrow \Box\alpha$,

where $A \in$ PROP and $\alpha, \beta \in$ WFF. – Note that two of the expected schemata are missing in this list, viz $\mathsf{O}\alpha \rightarrow \alpha$ (see Proposition 3.2) and $(A \rightarrow \Box A) \wedge (\Diamond A \rightarrow A)$. This is because all instances of these schemata are derivable by means of the calculus we are going to define right now.

We obtain a logical system from the above list of axioms by adding the standard proof rules of modal logic, i.e., *modus ponens* and *necessitation with respect to each modality*. We call the system $\mathsf{ET}$, indicating *extended* topologic.

DEFINITION 6 (The logic). Let $\mathsf{ET}$ be the smallest set of formulas containing the axiom schemata 1 – 12 and closed under application of the following rule schemata:

$$(\text{MODUS PONENS}) \quad \frac{\alpha \rightarrow \beta, \alpha}{\beta} \qquad (\Delta\text{–NECESSITATION}) \quad \frac{\alpha}{\Delta\alpha},$$

where $\alpha, \beta \in$ WFF and $\Delta \in \{\mathsf{K}, \Box, \mathsf{O}\}$.

The following result is the first of the main issues of this paper.

THEOREM 7 (Soundness and completeness). *A formula* $\alpha \in$ WFF *is valid in all set spaces, iff it is* $\mathsf{ET}$*–derivable.*

**Proof.** (Sketch.) The soundness part of Theorem 7 is easy to prove. Thus we need not go into that in more detail. The proof of completeness is both an adaption to the new system and a suitable extension of the corresponding proof for topologic; cf [3], Sec. 2.2. Only the part concerning the modifications is given detailedly.

Let $\alpha \in$ WFF be not $\mathsf{ET}$–derivable. We attain a set space falsifying $\alpha$ by an infinite 'three-dimensional' step-by-step construction. In each step, an approximation to the final model is defined. In order to ensure that this 'limit structure' behaves as desired, several requirements on the intermediate models have to be kept under control.

Let $\mathcal{C}$ be the set of all maximal $\mathsf{ET}$–consistent sets of formulas and $\xrightarrow{\mathsf{K}}$, $\xrightarrow{\Box}$, and $\xrightarrow{\mathsf{O}}$ the accessibility relations on $\mathcal{C}$ induced by the modalities $K$, $\Box$ and $\mathsf{O}$, respectively. Suppose that $\Gamma_0 \in \mathcal{C}$ is to be realized (i.e., $\Gamma_0$ contains $\neg\alpha$). We choose a denumerably infinite set of points, $Y$, fix an element $x_0 \in Y$, and construct inductively a sequence of quadruples $(X_n, P_n, i_n, t_n)$ such that $P_0 = \{\bot\}$ and, for every $n \in \mathbb{N}$,

1. $X_n \subseteq Y$ is a finite set containing $x_0$,
2. $P_n$ is a finite disjoint union of finite trees,[2]
3. $i_n : P_n \longrightarrow \mathcal{P}(X_n)$ is an injective function from $P_n$ into the set of all non-singleton subsets of $X_n$ such that $p \leq q \iff i_n(p) \supseteq i_n(q)$ holds for all $p, q \in P_n$, and
4. $t_n : X_n \times P_n \longrightarrow \mathcal{C}$ is a *partial* function assigning a maximal $\mathsf{ET}$–consistent set to some pairs from $X_n \times P_n$ such that, whenever $x, y \in X_n$ and $p, q \in P_n$, then
   (a) $t_n(x, p)$ is defined iff $x \in i_n(p)$; in this case it holds that
      i. if $y \in i_n(p)$, then $t_n(x, p) \xrightarrow{\mathsf{K}} t_n(y, p)$,
      ii. if $q \geq p$, then $t_n(x, p) \xrightarrow{\Box} t_n(x, q)$ and
      iii. if $x \in i_n(q)$, then $t_n(x, p) \xrightarrow{\mathsf{O}} t_n(x, q)$,
   (b) $t_n(x_0, \bot) = \Gamma_0$.

We now explain to what extent the intermediate structures $(X_n, P_n, i_n, t_n)$ approximate the desired model. Actually, it can be guaranteed that, for all $n \in \mathbb{N}$,

5. $X_n \subseteq X_{n+1}$,

[2]The induced partial order on $P_n$ is denoted $\leq$.

6. $P_{n+1}$ is an *end extension* of $P_n$ (i.e., a sup-structure of $P_n$ such that no element of $P_{n+1} \setminus P_n$ is strictly smaller than some element of $P_n$),
7. $i_{n+1}(p) \cap X_n = i_n(p)$ for all $p \in P_n$, and
8. $t_{n+1} \mid_{X_n \times P_n} = t_n$.

