# ML is not finitely axiomatizable over Cheq

GAËLLE FONTAINE

ABSTRACT. We show that the Medvedev logic **ML** is not finitely axiomatizable over the logic **Cheq** of chequered subsets of $\mathbb{R}^\infty$. This gives a negative solution to one of the questions raised by Litak [2].

## 1 Introduction

In 1962 Medvedev [4] introduced the intermediate logic of "finite problems". It became known as the Medvedev logic **ML**. It is known that **ML** has the finite model property, the disjunction property, contains the Kreisel-Putnam and Scott logics, and is contained in the logic of weak excluded middle (see, e.g., [1]). In the late 1970's Maksimova et al. [3] showed that **ML** is not finitely axiomatizable by any set of formulas in finitely many variables. The question whether **ML** is decidable is one of the most long-standing open problems in the field of intermediate logics; see, e.g., [1, §16] for further discussion.

Recently, van Benthem et al. [5] introduced the modal logic of chequered subsets of $\mathbb{R}^\infty$ and showed that it has the finite model property. Litak [2] denoted the intermediate companion of this modal logic by **Cheq** and showed that it has the disjunction property, contains the Scott logic and is contained in the Medvedev logic **ML**. In fact, every finite rooted **Cheq**-frame is a $p$-morphic image of a finite rooted **ML**-frame. However, it is still unknown whether **Cheq** is finitely axiomatizable and/or decidable.

Litak [2] raised a question whether **ML** is finitely axiomatizable over **Cheq**. If this were the case, it would imply that **Cheq** is not finitely axiomatizable. In this note we give a negative solution to Litak's question by proving that **ML** is not finitely axiomatizable over **Cheq**. Thus, the connection between the Medvedev logic and **Cheq** is not as strong as it first appeared. It still remains an open problem whether **Cheq** is finitely axiomatizable.

## 2 ML is not finitely axiomatizable

We assume the reader's familiarity with basics of Kripke semantics for intermediate logics and refer to Chagrov and Zakharyaschev [1] for the details.

DEFINITION 1 (Maksimova et al. [3]). For a finite non-empty set $D$, let $\mathcal{P}^0(D)$ denote the Kripke frame

$$\mathcal{P}^0(D) = \langle \{X \subseteq D | X \neq \emptyset\}, \supseteq \rangle.$$

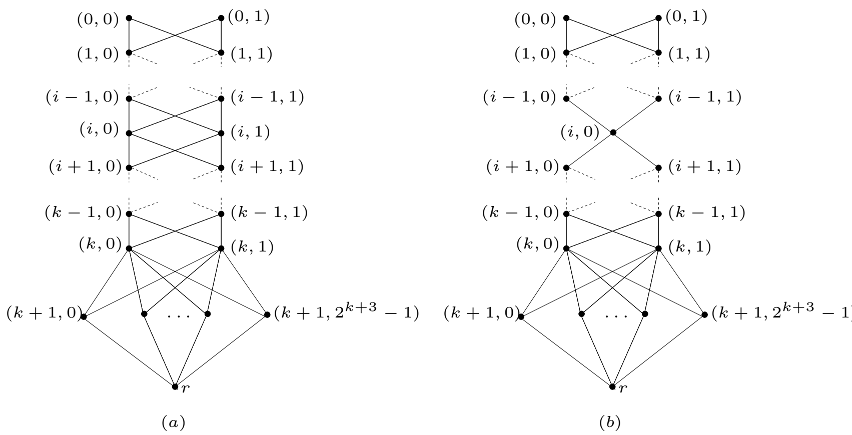


Figure 1. The frames $\mathcal{G}_k$ and $\mathcal{G}_k^i$.

We call $\mathcal{P}^0(D)$ a *Medvedev frame.* The intermediate logic **ML** is the logic of all Medvedev frames. As usual, a frame $\mathcal{F}$ is called an **ML**-*frame* if all the theorems of **ML** are valid in $\mathcal{F}$.

For each natural number $k \neq 0$ and each $i \leq k$, let $\mathcal{G}_k$ and $\mathcal{G}_k^i$ be the frames shown in Figure 1 (a) and (b), respectively. The following lemma is proved in [3].

LEMMA 2.

