A decidable modal logic that is finitely undecidable

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ABSTRACT. The aim of this paper is to construct an example of a decidable modal logic above $K4$ such that the logic of its finite Kripke frames is undecidable.

1 Introduction

A theory is said to be *finitely decidable* if the theory of its finite models is decidable. In this paper we consider the following question: is it the case that if a theory is decidable then it is finitely decidable? That is, whether the following implication holds:

$$\text{DECIDABILITY} \quad \Rightarrow \quad \text{FINITE DECIDABILITY}$$

For some theories, there is a correlation between decidability and finite decidability. For example, the implication above holds for the equational group theory, the equational ring theory, and some others (see, e.g., [5, 4, 2]). However, in general this implication does not hold. Examples of a decidable first-order theory and an equational theory that are finitely undecidable were given by Joohee Jeong [3]. The question whether the above implication holds for propositional (normal uni)modal logics was asked by Michael Zakharyaschev.

Here we give a 'negative' answer to this question by proving the following

THEOREM 1. *There exists a decidable normal unimodal logic L above $K4$ such that the logic of its finite frames is undecidable.*

2 Definitions

We deal with the standard modal language containing the connectives $\wedge$, $\vee$, $\rightarrow$, $\Box$, and the propositional constant $\bot$ (falsehood). The connectives $\neg$ and $\Diamond$ are defined as abbreviations: $\neg\varphi = \varphi \rightarrow \bot$, $\Diamond\varphi = \neg\Box\neg\varphi$. We also write $\Box^+\varphi$ for $\Box\varphi \wedge \varphi$.

The following definitions are required for the proof of our theorem. A *Kripke frame* $\mathfrak{F}$ is a pair $\langle W, R\rangle$, where W is a non-empty set of worlds and R is a binary relation on W. A world y is *accessible from a world x* if the relation xRy holds. A set $G \subseteq W$ is *accessible from a world w* if every world

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of G is accessible from w. A frame $\mathfrak{F} = \langle W, R\rangle$ is called *transitive* if R is a transitive relation. A transitive frame is *rooted* if there exists a world d such that the set $W \setminus \{d\}$ is accessible from d. In this case the world d is called a *root* of the frame.

Let $\mathfrak{F} = \langle W, R\rangle$ be a frame and ν a valuation in $\mathfrak{F}$, i.e., $\nu(p) \subseteq W$, for every propositional variable p. Let x be a world in $\mathfrak{F}$. We define a *truth-relation* $x \models^\nu \varphi$, "φ is true at x under ν", by taking

$$\begin{array}{lll}
x \not\models^\nu \bot; & & \\
x \models^\nu p & \text{iff} & x \in \nu(p), \text{ for every variable } p; \\
x \models^\nu \psi \wedge \chi & \text{iff} & x \models^\nu \psi \text{ and } x \models^\nu \chi; \\
x \models^\nu \psi \vee \chi & \text{iff} & x \models^\nu \psi \text{ or } x \models^\nu \chi; \\
x \models^\nu \psi \to \chi & \text{iff} & x \models^\nu \psi \text{ implies } x \models^\nu \chi; \\
x \models^\nu \Box\psi & \text{iff} & y \models^\nu \psi \text{ for all } y \in W \text{ such that } xRy.
\end{array}$$

If $x \not\models^\nu \varphi$ then we say φ is *false* at x under ν.

All frames we consider below are assumed to be transitive. A world u is called a *dead end* if uRw for no $w \in W$. A set $\mathfrak{a} \subseteq W$ is called an *antichain* if, for any distinct worlds $w, v \in \mathfrak{a}$, we have $\neg wRv$ and $\neg vRw$. The cardinality of a set G is denoted by $\|G\|$. An antichain $\mathfrak{a}$ is called a *singleton* if $\|\mathfrak{a}\| = 1$. A set $\mathfrak{a} \subseteq W$ is called a *chain* if for every pair of worlds $w, v \in \mathfrak{a}$, $w \neq v$, we have wRv or vRw.

For a set $X \subseteq W$, let

- $(X){\uparrow} = \{w \in W |\ \exists x \in X, xRw\}$;
- $(X)\underline{\uparrow} = X \cup (X){\uparrow}$;
- $(X){\downarrow} = \{w \in W |\ \exists x \in X, wRx\}$;
- $(X)\overline{\downarrow} = X \cup (X){\downarrow}$.

We also write $x{\uparrow}$, $x\underline{\uparrow}$, and $x{\downarrow}$ instead of $\{x\}{\uparrow}$, $\{x\}\underline{\uparrow}$, and $\{x\}{\downarrow}$, respectively.

