A General Semantics for Quantified Modal Logic[1]

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abstract. In [9] we developed a semantics for quantified relevant logic that uses general frames. In this paper, we adapt that model theory to treat quantified modal logics, giving a complete semantics to the quantified extensions, both with and without the Barcan formula, of every propositional modal logic S. If S is *canonical* our models are based on propositional frames that validate S. We employ frames in which not every set of worlds is an admissible proposition, and an alternative interpretation of the universal quantifier using greatest lower bounds in the lattice of admissible propositions. Our models have a fixed domain of individuals, even in the absence of the Barcan formula.

For systems with the Barcan formula it is possible to preserve the usual Tarskian reading of the quantifier, at the expensive of sometimes losing validity of S in the underlying propositional frames. We apply our results to a number of logics, including S4.2, S4M and KW, whose quantified extensions are incomplete for the standard semantics.

Keywords: Quantified modal logic, general frame, Tarskian, canonical logic, completeness, incompleteness, Barcan formula.

1 Introduction

A general semantics for quantified relevant logic is developed in [14]. In this paper, we adapt this to treat quantified modal logic, providing a complete semantics for the standard predicate extension QS of any propositional modal logic S, as well as for the logic QSB obtained by adding the Barcan formula to QS. In particular we provide complete semantics for modal predicate logics known to be incomplete for their usual possible worlds model theories. We employ *general* frames, in which not every set of worlds is admissible as a proposition, and use the set of admissible propositions to give an alternative interpretation of the universal quantifier $\forall$. Our results are applied to a number of logics

The idea behind the semantics for the universal quantifier is the following. Suppose that a formula A has at most the variable x free. The formula $\forall xA$ is true at a world a if and only if there is some proposition X such that X entails every instantiation of A and X obtains at a. This needs some explanation. A proposition on our theory is a set of worlds. A proposition

[1] Supported by grant 05-VUW-079 from the Marsden Fund of the Royal Society of New Zealand.

obtains at a world if that world is in that proposition. A proposition X entails a formula A if every world in X makes A true. More formally we have the following:

$$\mathcal{M}, a \models_f \forall x_n A \text{ iff there is a proposition } X \text{ such that } a \in X \text{ and } X \subseteq |A|^{\mathcal{M}}_{f[j/n]} \text{ for all } j \in I.$$

A bit more explanation is in order. Here $\mathcal{M}$ is a model, a is a world, and f is an assignment to individual variables taking values in a domain I of individuals. The notation $f[j/n]$ refers to the function that is just like f except that it assigns the individual j to x_n. $|A|^{\mathcal{M}}_{f[j/n]}$ is the set of worlds at which formula A is true under $f[j/n]$, representing the proposition expressed by A when the variable x_n is instantiated to j. Note that the proposition X need not be $|\forall x_n A|^{\mathcal{M}}_f$ itself, so this is not the standard semantics for $\forall$.

The integrating of general propositions into our semantics for relevant logics can be justified by our attempt to obtain a theory of partial information. A general proposition is the information that informs one of the truth of a universal statement. But there are other philosophical projects that utilize general propositions. Recently David Armstrong has proposed his *truth-maker principle,* that every true statement is made true by some truth-maker (in the case of Armstrong, truth-makers are facts) (see [1] and [2]). Thus, to any true universal statement there must be some fact that corresponds to it – a general fact. In our parlance, a fact is just a true proposition. One way of understanding the model theory that we present in the current paper is as a modal theory of truth-makers that includes general propositions.

This paper shows that we can produce a complete semantics for any logic that results from adding the standard axioms and rules for quantification to a propositional modal logic. That there is a class of general frames over which these logics are complete is hardly surprising. General frames are essentially algebras in represented form, and every such logic has an algebraic semantics over which it is complete (i.e. the class consisting in just the Lindenbaum algebra of that logic). There is, however, something much more interesting about the present semantics. As we show, *for any* ***canonical*** *modal logic S, its quantified extension QS is complete over a class of general frames for which the underlying propositional frames are just the S-frames.* This means that our semantics for quantified logics just "sits gingerly on top of" the semantics for the corresponding canonical propositional logic.

We also explore some consequences of adding the Barcan formula to any of our logics QS. We show that the resulting system QSB is complete over a special class of our frames that we call "Tarskian", in which $\forall$ gets its classical reading of

$$\mathcal{M}, a \models_f \forall x_n A \quad \text{iff} \quad \text{for all } j \in I,\ \mathcal{M}, a \models_{f[j/n]} A.$$

But the price paid for this is that the underlying propositional frames of these Tarskian general frames may fail to validate S. On the other hand we also have another complete class of general frames for QSB which may be based on propositional frames for S while giving a non-Tarskian reading for $\forall$, *and yet still validating the Barcan formula.* This gives a new perspective on the semantics of the Barcan formula. Instead of seeing it as the axiom corresponding to constant domain models – indeed our models have a fixed domain I of individuals even for systems without the Barcan formula – we see it as the principle needed to ensure that we can confine ourselves to models that give the standard Tarskian interpretation of $\forall$. But at a price.

The range of possibilities here is illustrated by reference to models of a number of well-known logics whose quantified extensions are incomplete for the standard semantics, including S4.2, S4+ the McKinsey axiom, and the provability logic KW.

2 Logics

We use two languages in this paper. The first is a standard propositional modal language, $\mathcal{L}$, that includes an infinite set of propositional variables, the connectives $\supset$, $\Box$, and $\bot$, and parentheses. This language has the standard formation rules and the connectives $\neg$, $\wedge$, $\vee$, $\equiv$, and $\Diamond$ are defined in the usual way. We use lower case letters from the first part of the Greek alphabet as metavariables that range over formulas of $\mathcal{L}$.

The second language is a standard predicate language, $\mathcal{LQ}$, which contains a countable list of individual variables $(x_0, x_1, ...)$; some or all of: individual constants, predicate letters and function symbols; the same connectives as $\mathcal{L}$; and the universal quantifier $\forall$. The existential quantifier is defined in the usual way. We use upper case letters from the first part of the Roman alphabet as metavariables that range over formulas of $\mathcal{LQ}$.

Our paper concerns a class of normal propositional modal logics and their predicate extensions. Each of these propositional logics is a set of $\mathcal{L}$-formulas that includes the theorems of the logic K, that is, it contains all substitution instances of the theorems of the propositional calculus and all instances of the K-schema $(\Box(\alpha \supset \beta) \supset (\Box\alpha \supset \Box\beta))$, as well as being closed under the rules,

$$\text{MP}\ \frac{\vdash \alpha \supset \beta,\ \vdash \alpha}{\vdash \beta}, \qquad \text{N}\ \frac{\vdash \alpha}{\vdash \Box\alpha},$$

and the rule of uniform substitution.

Each of the predicate logics fulfils this same description but for $\mathcal{L}Q$-formulas, and includes all instances of the schema

$$\text{(UI)}\quad \forall x A \supset A[\tau/x], \quad \text{where } x \text{ is free for term } \tau \text{ in } A,$$

and is also closed under the rule of restricted generalization[1], viz.,

[1] In [14], we called the relevant version of this rule "RIC", for the "rule of intensional confinement". This name was useful in that context in order to link it with the schema $\forall x(A \to B) \to (A \to \forall x B)$, where x is not free in A. This schema is usually called "confinement" and can be derived in the logic R from RIC. Moreover, we wished to distinguish this sort of confinement from the more "extensional" variety that we find in the schema $\forall x(A \vee B) \to (A \vee \forall x B)$, where x is not free in A. This latter schema is not derivable in our base system QR.

$$\text{RGen} \quad \frac{\vdash A \supset B}{\vdash A \supset \forall x B}, \text{ where } x \text{ is not free in } A.$$

Where S is a propositional modal logic as just defined, we call QS the smallest predicate logic that contains all the $\mathcal{L}Q$-substitution-instances of theorems of S.

We assume from the outset that $\mathcal{L}Q$ has infinitely many individual constants. For otherwise we could add such constants in a way that is conservative with respect to our finitary proof theory, as is well known.

We also assume that the set of propositional variables of $\mathcal{L}$ is at least as large as the set of $\mathcal{L}Q$-formulas. This will be used in the proof of Lemma 5 (see also the Appendix).