(*a*) *For each natural number $k > 0$, the frame $\mathcal{G}_k$ is not an* **ML**-*frame.*

(*b*) *For each natural number $k > 0$ and each $i \leq k$, the frame $\mathcal{G}_k^i$ is an* **ML**-*frame.*

(*c*) *Let $\varphi$ be a formula with $k$ variables. There exists a natural number $i \leq k$ such that*

$$\mathcal{G}_k \Vdash \varphi \textit{ iff } \mathcal{G}_k^i \Vdash \varphi.$$

It is an easy corollary of Lemma 2 that **ML** is not finitely axiomatizable. Indeed, suppose there is a finite set of formulas axiomatizing **ML**. Without loss of generality we may assume that **ML** is axiomatized by a single formula $\varphi$ with $k$ variables (for some natural number $k$). By Lemma 2(c), there exists a natural number $i \leq k$ such that $\varphi$ is valid in $\mathcal{G}_k$ iff $\varphi$ is valid in $\mathcal{G}_k^i$. By Lemma 2(b), $\mathcal{G}_k^i$ is an **ML**-frame. Thus, $\varphi$ is valid in $\mathcal{G}_k^i$. Therefore, $\varphi$ is valid in $\mathcal{G}_k$. But $\mathcal{G}_k$ is not an **ML**-frame by Lemma 2(a). This contradiction proves that such a $\varphi$ does not exist. Thus, we arrive at the following theorem.

THEOREM 3 (Maksimova et al. [3]). *The logic* **ML** *is not finitely axiomatizable. Moreover, it is not axiomatizable by any set of formulas in finitely many variables.*

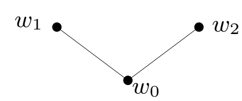


Figure 2. The frame $\mathcal{F}_1$.

## 3 ML is not finitely axiomatizable over Cheq

DEFINITION 4 (van Benthem et al. [5]). Let $\mathcal{F}$ denote the two-fork Kripke frame shown in Figure 2. Let $\mathcal{F}_n = \underbrace{\mathcal{F} \times \cdots \times \mathcal{F}}_{n \text{ times}}$. The intermediate logic **Cheq** is the logic of $\{\mathcal{F}_n : n \in \mathbb{N}\}$.

THEOREM 5 (Litak [2]). **ML** *is a proper extension of* **Cheq**.

Our main goal is to show that **ML** is not finitely axiomatizable over **Cheq**. For an $n$-tuple $x$, let $N_i(x)$ denote the number of $w_i$ that occur in $x$ ($i = 0, 1, 2$). We denote the $j$-th component of $x$ by $x(j-1)$. For a Kripke frame $\langle W, \leq \rangle$ and $w, v \in W$, we say that $v$ is an *immediate successor* of $w$ if $w \neq v$, $w \leq v$ and there is no $u \notin \{w, v\}$ such that $w \leq u$ and $u \leq v$. Note that if $x \in \mathcal{F}_n$ is an immediate successor of $(w_0, \ldots, w_0)$, it has only one component $x(i_0)$ that differs from $w_0$ and we denote it by $\delta(x)$.

For every $k > 1$ and $l > 0$, let $\mathcal{G}_{k,l}$ denote the frame shown in Figure 3 (note that $\mathcal{G}_k = \mathcal{G}_{k,2^{k+3}-1}$).

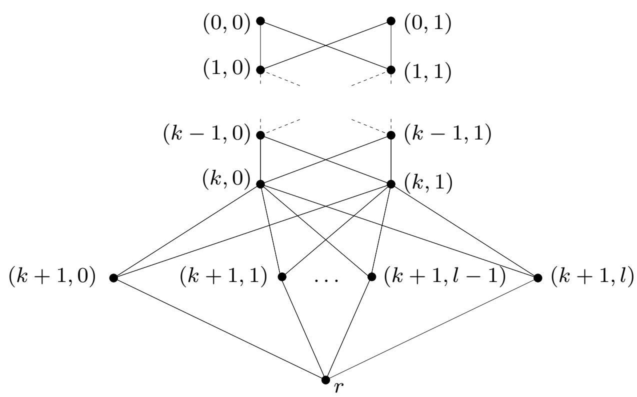


Figure 3. The frame $\mathcal{G}_{k,l}$.

