The Truth About Algorithmic Problems in Correspondence Theory

ALEXANDER CHAGROV AND LILIA CHAGROVA

ABSTRACT. We present proofs that the following three major algorithmic problems in Correspondence Theory are undecidable: first-order definability of propositional modal formulas, modal (propositional) definability of first-order (classical predicate) formulas, and the correspondence problem (equivalence on Kripke frames) for propositional modal formulas and first-order formulas. These results have been known since Chagrova's PhD thesis [4] published in 1989. However, [4] considered (propositional) intuitionistic formulas which are much harder to deal with than the modal ones. Although the proofs from [4] provided some extra subtle results (such as the undecidability of the set of propositional formulas that are first-order definable on the class of countable frames, but not on the class of all frames; see also [9]), the technique employed was rather involved and the intuition behind was not clear (which has been mentioned by many of our colleagues).

The main aim of this paper is to give simple and direct proofs for the modal case and to consider some variations of the three major problems (e.g., the case of countable frames). Our second goal is to show that in the proof of the undecidability of modal definability of first-order formulas, it is sufficient to use one simple modal formula, namely $\bot$.

1 Introduction

The question about the model-theoretic correspondence between nonclassical (modal, in the first place) propositional formulas and classical first-order formulas arose practically simultaneously with the creation of Kripke semantics: it turned out that models of first-order formulas in the signature of one binary predicate (and equality) can serve as 'model structures' (later called frames) for modal formulas. Already early papers in this area, say [13], observed that many natural conditions expressible in such structures by first-order formulas correspond to validity of some propositional modal formulas. We only remind the reader of two banal examples:

first-order formula	modal formula
$\forall x\, xRx$ (reflexivity)	$\Box p \to p$
$\forall x\, \forall y\, \forall z\, (xRy \;\&\; yRz \Rightarrow xRz)$ (transitivity)	$\Box p \to \Box\Box p.$

Those observations led to the creation of Correspondence Theory (see [1, 2]), which has become an integral part of the branch of mathematical logic known as 'Modal Logic.' Moreover, some authors consider modal logic as just one of the languages for describing properties of relational structures.[1]

We are not going to discuss here the role of Correspondence Theory in the general theory of modal logic; this topic has already been analysed by many authors; see, e.g., [1, 2]. The problems we consider here are much more modest: Can we, given a modal formula, recognize whether this formula has an equivalent first-order condition on modal frames? Conversely, is it possible, given a condition expressible by a first-order formula, to recognise whether it has a propositional modal equivalent? One more important problem is to recognise, for a first-order formula and a modal formula, whether they determine the same class of frames. (Here we take a rather wide view of modal formulas: they can be, for example, polymodal formulas or propositional dynamic formulas with the intended semantics.) It should also be noted that in this paper we only consider the so-called global variants of definability. However, our methods can easily be extended to the local cases.

A solution to the problems above for the (uni)modal language has been known since Chagrova's PhD thesis [4][2] where it was proved, in an indirect way, that all of them are undecidable. For example, [4] proves that definability of first-order formulas by intuitionistic propositional formulas (on the class of intuitionistic frames) is undecidable. To show that modal definability is undecidable as well, it suffices to replace the intuitionistic formulas used in the proof with their Gödel T-translations; for definitions see [10]. The resulting reduction of a certain undecidable problem to definability of first-order formulas by means of modal formulas, which contain as conjuncts the reflexivity and transitivity of the relation R, leads to the following dilemma: either the given first-order formula has no modal equivalent at all, or its modal equivalent is the T-translation of some intuitionistic formula with two extra conjuncts $\Box p \to p$ and $\Box p \to \Box\Box p$. We shall not go into further details here because this proof from [4] was a 'side effect' of rather cumbersome constructions required to solve more difficult and subtle problems such as first-order definability of intuitionistic formulas on the class of countable frames. In fact, as we show below, the direct proof of the undecidability of modal definability is not so complex. Note also that, according to the remark in [2] on page 398, the undecidability of modal definability of first-order sentences was proved by Frank Wolter.[3]

We were encouraged to write this paper by our friends and colleagues Ph. Balbiani, T. Tinchev and D. Vakarelov who discussed with us the algorithmic problems in Correspondence Theory during our stay at the Institut de recherche en informatique de Toulouse (Universite Paul Sabastier) in October–November 2003. Thus, Ph. Balbiani raised the algorithmic problem of definability of first-order formulas by propositional formulas on the

[1]See, for example, [3], p. 24: "So far we have been viewing modal languages as tools for talking about models."

[2]Parts of the thesis were also published as preprints [5, 6, 7].

[3]In fact, [2] refers to [15] which contains no such result. Moreover, Frank Wolter kindly informed us that he never published his proof, but discussed it in some seminars.

class of frames for propositional dynamic logic, D. Vakarelov was interested in definability of first-order formulas by variable-free modal formulas. The results of this paper are related to both extensions of modal languages and the number of variables used in modal formulas. T. Tinchev helped us to amend the original proofs. We are grateful to our colleagues. We would also like to thank the anonymous referees for their useful comments. Especially we are grateful to our friend Michael Zakharyaschev whose remarks and suggestions substantially helped to improve the presentation

2 Undecidability of first-order definability of modal formulas

In this section we prove the following:

THEOREM 1. *The problem of first-order definability of modal formulas is undecidable.*

Proof. Here and in what follows our 'master' problems are some standard undecidable problems for Minsky machines; for definitions see Section 5.1. We are going to construct a set of formulas denoted by

$$B(\mathcal{P}, (\alpha, m, n), (\beta, k, l))$$

with the following properties:

1. $B(\mathcal{P}, (\alpha, m, n), (\beta, k, l))$ is constructed effectively for a given Minsky machine program $\mathcal{P}$ and configurations (α, m, n), (β, k, l).

2. The following equivalence holds:

$$B(\mathcal{P}, (\alpha, m, n), (\beta, k, l)) \textit{ is first-order definable}$$
$$\Updownarrow$$
$$\mathcal{P} : (\alpha, m, n) \to (\beta, k, l).$$

As the problem '$\mathcal{P} : (\alpha, m, n) \to (\beta, k, l)$?' is undecidable, these two properties will prove Theorem 1.

According to the second item, if $\mathcal{P} : (\alpha, m, n) \to (\beta, k, l)$, then the formula $B(\mathcal{P}, (\alpha, m, n), (\beta, k, l))$ must be first-order definable. We ensure this by choosing our formula to be deductively equivalent in this case to the formula $\neg H$, where $H = \Diamond(\Box^2\bot \wedge \Diamond\neg p \wedge \Diamond p)$. This formula is clearly first-order definable. (Perhaps some readers would prefer the equivalent formula $\Box(\Box^2\bot \to \Box p \vee \Box\neg p)$.)