3 Frames and Models

As usual a frame for propositional modal logic is a pair $\mathcal{F} = \langle W, R\rangle$, where W is a non-empty set (of "worlds") and R is a binary relation on W. A *model* $\mathcal{M} = \langle W, R, v\rangle$ on $\mathcal{F}$ is given by a valuation v mapping each propositional variable to a subset of W. Each model determines a satisfaction relation $\mathcal{M}, a \models \alpha$, expressing truth/satisfaction of α at world a, as follows:

- $\mathcal{M}, a \models p$ iff $a \in v(p)$, for all propositional variables p;
- $\mathcal{M}, a \models \alpha \supset \beta$ iff $\mathcal{M}, a \not\models \alpha$ or $\mathcal{M}, a \models \beta$;
- $\mathcal{M}, a \not\models \bot$;
- $\mathcal{M}, a \models \Box\alpha$ iff $\forall b \in W(aRb$ implies $\mathcal{M}, b \models \alpha)$.

A formula A is *valid in* $\mathcal{M}$ if and only if it is satisfied at every world in $\mathcal{M}$, and *valid in* $\mathcal{F}$ if and only if it is valid in every model on $\mathcal{F}$. A logic S is *sound* over a class of frames if and only if every theorem of S is valid in that class of frames (i.e. valid in all members of the class). S is *complete* for a class when every formula valid in the class is an S-theorem; and *characterised* by a class when it both sound over and complete for that class. We say that a frame $\mathcal{F}$ is an *S-frame* if S is sound over $\{\mathcal{F}\}$.

Recall that $\langle W', R'\rangle$ is a *generated subframe* of $\langle W, R\rangle$ if it is a substructure of $\langle W, R\rangle$ that is R-closed in the sense that if $a \in W'$ and aRb, then $b \in W'$. In this case, any formula valid in $\langle W, R\rangle$ will be valid also in $\langle W', R'\rangle$.

A given frame $\langle W, R\rangle$ determines a function $[R] : \wp W \to \wp W$, where $\wp W$ is the powerset of W, having

$$[R]X = \{a \in W : \ \forall b \in W(aRb \text{ implies } b \in X)\}.$$

Then if $|\alpha|^{\mathcal{M}} = \{a : \mathcal{M}, a \models \alpha\}$ is the "truth set" of α defined by $\mathcal{M}$, we see that $[R]|\alpha|^{\mathcal{M}} = |\Box\alpha|^{\mathcal{M}}$. Also, if $X \Rightarrow Y = (W \backslash X) \cup Y$, then $|\alpha|^{\mathcal{M}} \Rightarrow |\beta|^{\mathcal{M}} = |\alpha \supset \beta|^{\mathcal{M}}$.

The operations $[R]$ and $\Rightarrow$ on $\wp W$ can be lifted to operations on functions of the form $I^{\omega} \to \wp W$, where I is any given set. If φ and ψ are two such functions, we define functions $[R]\varphi$ and $\varphi \Rightarrow \psi$ of the same form by putting $([R]\varphi)f = [R](\varphi f)$ and $(\varphi \Rightarrow \psi)f \ = \ (\varphi f \Rightarrow \psi f)$ for all $f \in I^\omega$.

These operations will be used in defining frames for predicate logics. For this purpose, fix a set $\mathit{Prop} \subseteq \wp W$. Then Prop determines an operation $\bigsqcap : \wp\wp W \to \wp W$ defined, for each $S \subseteq \wp W$, by putting

$$\bigsqcap S = \bigcup\{X \in \mathit{Prop} : X \subseteq \bigcap S\}.$$

In general $\bigsqcap S \subseteq \bigcap S$, so $\bigsqcap S$ is a lower bound of S in the partially-ordered set $(\wp W, \subseteq)$. If $S \subseteq \mathit{Prop}$ and $\bigsqcap S \in \mathit{Prop}$, then $\bigsqcap S$ is the *greatest lower bound* of S in $(\mathit{Prop}, \subseteq)$. Moreover, if $\bigcap S \in \mathit{Prop}$, then $\bigsqcap S = \bigcap S$. *In particular, this holds when $\mathit{Prop} = \wp W$.* We emphasise that the definition of $\bigsqcap$ depends on the particular set Prop, and it may be that $\bigsqcap S \notin \mathit{Prop}$ for some $S \subseteq \mathit{Prop}$.

We use $\bigsqcap$ to define functions $\forall_n$ that interpret the quantifiers $\forall x_n$ for each $n \in \omega$. Where $f \in I^\omega$, we write $f[j/n]$ to mean the function g such that $g(m) = f(m)$ for all $m \neq n$ and $g(n) = j$. Now, for any $\varphi : \ I^\omega \longrightarrow \wp W$, we set

$$(\forall_n \varphi)f \ = \ \textstyle\bigsqcap_{j \in I} \varphi(f[j/n]).$$

A *quantified general frame* (QG-frame) $\mathcal{F} = \langle W, R, I, \mathit{Prop}, \mathit{PropFun}\rangle$ is a structure in which W is a non-empty set, R is a binary relation on W, I is a non-empty set (of "individuals"), Prop is a set of subsets of W, and $\mathit{PropFun}$ is a set of *propositional functions*, i.e. functions from I^ω to Prop, such that the following conditions hold:

CProp: $\emptyset \in \mathit{Prop}$, and if X and Y are in Prop, then so are $X \Rightarrow Y$ and $[R]X$. Hence Prop is a modal algebra of subsets of W.

CFalse: The constant function $\varphi_\emptyset \in \mathit{PropFun}$, where $\varphi_\emptyset(f) = \emptyset$ for all $f \in I^\omega$.

CImp: If $\varphi, \psi \in \mathit{PropFun}$, then $\varphi \Rightarrow \psi \in \mathit{PropFun}$.

CMod: If $\varphi \in \mathit{PropFun}$, then $[R]\varphi \in \mathit{PropFun}$.

CAll: If $\varphi \in \mathit{PropFun}$, then $\forall_n \varphi \in \mathit{PropFun}$ for all $n \in \omega$.

A *valuation* for the language $\mathcal{L}Q$ on such a frame $\mathcal{F}$ is a function V that assigns to each individual constant c an element $V(c)$ of I, to each n-ary function symbol F a function $V(F) : I^n \to I$, and to each n-ary predicate letter P a function $V(P) : I^n \to \mathit{Prop}$. Put $\mathcal{M} = \langle \mathcal{F}, V\rangle$. Then each $\mathcal{L}Q$-term τ is interpreted by $\mathcal{M}$ as a function $\tau^{\mathcal{M}} : I^\omega \to I$, defined by putting $c^{\mathcal{M}}f = V(c)$ for each constant c; $(x_n)^{\mathcal{M}}f = fn$ for each variable x_n (so f functions as a *variable-assignment*); and inductively, $(F\tau_1 \ldots \tau_n)^{\mathcal{M}}f = V(F)(\tau_1^{\mathcal{M}}f, \ldots, \tau_n^{\mathcal{M}}f)$. Each *atomic* formula $P\tau_1 \ldots \tau_n$ gets interpreted as a function $|P\tau_1 \ldots \tau_n|^{\mathcal{M}} : I^\omega \to \mathit{Prop}$ defined by

$$|P\tau_1 \ldots \tau_n|^{\mathcal{M}}(f) = V(P)(\tau_1^{\mathcal{M}}f, \ldots, \tau_n^{\mathcal{M}}f).$$

The pair $\mathcal{M} = \langle F, V\rangle$ is called a *model* on the QG-frame $\mathcal{F}$ if:

$$|A|^{\mathcal{M}} \text{ belongs to } \mathit{PropFun} \text{ for all } \mathit{atomic} \text{ formulas } A.$$

Each model has a truth/satisfaction relation $\mathcal{M}, a \models_f A$ between worlds $a \in W$, variable-assignments $f \in I^\omega$ and formulas A. This has associated truth sets $|A|^{\mathcal{M}}_f =_{df} \{b \in W : \mathcal{M}, b \models_f A\}$.[2] The inductive definition of $\mathcal{M}, a \models_f A$ is as follows:

- $\mathcal{M}, a \models_f P\tau_1 \dots \tau_n$ if and only if $a \in |P\tau_1 \dots \tau_n|^{\mathcal{M}}(f)$;
- $\mathcal{M}, a \models_f A \supset B$ iff $\mathcal{M}, a \not\models_f A$ or $\mathcal{M}, a \models_f B$;
- $\mathcal{M}, a \not\models_f \bot$;
- $\mathcal{M}, a \models_f \Box A$ iff $\forall b \in W(aRb$ implies $\mathcal{M}, b \models_f A)$.
- $\mathcal{M}, a \models_f \forall x A$ iff there is an $X \in \mathit{Prop}$ such that $X \subseteq \bigcap_{j \in I} |A|^{\mathcal{M}}_{f[j/n]}$ and $a \in X$.