Denote by $Dom\, f$ and $Im\, f$ the domain and the range of a map f, respectively. Suppose that we have two frames $\mathfrak{F} = \langle W, R\rangle$ and $\mathfrak{G} = \langle V, S\rangle$. A partial map f from W onto V is called a *subreduction* (or a partial *p*-morphism) of $\mathfrak{F}$ to $\mathfrak{G}$ if, for any $x, y \in Dom\, f$, the following conditions hold:

- xRy implies $f(x)Sf(y)$;
- $f(x)Sf(y)$ implies $\exists z \in Dom\, f(xRz \wedge f(z) = f(y))$.

A subreduction f is called *cofinal* if, for every $z \in (Dom\, f){\uparrow} \setminus Dom\, f$, we have $z \in (Dom\, f){\downarrow}$.

Let $\mathfrak{G}$ be a finite frame and $\mathfrak{D}$ a set of antichains in $\mathfrak{G}$ different from reflexive singleton. A subreduction f is said to satisfy the *closed domain condition* (CDC) for $\mathfrak{D}$ if $f(x{\uparrow}) = (\mathfrak{a})\underline{\uparrow}$ implies $x \in Dom\, f$ for every $\mathfrak{a} \in D$ and every $x \in (Dom\, f){\uparrow}$.

Let $\mathfrak{F} = \langle W, R\rangle$ be a finite rooted (transitive) frame, $W = \{w_0, \ldots, w_n\}$, with w_0 being the root. The *canonical formula* $\alpha(\mathfrak{F}, \mathfrak{D}, \bot)$ associated with $\mathfrak{F}$ and $\mathfrak{D}$ is defined (see [6] and [1], p. 310) by taking

$$\alpha(\mathfrak{F}, \mathfrak{D}, \bot) = \bigwedge_{w_i R w_j} \varphi_{ij} \wedge \bigwedge_{i=0}^{n} \varphi_i \wedge \bigwedge_{\mathfrak{a} \in \mathfrak{D}} \varphi_{\mathfrak{a}} \wedge \varphi_{\bot} \to p_0,$$

where

$$\varphi_{ij} = \Box^+(\Box p_j \to p_i),$$

$$\varphi_i = \Box^+((\bigwedge_{\neg a_i R a_k} \Box p_k \wedge \bigwedge_{j=0, j\neq i}^{n} p_j \to p_i) \to p_i),$$

$$\varphi_{\mathfrak{a}} = \Box^+(\bigwedge_{w_i \in W \setminus \mathfrak{a}\underline{\uparrow}} \Box p_j \wedge \bigwedge_{i=0}^{n} p_i \to \bigvee_{w_j \in \mathfrak{a}} \Box p_j),$$

$$\varphi_{\bot} = \Box^+(\bigwedge_{i=0}^{n} \Box^+ p_i \to \bot).$$

We denote by $\mathfrak{D}^\sharp$ the set of all antichains in $\mathfrak{F}$ different from reflexive singletons. The formula $\alpha^\sharp(\mathfrak{F}, \bot) = \alpha(\mathfrak{F}, \mathfrak{D}^\sharp, \bot)$ is called the *frame formula for* $\mathfrak{F}$ (see [1], p. 312).

3 Logic L

In this section we construct a logic L satisfying the conditions of Theorem 1.

DEFINITION 2. Let $h : \omega \to \omega$ be some recursive function with a non-recursive range $Im\, h \subseteq \omega$. Let Y_{nm}, for $n, m \geq 0$, be the irreflexive transitive finite frame shown in Fig 1. Denote by Y the set of all such frames. (Note that $Y_{nm} = Y_{mn}$.)

Let la be the Gödel–Löb formula, i. e., $la = \Box(\Box p \to p) \to \Box p$. For any $n > 0$, denote by bw_n the formula

$$\bigwedge_{0 \leq i \leq n} \Diamond p_i \to \bigvee_{0 \leq i < j \leq n} \big(\Diamond(p_i \wedge (p_j \vee \Diamond p_j)) \vee \Diamond(p_j \wedge (p_i \vee \Diamond p_i))\big).$$

(The formula bw_n is called the *formula of width* n.)

Let $\Gamma = \{\alpha^\sharp(Y_{nm}, \bot) \mid h(n) \neq m, h(m) \neq n\}$. Now we can define the logic L we analyse in this paper.

DEFINITION 3. $L = K4 \oplus \{la, bw_2, \Box bw_1\} \oplus \Gamma$, where $\oplus$ assumes taking the closure under the standard inference rules of $K4$.

Since $bw_2 \in L$, it follows from Fine's theorem (see, e.g., [1], p. 354, p. 358) that L is Kripke complete .

DEFINITION 4. We say that a rooted frame $\mathfrak{F}$ is of *width* n if it contains an antichain of n worlds but no antichain of greater cardinality. A transitive rooted frame is called *pseudolinear* if its root is the only world from which

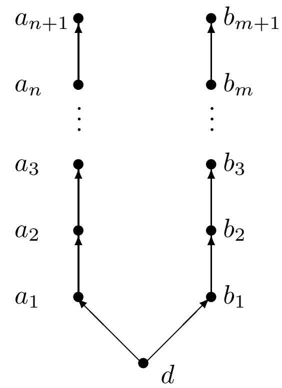

Figure 1

every antichain of the frame is accessible; in other words, in such a frame branching is possible only at the root.