PROPOSITION 6. *For each $l > 0$, there exists $n$ such that $\mathcal{G}_{2,l}$ is a p-morphic image of $\mathcal{F}_n$. Moreover, there is a p-morphism $f$ from $\mathcal{F}_n$ onto $\mathcal{G}_{2,l}$ such that $f^{-1}\{(3,i) | i \leq l\}$ is the set of all immediate successors of $(w_0, \ldots, w_0)$.*

**Proof.** Fix $l > 0$ and an arbitrary $n$ so that $2n \geq l+1$ and $n > 3$. We show that there is a $p$-morphism $f$ from $\mathcal{F}_n$ onto $\mathcal{G}_{2,l}$ such that $f^{-1}\{(3,i)|i \leq l\}$ is the set of all immediate successors of $(w_0, \ldots, w_0)$. Since $2n \geq l+1$, there is a map $g$ from the set of immediate successors of $(w_0, \ldots, w_0)$ onto $\{(3,i)|i \leq l\}$.

Define $f$ by

$$f(x) = \begin{cases} r & \text{if } x = (w_0, \ldots, w_0) \\ g(x) & \text{if } x \text{ is an immediate successor of } (w_0, \ldots, w_0) \\ (2,0) & \text{if } N_0(x) = n-2,\ x(i) = w_1,\ x(j) = w_2 \text{ and } i+j \text{ is even} \\ (2,1) & \text{if } N_0(x) = n-2,\ x(i) = w_1,\ x(j) = w_2 \text{ and } i+j \text{ is odd} \\ (1,0) & \text{if } x \text{ is not maximal, } N_1(x) > 1 \text{ and } N_2(x) \leq 1 \\ (1,1) & \text{if } x \text{ is not maximal, } N_2(x) > 1 \text{ and } N_1(x) \leq 1 \\ (0,0) & \text{if } x \text{ is maximal and either } N_1(x) = 1 \text{ or } N_2(x) = 1 \\ (0,1) & \text{otherwise.} \end{cases}$$

Obviously, $f$ is a well-defined onto map such that $f^{-1}\{(3,i)|i \leq l\}$ is the set of all immediate successors of $(w_0, \ldots, w_0)$. We show that $f$ is a $p$-morphism; that is, if $f(x) \leq u$, then there is $y$ such that $x \leq y$ and $f(y) = u$ and if $x \leq y$, then $f(x) \leq f(y)$. First, we verify the former condition.

For $x \in \mathcal{F}_n$ and $u \in \mathcal{G}_{2,l}$, let $f(x) \leq u$. Then we need to find a $y \in \mathcal{F}_n$ such that $x \leq y$ and $f(y) = u$. If $f(x) = u$, then define $y$ as $x$. From now on we assume that $f(x) \neq u$.

Since $\mathcal{G}_{2,l}$ is finite, there are $k \in \mathbb{N}$ and $u_0, \ldots, u_k \in \mathcal{G}_{2,l}$ such that $f(x) \leq u_0 \leq \cdots \leq u_k = u$, $u_0$ is an immediate successor of $f(x)$ and each $u_{i+1}$ is an immediate successor of $u_i$. We show the existence of $y$ by induction on $k$. If $k = 0$, $u$ is an immediate successor of $f(x)$ and there are nine cases possible.

1. $x = (w_0, \ldots, w_0)$. Take any $y$ such that $f(y) = u$.

2. $x$ is an immediate successor of $(w_0, \ldots, w_0)$ and $u = (2,0)$. Without loss of generality we may assume that $x(i_0) = w_1$. Since $n > 3$, there is an index $i_1 \neq i_0$ such that $i_0 + i_1$ is even. Then take $y$ such that $y(i_1) = w_2$ and $y(i) = x(i)$ for all $i \neq i_1$.

3. $x$ is an immediate successor of $(w_0, \ldots, w_0)$ and $u = (2,1)$. Then the argument is similar to case (2).

4. $N_0(x) = n-2$, $x(i) = w_1$, $x(j) = w_2$ and $u = (1,0)$. Since $n > 3$, there is an index $i_0$ such that $x(i_0) = w_0$. Then take $y$ such that $y(i_0) = w_1$ and $y(i) = x(i)$ for all $i \neq i_0$.

5. $N_0(x) = n-2$, $x(i) = w_1$, $x(j) = w_2$ and $u = (1,1)$. Then the argument is similar to case (4).

6. $N_1(x) > 1$, $N_2(x) \leq 1$ and $u = (0,0)$. If $N_2(x) = 1$, then there exists $i_0$ such that $x(i_0) = w_2$. Then take $y$ such that $y(i_0) = w_2$

and $y(i) = w_1$ for all $i \neq i_0$. If $N_2(x) = 0$, fix an index $i_0$ such that $x(i_0) = w_0$ and take $y$ such that $y(i_0) = w_2$ and $y(i) = w_1$ for all $i \neq i_0$.