The assignment $f \mapsto |A|^{\mathcal{M}}_f$ gives a function $|A|^{\mathcal{M}} : I^\omega \to \wp W$ for each formula A. These functions satisfy

- $|\bot|^{\mathcal{M}} = \varphi_\emptyset$;
- $|A \supset B|^{\mathcal{M}} = |A|^{\mathcal{M}} \Rightarrow |B|^{\mathcal{M}}$;
- $|\Box A|^{\mathcal{M}} = [R]|A|^{\mathcal{M}}$;
- $|\forall x_n A|^{\mathcal{M}} = \forall_n |A|^{\mathcal{M}}$.

These properties, and the closure properties of *PropFun*, ensure that we always have $|A|^{\mathcal{M}} \in \mathit{PropFun}$ and hence $|A|^{\mathcal{M}}_f \in \mathit{Prop}$, for any formula A. The satisfaction clause for $\forall x_n$ can be expressed as

$$|\forall x_n A|^{\mathcal{M}}_f = \bigsqcap_{j \in I} |A|^{\mathcal{M}}_{f[j/n]},$$

showing that $|\forall x_n A|^{\mathcal{M}}_f$ is the greatest lower bound of $\{|A|^{\mathcal{M}}_{f[j/n]} : j \in I\}$ in the partially ordered set $(\mathit{Prop}, \subseteq)$. This is the natural interpretation of $\forall$ in algebraic semantics. To reproduce the standard Tarskian semantics for $\forall$ we would need this greatest lower bound to be $\bigcap_{j \in I} |A|^{\mathcal{M}}_{f[j/n]}$. But this need not be so (see Section 6).

Taking $\exists x_n$ as abbreviating $\neg \forall x_n \neg$, we find that

$$|\exists x_n A|^{\mathcal{M}}_f = \bigsqcup_{j \in I} |A|^{\mathcal{M}}_{f[j/n]},$$

where $\bigsqcup$ is the operation defined, for $S \subseteq \wp W$, by

$$\bigsqcup S = \bigcap \{X \in \mathit{Prop} : \bigcup S \subseteq X\}.$$

[2]The symbols $\tau^M f$ and $|A|^{\mathcal{M}}_f$ were written $Vf\tau$ and $|A|_{Vf}$ in [14].

For $S \subseteq \mathit{Prop}$, if $\bigsqcup S$ belongs to Prop then it is the least upper bound of S in the partial-ordering $(\mathit{Prop}, \subseteq)$.

Now by a simple induction we can show that $|A|_f^{\mathcal{M}} = |A|_g^{\mathcal{M}}$ whenever f and g agree on all free variables of A [14, Lemma 4.4]. Hence if x_n is not free in A, $|A|_f^{\mathcal{M}} = |A|_{f[j/n]}^{\mathcal{M}}$ for all $j \in I$.

It can also be shown that if term τ is free for x_n in A, and $g = f[\tau^{\mathcal{M}} f/n]$, then $|A[\tau/x_n]|_f^{\mathcal{M}} = |A|_g^{\mathcal{M}} = |A|_{f[\tau^{\mathcal{M}} f/n]}^{\mathcal{M}}$ [14, Lemma 7.1].

LEMMA 1. *If $\mathcal{M}$ is any model on a QG-frame $\mathcal{F}$, then*

(i) *The K-schema $\Box(A \supset B) \supset (\Box A \supset \Box B)$ is valid in $\mathcal{M}$.*

(ii) *if A is valid in $\mathcal{M}$, then so is $\Box A$.*

(iii) *if $A \supset B$ and A are valid in $\mathcal{M}$, then so is B.*

(iv) *UI is valid is $\mathcal{M}$.*

(v) *if $A \supset B$ is valid in $\mathcal{M}$ and x is not free in A, then $A \supset \forall x B$ is valid in $\mathcal{M}$.*

Proof. (i)–(iii) are standard and straightforward. For (iv), if $\mathcal{M}, a \models_f \forall x_n A$, then the semantics of $\forall$ ensures that $\mathcal{M}, a \models_{f[j/n]} A$ for all $j \in I$. In particular $\mathcal{M}, a \models_{f[\tau^{\mathcal{M}} f/n]} A$, and hence $\mathcal{M}, a \models_f A[\tau/x_n]$ by the last observation before the Lemma.

For (v), if $A \supset B$ is valid in $\mathcal{M}$, then $|A|_g^{\mathcal{M}} \subseteq |B|_g^{\mathcal{M}}$ for any g. Thus if x_n is not free in A, for any f we get $|A|_f^{\mathcal{M}} = |A|_{f[j/n]}^{\mathcal{M}} \subseteq |B|_{f[j/n]}^{\mathcal{M}}$ for all $j \in I$. Hence if $\mathcal{M}, a \models_f A$, then $x \in |A|_f^{\mathcal{M}} \subseteq \bigcap_{j \in I} |B|_{f[j/n]}^{\mathcal{M}}$. Since $|A|_f^{\mathcal{M}} \in \mathit{Prop}$, this implies $\mathcal{M}, a \models_f \forall x_n B$. Thus $A \supset \forall x_n B$ is valid in $\mathcal{M}$. ■

A QG-frame $\langle W, R, I, \mathit{Prop}, \mathit{PropFun} \rangle$ will be said to be *based on* the propositional frame $\langle W, R \rangle$. Where $\mathcal{C}$ is a class of propositional frames, $Q(\mathcal{C})$ is the class of QG-frames based on members of $\mathcal{C}$.

THEOREM 2. *If $\mathcal{C}$ is a class of S-frames then QS is sound over $Q(\mathcal{C})$.*

Proof. In view of Lemma 1, it is enough to show that every substitution instance of a theorem of S is valid over $Q(\mathcal{C})$. Suppose that α is an S-theorem and that $p_1, \ldots, p_n$ are all the propositional variables that occur in α. We show that $\alpha[B_1/p_1, \ldots, B_n/p_n]$ is valid over $Q(\mathcal{C})$ for any $\mathcal{LQ}$-formulas $B_1, \ldots, B_n$.

Let $\mathcal{M}$ be any model on a QG-frame based on some S-frame $\langle W, R \rangle \in \mathcal{C}$. Then α is valid in $\langle W, R \rangle$. Given $f \in I^\omega$, define a propositional model $\mathcal{M}_f = \langle W, R, v \rangle$ by putting $v(p_i) = |B_i|_f^{\mathcal{M}}$ for all $i \leq n$ (and $v(p)$ can be arbitrary otherwise). Thus

$$\mathcal{M}_f, a \models p_i \quad \text{iff} \quad \mathcal{M}, a \models_f B_i$$

for all $a \in W$ and $i \leq n$. A simple induction then shows that

$$\mathcal{M}_f, a \models \beta \quad \text{iff} \quad \mathcal{M}, a \models_f \beta[B_1/p_1, \ldots, B_n/p_n]$$

in general, where β is any $\mathcal{L}$-formula whose variables are among $p_1, \ldots, p_n$. In particular, when $\beta = \alpha$ we get $\mathcal{M}, a \models_f \alpha[B_1/p_1, \ldots, B_n/p_n]$ as required, since α is valid in $\mathcal{M}_f$. ■

REMARK 3. The proof of this Theorem shows that if an $\mathcal{L}$-formula α is valid in a frame $\langle W, R\rangle$, then every $\mathcal{LQ}$-substitution instance of α is valid in every QG-frame based on $\langle W, R\rangle$. ■

Historical Remarks

Our notion of QG-frame combines two prior notions. The first, concerning *Prop*, is the notion of a *general* frame for propositional modal logic, in which not every set of worlds is admissible as a proposition. Frames of the form $\langle W, R, \mathit{Prop}\rangle$ were introduced by S. K. Thomason [22], and their mathematical theory systematically developed in the first author's thesis [7].