The semantic meaning of the formulas la and bw_n is given in [1] (p. 80, p. 83): la is valid in those transitive Kripke frames that contain no infinite ascending chains, bw_n is valid in those rooted frames that are of width $\leq n$. For $\Box bw_1$, one can easily show that $\Box bw_1$ is valid in a frame $\mathfrak{F}$ iff $\mathfrak{F}$ is pseudolinear (the corresponding proof is similar to the one for bw_n presented in [1]).

Thus we have

LEMMA 5. *If $\mathfrak{F} \models L$, then $\mathfrak{F}$ is a transitive, irreflexive, pseudolinear frame of width ≤ 2 without infinite ascending chains.*

DEFINITION 6. Transitive, irreflexive, pseudolinear frames of width ≤ 2 without infinite ascending chains will be called *M-frames*. Denote by U the set of all M-frames with two dead ends.

Note that the set of all finite frames from U coincides with Y.

4 Technical results

Let us consider some properties of the frames in the sets M, U, Y and some properties of reductions of frames in $M \cup U \cup Y$ to frames in the same set.

LEMMA 7. *If $\mathfrak{F} = \langle W, R \rangle$ is a frame from U with root d, and $b_1, b_2 \in W$ are the dead ends of the frame, then*

- *the sets $C(b_i) = \{w \in W \mid wRb_i \vee w = b_i\}$, $i = 1, 2$, are maximal chains of $\mathfrak{F}$;*

- *if $z \in W$ is not the root, then it belongs to exactly one chain in the frame;*
- *if $\mathfrak{c} \subseteq W$ is a chain of the frame $\mathfrak{F}$, then there exists a unique $i \in \{1,2\}$ such that $\mathfrak{c} \subseteq C(b_i)$;*
- *$\mathfrak{F}$ has only two maximal chains.*

Proof. Let us show that $C(b_i) = \{w \in W \mid wRb_i \vee w = b_i\}$, $i = 1,2$, is a chain. Without loss of generality, let us assume that $i = 1$. Suppose that the set $\{z_1, z_2\}$, where $z_1, z_2 \in C(b_1)$, $z_1 \neq z_2$, is an antichain. Since the frame $\mathfrak{F}$ is pseudolinear and $z_1, z_2 \neq d$, we obtain that $\{z_1, z_2, b_2\}$ is an antichain. Now let us prove the maximality. Suppose that, for some $z \in W$ such that $z \notin C(b_1)$, the set $C(b_1) \cup \{z\}$ is a chain. Since $\neg zRb_1$, we have b_1Rz.

If $z \in W$ and $z \notin C(b_i)$, for any $i = 1,2$, then $\{b_1, b_2, z\}$ is an antichain. If $z \in W$ and $z \in C(b_i)$, for any $i = 1,2$, then $z = d$.

If $\mathfrak{c}$ is a chain in $\mathfrak{F}$ and $\mathfrak{c} \nsubseteq C(b_i)$, for any $i = 1,2$, then there exist two worlds $z_1, z_2 \in \mathfrak{c}$, such that $z_1 \notin C(b_1)$ and $z_2 \notin C(b_2)$ $(z_1, z_2 \neq d)$. Therefore $z_1 \in C(b_2)$ and $z_2 \in C(b_1)$. If z_1Rz_2, then z_1Rb_1. If z_2Rz_1, then z_2Rb_2.

The last statement follows immediately from the previous one. ■

By the definition of subreduction we obtain the following technical results.

LEMMA 8. *Suppose $\mathfrak{F} = \langle W, R\rangle$ and $\mathfrak{G} = \langle V, S\rangle$ are frames, $\{u_1, \dots, u_n\}$ is an antichain in the frame $\mathfrak{G}$, $f : \mathfrak{F} \to \mathfrak{G}$ is a subreduction of $\mathfrak{F}$ to $\mathfrak{G}$, and $x_1 \in f^{-1}(u_1)$, …, $x_n \in f^{-1}(u_n)$. Then $\{x_1, \dots, x_n\}$ is an antichain in $\mathfrak{F}$.*

Proof. Let $x_i \in f^{-1}(u_i)$, $x_j \in f^{-1}(u_j)$, $i \neq j$, and x_iRx_j. It follows from the definition of subreduction that $f(x_i)Sf(x_j)$, and so u_iSu_j. ■

COROLLARY 9. *Suppose that $\mathfrak{F} = \langle W, R\rangle$ is an M-frame, $\mathfrak{G} = \langle V, S\rangle$ is an M-frame of width 2, and there exists a subreduction $f : \mathfrak{F} \to \mathfrak{G}$. Then $\mathfrak{F}$ is a frame of width 2.*