7. $N_2(x) > 1$, $N_1(x) \leq 1$ and $u = (0,0)$. Then the argument is similar to case (6).

8. $N_1(x) > 1$, $N_2(x) \leq 1$ and $u = (0,1)$. If $N_2(x) = 0$, then define $y$ as $(w_1, \ldots, w_1)$. If $N_2(x) = 1$, then there exists $i_0$ such that $x(i_0) = w_0$ and take $y$ such that $y(i_0) = w_2$ and $y(i) = x(i)$ for all $i \neq i_0$.

9. $N_2(x) > 1$, $N_1(x) \leq 1$ and $u = (0,1)$. Then the argument is similar to case (8).

Next suppose that $k = k' + 1$. By the induction hypothesis, there is $y' \in \mathcal{F}_n$ such that $x \leq y'$ and $f(y') = u_{k'}$. Recall that $u = u_{k'+1}$ is an immediate successor of $u_{k'} = f(y')$. In the same way as we showed above, we can prove that there is $y \in \mathcal{F}_n$ such that $y' \leq y$ and $f(y) = u$. Therefore, we obtain that $x \leq y' \leq y$ and $f(y) = u$.

Finally we verify that if $x \leq y$, then $f(x) \leq f(y)$. Suppose $x, y \in \mathcal{F}_n$ are two distinct points such that $x \leq y$. We show that $f(x) \leq f(y)$. There are six cases possible.

1. $x = (w_0, \ldots, w_0)$. Then $f(x) = r$ and $r \leq f(y)$.

2. $x$ is an immediate successor of $(w_0, \ldots, w_0)$. By the definition of $f$, we have $f(x)$ is equal to some $(3, i)$. Since $y$ is not an immediate successor of $(w_0, \ldots, w_0)$, $f(y)$ is also not an immediate successor of $r$. Hence, $f(x) \leq f(y)$.

3. $N_0(x) = n - 2$, $x(i) = w_1$ and $x(j) = w_2$. By the definition of $f$ $f(x)$ is either $(2,0)$ or $(2,1)$. Since $x \leq y$, we can deduce that either $N_1(y) > 1$ or $N_2(y) > 1$. In both cases this implies that $f(y)$ belongs to $\{(1,0), (1,1), (0,0), (0,1)\}$. So $f(x) \leq f(y)$.

4. $x$ is not maximal, $N_1(x) > 1$ and $N_2(x) \leq 1$. From the definition of $f$ it follows that $f(x) = (1,0)$. Moreover, since $x \leq y$, we also have that $N_1(y) > 1$. So $f(y)$ belongs to $\{(1,0), (0,0), (0,1)\}$. In any case, $f(x) \leq f(y)$.

5. $x$ is not maximal, $N_2(x) > 1$ and $N_1(x) \leq 1$. Then the argument is similar to case (4).

6. $N_1(x) > 1$ and $N_2(x) > 1$. By the definition of $f$, we have that $f(x) = (0,1)$. Moreover, $x \leq y$ implies $N_1(y) > 1$ and $N_2(y) > 1$. So $f(y)$ is also equal to $(0,1)$.

■

PROPOSITION 7. *For each $k > 1$ and $l > 0$, there exists $n > 2$ such that $\mathcal{G}_{k,l}$ is a p-morphic image of $\mathcal{F}_n$. Moreover, there is a p-morphism $f$ from $\mathcal{F}_n$ onto $\mathcal{G}_{k,l}$ such that $f^{-1}\{(k+1,i)|i \leq l\}$ is the set of all immediate successors of $(w_0,\dots,w_0)$.*

**Proof.** The proof is by induction on $k$. If $k = 2$, apply Proposition 6. Suppose $k = k'+1$ and there is a $p$-morphism $f$ from $\mathcal{F}_n$ onto $\mathcal{G}_{k',l}$ such that $f^{-1}\{(k'+1,i)|i \leq l\}$ is the set of all immediate successors of $(w_0,\dots,w_0)$.
If $x \in \mathcal{F}_{n+1}$, $x' = (x(0),\dots,x(n-1))$ and $j \in \{1,2\}$, define $g(x)$ by

$$g(x) = \begin{cases} r & \text{if } x = (w_0,\dots,w_0) \\ (k'+2,0) & \text{if } x = (w_0,\dots,w_0,w_j) \\ (k'+2,i) & \text{if } x \neq (w_0,\dots,w_0,w_j),\ N_0(x) = n \text{ and} \\ & f(x') = (k'+1,i) \\ (k'+1,0) & \text{if } N_0(x) = n-1,\ N_0(x') = n-1 \text{ and } \delta(x') = x(n) \\ (k'+1,1) & \text{if } N_0(x) = n-1,\ N_0(x') = n-1 \text{ and } \delta(x') \neq x(n) \\ f(x') & \text{if } N_0(x') < n-1. \end{cases}$$