The second precursor, concerning *PropFun*, is Halmos' notion of a *functional polyadic algebra* (see [12]). This is an algebra of "propositional" functions of the form $I^{\mathcal{V}} \to \mathbf{B}$ that is closed under operations corresponding to the standard connectives and quantifiers. Here $\mathbf{B}$ is a Boolean algebra, thought of as a collection of propositions, I is a domain of individuals, and $\mathcal{V}$ is a set of "variables". The operations corresponding to the quantifiers are defined using products (greatest lower bounds) and sums (least upper bounds) in $\mathbf{B}$. For QG-frames we have taken $\mathbf{B}$ to be a Boolean algebra of subsets of a Kripke frame $\langle W, R\rangle$, with the operation $[R]$ corresponding to $\Box$. But it is crucial to realise that even in this set-theoretic context, in which a binary product $X \sqcap Y$ is just the intersection $X \cap Y$, an infinite product $\bigsqcap S$ need not be the intersection $\bigcap S$, but in general is the operation that we have defined. This yields our alternative set-theoretic interpretation of the quantifier $\forall$.

Apparently it was Andrjez Mostowski [16] who introduced the method of interpreting an n-ary predicate as a lattice-valued function $I^n \to \mathbf{L}$, with the quantifiers interpreted by products and sums in $\mathbf{L}$. He took $\mathbf{L}$ to be a complete Brouwerian lattice for the application he was interested in and raised the question of whether his approach provided a complete semantics for intuitionistic logic. The idea was taken up by Rasiowa and Sikorski, who gave their famous algebraic proof of Gödel's completeness theorem for classical first-order logic [17], and systematically developed this method into an algebraic semantics for superintuitionistic predicate logics and for extensions of QS4 [18].

4 Canonical Frames and Models

In this section we remind the reader about certain facts concerning canonical models for propositional modal logics and give our construction of general canonical models for quantified logics.

Recall that for any logic L (propositional or quantified) a set Γ of formulas of the language of L is *L-consistent* if there are no formulas $A_1, ..., A_n \in \Gamma$ such that $(A_1 \wedge \cdots \wedge A_n) \supset \bot$ is an L-theorem. A *maximally L-consistent* set is one that is L-consistent and has no proper L-consistent extensions, or equivalently, is L-consistent and contains one of A and $\neg A$ for all formulas A. The intersection of the class of all maximally L-consistent sets is just the set of all L-theorems.

The *canonical frame* of a propositional modal logic S is $\mathcal{F}_{\mathrm{S}} = \langle W_{\mathrm{S}}, R_{\mathrm{S}} \rangle$, where W_{S} is the set of maximally S-consistent sets and R_{S} is the binary relation on W_{S} having $aR_{\mathrm{S}}b$ if and only if $\{\alpha : \Box\alpha \in a\} \subseteq b$. The *canonical S-model* $\mathcal{M}_{\mathrm{S}} = \langle \mathcal{F}_{\mathrm{S}}, v_{\mathrm{S}} \rangle$ has the valuation v_{S} defined by $v_{\mathrm{S}}(p) = \{a \in W_{\mathrm{S}} : p \in a\}$. This satisfies the "Truth Lemma"

$$\mathcal{M}_{\mathrm{S}}, a \models \alpha \quad \text{iff} \quad \alpha \in a,$$

from which it follows that a formula is valid in $\mathcal{M}_{\mathrm{S}}$ iff it is an S-theorem. Thus S is characterised by the model $\mathcal{M}_{\mathrm{S}}$, but not necessarily by the frame $\mathcal{F}_{\mathrm{S}}$. While any formula valid in $\mathcal{F}_{\mathrm{S}}$ is an S-theorem, the converse may not be true.

The *canonical QG-frame* of a quantified modal logic L is the structure

$$\mathcal{F}_{\mathrm{L}} = \langle W_{\mathrm{L}}, R_{\mathrm{L}}, I_{\mathrm{L}}, \mathit{Prop}_{\mathrm{L}}, \mathit{PropFun}_{\mathrm{L}} \rangle,$$

and the *canonical QG-model* is $\mathcal{M}_{\mathrm{L}} = \langle \mathcal{F}_{\mathrm{L}}, V_{\mathrm{L}} \rangle$, where

- W_{L} is the set of maximally L-consistent sets of $\mathcal{LQ}$-formulas;
- R_{L} is the binary relation on W_{L} defined as in the canonical logic for propositional modal logics;
- I_{L} is the set of closed terms of $\mathcal{LQ}$;
- $||A||_L$ is the set of maximally L-consistent sets a such that $A \in a$;
- $\mathit{Prop}_{\mathrm{L}}$ is the set $\{||A||_L : A \text{ is a closed formula of } \mathcal{LQ}\}$;
- For any $f \in I_{\mathrm{L}}^{\omega}$ and any formula A, $A^f = A[f0/x_0, \ldots, fn/x_n, \ldots] =$ the closed formula got by uniformly substituting the closed term fn for free occurrences of x_n in A;
- For each formula A, φ_A is the function from I_{L}^{ω} to $\mathit{Prop}_{\mathrm{L}}$ such that $\varphi_A f = ||A^f||_L$;
- $\mathit{PropFun}_{\mathrm{L}}$ is the set of functions φ_A for all formulas A;
- $V_{\mathrm{L}}(c) = c$, for all individual constants c.
- For F an n-ary function symbol, $V_{\mathrm{L}}(F)(\tau_1, \ldots, \tau_n) = F\tau_1 \ldots \tau_n$ for all closed terms $\tau_1, \ldots, \tau_n$.
- For P an n-ary predicate letter, $V_{\mathrm{L}}(P)(\tau_1, \ldots, \tau_n) = ||P\tau_1 \ldots \tau_n||_L$.

Note that we do not require that the worlds in this frame/model are $\forall$-complete (see Section 7).

Many properties of $\mathcal{F}_{\mathrm{L}}$ and $\mathcal{M}_{\mathrm{L}}$ can be derived just as in [14]. In particular, as in Lemmas 9.1–9.4 of [14] we can show that the canonical frame $\mathcal{F}_{\mathrm{L}}$ is in fact a QG-frame, that is, it satisfies CProp–CAll. This involves showing that $||A||_L \Rightarrow ||B||_L = ||A \supset B||_L$, $[R_{\mathrm{L}}]||A||_L = ||\Box A||_L$, $\varphi_\emptyset = ||\bot||_L$, $\varphi_A \Rightarrow \varphi_B = \varphi_{A\supset B}$, $[R_{\mathrm{L}}]\varphi_A = \varphi_{\Box A}$, $\forall_n \varphi_A = \varphi_{\forall x_n A}$.

The canonical model also satisfies a Truth Lemma in the form

(4.1) $\mathcal{M}_{\mathrm{L}}, a \models_f A$ iff $A^f \in a$.

Suppose that the variables that occur free in A are amongst $x_0, ..., x_n$, and that $c_0, ..., c_n$ are constants that do not occur in A (recall our assumption that $\mathcal{LQ}$ has infinitely many constants). We can then show that $A[c_0/x_0, ..., c_n/x_n]$ is an L-theorem if and only if A is an L-theorem. This implies that

(4.2) $\mathrm{L} \vdash A$ iff for all $f \in I^\omega_{\mathrm{L}}$, $L \vdash A^f$.

From (4.1) and (4.2) and the fact that $\bigcap W_L$ is the set of all L-theorems, it follows that a formula A is valid in $\mathcal{M}_{\mathrm{L}}$ if and only if it is an L-theorem (see [14], Lemma 9.6 and Theorem 9.7). Consequently, any formula valid in $\mathcal{F}_{\mathrm{L}}$ is an L-theorem, and so to show that $\mathcal{F}_{\mathrm{L}}$ characterises L it suffices to show that it validates L.

An immediate application of this construction is a completeness theorem for any logic of the form QS:

THEOREM 4. *For any propositional modal logic S, the quantified logic QS is characterised by its canonical general frame $\mathcal{F}_{QS}$, and hence is complete for the class of all its validating quantified general frames.*

Proof. Completeness for $\mathcal{F}_{\mathrm{QS}}$: if a QS-formula A is valid in $\mathcal{F}_{\mathrm{QS}}$, then it is valid in the model $\mathcal{M}_{\mathrm{QS}}$, and hence is a QS-theorem.