LEMMA 10. *Suppose $\mathfrak{F} = \langle W, R\rangle$ and $\mathfrak{G} = \langle V, S\rangle$ are frames, S is an irreflexive relation on V, $f : \mathfrak{F} \to \mathfrak{G}$ is a subreduction of $\mathfrak{F}$ to $\mathfrak{G}$, and $v \in V$. Then $f^{-1}(v)$ is an antichain.*

Proof. Let $x, y \in f^{-1}(v)$, $x \neq y$, and xRy. By the definition of subreduction, we have $f(x)Sf(y)$, and so vSv. ■

LEMMA 11. *Let $\mathfrak{F} = \langle W, R\rangle$ and $\mathfrak{G} = \langle V, S\rangle$ be frames in U and $f : \mathfrak{F} \to \mathfrak{G}$ be a subreduction of $\mathfrak{F}$ to $\mathfrak{G}$. If v is a root of the frame $\mathfrak{G}$, then $\|f^{-1}(v)\| = 1$ and $f^{-1}(v)$ is a root of the frame $\mathfrak{F}$.*

Proof. Let $\{z_1, z_2\}$ be an antichain of $\mathfrak{G}$. If $x \in f^{-1}(v)$ and x is not the root $\mathfrak{F}$, then there exist worlds $y_1 \in f^{-1}(z_1)$ and $y_2 \in f^{-1}(z_2)$ such that xRy_1 and xRy_2. Since $\{y_1, y_2\}$ is an antichain, we obtain a contradiction with $\mathfrak{F}$ being pseudolinear. ■

LEMMA 12. *If $\mathfrak{F} = \langle W, R\rangle$ and $\mathfrak{G} = \langle V, S\rangle$ are frames in U and $f : \mathfrak{F} \to \mathfrak{G}$ is a subreduction of $\mathfrak{F}$ to $\mathfrak{G}$, then for any world $v \in V$, we have $\|f^{-1}(v)\| = 1$.*

Proof. It is sufficient to consider the case when v is not a root of $\mathfrak{G}$. Suppose there are $x_1, x_2 \in W$ such that $x_1 \neq x_2, \{x_1, x_2\} \subseteq f^{-1}(v)$. Then $\{x_1, x_2\}$ is an antichain. Let b_1, b_2 be dead ends of the frame $\mathfrak{G}$ and $v \in C(b_1)$. Then $x_1, x_2 \in C(f^{-1}(b_1))$. Note that $f^{-1}(b_2) \neq \varnothing$. Let $y \in f^{-1}(b_2)$. Then $\{x_1, x_2, y\}$ is an antichain. It follows that $\mathfrak{F}$ is a frame of width 3. ■

COROLLARY 13. *If $\mathfrak{F} = \langle W, R\rangle$ and $\mathfrak{G} = \langle V, S\rangle$ are frames in U and $f : \mathfrak{F} \to \mathfrak{G}$ is a subreduction of $\mathfrak{F}$ to $\mathfrak{G}$, then for any worlds $w, v \in W$, we have $f(w)Sf(v)$ iff wRv.*

Proof. Let $x = f(w)$, $y = f(v)$, and xSy. It follows from the definition of subreduction that $f(w)Sf(v)$ implies $\exists u \in \mathit{Dom}\, f(wRu \wedge f(u) = f(v))$. If $u \neq v$ then $\|f^{-1}(y)\| > 1$. We obtain a contradiction with Lemma 12. Hence $u = v$, and so wRv. ■

COROLLARY 14. *If $\mathfrak{F} = \langle W, R\rangle$ and $\mathfrak{G} = \langle V, S\rangle$ are frames from U and $f : \mathfrak{F} \to \mathfrak{G}$ is a subreduction of $\mathfrak{F}$ to $\mathfrak{G}$, then f is an isomorphism between $\mathit{Dom}\, f$ and V.*

COROLLARY 15. *Suppose that $\mathfrak{F} = \langle W, R\rangle$ and $\mathfrak{G} = \langle V, S\rangle$ are frames from U, and $f : \mathfrak{F} \to \mathfrak{G}$ is a subreduction. Then we have:*

- *if $\mathfrak{c} \subseteq W$ is a chain such that $f(\mathfrak{c}) \neq \varnothing$, then $f(\mathfrak{c})$ is a chain of $\mathfrak{G}$;*
- *for any chain $\mathfrak{d} \subseteq V$, the set $f^{-1}(\mathfrak{d})$ is a chain of $\mathfrak{F}$.*

LEMMA 16. *Let $\mathfrak{F} = \langle W, R\rangle$ and $\mathfrak{G} = \langle V, S\rangle$ be frames and $f : \mathfrak{F} \to \mathfrak{G}$ be a cofinal subreduction of $\mathfrak{F}$ to $\mathfrak{G}$. If $b \in V$ is a dead end of $\mathfrak{G}$ then every world in $f^{-1}(b)$ is a dead end.*