Intuitively, the frame $\mathcal{G}_{k'+1,l}$ is obtained from the frame $\mathcal{G}_{k',l}$ by adding two points between the points of depth $k'+1$ and the points of depth $k'+2$. In general, if $x \in \mathcal{F}_{n+1}$ and if $x = (x',w)$, we just map $x$ to the same point that $x'$ was mapped to before. The only exceptions are when $w \neq w_0$ and $x'$ is either $(w_0,\dots,w_0)$ or an immediate successor of $(w_0,\dots,w_0)$. In the case $x'$ is equal to $(w_0,\dots,w_0)$ and $w$ is either $w_1$ or $w_2$, we map $x$ to an arbitrary immediate successor of $r$. In the case $x'$ is an immediate successor of $(w_0,\dots,w_0)$ and $w$ is either $w_1$ or $w_2$, we map $x$ to one of the two added points.

Obviously $g : \mathcal{F}_{n+1} \to \mathcal{G}_{k'+1,l}$ is a well-defined onto map such that $g^{-1}\{(k+1,i)|i \leq l\}$ is the set of all immediate successors of $(w_0,\dots,w_0)$. We check that $g$ is a $p$-morphism. For $x \in \mathcal{F}_{n+1}$ and $u \in \mathcal{G}_{k,l}$, let $g(x) \leq u$. Then we need to find a $y \in \mathcal{F}_{n+1}$ such that $x \leq y$ and $g(y) = u$. As in the previous proof we may assume that $u$ is an immediate successor of $g(x)$. There are eight cases possible.

1. $x = (w_0,\dots,w_0)$ and $u = (k'+2,i)$. By the induction hypothesis, there is a $t$ such that $N_0(t) = 1$ and $f(t) = (k'+1,i)$. Then put $y = (t,w_0)$.

2. $x = (x',w_0)$, $N_0(x) = n$ and $u = (k'+1,0)$. Then put $y = (x',\delta(x'))$.

3. $x = (x',w_0)$, $N_0(x) = n$ and $u = (k'+1,1)$. Then the argument is similar to case (2).

4. $x = (w_0,\dots,w_0,w_i)$, $i$ is either 1 or 2 and $u = (k'+1,0)$. Then put $y = (w_i,w_0,\dots,w_0,w_i)$.

5. $x = (w_0,\dots,w_0,w_i)$, $i$ is either 1 or 2 and $u = (k'+1,1)$. Then the argument is similar to case (4).

6. $x = (x', w_i)$, $i$ belongs to $\{1, 2\}$, $N_0(x') = n-1$, $u = (i_1, i_2)$ and $i_1 \leq k'$. Recall that $f(x')$ has to be equal to some $(k'+1, i)$. Since $f$ is a $p$-morphism, there is $s \in \mathcal{F}_n$ such that $x' \leq s$ and $f(s) = u$. We put $y = (s, w_i)$.
7. $x = (x', w_i)$, $N_0(x') < n-1$, $f(x') = (i_1, i_2)$ and $u = (i_1-1, 0)$. By the definition of $g$, we have that $g(x) = (i_1, i_2)$. Since $f$ is a $p$-morphism, there is $s \in \mathcal{F}_n$ such that $x' \leq s$ and $f(s) = u$. We put $y = (s, w_i)$.
8. $x = (x', w_i)$, $N_0(x') < n-1$, $f(x') = (i_1, i_2)$ and $u = (i_1-1, 1)$. Then the argument is similar to case (7).

Next suppose that $x, y \in \mathcal{F}_{n+1}$ are two distinct points such that $x \leq y$. We show that $g(x) \leq g(y)$. Let $x', y', i$ and $i'$ be such that $x = (x', w_i)$ and $y = (y', w_{i'})$. There are four cases possible.