Observe that to be able to refute $\neg H$ (or to satisfy H) a frame $\mathfrak{F}$ should satisfy the following condition (note a clear similarity between this condition and the formula $\neg H$ itself):

$$\mathfrak{F} \models \exists x \exists y\, (xRy \wedge \neg\exists z\, yR^2z \wedge \exists u \exists v(yRu \wedge yRv \wedge u \neq v)).$$

Conversely, if $\mathfrak{F}$ satisfies this condition, then one can define a valuation in $\mathfrak{F}$ satisfying H. Indeed, let x, y, u, v be the points the existence of which is asserted by this condition. Take $u \not\models p$ and $v \models p$. The condition $\neg\exists z\, yR^2z$ guarantees that $y \models \Box^2\bot$. Together with yRu and yRv, this gives $y \models \Box^2\bot \wedge \Diamond\neg p \wedge \Diamond p$, from which we obtain $x \models H$ because xRy.

Thus we have:

LEMMA 2. *$\neg H$ is first-order definable by the formula*

$$\neg\exists x\exists y\,(xRy \wedge \neg\exists z\, yR^2z \wedge \exists u\exists v(yRu \wedge yRv \wedge u \neq v)).$$

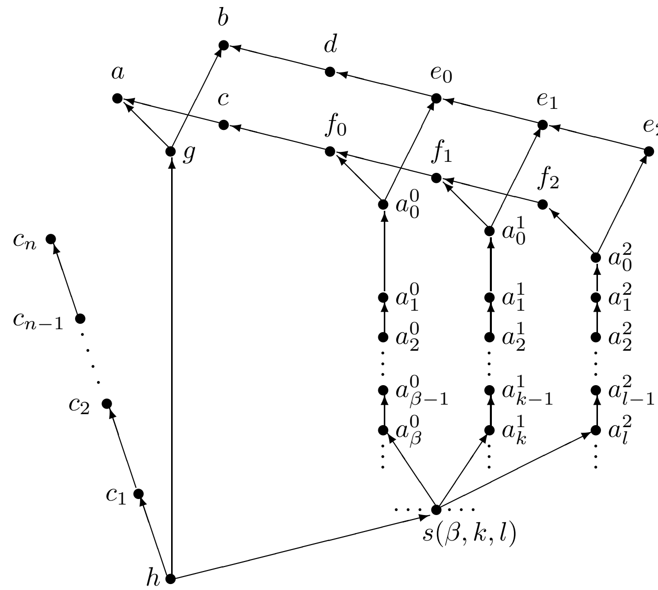

Figure 1. Frame $\mathfrak{F}_n$.

Now fix a Minsky machine $\mathcal{P}$ and a configuration (α, m, n). To understand the meaning of the formulas below the reader should consult the frame $\mathfrak{F}_n$ depicted in Fig. 1. We briefly explain its construction. First, all the points $\bullet$ are supposed to be irreflexive. The accessibility relation xRy in $\mathfrak{F}_n$ is the transitive closure of the relation defined by the arrows $\rightarrow$. The points of the form a^i_0, a^i_1, a^i_2, ... have natural numbers as subscripts. The meaning of these subscripts is as follows: a^0_β represents the state of $\mathcal{P}$, a^1_k represents the information on the first tape, and a^2_l the information on the second tape. The points $s(\beta,k,l)$ form the set $\{s(\beta,k,l) \mid \mathcal{P} : (\alpha,m,n) \rightarrow (\beta,k,l)\}$, that is, we include in $\mathfrak{F}_n$ the point $s(\beta,k,l)$ only if the configuration (β,k,l) is reachable from (α,m,n) by $\mathcal{P}$. We also assume that

$$s(\beta,k,l)Ra^0_\beta, \quad s(\beta,k,l)Ra^1_k, \quad s(\beta,k,l)Ra^2_l,$$

but none of the relations

$$s(\beta,k,l)Ra^0_{\beta+1}, \quad s(\beta,k,l)Ra^1_{k+1}, \quad s(\beta,k,l)Ra^2_{l+1}$$

holds. Note that $\mathfrak{F}_n$ refutes the formula $\neg H$. Moreover, if $\mathfrak{F}_n \not\models \neg H$ then

$$\text{either} \quad a \models \Box\bot \wedge p,\ b \models \Box\bot \wedge \neg p \quad \text{or} \quad a \models \Box\bot \wedge \neg p,\ b \models \Box\bot \wedge p.$$

Since these two cases are symmetric, we will only consider the former.

Now define the formulas:

$$A = \Box\bot \wedge p\,,\ B = \Box\bot \wedge \neg p\,,$$

$$C = \Diamond A \wedge \neg\Diamond\Diamond A \wedge \neg\Diamond B\,,\ D = \Diamond B \wedge \neg\Diamond\Diamond B \wedge \neg\Diamond A\,,$$

$$G = \Box^2\bot \wedge \Diamond\neg p \wedge \Diamond p\,,\ H = \Diamond G\,,$$

$$E_0 = \Diamond D \wedge \neg\Diamond\Diamond D,\ E_1 = \Diamond E_0 \wedge \neg\Diamond\Diamond E_0 \wedge \neg\Diamond C,$$

$$E_2 = \Diamond E_1 \wedge \neg\Diamond\Diamond E_1 \wedge \neg\Diamond C,\ F_0 = \Diamond C \wedge \neg\Diamond\Diamond C,$$

$$F_1 = \Diamond F_0 \wedge \neg\Diamond\Diamond F_0 \wedge \neg\Diamond D,\ F_2 = \Diamond F_1 \wedge \neg\Diamond\Diamond F_1 \wedge \neg\Diamond D,$$

$$A_0^0 = \Diamond E_0 \wedge \Diamond F_0 \wedge \neg\Diamond\Diamond E_0 \wedge \neg\Diamond\Diamond F_0,\ A_0^1 = \Diamond E_1 \wedge \Diamond F_1 \wedge \neg\Diamond\Diamond E_1 \wedge \neg\Diamond\Diamond F_1,$$

$$A_0^2 = \Diamond E_2 \wedge \Diamond F_2 \wedge \neg\Diamond\Diamond E_2 \wedge \neg\Diamond\Diamond F_2,$$

$$A_{j+1}^i = \Diamond A_j^i \wedge \neg\Diamond^2 A_j^i \wedge \bigwedge_{i\neq k=0}^{2} \neg\Diamond A_0^k,$$

where $i \in \{0,1,2\}$, $j \geq 0$, and let

$$S(\beta,\varphi,\psi) = \Diamond A_\beta^0 \wedge \neg\Diamond A_{\beta+1}^0 \wedge \Diamond\varphi \wedge \neg\Diamond\Diamond\varphi \wedge \Diamond\psi \wedge \neg\Diamond\Diamond\psi$$

for any formulas φ, ψ, and $\beta \geq 0$.