Furthermore, the construction complies with the following requirements on existential formulas. For all $n \in \mathbb{N}$,

9. if $\mathsf{L}\beta \in t_n(x,p)$, then there are $k \in \mathbb{N}$, $n < k$, and $y \in i_k(p)$ such that $\beta \in t_k(y,p)$,
10. if $\Diamond\beta \in t_n(x,p)$, then there are $k \in \mathbb{N}$, $n < k$, and $q \in P_k$ such that $p \leq q$ and $\beta \in t_k(x,q)$, and
11. if $\mathsf{P}\beta \in t_n(x,p)$, then there are $k \in \mathbb{N}$, $n < k$, and $q \in P_k$ such that $\beta \in t_k(x,q)$.

Let us assume for the moment that the construction has been carried out successfully, meeting all these requirements. Let $(X, P, i, t)$ be the *limit* of the structures $(X_n, P_n, i_n, t_n)$, i.e.,

- $X = \bigcup\limits_{n \in \mathbb{N}} X_n$,
- $P = \bigcup\limits_{n \in \mathbb{N}} P_n$,
- $i$ is given by $i(p) = \bigcup\limits_{m \geq n} i_m(p)$, where $n$ is the smallest number $l$ such that $p \in P_l$, and
- $t$ is given by $t(x,p) := t_n(x,p)$, where $n$ is the smallest number $l$ such that $t_l(x,p)$ is defined.

Now let $\mathcal{O} := \mathrm{Im}(i)$ and $\mathcal{F} := (X, \mathcal{O})$. We define an $\mathcal{F}$–valuation $V$ by

$$V(A) := \{x \in X \mid A \in t(x,p) \text{ for some minimal element } p \in P\},$$

for all $A \in \mathrm{PROP}$. Then, $\mathcal{M} := (X, \mathcal{O}, V)$ is obviously a set space. Moreover, the following properties can (among other things) easily be deduced from the above requirements:

12. for all $x \in X$ and $p \in P$ we have that $t(x,p)$ is defined iff $x \in i(p)$; in this case it holds that
    (a) if $q \in P$ and $x \in i(q)$, then $t(x,p) \xrightarrow{\mathsf{O}} t(x,q)$, and
    (b) if $\beta \in \mathrm{WFF}$ and $\mathsf{O}\beta \in t(x,p)$, then there is some $q \in P$ such that $\beta \in t(x,q)$;
13. $t(x_0, \perp) = \Gamma_0$, where $x_0$ and $\Gamma_0$ are as above.

With that, the following *Truth Lemma* can be proved by induction on the structure of formulas.

LEMMA 8 (Truth Lemma). *For all formulas* $\beta \in \mathrm{WFF}$ *and neighbourhood situations* $(x, i(p)) \in \mathcal{N}_{\mathcal{F}}$*, we have that* $(x, i(p) \models_{\mathcal{M}} \beta \iff \beta \in t(x,p))$.

**Proof.** We consider only the case that $\beta = \mathsf{P}\gamma$. First, assume that $x, i(p) \models_{\mathcal{M}} \mathsf{P}\gamma$. (Note for later purposes that this means, in particular, that $x \in i(p)$, i.e., $t(x,p)$ is defined.) Then, by Definition 2 and the definition of $\mathcal{F}$ there exists $q \in P$ such that $x \in i(q)$ and $x, i(q) \models_{\mathcal{M}} \gamma$. The induction hypothesis now gives us $\gamma \in t(x,q)$. We then conclude $t(x,p) \xrightarrow{\mathsf{O}} t(x,q)$ with the aid of item 12 (a). Therefore, $\mathsf{P}\gamma \in t(x,p)$. – The converse is proven similarly, but item 12 (b) has to be used in this case instead of 12 (a). ■

Letting now $x := x_0$, $p := \perp$ and $\beta := \neg\alpha$, then Theorem 7 follows immediately from Lemma 8 and item 13.

It remains to define $(X_n, P_n, i_n, t_n)$, for all $n \in \mathbb{N}$. The case $n = 0$ is easy. In the induction step, some existential formula contained in some maximal $\mathsf{ET}$–consistent set $t_n(x,p)$, where $x \in X_n$ and $p \in P_n$, must be realized in a way meeting all the above requirements. Again, we confine ourselves to the case of the overlap operator.

So, let $\mathsf{P}\beta \in t_n(x,p)$. We choose some element $y \in Y$ not yet processed, and we take some $q \notin P_n$. Then we let $X_{n+1} := X_n \cup \{y\}$, $P_{n+1} := P_n \cup \{q\}$, and

$$i_{n+1}(p') := \begin{cases} i_n(p') & \text{if } p' \in P_n \\ \{x, y\} & \text{if } p' = q, \end{cases}$$

for all $p' \in P_{n+1}$; furthermore, we let $q$ be incomparable with any element of $P_n$. Finally, we define $t_{n+1}$ as follows. Let $\Gamma$ be a maximal $\mathsf{ET}$–consistent set such that $t_n(x,p) \xrightarrow{\mathsf{O}} \Gamma$ and $\beta \in \Gamma$. Then,

$$t_{n+1}(x', p') := \begin{cases} t_n(x', p') & \text{if } x' \in X_n \text{ and } p' \in P_n \\ \Gamma & \text{if } (x' = x \text{ or } x' = y) \text{ and } p' = q \\ \text{undefined} & \text{otherwise,} \end{cases}$$