1. $x = (w_0, \ldots, w_0)$. Then $g(x) = r$ and $r \leq g(y)$.
2. $x$ is an immediate successor of $(w_0, \ldots, w_0)$. It is easy to see that $g^{-1}\{(k'+2, i) | i \leq l\}$ is the set of all immediate successors of $(w_0, \ldots, w_0)$. Thus, $g(x) \leq g(y)$.
3. $N_0(x) = n-2$ and $i \in \{1, 2\}$. By the definition of $g$, $g(x)$ is either $(k'+1, 0)$ or $(k'+1, 1)$. Now since $x' \leq y'$ and $x' \neq y'$, we also have that $N_0(y') < n-1$. So $g(y)$ is equal to $f(y')$ and from our assumption on $f$, we can deduce that $f(y')$ is equal to some $(i_1, i_2)$, where $i_1 \leq k'$. It follows that $g(x) \leq f(y')$.
4. $N_0(y) < n-1$. By the definition of $g$, $g(x)$ is equal to $f(x')$. Also since $x' \leq y'$, we have that $N_0(y') < n-1$ and so $g(y) = f(y')$. Using the fact that $f$ is a $p$-morphism, we obtain that $f(x') \leq f(y')$.

■

COROLLARY 8.

(*a*) *For each $k > 1$, the frame $\mathcal{G}_k$ is a $p$-morphic image of some $\mathcal{F}_n$.*

(*b*) *For each $k > 1$, $\mathcal{G}_k$ is a* **Cheq**-*frame.*

**Proof.** The result follows from Proposition 7. ■

THEOREM 9. *The logic* **ML** *is not finitely axiomatizable over* **Cheq**. *In fact,* **ML** *is not axiomatizable over* **Cheq** *by any set of formulas in finitely many variables.*

**Proof.** Suppose there is a finite set of formulas that axiomatizes **ML** over **Cheq**. Without loss of generality we may assume that there is a single formula $\varphi$ with $k$ variables such that **ML** = **Cheq** + $\varphi$. By Lemma 2(c), there exists a natural number $i \leq k$ such that $\varphi$ is valid in $\mathcal{G}_k$ iff $\varphi$ is valid in $\mathcal{G}_k^i$. By Lemma 2(b), $\mathcal{G}_k^i$ is an **ML**-frame. Thus, $\varphi$ is valid in $\mathcal{G}_k^i$. Therefore, $\varphi$ is valid in $\mathcal{G}_k$. By Corollary 8, $\mathcal{G}_k$ is a **Cheq**-frame. Thus, $\mathcal{G}_k$ is an **ML**-frame. But this contradicts Lemma 2(a). ■

## 4 Conclusion

We proved that **ML** is not finitely axiomatizable over **Cheq**. Thus, the two logics **ML** and **Cheq** are not as closely related as previously thought. It still remains an open problem whether **Cheq** is finitely axiomatizable and/or decidable. At present we can only show that **Cheq** is not axiomatizable in four variables. Of course, the decidability of **ML** still remains an interesting (but difficult) open problem.

**Acknowledgments.** I would like to thank Nick Bezhanishvili for introducing me to the subject and supervizing me, Guram Bezhanishvili for his help and support in writing this paper and Tadeusz Litak for many valuable suggestions. I would also like to thank the anonymous referees for their comments and the *Fonds national de la recherche scientifique en Belgique* for supporting this research.

## BIBLIOGRAPHY

[1] A. Chagrov and M. Zakharyaschev. *Modal Logic*. Clarendon Press, 1997.

[2] T. Litak. Some notes on the superintuitionistic logic of chequered subsets of $\mathbb{R}^\infty$. *Bulletin of the Section of Logic*, 33:81–86, 2004.

[3] L. Maksimova, V. Shehtman, and D. Skvorcov. The impossibility of a finite axiomatization of Medvedev's logic of finitary problems. *Soviet Math. Dokl.*, 20:394–398, 1979.

[4] Ju. T. Medvedev. Finite problems. *Soviet Math. Dokl.*, 3:227–230, 1962.

[5] J. van Benthem, G. Bezhanishvili, and M. Gehrke. Euclidian hierarchy in modal logic. *Studia Logica*, 75:327–344, 2003.

Gaëlle Fontaine

ILLC (Institute for Logic, Language and Computation)
Plantage Muidergracht 24
1018 TV Amsterdam
The Netherlands.

gfontain@science.uva.nl