LEMMA 3. *Let a valuation in $\mathfrak{F}_n$ be such that $\mathfrak{F}_n \not\models \neg H$. Then the following hold:*

1. $\{x \mid x \models H\} = \{h\}$*;*
2. $\{x \mid x \models A\} = \{a\}$*,* $\{x \mid x \models B\} = \{b\}$*;*
3. $\{x \mid x \models C\} = \{c\}$*,* $\{x \mid x \models D\} = \{d\}$*;*
4. $\{x \mid x \models E_i\} = \{e_i\}$*,* $\{x \mid x \models F_i\} = \{f_i\}$ *for* $i \in \{0,1,2\}$*;*
5. $\{x \mid x \models A_0^i\} = \{a_0^i\}$ *for* $i \in \{0,1,2\}$*;*
6. $\{x \mid x \models A_j^i\} = \{a_j^i\}$ *for* $i \in \{0,1,2\}$*,* $j > 0$*;*
7. *for* $\beta, k, l \geq 0$

$$\{x \mid x \models S(\beta, A_k^1, A_l^2)\} = \begin{cases} \{s(\beta,k,l)\}, & \textit{if } \mathcal{P} : \langle\alpha,m,n\rangle \to \langle\beta,k,l\rangle; \\ \emptyset, & \textit{if } \mathcal{P} : \langle\alpha,m,n\rangle \not\to \langle\beta,k,l\rangle\ . \end{cases}$$

Proof. The straightforward proof is left to the reader (the proof of item 6 is by induction on j). ■

REMARK 4. If $\neg H$ is refuted under a valuation with $a \models \Box\bot \wedge \neg p$ and $b \models \Box\bot \wedge p$, then items 2, 3, 4 of the lemma above should be rewritten as follows:

2. $\{x \mid x \models A\} = \{b\}$, $\{x \mid x \models B\} = \{a\}$;

3. $\{x \mid x \models C\} = \{d\}$, $\{x \mid x \models D\} = \{c\}$;

4. $\{x \mid x \models E_i\} = \{f_i\}$, $\{x \mid x \models F_i\} = \{e_i\}$ *for* $i \in \{0, 1, 2\}$.

These changes have no influence on what follows.

Now define the following formulas:

$$\begin{aligned}
Q_1 &= (\Diamond A_0^1 \vee A_0^1) \wedge \neg\Diamond A_0^0 \wedge \neg\Diamond A_0^2 \wedge p_1 \wedge \neg\Diamond p_1,\\
Q_2 &= \Diamond A_0^1 \wedge \neg\Diamond A_0^0 \wedge \neg\Diamond A_0^2 \wedge \Diamond p_1 \wedge \neg\Diamond\Diamond p_1,\\
R_1 &= (\Diamond A_0^2 \vee A_0^2) \wedge \neg\Diamond A_0^0 \wedge \neg\Diamond A_0^1 \wedge p_2 \wedge \neg\Diamond p_2,\\
R_2 &= \Diamond A_0^2 \wedge \neg\Diamond A_0^0 \wedge \neg\Diamond A_0^1 \wedge \Diamond p_2 \wedge \neg\Diamond\Diamond p_2\,.
\end{aligned}$$

Some properties of these formulas are formulated in the following lemma.

LEMMA 5. (i) *Let a valuation in* $\mathfrak{F}_n$ *be such that* $x \models Q_1$ *for some point* x. *Then, for some* $i \geq 0$,

$$\{x \mid\, x \models Q_1\} = \{a_i^1\},\ \{x \mid\, x \models Q_2\} = \{a_{i+1}^1\}\,.$$

(ii) *Let a valuation in* $\mathfrak{F}_n$ *be such that* $x \models Q_2$ *for some point* x. *Then, for some* $i \geq 1$,

$$\{x \mid\, x \models Q_2\} = \{a_i^1\},\ \{x \mid\, x \models Q_1\} = \{a_{i-1}^1\}\,.$$

(iii) *Let a valuation in* $\mathfrak{F}_n$ *be such that* $x \models R_1$ *for some point* x. *Then, for some* $i \geq 0$,

$$\{x \mid\, x \models R_1\} = \{a_i^2\},\ \{x \mid\, x \models R_2\} = \{a_{i+1}^2\}.$$

(iv) *Let a valuation in* $\mathfrak{F}_n$ *be such that* $x \models R_2$ *for some point* x. *Then, for some* $i \geq 1$

$$\{x \mid\, x \models R_2\} = \{a_i^2\},\ \{x \mid\, x \models R_1\} = \{a_{i-1}^2\}.$$

Proof. Note first that if a valuation satisfies one of (i)–(iv) then $\mathfrak{F}_n \not\models \neg H$, more precisely $h \models H$.

As the proofs of all four claims are similar, we only consider (iv). By Lemma 3, $x \models R_2$ means that

$$xRa_0^2 \ \text{ and } \ \neg xRa_0^0, \neg xRa_0^1.$$

Therefore, by the structure of $\mathfrak{F}_n$, we obtain $x \in \{a_u^2 \mid u \geq 1\}$. Let i be such that $x = a_i^2$. Again, $x \models R_2$ implies

$$a_i^2 \models \Diamond p_2 \ \text{ and } \ a_i^2 \not\models \Diamond\Diamond p_2,$$

which means that there is a point accessible from a_i^2 where p_2 is true, and p_2 is false at any point that is accessible from a_i^2 in two (or more) steps. By the structure of $\mathfrak{F}_n$, it follows that a_{i-1}^2 is the only point in the frame accessible from a_i^2 where p_2 is true. This means, apart from other things, that a_i^2 is the only point in the set $\{a_u^2 \mid u \geq 1\}$ such that $a_u^2 \models \Diamond p_2 \land \neg\Diamond\Diamond p_2$ and a_{i-1}^2 is the only point in the set $\{a_u^2 \mid u \geq 1\}$ such that $a_u^2 \models p_2 \land \neg\Diamond p_2$. Therefore, by Lemma 3, we are done. ∎

We are now in a position to define a set of formulas AxI which corresponds to the instruction set of the program $\mathcal{P}$:

- if $I = \gamma \to \langle \delta, 1, 0\rangle$ then
$$AxI = H \land \Diamond S(\gamma, Q_1, R_1) \to H \land \Diamond S(\delta, Q_2, R_1)\ ;$$
- if $I = \gamma \to \langle \delta, 0, 1\rangle$ then
$$AxI = H \land \Diamond S(\gamma, Q_1, R_1) \to H \land \Diamond S(\delta, Q_1, R_2)\ ;$$
- if $I = \gamma \to \langle \delta_1, -1, 0\rangle\,(\langle \delta_2, 0, 0\rangle)$ then
$$\begin{aligned} AxI = \quad & (H \land \Diamond S(\gamma, Q_2, R_1) \to H \land \Diamond S(\delta_1, Q_1, R_1)) \land \\ & \land (H \land \Diamond S(\gamma, A_0^1, R_1) \to H \land \Diamond S(\delta_2, A_0^1, R_1))\ ; \end{aligned}$$
- if $I = \gamma \to \langle \delta_1, -1, 0\rangle\,(\langle \delta_2, 0, 0\rangle)$ then
$$\begin{aligned} AxI = \quad & (H \land \Diamond S(\gamma, Q_1, R_2) \to H \land \Diamond S(\delta_1, Q_1, R_1)) \land \\ & \land (H \land \Diamond S(\gamma, Q_1, A_0^2) \to H \land \Diamond S(\delta_2, Q_1, A_0^2))\ . \end{aligned}$$

Finally, define
$$Ax\mathcal{P} = \bigwedge_{I \in \mathcal{P}} AxI\ .$$

Denote by $\mathbf{K} \oplus Ax\mathcal{P}$ the closure of $\mathbf{K} \cup \{Ax\mathcal{P}\}$ under modus ponens, substitution and necessitation (and use a similar notation in similar cases). $L \vdash \varphi$ and $\varphi \in L$ are regarded to be synonymous.

LEMMA 6. *If*
$$P : (\alpha, m, n) \to (\beta, k, l)$$
then
$$\mathbf{K} \oplus Ax\mathcal{P} \vdash H \land \Diamond S(\alpha, A_m^1, A_n^2) \to H \land \Diamond S(\beta, A_k^1, A_l^2)\,.$$

Proof. The proof is by induction on the length of the computation from (α, m, n) to (β, k, l).

For example, consider a step of the computation where we use the instruction I of the form 'in state q_γ, add 1 to s_1, and go to state q_δ.' The resulting configuration will be (δ, s_1+1, s_2). Using substitution, from the formula AxI corresponding to I, that is

$$H \wedge \Diamond S(\gamma, Q_1, R_1) \to H \wedge \Diamond S(\delta, Q_2, R_1),$$

we obtain a formula that is equivalent to

$$H \wedge \Diamond S(\gamma, A^1_{s_1}, A^2_{s_2}) \to H \wedge \Diamond S(\delta, A^1_{s_1+1}, A^2_{s_2}).$$

Therefore,

$$\mathbf{K} \oplus Ax\mathcal{P} \vdash H \wedge \Diamond S(\gamma, A^1_{s_1}, A^2_{s_2}) \to H \wedge \Diamond S(\delta, A^1_{s_1+1}, A^2_{s_2}).$$

Other instructions are considered in the same way. ∎

LEMMA 7. *For any* $0 < n < \omega$, $\mathfrak{F}_n \models Ax\mathcal{P}$.

Proof. A straightforward proof using Lemmas 3 and 5 is left to the reader. ∎

Define the following formula

$$\begin{aligned} B(\mathcal{P}, (\alpha, m, n), (\beta, k, l)) \; = \; & Ax\mathcal{P} \wedge \\ & ((H \wedge \Diamond S(\alpha, A^1_m, A^2_n) \to H \wedge \Diamond S(\beta, A^1_k, A^2_l)) \to \neg H) \wedge \\ & (\neg H \vee (\Box(\Box q \to q) \to \Box q)). \end{aligned}$$

The definition of $B(\mathcal{P}, (\alpha, m, n), (\beta, k, l))$ is clearly constructive, which gives property 1 on page 123. Let us show that property 2 holds as well.

LEMMA 8. *If* $\mathcal{P} : (\alpha, m, n) \to (\beta, k, l)$ *then* $B(\mathcal{P}, (\alpha, m, n), (\beta, k, l))$ *is first-order definable.*

Proof. If $\mathcal{P} : (\alpha, m, n) \to (\beta, k, l)$ then, by Lemma 6, we have

$$\mathbf{K} \oplus Ax\mathcal{P} \vdash H \wedge \Diamond S(\alpha, A^1_m, A^2_n) \to H \wedge \Diamond S(\beta, A^1_k, A^2_l),$$

and, by modus ponens,

$$\mathbf{K} \oplus B(\mathcal{P}, (\alpha, m, n), (\beta, k, l)) \vdash \neg H.$$

Besides, we have

$$\mathbf{K} \oplus \neg H \vdash B(\mathcal{P}, (\alpha, m, n), (\beta, k, l)).$$

This means that $B(\mathcal{P}, (\alpha, m, n), (\beta, k, l))$ is deductively equivalent to $\neg H$ and, therefore, is first-order definable by Lemma 2. ∎

LEMMA 9. *If* $\mathcal{P} : (\alpha, m, n) \not\to (\beta, k, l)$ *then* $B(\mathcal{P}, (\alpha, m, n), (\beta, k, l))$ *is not first-order definable.*

Proof. Let $\mathcal{P} : (\alpha, m, n) \not\to (\beta, k, l)$. Then $B(\mathcal{P}, (\alpha, m, n), (\beta, k, l))$ is valid in the frame $\mathfrak{F}_n$, for any $0 < n < \omega$. Indeed,

$\mathfrak{F}_n \models Ax\mathcal{P}$ by Lemma 7;

$\mathfrak{F}_n \models (H \wedge \Diamond S(\alpha, A^1_m, A^2_n) \to H \wedge \Diamond S(\beta, A^1_k, A^2_l)) \to \neg H$ by Lemma 3;

$\mathfrak{F}_n \models \neg H \vee (\Box(\Box q \to q) \to \Box q)$ because there is no infinite ascending chain in this frame.

Suppose that $B(\mathcal{P}, (\alpha, m, n), (\beta, k, l))$ is equivalent to some first-order formula φ. Then we have $\mathfrak{F}_n \models \varphi$, for any $0 < n < \omega$.

Consider the formula

$$G(x, y) = xRy \wedge \neg\exists z\, yR^2 z \wedge \exists u \exists v (yRu \wedge yRv \wedge u \neq v).$$

In the frame $\mathfrak{F}_n$, this formula can select the points h and g; more precisely, we have

$$\mathfrak{F}_n \models G(x, y) \iff x = h \text{ and } y = g.$$

Using this fact, we obtain that the following formulas are true in the $\mathfrak{F}_n$, for any n:

$\forall x\ \neg xRx,$ (1)
$\forall x\, \forall y\, \forall z\, (xRy \wedge yRz \to xRz),$ (2)
$\exists x\, \exists y\, G(x, y),$ (3)
$\forall x\, \forall y\, (G(x, y) \to \exists! z\, G(x, z)),$ (4)
$\forall x\, (\exists y\, G(x, y) \to \forall z\, (x = z \vee xRz)).$ (5)

Consider the formulas

$$\begin{aligned} S(k) \ = \ & \forall x\, \forall y\, \forall u\, \forall v\, (xRy \wedge \neg\exists z yR^2 z \wedge yRu \wedge yRv \wedge u \neq v \to \\ & \exists w_1 \ldots \exists w_k\, (xRw_1 \ldots w_{k-1}Rw_k \wedge \neg(w_kRu \vee w_kRv \vee w_k = u \vee w_k = v) \wedge \\ & \forall z\, (\neg(zRu \vee zRv \vee z = u \vee z = v) \to z = w_1 \vee \cdots \vee z = w_k \vee w_kRz))). \end{aligned}$$

We have $\mathfrak{F}_n \models S(k)$, for any $n \geq k$.