for all $x' \in X_{n+1}$ and $p' \in P_{n+1}$. This completes the definition of the approximating structure in the induction step, and we must now check that properties 1 – 8 and 11 remain valid (9 and 10 are irrelevant to the present case). Apart from 4 all items are more or less obvious from the construction. (Note, however, that we need that every subset has at least two elements in order to establish property 3 for $i_{n+1}$.) Thus only the verification of property 4 is left. As 4 (b) is also obvious we concentrate on 4 (a). First, it follows from the definition of $t_{n+1}$ that the condition on the domain of this function is satisfied. Then, (i) is clear from the definition of $t_{n+1}$ and the validity of this condition for $n$. (Note that $\xrightarrow{\mathsf{K}}$ is a reflexive relation.) And (ii) too is valid for corresponding reasons ($\xrightarrow{\Box}$ is reflexive as well). Only for (iii) some arguments are needed. So, assume first that

$x \in i_{n+1}(p')$ and let $q' \in P_{n+1}$ be such that $x \in i_{n+1}(q')$ ($x$ from above). We must then show that $(*)$ $t_{n+1}(x,p') \xrightarrow{\mathsf{O}} t_{n+1}(x,q')$. If $p', q' \in P_n$, then this follows from the induction hypothesis. If $p', q' \notin P_n$, then $(*)$ is valid since $\xrightarrow{\mathsf{O}}$ is a reflexive relation. If $p' \in P_n$ and $q' \notin P_n$, then we obtain $t_{n+1}(x,p') \xrightarrow{\mathsf{O}} t_{n+1}(x,p)$ from the induction hypothesis, and as we also have $t_{n+1}(x,p) \xrightarrow{\mathsf{O}} t_{n+1}(x,q)$, the assertion follows in this case because of the transitivity of the relation $\xrightarrow{\mathsf{O}}$. Finally, if $p' \notin P_n$ and $q' \in P_n$, the symmetry of $\xrightarrow{\mathsf{O}}$ is used in addition. – Second, if $x' \in X_{n+1}$ is a point different from $x$, then either $x' = y$ holds from which we conclude $p' = q' = q$ and, therefore, the validity of $(*)$, or $x' \in X_n$ thus $p', q' \in P_n$, and $(*)$ follows from the induction hypothesis. In this way, property 4 is proved for $t_{n+1}$.

In order to ensure that *all* possible cases are eventually exhausted, processing has to be suitably scheduled with regard to each of the modalities involved. This can be done by means of appropriate enumerations. Concerning this and the construction in case of a modality of topologic, the reader is referred to the paper [3] for further details. However, it should be mentioned here that Axiom 12 is additionally needed to establish property 4 (a) (iii) in the induction step for $\mathsf{L}\beta$ and $\Diamond\beta$, respectively. – This completes the proof sketch of Theorem 7. ∎

We now show that the set of all ET–theorems is decidable. As the logic does not satisfy the finite model property with respect to the class of all set spaces (cf [3], Sec. 1.3), we make a little detour in order to obtain this result. We introduce a certain class of auxiliary Kripke models and prove that ET (is sound and complete and) satisfies the finite model property with respect to *this* class of models. This gives us the desired decidability result in a standard manner.

In the following, $R$ and $\mathsf{K}$, $S$ and $\Box$, and $T$ and $\mathsf{O}$, respectively, correspond to each other.

DEFINITION 9 (ECA–model). Let $M := (W, R, S, T, V)$ be a trimodal model, where $R, S, T \subseteq W \times W$ are binary relations and $V$ is a valuation. Then, $M$ is called an *extended cross axiom model* (or, in short, an *ECA–model*) iff the following conditions are satisfied:[3]

1. $R$ and $T$ are equivalence relations, and $S$ is reflexive and transitive,
2. $S \subseteq T$,
3. $S \circ R \subseteq R \circ S$, and
4. for all $w, w' \in W$ and $A \in$ PROP : $(w \in V(A) \iff w' \in V(A))$, if $w\,T\,w'$.

[3]The term 'cross axiom model' was introduced in [3]; see Sec. 2.3 there.

Note that every set space induces a semantically equivalent ECA–model in a rather straightforward manner.

It is easy to see that all the axioms considered in the first part of this section are sound with respect to the class of all ECA–models. Furthermore, the canonical model of $\mathsf{ET}$ is an example of an ECA–model. This gives us the following theorem.