Proof. If $x \in f^{-1}(b)$ and x is not a dead end, then there is $y \in W$ such that xRy. Therefore $y \in (\mathit{Dom}\, f){\uparrow}$. Now, by the definition of cofinal subreduction, we obtain the either $y \in \mathit{Dom} f$ or there exists $z \in \mathit{Dom}\, f$ such that $xRyRz$. Therefore, it follows that there is a world $w \in \{f(y), f(z)\}$ such that bSw. Hence b is not a dead end of the frame $\mathfrak{G}$. ■

Since any frame from U has two dead ends, we obtain

COROLLARY 17. *If $\mathfrak{F} = \langle W, R\rangle$ is an M-frame, $\mathfrak{G} = \langle V, S\rangle$ is a frame from U, $f : \mathfrak{F} \to \mathfrak{G}$ is a cofinal subreduction, then $\mathfrak{F} \in U$.*

LEMMA 18. *Let $\mathfrak{F} = \langle W, R\rangle$ be a frame from U, $\mathfrak{G} = \langle V, S\rangle \in Y$, a world $b_1 \in V$ be a dead end, and $\mathfrak{c} \subseteq W$ be a maximal chain of $\mathfrak{F}$. Let $f : \mathfrak{F} \to \mathfrak{G}$ be a cofinal subreduction such that $f^{-1}(C(b_1)) \subseteq \mathfrak{c}$. Let $\mathfrak{D}$ be a set of antichains in $\mathfrak{G}$ and all singletons belong to $\mathfrak{D}$. If $\|C(b_1)\| < \|\mathfrak{c}\|$, then f does not satisfy (CDC) for $\mathfrak{D}$.*

Proof. Since f is one-to-one mapping, we have $\|f^{-1}(C(b_1))\| < \|\mathfrak{c}\|$. So there exists $x \in \mathfrak{c} \setminus f^{-1}(C(b_1))$ such that $x \notin Dom\, f$. Since x is not a dead end, we obtain that $x \in (Dom\, f)\downarrow$.

Let us denote by a_1 a dead end of $\mathfrak{F}$ such that $f(a_1) = b_1$. Note that $a_1 \in x\uparrow \cap Dom\, f$, and hence $x\uparrow \cap Dom\, f \neq \varnothing$. Since $\mathfrak{G}$ is a finite frame, it follows that there is a world $y \in x\uparrow \cap Dom\, f$ such that for any $z \in \mathfrak{c}$, the relation $xRzRy$ implies $z \notin Dom\, f$.

Let us prove that $f(x\uparrow) = f(y)\underline{\uparrow}$. Since xRy, we have $f(y)\underline{\uparrow} \subseteq f(x\uparrow)$. Suppose $w \neq f(y)$ and $w \in f(x\uparrow)$. Then there exists $z \in x\uparrow$, such that $f(z) = w$. Since $x\uparrow$ is a chain, we obtain yRz. This means that $f(y)Sw$. Therefore, we have $f(x\uparrow) = f(y)\underline{\uparrow}$. Since $\{f(y)\} \in D$, we obtain a contradiction with (CDC) for $\mathfrak{D}$. ■

LEMMA 19. *Let $\mathfrak{F} = \langle W, R\rangle$ be a frame in U, and worlds b_1, b_2 be the dead ends of $\mathfrak{F}$. If for some $n, m, \geq 0$ we have either $\|C(b_1)\| \neq n+1$ and $\|C(b_1)\| \neq m+1$, or $\|C(b_2)\| \neq n+1$ and $\|C(b_2)\| \neq m+1$, then there is no cofinal subreduction $f : \mathfrak{F} \to Y_{n,m}$ satisfying (CDC) for $\mathfrak{D}$, where $\mathfrak{D}$ is a set of antichains in the frame $Y_{n,m}$ and all singletons belong to $\mathfrak{D}$.*

Proof. Suppose that w_1, w_2 are dead ends of $Y_{n,m}$ and $\|C(b_1)\| = r$, $\|C(w_1)\| = n+1$, $\|C(w_2)\| = m+1$. Without loss of generality, we may assume that $r < n+1$, $r > m+1$. In this case we have $\|C(b_1)\| < \|C(w_1)\|$. By Corollary 14, there is no subreduction f such that $f(C(b_1)) \subseteq C(w_1)$, for otherwise we would have $\|C(b_1)\| > \|C(w_2)\|$. By taking into account Lemma 18, we obtain that a subreduction f with $f(C(b_1)) \subseteq C(w_2)$, does not satisfy (CDC) for $\mathfrak{D}$. ■

LEMMA 20. *Let $\mathfrak{F} = \langle W, R\rangle$ be a frame from U, $\mathfrak{G} = \langle V, S\rangle \in Y$, $f : \mathfrak{F} \to \mathfrak{G}$ a subreduction. The subreduction f satisfies (CDC) for any antichain $\{u_1, u_2\}$ of the frame.*