Now we form the ultraproduct of all $\mathfrak{F}_n$ with respect to some nonprincipal ultrafilter over $\omega - \{0\}$ and denote it by $\mathfrak{F}$. By the Łoś theorem[4], for any $k \in \omega - \{0\}$,

$$\mathfrak{F} \models \varphi \wedge (1) \wedge (2) \wedge (3) \wedge (4) \wedge (5) \wedge S(k).$$

By (3), we have in $\mathfrak{F}$ four points h, g, a, b such that hRg, gRa, gRb (there is no z such that gR^2z), and h, g are unique with these properties by (4) and (5).

By $S(k)$, we obtain a sequence $c_1, c_2, \ldots, c_k, \ldots$ such that $\neg c_kRa$, $\neg c_kRb$, $c_k \neq a$, $c_k \neq b$, and

$$c_1Rc_2Rc_3 \ldots c_{k-1}Rc_kRc_{k+1} \ldots .$$

[4]For notions and results of classical model theory one can consult [11].

Define V on $\mathfrak{F}$ as follows: $V(p) = \{a\}$, $V(q) = W - \{c_1, c_2, \dots\}$. Then we have $a \models \Box\bot \wedge p$, $b \models \Box\bot \wedge \neg p$, $g \models \Box^2\bot \wedge \Diamond p \wedge \Diamond\neg p$, $h \models H$, so

$$h \not\models \neg H.$$

Besides,

$$h \not\models \Box(\Box q \to q) \to \Box q.$$

Therefore,

$$\mathfrak{F} \not\models B(\mathcal{P}, (\alpha, m, n), (\beta, k, l)),$$

contrary to $\mathfrak{F} \models \varphi$. ■

By Lemmas 8 and 9, we obtain that property 2 on page 123 is satisfied. This completes the proof of Theorem 1. ■

3 Undecidability of modal definability

In this section we give a solution to the problem that is in some sense inverse to the problem considered in Section 2: is it possible to check algorithmically whether a given first-order formula is modally definable? In other words, is modal definability of first-order formulas decidable? However, this case is considerably simpler, since the language of first-order logic is more natural and, therefore, easier to use than the language of propositional modal logic.

THEOREM 10. *Modal definability of first-order formulas in the signature of one binary predicate and equality is algorithmically undecidable.*

Proof. We will use the transitive irreflexive frame $\mathfrak{F}$ (which is also a model for first-order predicate logic) that is obtained from $\mathfrak{F}_n$ in Fig. 1 by replacing the finite chain $c_1 R c_2 R \dots c_{n-1} R c_n$ with the infinite chain $c_1 R c_2 R \dots c_{n-1} R c_n \dots$; see Fig. 2.

The first-order formulas associated with this frame can be found in Section 5.2. Here we only give their English translations. The first-order formula AxI imitating an instruction I of a Minsky machine means 'given a configuration, apply I and move to the next configuration.' It can be constructed analogously to its modal counterpart AxI from Section 2. The formula $Ax\mathcal{P}$ contains some additional conjuncts expressing some properties of $\mathfrak{F}$:

$$Ax\mathcal{P} = \bigwedge_{I \in \mathcal{P}} AxI \wedge IRR \wedge TRANS \wedge LIN_1 \wedge LIN_2,$$

where

- IRR says that the accessibility relation is irreflexive,
- $TRANS$ says that the accessibility relation is transitive,
- LIN_1 says that the set $\{a_i^1 \mid i \geq 0\}$ is linearly ordered by the accessibility relation,

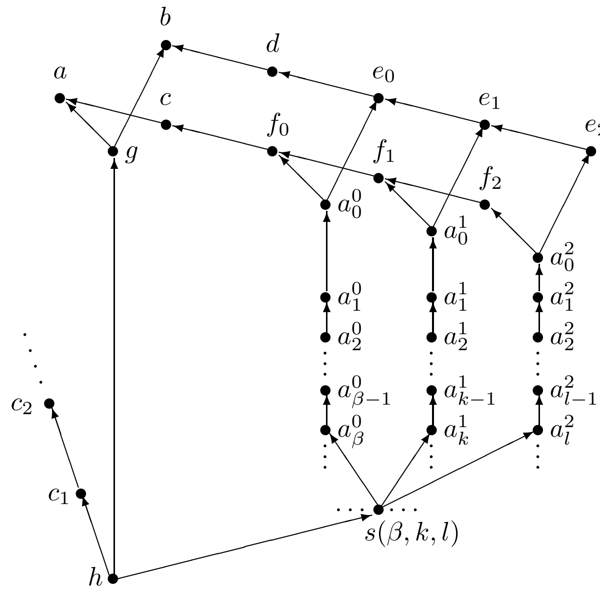

Figure 2. Frame $\mathfrak{F}$.

- LIN_2 says that the set $\{a_i^2 \mid i \geq 0\}$ is linearly ordered by the accessibility relation.

In Lemma 11 below we only need the first conjunct $\bigwedge_{I \in \mathcal{P}} AxI$ of $Ax\mathcal{P}$ and a first-order formula $A(\langle s_1, i_1, j_1 \rangle, \langle s_n, i_n, j_n \rangle)$ which says that if there is a point of the form $s(s_1, i_1, j_1)$ in $\mathfrak{F}$, then there is a point of the form $s(s_n, i_n, j_n)$ as well.

LEMMA 11. *If*

$$\mathcal{P} : \langle s_1, i_1, j_1 \rangle \to \langle s_n, i_n, j_n \rangle ,$$

then

$$Ax\mathcal{P} \models A(\langle s_1, i_1, j_1 \rangle, \langle s_n, i_n, j_n \rangle).$$

Proof. The straightforward proof by induction on n (the key case is $n = 2$) is left to the reader. ■

We are now in a position to prove Theorem 10, where the following formula is required.

