Connexive modal logic

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ABSTRACT. Connexive logic is a neglected direction in non-classical logic. In the present paper, first an axiomatic system of connexive propositional logic is presented. This logic, C, is shown to be sound and complete with respect to a class of relational models. It seems that this semantics is, in fact, the first known *intuitively plausible* interpretation of a system of connexive logic. The presentation of C suggests that connexive logic is *constructive*. It is a variant of David Nelson's constructive logics with strong negation. In Nelson's logics the verification conditions of implications are dynamic, whereas all falsification conditions are static conditions of falsification on the spot. In C, both the verification and the falsification conditions of implications are dynamic. This is enough to ensure that C is connexive and can be given a comprehensible and clear interpretation in terms of information states.

In a second step, the language of the system C is extended by the modal operators $\Box$ and $\Diamond$ to obtain a connexive analogue of the smallest normal modal propositional logic K. Aiming at a connexive analogue of K that can be faithfully embedded by a modal translation into QC, quantified C, we arrive at a system that will be called CK, connexive K. The system CK is a connexive version of the constructive modal logic FSK^d characterized in [12]. CK is shown to be sound and complete with respect to relational models and to be decidable.

We shall also critically discuss the evaluation clauses for the modal operators that are induced by the *standard* translation from modal propositional logic into first-order logic. In the context of the connexive base logic, the falsification clauses of formulas $\Box A$ induced by the standard translation appear to be intuitively implausible. In any case, both syntactic duality axioms $\sim \Box A \leftrightarrow \Diamond \sim A$ and $\sim \Diamond \leftrightarrow \Box \sim A$ fail to hold.

It seems that CK is the first system of connexive modal logic considered in the modal logic literature. This paper may therefore be seen as a contribution to establishing connexive modal logic as a respectable branch of modal logic.

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1 Aristotle's Theses and Boethius' Theses

The following principle is well-known as "Aristotle's Thesis":

$$AT \quad \sim (\sim A \rightarrow A),$$

see, for example, [11]. AT is of interest, because although AT is not a theorem of classical logic, nevertheless in the history of logic, AT has sometimes been found plausible, if it is viewed as a formalization of "it is not the case that not-A implies A". In classical logic, AT is not logically equivalent with the non-theorem:

$$AT' \quad \sim (A \rightarrow \sim A).$$[1]

But also against the background of logics in which AT and AT' fail to be logically equivalent, they would seem to constitute a pair of equally intuitive (or unintuitive) theses. There exists, thus, a tension between the supraclassicality (the fact that AT and AT' are not classically valid) and some intuitive plausibility of AT and AT'.

One may observe that, using intuitionistically acceptable means only, the pair of theses AT and AT' is equivalent in deductive power with another pair of schemata, which in accordance with previously introduced terminology are here called (Strong) "Boethius' Theses" BT and BT'. Boethius' theses are:

$$BT \quad (A \rightarrow B) \rightarrow \sim (A \rightarrow \sim B); \qquad BT' \quad (A \rightarrow \sim B) \rightarrow \sim (A \rightarrow B).$$

More precisely, BT and BT' each allow one to intuitionistically derive both AT and AT', and AT' allows one to intuitionistically derive both BT and BT', whereas AT neither classically implies BT nor classically implies BT'. Moreover, the theses BT and BT' can easily be shown to be intuitionistically equivalent. Whereas the converses

$$BT_c \quad \sim (A \rightarrow \sim B) \rightarrow (A \rightarrow B), \qquad BT'_c \quad \sim (A \rightarrow B) \rightarrow (A \rightarrow \sim B)$$

of BT and BT' are classically valid, only BT'_c is also intuitionistically valid.

2 Connexive logic

Let $\mathcal{L}$ be a language containing a unary connective $\sim$ (negation) and a binary connective $\rightarrow$ (implication). A logical system in the language $\mathcal{L}$ is called a *connexive logic*, if AT, AT', BT, and BT' are valid schemata and,

[1] Since classical propositional logic is Post complete, neither AT nor AT' can consistently be added to it.

moreover, $(A \to B) \to (B \to A)$ fails to be a valid schema.[2] The connective $\to$ in a system of connexive logic is said to be a *connexive implication*.