THEOREM 10 (Kripke completeness I). *The system* $\mathsf{ET}$ *is sound and complete with respect to the class of all ECA–models.*

We use the method of *filtration* in order to prove the finite model property of $\mathsf{ET}$ with respect to ECA–models. For a given $\mathsf{ET}$–consistent formula $\alpha \in \mathrm{WFF}$, we define a filter set $\Sigma \subseteq \mathrm{WFF}$ as follows. We first let

$$\Sigma_0 := \mathrm{sf}(\alpha) \cup \{\neg\beta \mid \beta \in \mathrm{sf}(\alpha)\} \cup \{\mathsf{O}A \mid A \in \mathrm{PROP} \cap \mathrm{sf}(\alpha)\},$$

where $\mathrm{sf}(\alpha)$ designates the set of all subformulas of $\alpha$. Second, we form the closure of $\Sigma_0$ under finite conjunctions of pairwise distinct elements of $\Sigma_0$. Third, we close under single applications of the operator $\mathsf{L}$. And finally, we form the set of all subformulas of elements of the set obtained last. Let $\Sigma$ denote the resulting set of formulas.

We now consider the respective *smallest* filtrations of the accessibility relations $\xrightarrow{\mathsf{K}}$ and $\xrightarrow{\Box}$ of the canonical model, and the *symmetric and transitive* filtration of the relation $\xrightarrow{\mathsf{O}}$; cf [7], Sec. 1.4. Let $M := (W, R, S, T, V)$ be the corresponding filtration of a suitably generated submodel of the canonical model, for which the valuation $V$ assigns the empty set to all proposition letters not in $\Sigma$. Then we have the following lemma.

LEMMA 11. *The structure* $M$ *is a finite ECA–model of which the size depends computably on the length of* $\alpha$*.*

**Proof.** Most of it is clear from the definitions and the proof of [3], Theorem 2.11. Only items 2 and 4 of Definition 9 have to be checked.

For item 2, let $\Gamma, \Theta$ be two points of the canonical model such that $[\Gamma]\ S\ [\Theta]$, where the brackets $[\dots]$ indicate the respective classes. Then there are $\Gamma' \in [\Gamma]$ and $\Theta' \in [\Theta]$ such that $\Gamma' \xrightarrow{\Box} \Theta'$. Since $\xrightarrow{\Box} \subseteq \xrightarrow{\mathsf{O}}$ holds because of Axiom 12, we conclude $\Gamma' \xrightarrow{\mathsf{O}} \Theta'$ from that. Due to the first filtration condition (marked (F1) in [7], Sec. 1.4) it follows that $[\Gamma]\ T\ [\Theta]$, as desired.

For item 4, let $\Gamma, \Theta$ be as above and assume now that $[\Gamma]\ T\ [\Theta]$. Moreover, let $A \in \Sigma$ and $[\Gamma] \in V(A)$. It follows that $A \in \Gamma$. From this we infer $\mathsf{O}A \in \Gamma$ with the aid of Axiom 9. Due to the definition of $\Sigma_0$ we obtain $\mathsf{O}A \in [\Gamma]$. The usual filtration lemma then gives us $M, [\Gamma] \models \mathsf{O}A$. Consequently, $M, [\Theta] \models A$, i.e., $[\Theta] \in V(A)$. The desired assertion now follows for symmetry reasons. ■

For later purposes, note that the completeness theorem of $\mathsf{topologic}$ is used in [3] for proving the transitivity of the relation $S$.

Since the model $M$ from Lemma 11 realizes $\alpha$, the claimed decidability result follows readily.

THEOREM 12 (Decidability). *The set of all* $\mathsf{ET}$*–theorems is decidable.*

## 4 Towards coverings

We now introduce an 'iterated' overlap operator, $\mathsf{C}$, for set spaces and touch on the question how coverings can be dealt with by means of this modality. We then show that $\mathsf{C}$ is strongly related to the *common knowledge operator* of the usual logic of knowledge (see [4], Sec. 3.3). This yields, in particular, that interpreting $\mathsf{C}$ in ECA–models leads to a finitely axiomatizable and decidable logic. Finally in this section, we mention a connection between our logic and reasoning about knowledge of agents in so-called *systems with perfect recall;* cf [4], Sec. 4.4 and 8.2.

DEFINITION 13 (Semantics of $\mathsf{C}$). Let $\mathcal{M} = (X, \mathcal{O}, V)$ be a set space, $\alpha \in$ WFF a formula and $x, U$ a neighbourhood situation. Then,

$$x, U \models_{\mathcal{M}} \mathsf{C}\alpha :\Longleftrightarrow \begin{cases} \forall n \in \mathbb{N} \text{ and } y \in X: \text{ if } U_0, \ldots, U_n \in \mathcal{O} \text{ satisfy} \\ U = U_0,\ U_i \cap U_{i+1} \neq \emptyset \text{ for } i = 0, \ldots, n-1, \\ \text{and } y \in U_n,\ \text{ then } y, U_n \models_{\mathcal{M}} \alpha. \end{cases}$$

Thus all the formulas of the form

$$\underbrace{\mathsf{KO}\ldots\mathsf{KO}}_{n \text{ times}}\alpha$$

(for some $n \in \mathbb{N}$) must be true at $x, U$ in order to make $\mathsf{C}\alpha$ true at the neighbourhood situation $x, U$. It follows that the relation belonging to $\mathsf{C}$ amounts to the reflexive and transitive closure of the union of the relations belonging to $\mathsf{K}$ and $\mathsf{O}$ in ECA–models; see below.