Soundness for $\mathcal{F}_{\mathrm{QS}}$: it is enough to show that if $\mathcal{LQ}$-formula A is a substitution instance of a propositional S-theorem β, then A is valid in any model $\mathcal{M} = \langle \mathcal{F}_{\mathrm{QS}}, V\rangle$ on $\mathcal{F}_{\mathrm{QS}}$. Suppose A is obtained by uniformly substituting $\mathcal{LQ}$-formulas B_p for certain propositional variables p in β. Now the function $|B_p|^{\mathcal{M}}$ interpreting B_p in $\mathcal{M}$ belongs to $PropFun_{\mathrm{QS}}$, and so is equal to φ_{A_p} for some $\mathcal{LQ}$-formula A_p. Hence $|B_p|^{\mathcal{M}}_f = ||A^f_p||_L = \{a : \mathcal{M}_{\mathrm{QS}}, a \models_f A_p\}$ by the Truth Lemma.

A simple induction on $\mathcal{L}$-formulas α then shows that in general

$$\mathcal{M}, a \models_f \alpha[\dots, B_p/p, \dots] \quad \text{iff} \quad \mathcal{M}_{\mathrm{QS}}, a \models_f \alpha[\dots, A_p/p, \dots].$$

But then when $\alpha = \beta$, $\beta[\dots, A_p/p, \dots]$ is an instance of an S-theorem, hence valid in $\mathcal{M}_{\mathrm{QS}}$, so $A = \beta[\dots, B_p/p, \dots]$ is valid in $\mathcal{M}$.

■

5 Completeness for Canonical Logics

Although $\mathcal{F}_{\mathrm{QS}}$ validates QS, its underlying propositional frame $\langle W_{\mathrm{QS}}, R_{\mathrm{QS}}\rangle$ need not validate S. There may be a propositional model $\langle W_{\mathrm{QS}}, R_{\mathrm{QS}}, v\rangle$ falsifying some S-theorem (this would require that $v(p) \notin Prop_{\mathrm{QS}}$ for some variable p). Examples will be given later.

We now show that this situation cannot arise if S is *canonical*, which means that it is valid in the canonical propositional frame $\mathcal{F}_{\mathrm{S}} =< W_{\mathrm{S}}, R_{\mathrm{S}} >$, and hence is characterised by this frame. The class of canonical logics is wide: it includes every logic that is characterised by some first-order definable class of frames $\langle W, R\rangle$ [5], and many others besides [11, 10].

LEMMA 5. *If L is any quantified modal logic extending QS, then $\langle W_L, R_L\rangle$ is isomorphic to a generated subframe of $\mathcal{F}_S$.* ■

A direct proof of this result appears in the Appendix. Here we give a brief explanation of it by invoking the duality between algebraic models and frames. The Lindenbaum algebra $\mathbf{A}^{\mathrm{S}}$ of S is a free S-algebra, freely generated by the (equivalence classes of) propositional variables. The Lindenbaum algebra $\mathbf{A}^{\mathrm{L}}$ of L is also an S-algebra, and is no bigger than the set of generators of $\mathbf{A}^{\mathrm{S}}$, by the assumption that there are at least as many variables in $\mathcal{L}$ as there are $\mathcal{L}Q$-formulas. Hence there is a surjective homomorphism $f : \mathbf{A}^S \twoheadrightarrow \mathbf{A}^{\mathrm{L}}$. By duality, this induces an injective bounded morphism $f_+ : \mathbf{A}^{\mathrm{L}}_+ \rightarrowtail \mathbf{A}^{\mathrm{S}}_+$ in the reverse direction, between the canonical structures of these algebras. The image of f_+ is a generated subframe of $\mathbf{A}^{\mathrm{S}}_+$ isomorphic to $\mathbf{A}^{\mathrm{L}}_+$. But the points of $\mathbf{A}^{\mathrm{S}}_+$ are the ultrafilters of $\mathbf{A}^{\mathrm{S}}$, which can be identified with maximally S-consistent sets, and so $\mathbf{A}^{\mathrm{S}}_+$ can be identified with $\langle W_{\mathrm{S}}, R_{\mathrm{S}}\rangle$. Similarly $\mathbf{A}^{\mathrm{L}}_+$ can be identified with $\langle W_{\mathrm{L}}, R_{\mathrm{L}}\rangle$.

Since validity of formulas is preserved by generated subframes and isomorphism, Lemma 5 and Theorem 4 immediately give

THEOREM 6. *If S is a canonical propositional logic and L is a quantified logic extending QS, then $\langle W_L, R_L\rangle$ is an S-frame. In particular $\langle W_{QS}, R_{QS}\rangle$ is an S-frame and so QS is characterised by the class of all QG-frames whose underlying propositional frame validates S.* ■

Now let Φ be any set of conditions on proposition frames such that (i) S is validated by all frames satisfying Φ, (ii) $\mathcal{F}_{\mathrm{S}}$ satisfies Φ, and (iii) Φ is preserved by generated subframes and isomorphism. Then S is canonical and $\langle W_{\mathrm{QS}}, R_{\mathrm{QS}}\rangle$ satisfies Φ. It follows from Theorems 2 and 6 that QS is characterised by the class of all QG-frames whose underlying propositional frame satisfies Φ.

For example, let S=S4M, the extension of S4 by the McKinsey axiom $\Box\Diamond A \supset \Diamond\Box A$. The S4M-frames are precisely those that are reflexive, transitive frames and *final* in the sense that every world has an accessible "endpoint":

$$\forall x \exists y (xRy \wedge \forall z (yRz \supset y = z)).$$

These conditions are preserved by generated subframes, and possessed by the canonical frame of S4M. It follows that $\langle W_{QS4M}, R_{QS4M}\rangle$ is reflexive, transitive and final, and that QS4M is characterised by the class of all quantified general frames based on S4M-frames.

The significance of this is that QS4M is incomplete for its standard quantificational semantics based on S4M-frames [13, p. 283].

6 Tarskian Frames

The *converse Barcan formula* $\Box\forall xA \supset \forall x\Box A$ is derivable as a schema in the proof theory of any system QS. In the standard semantics for quantified modal logics, this schema is usually validated by constraining models to have "increasing domains": each world a is assigned a domain Ia of individuals such that aRb implies $Ia \subseteq Ib$, and the satisfaction of a formula $\forall xA$ at world a is evaluated by having x range over the members of Ia. Our semantics also validates the converse Barcan formula, but is closer to what is usually called a *constant domain* semantics, in that quantified variables range over the one domain I of individuals relative to all worlds.

In the standard semantics, constant domain models validate the *Barcan formula*

$$\text{(BF) } \forall x\Box A \supset \Box\forall xA.$$

A standard constant domain model has the form $\mathcal{M} = \langle W, R, I, \nu\rangle$, where the valuation ν assigns to each constant an element of I, to each n-ary function letter an n-ary function on I, and to each n-ary predicate letter P and each element a of W an n-ary relation $\nu(P, a) \subseteq I^n$. The *standard satisfaction relation* $\mathcal{M}, a \models_f A$ is defined inductively, with the atomic case given by

$$\mathcal{M}, a \models_f P\tau_1 \dots \tau_n \quad \text{iff} \quad \langle \tau_1^{\mathcal{M}} f, \dots, \tau_n^{\mathcal{M}} f\rangle \in \nu(P, a);$$

the propositional connectives treated as usual; and

(6.1) $\mathcal{M}, a \models_f \forall x_n A$ iff for all $j \in I$, $\mathcal{M}, a \models_{f[j/n]} A$.

Our general semantics does not validate the Barcan formula for most logics (QS5 is a notable exception), despite the fact that it has constant domains. To obtain a condition validating BF we say that QG-frame $\mathcal{F} = \langle W, R, I, \mathit{Prop}, \mathit{PropFun}\rangle$ is *Tarskian* if

$$\bigcap_{j\in I} \varphi(f[j/n]) \in \mathit{Prop}$$

for all $\varphi \in \mathit{PropFun}$, $n \in \omega$, and $f \in I^\omega$. When this holds, we have

$$\bigcap_{j\in I} \varphi(f[j/n]) = \prod_{j\in I} \varphi(f[j/n]) = (\forall_n\varphi)f,$$

so an equivalent definition of "Tarskian" is that in general

$$(\forall_n\varphi)f = \bigcap_{j\in I} \varphi(f[j/n]).$$

This condition implies that in any model on $\mathcal{F}$, the satisfaction clause for $\forall x_n A$ simplifies to the standard Tarskian clause (6.1), from which validity of BF follows readily.