In [11] McCall presents an axiomatization of a system of propositional connexive logic introduced by Angell [1] in terms of certain four-valued matrices. The language of McCall's logic $CC1$ contains as primitive (notation adjusted) a unary connective $\sim$ (negation) and the binary connectives $\wedge$ (conjunction) and $\to$ (implication). Disjunction $\vee$ and equivalence $\leftrightarrow$ are defined in the usual way. The schematic axioms and the rules of $CC1$ are as follows:

$A1$ $(A \to B) \to ((B \to C) \to (A \to C))$
$A2$ $((A \to A) \to B) \to B$
$A3$ $(A \to B) \to ((A \wedge C) \to (B \wedge C))$
$A4$ $(A \wedge A) \to (B \to B)$
$A5$ $(A \wedge (B \wedge C)) \to (B \wedge (A \wedge C))$
$A6$ $(A \wedge A) \to ((A \to A) \to (A \wedge A))$
$A7$ $A \to (A \wedge (A \wedge A))$
$A8$ $((A \to \sim B) \wedge B) \to \sim A$
$A9$ $(A \wedge \sim (A \wedge \sim B)) \to B$
$A10$ $\sim (A \wedge \sim (A \wedge A))$
$A11$ $(\sim A \vee ((A \to A) \to A)) \vee (((A \to A) \vee (A \to A)) \to A)$
$A12$ $(A \to A) \to \sim (A \to \sim A)$
$R1$ modus ponens
$R2$ adjunction

Note that among these axiom schemata, only A12 is supraclassical. The system $CC1$ is characterized by the following four-valued truth tables with designated values 1 and 2:

$\sim$	
1	4
2	3
3	2
4	1

$\wedge$	1	2	3	4
1	1	2	3	4
2	2	1	4	3
3	3	4	3	4
4	4	3	4	3

$\to$	1	2	3	4
1	1	4	3	4
2	4	1	4	3
3	1	4	1	4
4	4	1	4	1

McCall emphasizes that the logic $CC1$ is only one among many possible systems satisfying the theses of Aristotle and Boethius and that he "does

[2] According to McCall [11, p. 416], a "system of connexive logic may range from one in which no proposition implies or is implied by its own negation to one in which Boethius' thesis [i.e., BT] is asserted." But, if "it is to be of any interest, it must exclude the characteristic thesis $(p \to q) \to (q \to p)$ of equivalence" [11, p. 417, notation adjusted]

not wish it to be regarded as *the* system of connexive logic" [11, p. 418]. Indeed, McCall does not suggest any intuitive interpretation of the four truth values employed in the above matrices and the fact hat the values 1 and 2 are designated, and hence the semantics appears to be a purely formal method with little explanatory power.[3] In contrast to this, the system C of the present paper is meant to be a suitable candidate for the title 'the basic system of connexive logic'.

Moreover, McCall points out that $CC1$ has some properties that are difficult to justify, if the name connexive logic is meant to reflect the fact that in a valid implication $A \to B$ there exists a connection between the antecedent A and the succedent B. Axiom A4, for example, is bad in this respect. On the other hand, $CC1$ might be said to undergenerate, since $(A \wedge A) \to A$ and $A \to (A \wedge A)$ fail to be theorems of $CC1$. If the validity of Aristotle's and Boethius' theses is distinctive of connexive logics, it is not quite clear, however, how damaging the above criticism is.

More serious doubts have been cast on $CC1$ by Routley and Montgomery [22] and, as these authors have claimed, on connexive logics in general. The starting point of their criticism, however, is a certain subsystem $Z1$ of $CC1$ containing the contraposition theorem $(\sim A \to B) \to (\sim B \to A)$. This troublesome schema fails to be a theorem of the connexive logic C, defined in Section 3.

3 A basic system of connexive logic

The key observation for obtaining a connexive logic admitting a transparent and illuminating semantical characterization is simple: in the presence of the double negation introduction law, it suffices to validate both BT' and BT'_c. In other words, an interpretation of the falsification conditions of implications is called for, which deviates from the standard conditions. In Nelson's systems of constructive logic, the double negation laws hold, and the relational semantics for these logics is such that falsification and verification of formulas are dealt with separately. However, the falsification conditions of implications are the classical ones expressed by the schema $\sim (A \to B) \leftrightarrow (A \wedge \sim B)$.[4] To obtain a connexive implication, it is there-

[3]Note that the constant truth functions **1**, **2**, **3**, and **4** can be defined as follows [11, p. 421]: $\mathbf{1} := (p \to p)$, $\mathbf{2} := \sim (p \leftrightarrow \sim p)$, $\mathbf{3} := (p \leftrightarrow \sim p)$, $\mathbf{4} := \sim (p \to p)$, for some sentence letter p. Routley and Montgomery [22, p. 95] point out that $CC1$ "can be given a semantics by associating the matrix value 1 with logical necessity, value 4 with logical impossibility, value 2 with contingent truth, and value 3 with contingent falsehood. However, many anomalies result; e.g. the conjunction of two contingent truths yields a necessary truth".