Proof. Let us show that for every $x \in W$ if $f(x\uparrow) = \{u_1, u_2\}\underline{\uparrow}$ then $x \in Dom\, f$. Suppose $f(y_1) = u_1$, $f(y_2) = u_2$. Then $y_1, y_2 \in x\uparrow$. It follows from Lemma 7 that $\{y_1, y_2\}$ is antichain. Hence x is a root of $\mathfrak{F}$. Using Lemma 11, we obtain $x \in Dom\, f$. ■

From Lemma 19 and Lemma 20 we obtain the following

LEMMA 21. *Suppose that $\mathfrak{D}$ is a set of antichains in the frame $Y_{n,m}$ and that all singletons belong $\mathfrak{D}$. There exists a cofinal subreduction from Y_{rs} to Y_{nm} satisfying (CDC) for $\mathfrak{D}$ iff $r = n$ and $s = m$ or $r = m$ and $s = n$.*

5 Undecidability of L_{fin}

Let L_{fin} be the logic of all finite frames for the logic L defined above. Let us prove that L_{fin} is undecidable. For the proof we need the following formulas:

$$\gamma_{nm} = (\Diamond(\Box\bot \wedge p) \wedge \Diamond(\Box\bot \wedge \neg p)) \to ((\Diamond^{n+1}(\Box\bot \wedge p) \wedge \neg\Diamond^{n+2}(\Box\bot \wedge p)) \to (\Diamond^{m+1}(\Box\bot \wedge \neg p) \wedge \neg\Diamond^{m+2}(\Box\bot \wedge \neg p))),$$

where n, m are arbitrary natural numbers.

LEMMA 22. *The formula γ_{nm} is refuted in a finite frame $\mathfrak{F} = \langle W, R\rangle$ for the logic L iff $\mathfrak{F}$ is isomorphic to Y_{rs}, where $n = s$ and $m \neq r$.*

Proof. ($\Rightarrow$) Suppose that, for some valuation ν, we have $w \not\models^{\nu} \gamma_{nm}$. Then $w \models^{\nu} \Diamond(\Box\bot \wedge p) \wedge \Diamond(\Box\bot \wedge \neg p)$. Thus two different dead ends of $\mathfrak{F}$ are accessible from the world w. It follows that $\mathfrak{F} \in U$ and the world w is a root of $\mathfrak{F}$. Since $\mathfrak{F}$ is finite, we obtain that $\mathfrak{F} \in Y$.

Denote by b_1, b_2 the dead ends in $\mathfrak{F}$. Suppose $b_1 \in \nu(p)$ and $b_2 \notin \nu(p)$. As $(\Diamond^{n+1}(\Box\bot \wedge p) \wedge \neg\Diamond^{n+2}(\Box\bot \wedge p)) \to (\Diamond^{m+1}(\Box\bot \wedge \neg p) \wedge \neg\Diamond^{m+2}(\Box\bot \wedge \neg p))$ is refuted at the root w, we obtain $w \models^{\nu} \Diamond^{n+1}(\Box\bot \wedge p) \wedge \neg\Diamond^{n+2}(\Box\bot \wedge p)$. Hence the world b_1 is accessible from the root in $n+1$ steps, that is, there is a maximal chain $x_1 R x_2 R \ldots R x_{n+2}$, where $x_1 = w$ and $x_{n+2} = b_1$. Otherwise we have $w \not\models^{\nu} \Diamond^{m+1}(\Box\bot \wedge \neg p) \wedge \neg\Diamond^{m+2}(\Box\bot \wedge \neg p)$. Therefore b_2 is not accessible in $m+1$ steps and $\|C(b_2)\| \neq m+2$.

($\Leftarrow$) This direction is trivial. ■

Observe that the frame Y_{nm} is a frame for the logic L (this means that Y_{nm} is a frame for L_{fin} as well) iff $m = h(n)$. Thus we have

LEMMA 23. *We have $\gamma_{nh(n)} \notin L_{fin}$ iff $n \in Im\, h$, for any $n \geq 0$ such that $n \neq h(n)$ and $n \neq h^2(n)$.*

Proof. ($\Leftarrow$) Suppose $n \in Im\, h$. Then there exists $k \geq 0$ such that $h(k) = n$. Since $n \neq h(n)$ and $h^2(n) \neq n$, we obtain $k \neq n$ and $h(n) \neq k$. Thus the formula $\gamma_{nh(n)}$ is refuted in the frame Y_{kn}. Otherwise, since $h(k) = n$, we have that the frame Y_{kn} is a frame for the logic L_{fin}.