What about coverings? – Well, suppose at the moment that we have the *global modality* $\mathsf{A}$ to hand (cf [1], Sec. 7.1). Then, the formula $\mathsf{C}\alpha \rightarrow \mathsf{A}\alpha$ obviously describes the fact that the set system $\mathcal{O}$ of a given set frame $(X, \mathcal{O})$ represents a 'connected covering' of the domain $X$ of the frame. (In other words, every neighbourhood situation is reachable from the actual one via a finite sequence of overlapping sets.)

Unfortunately, we do not have nice meta-theorems for the logic of set spaces regarding the richer language up to now. The situation changes, however, if we restrict attention to ECA–models.

DEFINITION 14 (Enriched ECA–model). A tuple $M := (W, R, S, T, Q, V)$ is called an *enriched ECA–model*, iff $(W, R, S, T, V)$ is an ECA–model and $Q \subseteq W \times W$ the reflexive and transitive closure of the relation $R \cup T$.

Again, every set space induces a semantically equivalent enriched ECA–model; see the remark right after Definition 9.

Adding now the axiom schema

$$\mathsf{C}\alpha \rightarrow \mathsf{K}(\alpha \wedge \mathsf{C}\alpha) \wedge \mathsf{O}(\alpha \wedge \mathsf{C}\alpha)$$

and the rule schema

$$\frac{\alpha \to \mathsf{K}(\alpha \wedge \beta) \wedge \mathsf{O}(\alpha \wedge \beta)}{\alpha \to \mathsf{C}\beta}$$

to the system from Section 3, yields a logic we designate $\mathsf{ETC}$. It turns out that $\mathsf{ETC}$ is sound and complete, and satisfies the finite model property, with respect to the class of all enriched ECA–models.

THEOREM 15 (Kripke completeness II). *The system* $\mathsf{ETC}$ *is sound and complete with respect to the class of all finite enriched ECA–models.*

**Proof.** Soundness is again easy. For completeness, we start off with the structure $M = (W, R, S, T, V)$ considered in Lemma 11. First note that both the new axiom and the rule schema are precisely those for common knowledge in case two agents are involved in the group; cf [4], Sec 3.3. From this we get in a standard manner that $Q := (R \cup T)^*$ is a filtration of the accessibility relation $\xrightarrow{\mathsf{C}}$, belonging to the operator $\mathsf{C}$, of the canonical model of $\mathsf{ETC}$. Note that only the *definability lemma* (cf [7], 9.7), the new axiom and rule schemata, respectively, and the conditions defining a filtration are used for that.

However, we cannot yet conclude that the theorem is true since we do not have completeness of the system $\mathsf{ETC}$ with respect to set spaces; cf the remark following the proof of Lemma 11. Nevertheless, there is a way out. We have to change the filtration of $\xrightarrow{\Box}$ and consider the *transitive closure of the smallest filtration* (cf [2], p 141 f) instead of the smallest one. Then, the main result of [16] assures us that the structure $M' := (W, R, S, T, Q, V)$ is really a finite enriched ECA–model. The desired completeness result now follows easily. ■

The following corollary is immediate from Theorem 15.

COROLLARY 16. *The set of all* $\mathsf{ETC}$*–theorems is decidable.*

We conclude this section with an additional remark on the relation of the systems $\mathsf{ET(C)}$ to the usual logic of knowledge. We first state that the system $\mathsf{ET}$ contains the logic of knowledge of two agents having perfect recall as a sublogic. In fact, the Cross Axiom may be viewed as the schema corresponding to perfect recall in our context; cf the discussion in [4], Sec. 8.2. And the Cross Axiom holds for the second 'knower' as well since

$$(*) \quad \mathsf{O}\Box\alpha \to \Box\mathsf{O}\alpha$$

is $\mathsf{ET}$–derivable. Calling the logic where Axiom 9 is skipped and Axiom 12 is replaced with $(*)$, therefore, $\mathsf{L}^{\mathsf{pr}}$, we have that $\mathsf{L}^{\mathsf{pr}} \subseteq \mathsf{ET}$. The same is true if common knowledge is included: $\mathsf{LC}^{\mathsf{pr}} \subseteq \mathsf{ETC}$. The question now arises whether $\mathsf{LC}^{\mathsf{pr}}$ is decidable. Unfortunately, we cannot answer this question positively from our previous techniques and results. Rather on the contrary, undecidability seems more likely.[4]

[4] The question just raised is very natural in view of [9], Corollary 3.6, and the remark

## 5 Hybridization

Concluding the technical part of the paper, we now *hybridize* the logic of overlapping. Although it is true that we cannot solve the problems left over from the preceding section in this way (because of the first-order nature of hybrid logic), this method is advantageous to us in several respects. First, we obtain much more expressive power concerning properties of relations, and this can already be achieved by simply adding suitable sets of nominals to the ground language. Second, we are able to do without the global modality, which was mainly needed for technical reasons in previous hybridizations of $\mathsf{topologic}$; cf [14, 15]. Actually, the global modality appears somewhat artificial in this context since its semantics lacks to be typical of spatial structures. Third, other special purpose hybrid connectives like the 'in'–operators utilized in [15] become definable in the new language. And finally, all the meta-theorems proved in the papers just cited remain valid for the new system.