[4]As a result, provable equivalence fails to be a congruence relation. Otherwise, the formula $\sim\sim (A \to B) \leftrightarrow \sim (A \wedge \sim B)$ and hence, due to the double negation and

fore enough to assume another interpretation of the falsification conditions of implications expressed by the schema $\sim (A \to B) \leftrightarrow (A \to \sim B)$.

Consider the language $\mathcal{L} := \{\wedge, \vee, \to, \sim\}$ based on the denumerable set $At_{\mathcal{L}}$ of propositional variables. Equivalence $\leftrightarrow$ is defined as usual. The schematic axioms and rules of the logic C are:

$a1$	the axioms of intuitionistic positive logic
$a2$	$\sim\sim A \leftrightarrow A$
$a3$	$\sim (A \vee B) \leftrightarrow (\sim A \wedge \sim B)$
$a4$	$\sim (A \wedge B) \leftrightarrow (\sim A \vee \sim B)$
$a5$	$\sim (A \to B) \leftrightarrow (A \to \sim B)\}$
$R1$	modus ponens

Clearly, a5 is the only supraclassical axiom of C. Deducibility in C and the consequence relation $\vdash_{\mathsf{C}}$ are defined as usual.

A C-frame is a pair $\mathcal{F} = \langle W, \leq \rangle$, where $\leq$ is a reflexive and transitive binary relation on the non-empty set W. Let $\langle W, \leq \rangle^+$ be the set of all $X \subseteq W$ such that if $u \in X$ and $u \leq w$, then $w \in X$. A C-model is a structure $\mathcal{M} = \langle W, \leq, v^+, v^- \rangle$, where $\langle W, \leq \rangle$ is a C-frame and v^+ and v^- are valuation functions from $At_{\mathcal{L}}$ into $\langle W, \leq \rangle^+$. Intuitively, W is a set of information states. The function v^+ sends an atom p to the states in W that support the truth of p, whereas v^- sends p to the states that support the falsity of p. $\mathcal{M} = \langle W, \leq, v^+, v^- \rangle$ is said to be the model based on the frame $\langle W, \leq \rangle$. The relations $\mathcal{M}, t \models^+ A$ ($\mathcal{M}$ supports the truth of A at t) and $\mathcal{M}, t \models^- A$ ($\mathcal{M}$ supports the falsity of A at t) are inductively defined as follows:

$\mathcal{M}, t \models^+ p$	iff	$t \in v^+(p)$
$\mathcal{M}, t \models^- p$	iff	$t \in v^-(p)$
$\mathcal{M}, t \models^+ (A \wedge B)$	iff	$\mathcal{M}, t \models^+ A$ and $\mathcal{M}, t \models^+ B$ t
$\mathcal{M}, t \models^- (A \wedge B)$	iff	$\mathcal{M}, t \models^- A$ or $\mathcal{M}, t \models^- B$ t
$\mathcal{M}, t \models^+ (A \vee B)$	iff	$\mathcal{M}, t \models^+ A$ or $\mathcal{M}, t \models^+ B$ t
$\mathcal{M}, t \models^- (A \vee B)$	iff	$\mathcal{M}, t \models^- A$ and $\mathcal{M}, t \models^- B$ t
$\mathcal{M}, t \models^+ (A \to B)$	iff	$\forall v \geq t\ (\mathcal{M}, v \models^+ A$ implies $\mathcal{M}, v \models^+ B)$
$\mathcal{M}, t \models^- (A \to B)$	iff	$\forall v \geq t\ (\mathcal{M}, v \models^+ A$ implies $\mathcal{M}, v \models^- B)$
$\mathcal{M}, t \models^+ \sim A$	iff	$\mathcal{M}, t \models^- A$
$\mathcal{M}, t \models^- \sim A$	iff	$\mathcal{M}, t \models^+ A$

DeMorgan laws, $(A \to B) \leftrightarrow (\sim A \vee B)$ would be provable. But this is not the case in a constructive logic, where the verification conditions of implications are those of intuitionistic implications.