Note that a QG-frame must be Tarskian if it has *Prop* $= \wp W$, since then $\bigcap S \in$ *Prop* for all $S \subseteq$ *Prop*. Also a frame in which *Prop* is finite must be Tarskian, since *Prop* is closed under finite intersection. Finally, a frame must be Tarskian if I is finite, for then so is $\{\varphi(f[j/n]) : j \in I\}$.

If $\mathcal{M} = \langle \mathcal{F}, V \rangle$ is any model, in our sense, on a QG-frame $\mathcal{F}$, we obtain a standard model $\mathcal{M}^s = \langle W, R, I, \nu_{\mathcal{M}} \rangle$ by defining $\nu_{\mathcal{M}}$ to agree with V on constants and function letters, and putting

$$\nu_{\mathcal{M}}(P, a) = \{\langle j_1, \dots, j_n \rangle : a \in V(P)(j_1, \dots, j_n)\}.$$

Then we get $\tau^{\mathcal{M}^s} = \tau^{\mathcal{M}}$ for all $\mathcal{LQ}$-terms τ, because of the agreement of $\nu_{\mathcal{M}}$ and V on constants and function letters. Moreover, *if $\mathcal{F}$ is Tarskian*, then in general $\mathcal{M}, a \models_f A$ iff $\mathcal{M}^s, a \models_f A$, and so the two models validate the same formulas.

In the converse direction, given $\mathcal{M} = \langle W, R, I, \nu \rangle$ a standard constant domain model, we define the QG-model $\mathcal{M}^* = \langle W, R, I, \wp W, (\wp W)^I, V_{\mathcal{M}} \rangle$ by taking $V_{\mathcal{M}}$ to agree with ν on constants and function letters, and putting

$$V_{\mathcal{M}}(P)(j_1, \dots, j_n) = \{a \in W : \langle j_1, \dots, j_n \rangle \in \nu(P, a)\}.$$

Thus the underlying frame of $\mathcal{M}^*$ is the *full* frame based on $\langle W, R, I \rangle$, with *Prop* containing every subset of W, and *PropFun* every function $I \to \wp W$. This makes it immediately a QG-frame (i.e. CProp–CAll hold) *that is Tarskian*. Because of the agreement of $V_{\mathcal{M}}$ with ν we get $\tau^{\mathcal{M}^*} = \tau^{\mathcal{M}}$ for all terms τ. In general $\mathcal{M}^*, a \models_f A$ iff $\mathcal{M}, a \models_f A$, and so these two models also validate the same formulas.

Now if $\mathcal{M}^{*s} = \langle W, R, I, \nu_{\mathcal{M}^*} \rangle$ is the standard constant domain model constructed from the QG-model $\mathcal{M}^*$, then in fact $\nu_{\mathcal{M}^*} = \nu$, and so $\mathcal{M}^{*s} = \mathcal{M}$. Conversely, if $\mathcal{M}^{s*} = \langle W, R, I, \wp W, (\wp W)^I, V_{\mathcal{M}^s} \rangle$ is the Tarskian QG-model constructed from the standard model $\mathcal{M}^s = \langle W, R, I, \nu_{\mathcal{M}} \rangle$ associated with a QG-model $\mathcal{M} = \langle \mathcal{F}, V \rangle$, then $V_{\mathcal{M}^s} = V$. It does not follow however that $\mathcal{M}^{s*} = \mathcal{M}$, since the underlying frame $\mathcal{F}$ of $\mathcal{M}$ need not be full (i.e. maybe *Prop* $\neq \wp W$ in $\mathcal{F}$), whereas that of $\mathcal{M}^{s*}$ *is* full by definition. What we do get is that if $\mathcal{F}$ is Tarskian, then $\mathcal{M}, a \models_f A$ iff $\mathcal{M}^{s*}, a \models_f A$, so the satisfaction relations of the two models are identical.

The upshot of all this is an equivalence between the standard constant domain semantics and the semantics of our Tarskian general frames. In particular:

THEOREM 7. *Let $\mathcal{C}$ be a class of propositional frames of the form $\langle W, R \rangle$. Then for any $\mathcal{LQ}$-formula A, then following are equivalent.*

- *A is valid in all standard constant domain models based on members of $\mathcal{C}$.*
- *A is valid in all Tarskian QG-frames based on members of $\mathcal{C}$.*
- *A is valid in all full QG-frames based on members of $\mathcal{C}$.* ■

7 Tarskian Canonical Frames

Let LB be the extension of quantified logic L by the Barcan formula BF. The canonical QG-frame $\mathcal{F}_{\mathrm{LB}}$ for LB of Section 4 validates LB, but it need not be Tarskian. Thus the Tarskian condition is sufficient to ensure validity of BF, but it is not necessary.

To see this, we first show

LEMMA 8. *If L is any quantified logic containing the schema BF, then the canonical frame $\mathcal{F}_L$ validates BF.*

Proof. In any model $\mathcal{M}$ on $\mathcal{F}_{\mathrm{L}}$, each formula A is interpreted as a function $|A|^{\mathcal{M}}$ that belongs to $\mathit{PropFun}_{\mathrm{L}}$, and hence is equal to φ_B for some B. Thus to show that $\mathcal{M}$ validates $\forall x_n \Box A \supset \Box \forall x_n A$ it is enough to show that $\forall_n [R_{\mathrm{L}}]\varphi_B(f) \subseteq [R_{\mathrm{L}}]\forall_n \varphi_B(f)$ for all $f \in I_{\mathrm{L}}^{\omega}$, i.e. that $\varphi_{\forall x_n \Box B}(f) \subseteq \varphi_{\Box \forall x_n B}(f)$. This means that $||(\forall x_n \Box B)^f||_L \subseteq ||(\Box \forall x_n B)^f||_L$. But that follows by properties of maximally L-consistent sets, since $(\forall x_n \Box B)^f \supset (\Box \forall x_n B)^f$ is an L-theorem derivable from BF. ■

COROLLARY 9. *For any proposition modal logic S, QSB is characterised by its canonical frame $\mathcal{F}_{QSB}$.* ■

Now consider the well-known logic S4.2, the extension of S4 by the schema $\Diamond\Box A \supset \Box\Diamond A$. S4.2 is a canonical logic whose frames are precisely those reflexive, transitive frames that are *convergent*, i.e.

$$\forall x, y, z(xRy \wedge xRz \supset \exists w(yRw \wedge zRw)).$$

Hence by Theorem 6, the canonical QG-frame for QS4.2B is based on an S4.2 frame. If this canonical frame were Tarskian, then QS4.2B would be characterised by the class of all Tarskian QG-frames based on S4.2-frames, and hence by Theorem 7 would be characterised by the class of all standard constant domain models based on S4.2-frames. But in fact QS4.2B is *incomplete* for the class of all standard constant domain models based on S4.2-frames [13, p. 271].[3]

Thus $\mathcal{F}_{\mathrm{QS4.2B}}$ is a characterising frame for QS4.2B that is reflexive, transitive and convergent; validates BF; but is non-Tarskian. We have found a semantics for this standardly-incomplete logic that has the standard semantics for its propositional part, but an alternative interpretation of the quantifier $\forall$.

[3] In the notation of [13], QS is LPC+S, and QSB is S+BF.

Now for any logic L that includes the schema BF there is also a canonical model construction based on $\forall$-*complete* sets. A set Γ of formulas is $\forall$-complete if $\forall x A \in \Gamma$ whenever $A[\tau/x]$ for all closed terms τ. Let W^T_{L} be the set of all maximally L-consistent $\forall$-complete sets. The presence of infinitely many constants in $\mathcal{LQ}$ ensures that any L-consistent formula can be shown to belong to some member of W^T_{L}, and hence that the intersection of all members of W^T_{L} is just the set of L-theorems. A frame $\mathcal{F}^T_{\mathrm{L}}$ and model $\mathcal{M}^T_{\mathrm{L}} = \langle \mathcal{F}^T_{\mathrm{L}}, V^T_{\mathrm{L}} \rangle$ can be constructed just as for $\mathcal{F}_{\mathrm{L}}$ and $\mathcal{M}_{\mathrm{L}}$ but using W^T_{L} in place of W_{L}. The crucial role of BF here, as first shown in [21], is to allow a proof that for any $\Gamma \in W^T_{\mathrm{L}}$,

$$\Box A \in \Gamma \quad \text{iff} \quad \text{for all } \Delta \in W^T_{\mathrm{L}},\ \Gamma R_{\mathrm{L}} \Delta \text{ implies } A \in \Delta.$$

This can then be used to show that the Truth Lemma

$$\mathcal{M}^T_{\mathrm{L}}, a \models_f A \quad \text{iff} \quad A^f \in a$$

holds in general, and hence that the formulas valid in $\mathcal{M}^T_{\mathrm{L}}$ are just the L-theorems. But now the $\forall$-completeness of members of W^T_{L} ensures that if A has only x free, then $||\forall x A||_L = \bigcap\{||A[\tau/x]||_L : \tau \in I_{\mathrm{L}}\}$. This implies that $\mathcal{F}^T_{\mathrm{L}}$ is Tarskian (see Lemmas 9.3 and 9.4 of [14].) We call it the *Tarskian canonical frame* for L.