We now verify all these points more or (mostly) less detailedly. For a start, we define the extended language. As we already indicated above, we merely add two sets of nominals to the language underlying $\mathsf{ET}$. If the denotation of a nominal is non-empty, then it is to be either a unique state or a distinguished set of states. Let $\mathrm{N}_{stat} = \{i, j, \dots\}$ and $\mathrm{N}_{sets} = \{I, J, \dots\}$ be the corresponding sets of symbols, which we call *names of states* and *names of sets,* respectively.

DEFINITION 17 (Set spaces with names). Let $\mathcal{F} = (X, \mathcal{O})$ be a set frame such that $X, \emptyset \in \mathcal{O}$.

1. A *hybrid $\mathcal{F}$–valuation* is a mapping

$$V : \mathrm{PROP} \cup \mathrm{N}_{stat} \cup \mathrm{N}_{sets} \longrightarrow \mathcal{P}(X)$$

   such that

   (a) $V(i)$ is either $\emptyset$ or a singleton subset of $X$ for every $i \in \mathrm{N}_{stat}$,

   (b) $V(A) \in \mathcal{O}$ for every $A \in \mathrm{N}_{sets}$.

2. A *set space with names* (or, in short, an *SSN*) is a triple $(X, \mathcal{O}, V)$, where $\mathcal{F} = (X, \mathcal{O})$ is a set frame and $V$ a hybrid $\mathcal{F}$–valuation.

Note that $X$ and the empty set are now included in $\mathcal{O}$. The inclusion of $X$ is mainly because of technical reasons. Actually, a smooth proof of Theorem 19 is facilitated in this way. The inclusion of $\emptyset$ takes into account that nominals may have an empty denotation. This is appropriate for our purposes, but not common in standard hybrid logic.

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at the end of Sec. 5 there, respectively. That corollary tells us that the usual logic of knowledge of two agents with perfect recall and common knowledge is undecidable. And in the remark, the authors suspect that the complexity of the logic of knowledge where $\Box$ is the only temporal modality will significantly decrease. Now note that we have a reduction, induced by $\mathsf{O}\alpha \mapsto \mathsf{O}\Box\alpha$, of the satisfiability problem of $\mathsf{ETC}$ to the satisfiability problem of $\mathsf{LC}^{\mathsf{pr}}$.

DEFINITION 18 (Satisfaction for nominals). Let $\mathcal{M} := (X, \mathcal{O}, V)$ be an SSN and $x, U$ a neighbourhood situation of $(X, \mathcal{O})$. Then

$$\begin{array}{lcl} x, U \models_{\mathcal{M}} i & :\Longleftrightarrow & x \in V(i) \\ x, U \models_{\mathcal{M}} I & :\Longleftrightarrow & V(I) = U, \end{array}$$

for all $i \in \mathrm{N}_{stat}$ and $I \in \mathrm{N}_{sets}$.

In Proposition 5, we gave an example of a frame property that can be captured with the aid of the overlap operator. Here is another one: *weak connectedness of neighbourhood situations*. This means that for any two overlapping sets, one of them is contained in the other. An appropriate correspondent reads

$$(\#) \quad (\alpha \to \Diamond\beta) \vee \mathsf{O}(\beta \to \Diamond\alpha).$$

(By the way, the modal analogue $\Box(\Box\alpha \to \beta) \vee \Box(\Box\beta \to \alpha)$ is in turn derivable from (#) in the system $\mathsf{ET}$.) However, certain desirable frame properties are *not* expressible in the language from Section 2, eg, that

(P) any two overlapping sets are contained in a common superset

(which is obviously true for all frames $(X, \mathcal{O})$ SSNs are based on since $X \in \mathcal{O}$; cf Definition 17). This deficiency can be remedied by using nominals. Eg, the just mentioned property is captured by the formula schema

$$(\&) \quad I \wedge \mathsf{P}J \to \mathsf{P}(\Diamond I \wedge \Diamond J),$$

where $I, J \in \mathrm{N}_{sets}$.

This example is really important because it leads us to a sound and complete axiomatization of *hybrid* $\mathsf{ET}$ without using the global modality. In fact, a suitable variant of the schema (&) serves as a substitute for Axiom 15 from [14] which likewise corresponds to (P) and is crucial to the completeness proof there. Now, letting $\mathsf{ETH}$ denote the augmentation of $\mathsf{ET}$ by the appropriately modified axioms and rules for names in hybrid topologic, cf [14] (or [15]), we obtain the following theorem.[5]

THEOREM 19 (Hybrid completeness). $\mathsf{ETH}$ *is sound and complete with respect to the class of all set spaces with names.*

**Proof.** (Sketch.) The proof of Theorem 19 goes along the lines of the proof of Theorem 3.2 from [14], but some modifications are required during the construction of the model refuting a given non-derivable formula. Again, we start off with the canonical model (of $\mathsf{ETH}$ this time). Let $\mathcal{C}$, $\xrightarrow{\mathsf{K}}$, $\xrightarrow{\Box}$, $\xrightarrow{\mathsf{O}}$, and $\Gamma_0$, be according to the previous proof.