Consider a machine $\mathcal{P}$ with configurations $\langle s_1, i_1, j_1 \rangle$ and $\langle s_n, i_n, j_n \rangle$, and let

$$B(\mathcal{P}, \langle s_1, i_1, j_1 \rangle, \langle s_n, i_n, j_n \rangle) = Ax\mathcal{P} \wedge \neg A(\langle s_1, i_1, j_1 \rangle, \langle s_n, i_n, j_n \rangle).$$

Since this formula is constructed effectively, our theorem follows immediately from the undecidability of the configuration problem and the following lemma:

LEMMA 12. *For every Minsky machine $\mathcal{P}$ and all configurations $\langle s_1, i_1, j_1\rangle$, $\langle s_n, i_n, j_n\rangle$ the following equivalence holds:*

$$B(\mathcal{P}, \langle s_1, i_1, j_1\rangle, \langle s_n, i_n, j_n\rangle) \text{ is modally definable}$$
$$\Updownarrow$$
$$\mathcal{P} : \langle s_1, i_1, j_1\rangle \to \langle s_n, i_n, j_n\rangle .$$

Proof. ($\Uparrow$) follows from Lemma 11. In the case $\mathcal{P} : \langle s_1, i_1, j_1\rangle \to \langle s_n, i_n, j_n\rangle$ the formula $B(\mathcal{P}, \langle s_1, i_1, j_1\rangle, \langle s_n, i_n, j_n\rangle)$ is trivially modally definable by $\bot$.

To show ($\Downarrow$), suppose the contrary: we assume that φ is a modal equivalent of $B(\mathcal{P}, \langle s_1, i_1, j_1\rangle, \langle s_n, i_n, j_n\rangle)$, yet $\mathcal{P} : \langle s_1, i_1, j_1\rangle \not\to \langle s_n, i_n, j_n\rangle$.

Consider the frame $\mathfrak{F}$ in Fig. 2. In this frame, all points $s(\beta, k, l)$ are such that $\mathcal{P} : \langle s_1, i_1, j_1\rangle \to \langle \beta, k, l\rangle$. One can easily check that

$$\mathfrak{F} \models B(\mathcal{P}, \langle s_1, i_1, j_1\rangle, \langle s_n, i_n, j_n\rangle),$$

and so

$$\mathfrak{F} \models \varphi.$$

Now take the frame $\mathfrak{F}'$ that is reconstructed from $\mathfrak{F}$ by replacing the infinite chain $c_1 R c_2 R \dots$ with a single reflexive point c. Clearly, $\mathfrak{F}'$ is a p-morphic image of $\mathfrak{F}$, from which

$$\mathfrak{F}' \models \varphi.$$

On the other hand, by the conjunct IRR, we have

$$\mathfrak{F}' \not\models B(\mathcal{P}, \langle s_1, i_1, j_1\rangle, \langle s_n, i_n, j_n\rangle),$$

which is a contradiction. ■

This completes the proof of Theorem 10. ■

4 Discussion

Since the main problem solved in Section 2 was concerned with the model-theoretic (equivalence) relation between modal formulas and first-order formulas, it would be useful to recall that, for first-order sentences φ and ψ, the following statements are equivalent (by the well-known Gödel completeness theorem and Löwenheim-Skolem-Tarski theorem):

- φ and ψ are true in the same models;
- φ and ψ are true in the same countable models;
- φ and ψ are deductively equivalent;
- the formula $\varphi \leftrightarrow \psi$ is derivable in the predicate calculus.

It is now natural to ask whether the same holds for modal formulas. The answer has been known for some time now: the behaviour of modal formulas is quite different. Here we mean various incompleteness effects (for the exact definitions and references see [10]): Kripke incompleteness of some modal calculi and the fact that countable frames are not sufficient for some Kripke complete modal logics. It would be fair to mention that the last statement is still proved only for logics: there are a normal modal logic L and a formula φ such that L is Kripke complete, $L \not\vdash \varphi$, and φ is true in all countable frames of L. It is not known whether a *logic* in this statement can be replaced with a *calculus* (a finitely axiomatisable logic).

In the context of this paper it would be relevant to investigate first-order definability of modal formulas on frames of bounded cardinality. In particular, what can be said about (the algorithmic problem of) first-order definability of modal formulas on the class of countable frames?

The answer to this question is almost obvious; the situation with such definability is the same as with first-order definability in the general case.

THEOREM 13. *The problem of first-order definability of modal formulas on countable frames is undecidable.*

Proof. First, we do not have to change anything in the proof of Theorem 1 in the case when $\mathfrak{F}$ is not just an ultraproduct of the frames $\mathfrak{F}_n$ with respect to some nonprincipal ultrafilter over $\omega - \{0\}$, but some countable elementary submodel of it. Such a submodel exists by the Löwenheim-Skolem-Tarski theorem, since the formula $\varphi \wedge (1) \wedge (2) \wedge (3) \wedge (4) \wedge (5) \wedge S(k)$ is still true.

Now, Theorem 13 can be proved simultaneously with Theorem 1. Indeed, in that proof, uncountable frames can be used (and are used) only in one place, where the ultraproduct is considered in Lemma 9. However, instead of the ultraproduct we can use its countable elementary submodel. Therefore, Lemma 9 can be reformulated in a stronger form: *if* $\mathcal{P} : (\alpha, m, n) \not\rightarrow (\beta, k, l)$ *then* $B(\mathcal{P}, (\alpha, m, n), (\beta, k, l))$ *is not first-order definable on the class of all countable frames.*

This change is enough to prove Theorem 13. ■

The reader may wonder whether Theorem 1 and Theorem 13 are equivalent. Can we obtain either theorem as a simple corollary of the other one (or is at least one of the theorems a corollary of the other one)? The answer to this question is negative, and the following example demonstrates why.

EXAMPLE 14. It is shown in K. Doets' PhD thesis [12] that K. Fine's formula

$$\Diamond\Box(p \vee q) \rightarrow \Diamond(\Box p \vee \Box q)$$

is not first-order definable, although it is first-order definable on countable frames. An analogous intuitionistic formula was constructed in the PhD thesis [4]. It was shown that the set of intuitionistic formulas that have this property is algorithmically undecidable.

Let us now analyse the proof of Theorem 10. The fact that we were dealing with *modal* formulas was actually used only once in this proof, namely, at the very end we used persistence of modal formulas under p-morphisms. Analogous effects can be obtained by using other model-theoretic constructions. For example, we can use persistence with respect to generated submodels: just replace the increasing chain $hRc_1Rc_2R\dots$ with the decreasing chain $\dots Rc_2Rc_1Rh$ and the conjunct IRR of $Ax\mathcal{P}$ with the formula $\forall x\,\exists y\, yRx$. Besides, we can replace the condition of irreflexivity with the condition of reflexivity. These modifications give numerous analogs of Theorem 10: for definability of first-order formulas by (poly)modal formulas on the classes of **Grz**-frames, **GL**-frames, **PDL**-frames, etc., for definability of first-order formulas by intuitionistic formulas, and many others.

Note also that in the proof of Theorem 10 we used only one modal formula: the formula $\bot$. Therefore, we can replace 'modal definability' with 'definability by variable-free modal formulas' or even 'definability by modal formula $\bot$' everywhere in the proof. This immediately gives us the following two results.