THEOREM 10. *For any propositional modal logic S, the logic QSB is characterised by its Tarskian canonical frame $\mathcal{F}^T_{QSB}$, and hence is complete for the class of all its validating Tarskian frames.*

Proof. As usual, any formula valid in $\mathcal{F}^T_{\mathrm{QSB}}$ is valid in $\mathcal{M}^T_{\mathrm{QSB}}$ and hence is a QSB-theorem. But $\mathcal{F}^T_{\mathrm{QSB}}$ validates all instances of S-theorems by the same proof as for Theorem 4, and validates BF because it is Tarskian, so validates QSB. ■

Corollary 9 and Theorem 10 now show that QSB has two characteristic canonical general frames: $\mathcal{F}_{\mathrm{QSB}}$ and $\mathcal{F}^T_{\mathrm{QSB}}$. If S is canonical, then $\mathcal{F}_{\mathrm{QSB}}$ will be based on an S-frame but may not be Tarskian, as we saw with S4.2. Vice versa, $\mathcal{F}^T_{\mathrm{QSB}}$ will be Tarskian, but may not be based on an S-frame. S4.2 shows this again: if $\mathcal{F}^T_{\mathrm{QS4.2B}}$ were based on an S4.2 frame, then QS4.2B would be characterised by the class of all Tarskian QG-frames based on S4.2-frames, which we have already observed to be false.

Now $\mathcal{F}^T_{\mathrm{QS4.2B}}$ is reflexive and transitive, by the usual application of S4-axioms, so the upshot of this discussion is that it cannot be convergent. It follows that $\langle W^T_{\mathrm{QS4.2}B}, R^T_{\mathrm{QS4.2}B} \rangle$ is not a generated subframe of the convergent frame underlying $\mathcal{F}_{\mathrm{QS4.2}B}$.

8 Non-Canonical Cases

We have just seen that $\mathcal{F}^T_{\mathrm{QSB}}$ need not be based on an S-frame even when S is canonical. In fact there is little gain when $\langle W^T_{\mathrm{QSB}}, R^T_{\mathrm{QSB}} \rangle$ *does* validate S, because in that case QSB is complete for Tarskian general frames based on S-frames, and hence by Theorem 7 is complete for the standard constant domain semantics based on S-frames. Our theory comes into its own in cases where QSB is incomplete for the standard semantics, by providing a complete semantics based on Tarskian general frames - albeit one in which $\langle W^T_{\mathrm{QSB}}, R^T_{\mathrm{QSB}}\rangle$ is not an S-frame.

There is one useful observation we can make about properties that are inherited by $\langle W^T_{\mathrm{QSB}}, R^T_{\mathrm{QSB}}\rangle$ from $\mathcal{F}_{\mathrm{S}}$. $\langle W^T_{\mathrm{QSB}}, R^T_{\mathrm{QSB}}\rangle$ is a *substructure* of $\langle W_{\mathrm{QSB}}, R_{\mathrm{QSB}}\rangle$, in that R^T_{QSB} is the restriction of R_{QSB} to the subset W^T_{QSB} of W_{QSB}, and hence is isomorphic to a substructure of $\mathcal{F}_{\mathrm{S}}$ (Lemma 5). If S includes the modal axiom(s) corresponding to a universal property, then that property will be possessed by the propositional frame underlying $\mathcal{F}^T_{\mathrm{QSB}}$.

What if S is not canonical? Can QS or QSB still be complete for a class of quantified general frames based on S-frames? Of course we should not expect this if S itself is incomplete for S-frames. Take the case that S is the incomplete logic GH from [3]. Every frame for GH has transitive R so validates the axiom 4: $\Box p \to \Box\Box p$, but this axiom is not derivable in GH. Now if QGH or QGHB were complete for a class of quantified general frames based on GH-frames, then these structures would have transitive R and validate all $\mathcal{LQ}$-instances of 4, which would then be derivable in the relevant quantified logic, and in particular would be derivable in QGHB. But this is not so, since the GH-model of [3] falsifying 4 can readily be turned into a QGHB-model falsifying $\Box Px \to \Box\Box Px$, where P is a unary predicate letter.

This situation is general: if S is any incomplete propositional logic, then there is some $\mathcal{L}$-formula α that is valid in all S-frames but not an S-theorem. By [13, Corollary 14.6] there must then be some $\mathcal{LQ}$-substitition instance A of α that is not a QSB-theorem. But if $\langle W, R\rangle$ is an S-frame, then by our Remark 3, A is valid in every QG-frame based on $\langle W, R\rangle$. It follows, similarly to the previous paragraph, that neither QS nor QSB can be complete for any class of QG-frames based on S-frames.

But even when S is S-frame complete, the desired property can fail for QS and QSB. This is exemplified by the propositional modal logic KW, the smallest normal modal logic containing the schema $\Box(\Box\alpha \supset \alpha) \supset \Box\alpha$. The KW-frames are precisely those transitive frames that have no infinite R-chains $b_0 R b_1 R \cdots R b_n R \cdots$. KW is characterised by these frames, but is not canonical.

The logic QKW was shown in [15] not to be complete for its standard (expanding domain) semantics. We adapt that example to our present semantics, by considering the formula

$$(\ddagger) \quad \forall x \Box (Px \to \Diamond PFx) \to \forall x \neg \Diamond Px,$$

where P is a unary predicate letter, and F a unary function symbol. Inspection of the QKW-model defined in Theorem 2 of [15], based on a non-standard model of arithmetic, shows that this model falsifies ($\ddagger$) with F interpreted as the successor function. Hence ($\ddagger$) is not a QKW-theorem.

But (‡) is valid in any quantified general frame $\mathcal{F}$ that is based on a KW-frame. For if not, there would be a model $\mathcal{M}$ on such an $\mathcal{F}$ having a point a at which (‡) was false. Then $\mathcal{M}, a \models \forall x \Box (Px \to \Diamond PFx)$ and $\mathcal{M}, a \not\models \forall x \neg \Diamond Px$. By our semantics for $\forall$, there must be some $X \in \mathit{Prop}$ with $a \in X$ and $X \subseteq |\Box(Pj \to \Diamond PFj)|^{\mathcal{M}}$ for all $j \in I$ (treating j as an individual constant). Since $\forall x \neg \Diamond Px$ is false at a, $X \nsubseteq |\neg \Diamond Pj^*|^{\mathcal{M}}$ for some j^*. Hence there is some world $b_0 \in X$ at which $b_0 \models \Diamond Pj^*$ in $\mathcal{M}$, so there is some b_1 such that $b_0 R b_1$ and $b_1 \models Pj^*$. Now suppose inductively that we have defined b_n ($n \geq 1$) such that $b_n \models PF^{n-1}j^*$ and $b_0 R b_n$. Here F^n is the n-th iteration of F, with $F^n j = F(F^{n-1}j)$ and $F^0 j = j$. Then

$$b_0 \in X \subseteq |\Box(PF^{n-1}j^* \to \Diamond PF^n j^*)|^{\mathcal{M}},$$

so $b_n \models PF^{n-1}j^* \to \Diamond PF^n j^*$. Hence there is some b_{n+1} with $b_n R b_{n+1}$ and $b_{n+1} \models PF^n j^*$. Then $b_0 R b_{n+1}$ by transitivity of R. This shows, by induction, that there is an infinite R-chain $b_0 R b_1 R \cdots R b_n R \cdots$ in $\mathcal{F}$, which contradicts the assumption that $\mathcal{F}$ is based on a KW-frame.