As it was already indicated above, we will make decisive use of the following axioms:

[5] Note that the global modality is involved in another schema for names, viz Axiom 14 from [14]. This has to be handled as follows: replace $\mathsf{A}$ with $\mathsf{K}\Box$ therein. Furthermore, note that the difference between $\mathsf{A}$ and $\mathsf{O}$ concerning axioms, lies in the schemata 9 and 12 from Section 3.

(A1) $i \wedge I \wedge \mathsf{P}\,(j \wedge J) \rightarrow \mathsf{P}\,(\Diamond(i \wedge I) \wedge \mathsf{L}\Diamond(j \wedge J))$

(A2) $\mathsf{A}\,(I \wedge \alpha \rightarrow \mathsf{L}\beta) \vee \mathsf{A}\,(I \wedge \beta \rightarrow \mathsf{L}\alpha)\,,$

where $i \in \mathrm{N}_{stat}$, $I \in \mathrm{N}_{sets}$ and $\alpha, \beta \in$ WFF. Note that (A1) is related to the fact that $X \in \mathcal{O}$ for every SSN $(X, \mathcal{O}, V)$.

Moreover, the ENRICHMENT rules are not only different, but also more complicated compared to those from [14]. Consider the following set of strings:

$$L := (\mathsf{PL})^* \cup (\mathsf{PL})^*\mathsf{P} \cup (\mathsf{LP})^* \cup (\mathsf{LP})^*\mathsf{L}.$$

Then, for all $W \in L$ and $\nabla \in \{\mathsf{L}, \Diamond, \mathsf{P}\}$ we have an $\mathsf{ETH}$–rule schema called $W_\nabla$–ENRICHMENT which reads

$$\frac{W\,(i \wedge I \wedge \nabla(j \wedge J \wedge \alpha)) \rightarrow \beta}{W\,(i \wedge I \wedge \nabla\alpha) \rightarrow \beta}\,,$$

where $\alpha, \beta \in$ WFF, $i, j \in \mathrm{N}_{stat}$, $I, J \in \mathrm{N}_{sets}$, and $j, J$ are new each time.

We then call a maximal consistent set $\Gamma$ of formulas

- *named* iff $\Gamma$ contains some $i \in \mathrm{N}_{stat}$ and some $I \in \mathrm{N}_{sets}$, and
- *enriched* iff for all $W \in L$ and every $\nabla \in \{\mathsf{L}, \Diamond, \mathsf{P}\}$ we have that $W(i \wedge I \wedge \nabla\alpha) \in \Gamma$ implies $W\,(i \wedge I \wedge \nabla(j \wedge J \wedge \alpha)) \in \Gamma$, for some $j \in \mathrm{N}_{stat}$ and $J \in \mathrm{N}_{sets}$.

Our first task now is extending $\Gamma_0$ to a named and enriched maximal consistent set $\Gamma'$ in the language which has been increased by enough new constants. This can be done by means of a suitable *Extended Lindenbaum Lemma* in the usual way (i.e., by using, in particular, the above rules); cf [1], 7.25, and [14], Lemma 3.3.

Let $\mathcal{M}_0$ be the submodel, generated by $\Gamma'$, of the canonical model of $\mathsf{ETH}$ in the extended language. Moreover, let $\mathcal{M}'$ be the substructure of $\mathcal{M}_0$ of which the domain, $D$, consists of all points that are named. $\mathcal{M}'$ is the model we want to operate in. We focus only on one of the points enabling us to do that successfully here, viz *uniqueness of the interpretation of set names.*

LEMMA 20. *Let $I$ be a set name. Assume that $\Gamma, \Gamma' \in D$ both contain $I$. Then $\Gamma \xrightarrow{\mathsf{K}} \Gamma'$.*

**Proof.** Assume on the contrary that there are two points $\Gamma, \Gamma' \in D$ such that $I \in \Gamma \cap \Gamma'$, but not $\Gamma \xrightarrow{\mathsf{K}} \Gamma'$. Then there is some $n \in \mathbb{N}$ and a sequence $\Gamma^0, \Gamma^1, \ldots, \Gamma^n$ of elements of $D$ such that $\Gamma^0 = \Gamma$, $\Gamma^n = \Gamma'$, and $\Gamma^i, \Gamma^{i+1}$ are connected by either $\xrightarrow{\mathsf{K}}$ or $\xrightarrow{\mathsf{O}}$ for all $i = 0 \ldots n-1$. This is true because $\mathcal{M}_0$ is generated and each $\Gamma^i$ can in fact be chosen as named (which is proved with the aid of our special notion of enrichment). The assertion of the lemma now follows by an iterated application of the schema (A1), since the desired contradiction is then yielded by the schema (A2). ■

The subsequent *Existence Lemma* can be proved with the aid of Lemma 20 (among other things).