THEOREM 15. *Definability of first-order formulas in the signature of one binary predicate and equality by variable-free modal formulas is undecidable.*

THEOREM 16. *Definability of first-order formulas in the signature of one binary predicate and equality by the modal formula $\bot$ is undecidable.*

Furthermore, in these theorems we can replace 'decidability' with 'recursive separability.' Thus, the following variant of Theorem 16 holds:

THEOREM 17. *Let A be the set of first-order formulas in the signature of one binary predicate and equality which are definable by modal formula $\bot$, and let B be the set of first-order formulas in the signature of one binary predicate and equality that are not modally definable. Then the sets A and B are not recursively separable.*

Other theorems above can also be reformulated in the same way. (We only did it for Theorem 16 because it is an immediate consequence of Church's theorem on the undecidability of the first-order predicate logic in the given signature.)

Using a slight modification of the frame $\mathfrak{F}$ in Fig. 1 and first-order equivalents of modal formulas of the form (see Lemma 2 for $n = 1$)

$$\Box(\Box^2\bot \to \bigvee_{\langle i_1,\dots,i_n\rangle\in\{0,1\}^n} \Box(\neg^{i_1}p_1 \wedge \cdots \wedge \neg^{i_n}p_n))$$

where $\neg^1 = \neg$ and $\neg^0$ is blank, we can prove some extension of Theorem 15.

THEOREM 18. *For any $0 < n < \omega$, the set of first-order formulas in the signature of one binary predicate and equality which are definable by modal formulas in n variables, but are not definable by modal formulas in $n-1$ variables is undecidable.*

Finally, we observe that the proofs of Theorems 1 and 10 give a negative solution to the correspondence problem, that is the question whether a given modal formula and a given first-order formula are equivalent on Kripke frames:

THEOREM 19. *The correspondence problem of modal formulas and first-order formulas is undecidable.*

5 Appendix

5.1 Minsky machines

A two-tape *Minsky machine* (today a Minsky machine would be described as a register machine with two registers) is a finite set of instructions (program $\mathcal{P}$) for transforming triples $\langle s, m, n\rangle$ of natural numbers, called *configurations*. The intended meaning of the components in the current configuration $\langle s, m, n\rangle$ is as follows: s is the number (label) of the current machine state, i.e. the number of the instruction to be executed at the next step, and m, n represent the current state of information (i.e. records on the first and the second tapes). Each instruction has one of the four possible forms:

$$s \to \langle t, 1, 0\rangle\,,\ s \to \langle t, 0, 1\rangle\,,\ s \to \langle t, -1, 0\rangle\,(\langle t', 0, 0\rangle),\ s \to \langle t, 0, -1\rangle\,(\langle t', 0, 0\rangle).$$

The last of them, for instance, means: transform $\langle s, m, n\rangle$ into $\langle t, m, n-1\rangle$ if $n > 0$ and into $\langle t', m, n\rangle$ if $n = 0$. The meaning of the others is defined analogously. So, the instruction $s \to \langle t, 1, 0\rangle$ means: transform $\langle s, m, n\rangle$ into $\langle t, m+1, n\rangle$.

If $\mathcal{P}$ is a Minsky machine then the notation $\mathcal{P} : \langle s, m, n\rangle \to \langle t, k, l\rangle$ means that starting with $\langle s, m, n\rangle$ and applying the instructions in $\mathcal{P}$, in finitely many steps we can reach the configuration $\langle t, k, l\rangle$.

As is well-known, the configuration problem for Minsky machines is undecidable: *there is no algorithm which, given a program $\mathcal{P}$ and configurations $\langle s, m, n\rangle$ and $\langle t, k, l\rangle$, can decide whether $\mathcal{P} : \langle s, m, n\rangle \to \langle t, k, l\rangle$ holds.*

This problem may be used for establishing a lot of undecidability results, but not all of them. Sometimes it is much more convenient to use a variant of the configuration problem with fixed suitable $\mathcal{P}$ and $\langle s, m, n\rangle$. It is called the *second configuration problem*: *there exist a program $\mathcal{P}$ and a configuration $\langle s, m, n\rangle$ such that there is no algorithm which is capable of deciding, given a configuration $\langle t, k, l\rangle$, whether $\mathcal{P} : \langle s, m, n\rangle \to \langle t, k, l\rangle$.*

For more details consult [10] and/or [14].

5.2 First-order formulas used in the proof of theorem 10

The following formulas AxI simulate the instructions I of a Minsky machine in the frame $\mathfrak{F}$ in Fig. 2:

- if $I = \gamma \to \langle \delta, 1, 0\rangle$, then

$$\begin{aligned} AxI \ = \ & \forall h, g, a, b, y_1, y_2 (hR_1 g \wedge \neg\exists z\, gR_2 z \wedge gR_1 a \wedge gR_1 b \wedge a \neq b \wedge \\ & P(y_1) \wedge Q(y_2) \wedge \exists x \exists y_0 (xR_1 y_0 \wedge xR_1 y_1 \wedge xR_1 y_2 \wedge A^0_\gamma(y_0)) \to \\ & \exists x, y_0, z (xR_1 y_0 \wedge xR_1 z R_1 y_1 \wedge xR_1 y_2 \wedge A^0_\delta(y_0) \wedge P(z))); \end{aligned}$$

- if $I = \gamma \to \langle \delta, 0, 1\rangle$, then

$$\begin{aligned}AxI \;=\; &\forall h, g, a, b, y_1, y_2(hR_1g \wedge \neg\exists z\, gR_2z \wedge gR_1a \wedge gR_1b \wedge a \neq b \wedge \\ &P(y_1) \wedge Q(y_2) \wedge \exists x \exists y_0(xR_1y_0 \wedge xR_1y_1 \wedge xR_1y_2 \wedge A^0_\gamma(y_0)) \to \\ &\exists x, y_0, z(xR_1y_0 \wedge xR_1y_1 \wedge xR_1zR_1y_2 \wedge A^0_\delta(y_0) \wedge Q(z)));\end{aligned}$$