Thus QKW, while being characterised by $\mathcal{F}_{\mathrm{QKW}}$ and by the class of its validating quantified general frames, is not complete for any such class of general frames that are based on KW-frames. Hence $\mathcal{F}_{\mathrm{QKW}}$ is not based on a KW-frame (and so must contain an infinite R-chain).

The logic QKWB with the Barcan formula is, similarly, characterised by its Tarskian canonical frame $\mathcal{F}^T_{\mathrm{QKWB}}$ and complete for its class of validating Tarskian frames, but is not complete for the class of Tarskian frames that are based on KW-frames. If it were, then it would be characterised by the class of all standard constant domain models based on KW-frames (Theorem 7). But the logic characterised by the standard constant domain models based on KW-frames was shown in [4] not to be recursively axiomatisable, so cannot be equal to QKWB.

9 Conclusion and Further Work

We have constructed a canonical general frame $\mathcal{F}_{\mathrm{QS}}$ for every quantified modal logic of the form QS, and shown that if S is a canonical propositional logic, then the Kripke frame underlying $\mathcal{F}_{\mathrm{QS}}$ validitates S. The semantics of $\forall$ in $\mathcal{F}_{\mathrm{QS}}$ may be non-Tarskian.

We also shed new light on the Barcan formula, by revealing that its role is to ensure that the standard Tarskian semantics for $\forall$ can be provided. For systems QSB including BF there is a second characterising canonical general frame $\mathcal{F}^T_{\mathrm{QSB}}$ in which $\forall$ gets the Tarskian semantics, but the underlying Kripke frame may not validate S. Thus there is some trade-off between these two desiderata.

There is much more to be done with our notion of quantified general frame. What about the case of a logic L that is not of the form QS or QSB? To show that $\mathcal{F}_{\mathrm{L}}$ characterises such an L may require us to assume that L has a rule of substitution of general formulas in place of atomic formulas. Extensions of QS4 with this rule have been studied by Shehtman and

Skvortsov [19, 20], using a "Kripke metaframe" semantics based on functorial constructions. They construct a characterising canonical metaframe for their quantified S4-extensions of a canonical propositional logic, both with and without BF.

In this context we should also mention the work of Ghilardi [6] showing that if S is any canonical superintuitionistic propositional logic, then the the quantified extension of S has a characteristic canonical functor frame based on an S-frame. He also notes (p. 211) that his result does not fully apply to the functor semantics of quantified extensions of S4.

To give a general semantics for systems without the converse Barcan formula will require us to consider models with varying (world-dependent) domains of individuals, as indicated at the start of Section 6. Then there is the question of adding the equality symbol to our languages and addressing the whole morass of issues around the interpretation of existence and identity in intensional logics. We intend to take these matters up in future work.

Appendix

Here is a direct construction to prove Lemma 5. Let $p \mapsto p^*$ be a mapping of the set of propositional variables of $\mathcal{L}$ *onto* the set of $\mathcal{L}Q$-formulas. Such a mapping exists by the assumption the the former set is at least as large as the latter (but see the end for further discussion of this assumption).

Now for any $\mathcal{L}$-formula α, let α^* be the result of uniformly substituting p^* for p in α, for all p that occur in α. This operation commutes with the $\mathcal{L}$-connectives: $\bot^* = \bot$, $(\alpha \supset \beta)^* = \alpha^* \supset \beta^*$, $(\Box\alpha)^* = \Box(\alpha^*)$ etc. Moreover, if α is an S-theorem, then α^* is a QS-theorem by definition, and hence is an L-theorem.

Define $f : W_{\mathrm{L}} \to W_{\mathrm{S}}$ by putting $f(a) = \{\alpha : \alpha^* \in a\}$ for all maximally L-consistent sets a. Of course it has to be checked that this $f(a)$ is maximally S-consistent. First, $f(a)$ is S-consistent, for otherwise there would be $\alpha_1, \dots, \alpha_n \in f(a)$ such that $(\alpha_1 \wedge \cdots \wedge \alpha_n) \supset \bot$ is an S-theorem, hence $(\alpha_1^* \wedge \cdots \wedge \alpha_n^*) \supset \bot$ is an L-theorem, contrary to the L-consistency of a. Second, a contains one of α^* and $\neg(\alpha^*) = (\neg\alpha)^*$ so $f(a)$ contains one of α and $\neg\alpha$, for all α, so $f(a)$ is negation complete, as required.

f is injective: if $a \neq b$ in W_{L}, then there is some formula $A = p^* \in a$ with $\neg A = (\neg p)^* \in b$, hence $p \in f(a)$ and $\neg p \in f(b)$, so $p \notin f(b)$ and $f(a) \neq f(b)$.

If $aR_{\mathrm{L}}b$, then $\Box\alpha \in f(a)$ implies $\Box(\alpha^*) = (\Box\alpha)^* \in a$, hence $\alpha^* \in b$ and $\alpha \in f(b)$; so $f(a)R_{\mathrm{S}}f(b)$.

Finally, to complete the proof that f is a bounded morphism, suppose $f(a)R_{\mathrm{S}}c$ in W_{S}. We have to show that $aR_{\mathrm{L}}b$ and $f(b) = c$ for some $b \in W_{\mathrm{L}}$. Put

$$b_0 = \{\alpha^* : \Box\alpha^* \in a\} \cup \{\beta^* : \beta \in c\}.$$

If b_0 were not L-consistent, then since the two sets that make up b_0 are each closed under finite conjunctions, there would be formulas α, β with $\Box\alpha^* \in a$ and $\beta \in c$, such that $\alpha^* \supset \neg\beta^*$ is an L-theorem. But then $\Box\alpha^* \supset \Box\neg\beta^*$ is an L-theorem, so belongs to a, implying that $\Box\neg\beta^* \in a$, hence $\Box\neg\beta \in f(a)$, and so $\neg b \in c$, contradicting the S-consistency of c.

Thus b_0 is L-consistent, and hence is included in some $b \in W_L$. Since $\{\alpha^* : \Box\alpha^* \in a\} \subseteq b$ we get $aR_{\mathrm{L}}b$, and since $\{\beta^* : \beta \in c\} \subseteq b$ we get $c \subseteq f(b)$, and so $c = f(b)$ as required, by *maximal* S-consistency of c.

Altogether then, f is an injective bounded morphism from $\langle W_{\mathrm{L}}, R_{\mathrm{L}}\rangle$ into $\mathcal{F}_{\mathrm{S}}$. The image of f is isomorphic to $\langle W_{\mathrm{L}}, R_{\mathrm{L}}\rangle$. But the image of a bounded morphism is always a generated subframe of its codomain. This proves Lemma 5.

In conclusion, we comment on the assumption that there are at least as many $\mathcal{L}$-variables as $\mathcal{L}Q$-formulas. While propositional modal languages are often based on a denumerable list of variables, we could assume a proper class $\{p_\lambda : \lambda \text{ is an ordinal}\}$ of such variables, generating a proper class *Fma* of formulas. For each infinite cardinal κ, let Fma^κ be the set of such formulas with variables from $\{p_\lambda : \lambda < \kappa\}$. A (normal propositional modal) logic S is a subclass of *Fma* defined as in Section 2. Then each $\mathrm{S}^\kappa = \mathrm{S} \cap Fma^\kappa$ is a logic in Fma^κ and has its own canonical frame $\mathcal{F}_{\mathrm{S}}^\kappa$ (of size 2^κ). We say that S is *canonical* if it is valid in these frames $\mathcal{F}_{\mathrm{S}}^\kappa$ for all κ.

Thus, given a predicate modal language $\mathcal{L}Q$ we can take any κ that is at least as large as the set of $\mathcal{L}Q$-formulas, and use S^κ as our version of S, and $\mathcal{F}_{\mathrm{S}}^\kappa$ as the canonical frame having a generated subframe isomorphic to $\langle W_{\mathrm{L}}, R_{\mathrm{L}}\rangle$. In this sense our assumption is innocuous.

Note that a logic S in *Fma* is uniquely determined by S^ω: S is the only logic in *Fma* whose restriction to Fma^ω is S^ω, and likewise S^κ is the only logic in Fma^κ whose restriction to Fma^ω is S^ω. For further discussion of this, see [9].

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