($\Rightarrow$) Suppose that $\gamma_{nh(n)} \notin L_{fin}$. Then there is a frame $\mathfrak{F}$ for L_{fin} such that $\mathfrak{F} \not\models \gamma_{nh(n)}$. Thus by Lemma 22 the frame $\mathfrak{F}$ is isomorphic to Y_{rs} with $r = n$ and $s \neq h(n)$. Otherwise, since Y_{rs} is a frame for L_{fin}, we have $s = h(r)$ or $r = h(s)$. Let $s = h(r)$, that is, $s = h(n)$. It follows that $r = h(s)$, and so $n = h(s)$ for any $s \geq 0$. Thus $n \in Im\, h$. ■

COROLLARY 24. *The logic L_{fin} is undecidable.*

Proof. Suppose the logic L_{fin} is decidable. Let us show that in this case the set $Im\, h$ is recursive. Indeed, for any $n \geq 0$, we can easily verify the conditions $n = h(n)$ and $n = h^2(n)$, from which it follows that $n \in Im\, h$. If these conditions are not satisfied, then we can use the formulas $\gamma_{nh(n)}$: if $\gamma_{nh(n)} \notin L_{fin}$, then $n \in Im\, h$. ■

6 Decidability of L

Let us prove that the logic L is decidable.

LEMMA 25. *Let $\mathfrak{F}$ be an M-frame and let $\mathfrak{G}$ be a frame satisfying one of the following conditions:*

- *it is of width ≥ 3;*
- *it contains a reflexive world;*
- *it is not pseudolinear.*

Then there is no subreduction of $\mathfrak{F}$ to $\mathfrak{G}$.

Proof. If the frame $\mathfrak{G}$ contains a three-element antichain, then by Lemma 8, its preimage must also contain an antichain of cardinality ≥ 3.

If $\mathfrak{G}$ contains a reflexive world, then its preimage is an infinite ascending chain.

If the frame $\mathfrak{G}$ contains an antichain which is accessible from some non-root world w, then the pseudolinearity of the $\mathfrak{F}$ implies that the preimage of w is a root of the frame $\mathfrak{F}$. ■

LEMMA 26. *Let $\mathfrak{D}$ be a set of antichains in $Y_{n,m}$. If $\mathfrak{D}$ does not contain all non-root singletons, then there is an infinite frame of the logic L such that it is cofinally subreducible to $Y_{n,m}$ and the corresponding subreduction satisfies (CDC) for $\mathfrak{D}$.*

Proof. A maximal chain in the set of all chains of the form $\{a_1, \ldots, a_s, \ldots, a_k\}$ (or $\{b_1, \ldots, b_s, \ldots, b_k\}$) from a frame $Y_{n,m}$ (where s takes every value in interval $[1, k]$, $1 \leq k \leq l$, $l \in \{n, m\}$; see Fig. 1) such that $\{a_i\} \in \mathfrak{D}$ (respectively, $\{b_i\} \in \mathfrak{D}$), $1 \leq i \leq s$, is called a *rooted segment* for the pair $\langle Y_{n,m}, \mathfrak{D}\rangle$. A rooted segment is called a *full rooted segment* if $s = l + 1$.

Let us consider first the case when neither of the rooted segments is full and one of them contains s worlds and the other one contains r worlds. Denote by U_{sr} the infinite frame shown in Fig. 2.

By using Lemma 22, it is not difficult to prove that the frame U_{sr} is a frame for L. A cofinal subreduction f of U_{sr} to Y_{nm} satisfying (CDC) for $\mathfrak{D}$ is defined as follows:

- $f(x_i) = a_i, f(v_k) = a_{n+2-k}, 1 \leq k \leq n + 1 - s, 1 \leq i \leq s;$
- $f(y_j) = b_j, f(u_l) = b_{m+2-l}, 1 \leq l \leq m + 1 - r, 1 \leq j \leq r;$
- $f(d_1) = d.$

If one of the rooted segment is full (i.e., $s = n + 1$), then consider the infinite frame for L shown in Fig. 3, which will be denoted by X_{nr}. The cofinal subreduction f of X_{nr} to Y_{nm} satisfying (CDC) for $\mathfrak{D}$ is defined as follows:

- $f(x_i) = a_i, 1 \leq i \leq n + 1;$

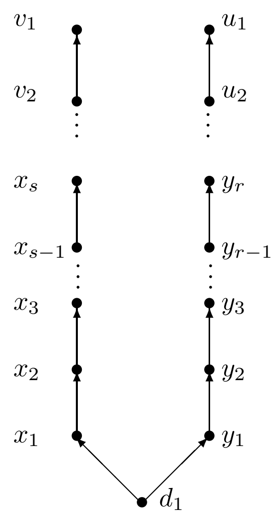

Figure 2

- $f(y_j) = b_j, f(u_l) = b_{m+2-l}, 1 \leq l \leq m+1-r, 1 \leq j \leq r$;
- $f(d_2) = d$.