- if $I = \gamma \to \langle\delta_1, -1, 0\rangle\,(\langle\delta_2, 0, 0\rangle)$, then

$$\begin{aligned}AxI = &\forall h, g, a, b, y_1, y_2(hR_1g \wedge \neg\exists z\, gR_2z \wedge gR_1a \wedge gR_1b \wedge a \neq b \wedge P(y_1) \wedge \\ &Q(y_2) \wedge \exists x, y_0, z(xR_1y_0 \wedge xR_1zR_1y_1 \wedge xR_1y_2 \wedge A^0_\gamma(y_0) \wedge P(z)) \to \\ &\exists x, y_0(xR_1y_0 \wedge xR_1y_1 \wedge xR_1y_2 \wedge A^0_{\delta_1}(y_0))) \wedge \\ &\forall h, g, a, b, y_1, y_2(hR_1g \wedge \neg\exists z\, gR_2z \wedge gR_1a \wedge gR_1b \wedge a \neq b \wedge A^1_0(y_1) \wedge \\ &Q(y_2) \wedge \exists x, y_0(xR_1y_0 \wedge xR_1y_1 \wedge xR_1y_2 \wedge A^0_\gamma(y_0)) \to \\ &\exists x, y_0(xR_1y_0 \wedge xR_1y_1 \wedge xR_1y_2 \wedge A^0_{\delta_2}(y_0)));\end{aligned}$$

- if $I = \gamma \to \langle\delta_1, 0, -1\rangle\,(\langle\delta_2, 0, 0\rangle)$, then

$$\begin{aligned}AxI = &\forall h, g, a, b, y_1, y_2(hR_1g \wedge \neg\exists z\, gR_2z \wedge gR_1a \wedge gR_1b \wedge a \neq b \wedge P(y_1) \wedge \\ &Q(y_2) \wedge \exists x, y_0\exists z(xR_1y_0 \wedge xR_1y_1 \wedge xR_1zR_1y_2 \wedge A^0_\gamma(y_0) \wedge Q(z)) \to \\ &\exists x\exists y_0(xR_1y_0 \wedge xR_1y_1 \wedge xR_1y_2 \wedge A^0_{\delta_1}(y_0))) \wedge \\ &\forall h, g, a, b, y_1, y_2(hR_1g \wedge \neg\exists z\, gR_2z \wedge gR_1a \wedge gR_1b \wedge a \neq b \wedge P(y_1) \wedge \\ &A^2_0(y_2) \wedge \exists x, y_0(xR_1y_0 \wedge xR_1y_1 \wedge xR_1y_2 \wedge A^0_\gamma(y_0)) \to \\ &\exists x, y_0(xR_1y_0 \wedge xR_1y_1 \wedge xR_1y_2 \wedge A^0_{\delta_2}(y_0))).\end{aligned}$$

Here we use the following abbreviations:

- 'accessible in exactly k steps:'

$$xR_ky = xR^ky \wedge \neg xR^{k+1}y, \quad (k \geq 1),$$

  where, as usual,

$$xR^0y = (x = y), \quad xR^{k+1}y = \exists z\,(xR^kz \wedge zRy);$$

- 'the point is of the type a^0_0:'

$$A^0_0(x) = \exists y(xR_1y \wedge yR_2a \wedge \neg yRb) \wedge \exists y(xR_1y \wedge yR_2b \wedge \neg yRa);$$

- 'the point is of the type a^1_0:'

$$A^1_0(x) = \exists y(xR_1y \wedge yR_3a \wedge \neg yRb) \wedge \exists y(xR_1y \wedge yR_3b \wedge \neg yRa);$$

- 'the point is of the type a^2_0:'

$$A^2_0(x) = \exists y(xR_1y \wedge yR_4a \wedge \neg yRb) \wedge \exists y(xR_1y \wedge yR_4b \wedge \neg yRa);$$

- 'the point is of the type a^0_s:'

$$A^0_s(x) = \exists y(xR_sy \wedge A^0_0(y)) \wedge \neg\exists y(xRy \wedge A^1_0(y)) \wedge \neg\exists y(xRy \wedge A^2_0(y));$$

- 'the point is of the type a^1_t:'

  $A^1_t(x) = \exists y(xR_t y \wedge A^1_0(y)) \wedge \neg\exists y(xRy \wedge A^0_0(y)) \wedge \neg\exists y(xRy \wedge A^2_0(y))$;

- 'the point is of the type a^2_u:'

  $A^2_u(x) = \exists y(xR_u y \wedge A^2_0(y)) \wedge \neg\exists y(xRy \wedge A^0_0(y)) \wedge \neg\exists y(xRy \wedge A^1_0(y))$;

- 'the point is of the type a^1_t with some arbitrary t:'

  $P(x) = (A^1_0(x) \vee \exists y(xRy \wedge A^1_0(y))) \wedge \neg\exists y(xRy \wedge A^0_0(y)) \wedge \neg\exists y(xRy \wedge A^2_0(y))$;

- 'the point is of the type a^2_u with some arbitrary u:'

  $Q(x) = (A^2_0(x) \vee \exists y(xRy \wedge A^2_0(y))) \wedge \neg\exists y(xRy \wedge A^0_0(y)) \wedge \neg\exists y(xRy \wedge A^1_0(y))$.

$Ax\mathcal{P}$ is defined as the conjunction of the following formulas:

- $IRR = \forall x\, \neg xRx$;
- $TRANS = \forall x, y, z\, (xRy \wedge yRz \to xRz)$;
- $LIN_1 = \forall h, g, a, b, y_1, y_2(hR_1 g \wedge \neg\exists z\, gR_2 z \wedge gR_1 a \wedge gR_1 b \wedge a \neq b \to$
  $\forall x \forall y(P_1(x) \wedge P_1(y) \to x = y \vee xRy \vee yRx))$;
- $LIN_2 = \forall h, g, a, b, y_1, y_2(hR_1 g \wedge \neg\exists z\, gR_2 z \wedge gR_1 a \wedge gR_1 b \wedge a \neq b \to$
  $\forall x, y(Q_1(x) \wedge Q_1(y) \to x = y \vee xRy \vee yRx))$;
- AxI, for every instruction I of a machine $\mathcal{P}$.

Finally, the formula saying that starting with a configuration we can eventually reach $\langle s_1, i_1, j_1\rangle$ is defined as follows:

$$\begin{aligned}
&A(\langle s_1, i_1, j_1\rangle, \langle s_n, i_n, j_n\rangle) = \\
&\quad \forall h, g, a, b(hR_1 g \wedge \neg\exists z\, gR_2 z \wedge gR_1 a \wedge gR_1 b \wedge a \neq b \wedge \\
&\quad\quad \exists x, y_0, y_1, y_2(xR_1 y_0 \wedge xR_1 y_1 \wedge xR_1 y_2 \wedge A^0_{s_1}(y_0) \wedge A^1_{i_1}(y_1) \wedge A^2_{j_1}(y_2)) \to \\
&\quad\quad\quad \exists x, y_0, y_1, y_2(xR_1 y_0 \wedge xR_1 y_1 \wedge xR_1 y_2 \wedge A^0_{s_n}(y_0) \wedge A^1_{i_n}(y_1) \wedge A^2_{j_n}(y_2))).
\end{aligned}$$

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Alexander Chagrov, Lilia Chagrova
Department of Mathematics
Tver State University
Zhelyabova Street, 33
Tver
170000
Russia
chagrov@mail.ru