Validity of a formula A in a C-model ($\mathcal{M} \models A$) and validity of A on a frame ($\mathcal{F} \models A$) are defined in the usual way. This means that if $\mathcal{M} = \langle W, \leq, v^+, v^- \rangle$ is a C-model, then $\mathcal{M} \models A$ iff for every $t \in W$, $\mathcal{M}, t \models^+ A$. $\mathcal{F} \models A$ holds iff $\mathcal{M} \models A$ for every model $\mathcal{M}$ based on $\mathcal{F}$. A formula is C-valid iff it is valid on every frame. Support of truth and support of falsity for arbitrary formulas are persistent with respect to the relation $\leq$ of possible expansion of information states. That is, for any C-model $\mathcal{M} = \langle W, \leq, v^+, v^- \rangle$, $s, t \in W$, and formula A, if $s \leq t$, then $\mathcal{M}, s \models^+ A$ implies $\mathcal{M}, t \models^+ A$ and $\mathcal{M}, s \models^- A$ implies $\mathcal{M}, t \models^- A$. It can easily be shown that a negation normal form theorem holds. Using familiar methods (see also Section 4), the following can be shown:

PROPOSITION 1. *For any $\mathcal{L}$-formula A, $\mathsf{C} \vdash A$ iff A is C-valid.*

PROPOSITION 2. *The logic C satisfies the disjunction property and the constructible falsity property. If $\mathsf{C} \vdash A \vee B$, then $\mathsf{C} \vdash A$ or $\mathsf{C} \vdash B$. If $\mathsf{C} \vdash \sim (A \wedge B)$, then $\mathsf{C} \vdash \sim A$ or $\mathsf{C} \vdash \sim B$.*

PROPOSITION 3. *The connexive logic C is decidable.*

It is obvious form the above presentation that C differs from Nelson's four-valued constructive logic $\mathsf{N4}$ only with respect to the falsification (or support of falsity) conditions of implications. As in $\mathsf{N4}$, provable strong equivalence is a congruence relation. (Note that, for instance, $(p \to p) \leftrightarrow (q \to q)$ is provable, though $\sim(p \to p) \leftrightarrow \sim(q \to q)$ is not.)

PROPOSITION 4. *The set $\{A \mid \mathsf{C} \vdash A\}$ is closed under the rule $A \leftrightarrow B$, $\sim A \leftrightarrow \sim B$ / $C(A) \leftrightarrow C(B)$.*

4 First-order connexive logic

In this section we define a first-order extension QC of C. The system differs from the first-order extension $\mathsf{QN4}$ of Nelson's four-valued constructive logic characterized in [12, Section 4.1] only with respect to the treatment of negated implications. In order to keep this paper self-contained, we shall here briefly present the axiomatization and semantic characterization of QC, following the presentation in [12].

We extend the propositional language $\mathcal{L}$ to a first-order language by adding denumerably many constant and predicate symbols. Let $T_{\mathcal{L}}$ ($CT_{\mathcal{L}}$) denote the set of $\mathcal{L}$-terms (closed $\mathcal{L}$-terms), and let $At_{\mathcal{L}}$ be the set of atomic formulas and $For_{\mathcal{L}}$ be the set of all $\mathcal{L}$-formulas. The schematic axioms and rules of QC are those of C together with:

$$\sim \exists x A \leftrightarrow \forall x \sim A; \quad \sim \forall x A \leftrightarrow \exists x \sim A$$

$$A(t) \to \exists x A(x) \ (t \text{ is free for } x \text{ in } A)$$

$\forall x A(x) \to A(t)$ (t is free for x in A)

$$\frac{A \to B(x)}{A \to \forall x B(x)} \ (x \text{ not free in } A); \quad \frac{A(x) \to B}{\exists x A(x) \to B} \ (x \text{ not free in } B)$$

Deducibility in QC and the consequence relation $\vdash_{\mathsf{QC}}$ are defined in the usual way.