Observe that $f(d_1) = d$, $f(d_2) = d$, $f(v_1) = a_{n+1}$, $f(u_1) = b_{m+1}$, and $f(x_s) = a_{n+1}$. Thus $w \in \mathit{Dom}\ f\uparrow \cap \overline{\mathit{Dom}\ f\downarrow}$, for any world w in U_{sr} or X_{nr}, i.e., f is cofinal. For (CDC), by Lemma 20, it is sufficient to observe that (CDC) holds for any one-element subset of $\mathfrak{D}$. Let $\{w\} \in \mathfrak{D}$. If w is in a rooted segment, then $f(z\uparrow) = \underline{w\uparrow}$ implies $z = x_i$ or $z = y_j$, where $1 \leq i \leq s$, $1 \leq j \leq r$; in both cases $z \in \overline{\mathit{Dom}}\ f$. If w is not in any rooted segment and $\{w\} \in \mathfrak{D}$ then $f(z\uparrow) = \underline{w\uparrow}$ implies $z = v_k$ or $z = u_l$, where $1 \leq k < n+1-s$, $1 \leq l < m+1-r$; again $z \in \mathit{Dom}\ f$. ■

LEMMA 27. *The logic L is decidable.*

Proof. Here we describe an algorithm deciding, for an arbitrary formula φ, whether it belongs to the logic L or not. It is proved in [6] (see also [1], p. 309) that there is an algorithm which for a given modal formula φ, returns a finite set of pairs $\{\langle \mathfrak{F}_1, \mathfrak{D}_1\rangle, \ldots, \langle \mathfrak{F}_n, \mathfrak{D}_n\rangle\}$ (where $\mathfrak{F}_i$ is a finite rooted frame and $\mathfrak{D}_i$ is a set of antichains in it) such that, for any frame $\mathfrak{F}$, $\mathfrak{F} \not\models \varphi$ iff there is a cofinal subreduction $f : \mathfrak{F} \to \mathfrak{F}_i$, for some $1 \leq i \leq n$, satisfying (CDC) for $\mathfrak{D}_i$.

For every frame of the list $\mathfrak{F}_1 \ldots \mathfrak{F}_n$ we check whether this frame is a finite M-frame or not. If M-frames do not belong to the list, then for any frame $\mathfrak{F}$ for L and any frame $\mathfrak{F}_i$ of the list, there is no subreduction of $\mathfrak{F}$ to $\mathfrak{F}_i$. Hence $\varphi \in L$.

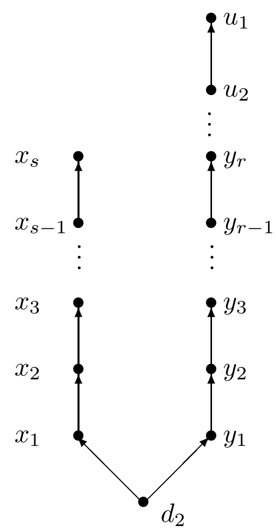

Figure 3

Suppose the list contains some M-frames. Denote them by $\mathfrak{F}_{\alpha_1} \dots \mathfrak{F}_{\alpha_m}$, $m \leq n$. Suppose $\mathfrak{F}_{\alpha_i} \notin Y$, for some $1 \leq i \leq m$. Then $\mathfrak{F}_{\alpha_i}$ is frame for L, and so $\varphi \notin L$.

Let $\{\mathfrak{F}_{\alpha_1} \dots \mathfrak{F}_{\alpha_m}\} \subseteq Y$. Then every frame $\mathfrak{F}_{\alpha_i}$ is equal to $Y_{n,m}$, for some pair (n, m). We can check that $m = h(n)$, for any such frame. If there is a pair (n, m) such that $m = h(n)$, then $Y_{n,m}$ is a frame of the logic L. Hence $\varphi \notin L$.

Suppose that the condition $m = h(n)$ does not hold for any frame of the list. Then by Lemma 20 and Lemma 26 it is enough to consider the rooted segments for the pairs $\langle \mathfrak{F}_{\alpha_i}, \mathfrak{D}_{\alpha_i} \rangle$ of the list. Since every set $\mathfrak{D}_{\alpha_i}$ of the list is finite, we can effectively check if the rooted segment corresponding to the pair $\langle \mathfrak{F}_{\alpha_i}, \mathfrak{D}_{\alpha_i} \rangle$ is full. If it is so for the pair $\langle \mathfrak{F}_{\alpha_i}, \mathfrak{D}_{\alpha_i} \rangle$ of the list, then any cofinal subreduction of $\mathfrak{F}_{\alpha_i}$ to some frame $\mathfrak{G}$ satisfies (CDC) for $\mathfrak{D}_{\alpha_i}$ only if $\mathfrak{G}$ is isomorphic to $\mathfrak{F}_{\alpha_i}$. Hence, if both rooted segments are full, for every pair $\langle \mathfrak{F}_{\alpha_i}, \mathfrak{D}_{\alpha_i} \rangle$ of the list, then $\varphi \in L$.

If it is not the case, then by Lemma 26, $\varphi \notin L$.

This completes our description of the decision algorithm. ■

Theorem 1 is follows immediately from Corollary 24 and Lemma 27.

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