LEMMA 21. *Assume that $\Gamma \in D$ contains the formula $\nabla\alpha$ (where $\nabla$ is as above). Then there exists some $\Theta \in D$ satisfying $\Gamma \xrightarrow{\nabla} \Theta$ and $\alpha \in \Theta$.*

Lemma 21 is fundamental to the further progress of the proof of Theorem 19, which now can be continued essentially as in [14], Sec. 3. ■

The proof of the following theorem is even more similar to the corresponding one from [14] (Theorem 4.4 there) and can be omitted here therefore.

THEOREM 22 (Hybrid decidability). *The set of all $\mathsf{ETH}$–derivable formulas is decidable.*

It is no accident that we succeeded in axiomatizing $\mathsf{ETH}$ without the global modality because the latter is definable in set spaces with names, actually through

$$\mathsf{A}\alpha :\equiv \mathsf{K}\Box\mathsf{O}\alpha.$$

Note that we again made use of the fact that $X \in \mathcal{O}$ for that (see above). Then, however, the usual hybrid *satisfaction operators* $@_{\ldots}$, cf [1], Sec. 7.3, can be defined as well. This can be seen as follows. The formulas of the form $i \wedge I$, where $i \in \mathrm{N}_{stat}$ and $I \in \mathrm{N}_{sets}$, are taken as names of neighbourhood situations. With that, the satisfaction operator associated with such a name reads

$$@_{i\wedge I}\alpha :\equiv \mathsf{E}(i \wedge I \wedge \alpha),$$

where $\alpha \in \mathrm{WFF}$ and $\mathsf{E}$ is the dual of $\mathsf{A}$.

And a final point should be stressed here: in SSNs, $\mathsf{O}$ is really *stronger* than the global modality. This is witnessed by the fact that even the 'in'–operators considered in the paper [15] become definable. (This is not the case with that system.) For convenience, we remind the reader of the semantics of these operators. Let $\mathcal{M} = (X, \mathcal{O}, V)$ be an SSN and $x, U$ a neighbourhood situation of $(X, \mathcal{O})$. Then

$$x, U \models_{\mathcal{M}} [\epsilon_I]\alpha :\Longleftrightarrow \text{ if } x \in V(I), \text{ then } x, V(I) \models_{\mathcal{M}} \alpha,$$

where $I \in \mathrm{N}_{sets}$ and $\alpha \in \mathrm{WFF}$. The 'in'–operators are, therefore, named variants of the overlap operator; cf Definition 2. Thus it is not surprising that they can be defined:

$$[\epsilon_I]\alpha :\equiv \mathsf{O}(I \rightarrow \alpha).$$

It follows that we obtain all the results from the paper [15] for the present system as well, in particular, that the (hybrid) logic of directed spaces (see Definition 4) is decidable; cf Theorem 20 and Corollary 21 there. (The decidability problem for the logic of directed spaces was left open in the paper [20]; cf the remark right after Definition 4.) And one can certainly expect similar results for other extensions of $\mathsf{ETH}$ by additional axioms, which are due to the applications one has in mind.

All in all, we may say that $\mathsf{ETH}$ seems to be the 'right' hybrid logic for set spaces with names.

## 6 Concluding remarks

In the paper, we first added an operator describing overlapping to Moss and Parikh's bi-modal language of set spaces. We then proved the soundness and completeness as well as the decidability of the arising logic, which we called extended topologic, ET. Subsequent to that, we discussed further-reaching modifications of ET. On the one hand we focussed on *coverings* of set spaces and their relation to common knowledge (system ETC), and on the other hand we dealt with a natural hybridization that turned out to be very neat in several respects (system ETH). In both cases we obtained some interesting results indicating the adequacy of our approach.

Some topics could only be touched on in the paper. This concerns the topological aspects we mentioned above as well as the connection between ET(C) and the usual logic of knowledge. In particular, a certain epistemic interpretation of the overlap operator suggested by one of the anonymous referees will have to be closer examined.

At this point we may also take a different view of topologic: consider the bi-modal system with knowledge and overlapping the basic one and build extensions of *this* system as needed. (Note that this system syntactically coincides with the usual logic of knowledge of two agents (apart from the rigidity condition for proposition letters; see Axiom 9 above).) Then, eg, introducing a modality complementary to the effort operator, i.e., quantifying 'upward' across $\mathcal{O}$, is very natural (among other things). Such an operator corresponds to *systems with no learning* in the usual logic of knowledge; cf [9] or [8].

Our final remark concerns complexity. Unfortunately, we do not have any satisfactory results regarding this (apart from the one proved in [11]). In particular, it seems to be difficult to establish reasonable upper bounds. Thus future research into the complexity of topologic and its extensions is urgently needed.

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Bernhard Heinemann
FernUniversität in Hagen
58084 Hagen, Germany
E-mail: Bernhard.Heinemann@fernuni-hagen.de