A QC-model is a structure $\langle W, \leq, \Delta, D, v^+, v^- \rangle$, where $\langle W, \leq \rangle$ is a C-frame; Δ is a set such that $CT_{\mathcal{L}} \subseteq \Delta \subseteq T_{\mathcal{L}}$, and D is a function from W to subsets of Δ such that (i) $CT_{\mathcal{L}} \subseteq D_u$ for all $u \in W$ and (ii) $D_u \subseteq D_t$ whenever $u \leq t$. Finally, v^+ and v^- are functions from $At_{\mathcal{L}}$ to $\langle W, \leq \rangle^+$ such that if $u \in v^{+(-)}(P(a_1, \dots, a_n))$, then $a_1, \dots, a_n \in D_u$. The function D assigns to every state $u \in W$ its domain D_u. If $\mathcal{M} = \langle W, \leq, \Delta, D, v^+, v^- \rangle$ is a model and $t \in W$, the notions $\mathcal{M}, t \models^+ A$ (state t supports the truth of A in $\mathcal{M}$) and $\mathcal{M}, t \models^- A$ (t supports the falsity of A in $\mathcal{M}$) are inductively defined as for C, except that in addition we have:

$\mathcal{M}, t \models^+ \forall x A(x)$ iff $(\forall u \in W)$ if $t \leq u$, then $(\forall c \in D_u)$ $\mathcal{M}, u \models^+ A(c)$

$\mathcal{M}, t \models^- \forall x A(x)$ iff $(\exists c \in D_t)$ $\mathcal{M}, t \models^- A(c)$

$\mathcal{M}, t \models^+ \exists x A(x)$ iff $(\exists c \in D_t)$ $\mathcal{M}, t \models^+ A(c)$

$\mathcal{M}, t \models^- \exists x A(x)$ iff $(\forall u \in W)$ if $t \leq u$, then $(\forall c \in D_u)$ $\mathcal{M}, u \models^- A(c)$

Again, support of truth and support of falsity for arbitrary formulas are persistent with respect to $\leq$. A formula $A \in For_{\mathcal{L}}$ is QC-valid if for any model $\mathcal{M} = \langle W, \leq, \Delta, D, v^+, v^- \rangle$ and $t \in W$, $\mathcal{M}, t \models^+ A$. Soundness can be shown by induction on the length of proofs.

PROPOSITION 5. *For any $A \in For_{\mathcal{L}}$, if $\mathsf{QC} \vdash A$, then A is QC-valid.*

As in the case of $\mathsf{QN4}$, completeness can be shown by a faithful embedding into QInt^+, positive first-order intuitionistic logic, see [2]. $\mathcal{L}^+$, the language of QInt^+, is the result of deleting $\sim$ from $\mathcal{L}$. A QInt^+-model is a structure $\langle W, \leq, \Delta, D, v \rangle$ defined like a QC-model, except that it has only one valuation function v from $At_{\mathcal{L}}$ to $\langle W, \leq \rangle^+$ such that if $u \in v(P(a_1, \dots, a_n))$, then $a_1, \dots, a_n \in D_u$.

Let $\mathcal{M} = \langle W, \leq, \Delta, D, v \rangle$ be a QInt^+-model, $t \in W$ and $A \in For_{\mathcal{L}^+}$. The relation $\mathcal{M}, t \models A$ (A is true at t in $\mathcal{M}$) is defined exactly as the relation $\mathcal{M}, t \models^+ A$ was defined for positive connectives and quantifies (with v^+ replaced by v in case of atomic formulas). A formula $A \in For_{\mathcal{L}^+}$ is said to be QInt^+-valid if for any QInt^+-model $\mathcal{M} = \langle W, \leq, \Delta, D, v \rangle$ and $t \in W$, A is true at t in $\mathcal{M}$.

PROPOSITION 6. *For any $A \in For_{\mathcal{L}^+}$, $\mathsf{QInt} \vdash A$ iff A is QInt^+-valid.*

The proof is essentially the same as the completeness proof for full first-order intuitionistic logic [2], omitting the part concerning negation.

We say that a formula $A \in For_{\mathcal{L}}$ is in *negation normal form* (nnf), if it contains negations only in front of atomic formulas, in other words, if $\sim B$ is a subformula of A, then B is atomic.

DEFINITION 7. We define the transformation $\overline{(\cdot)}$ on $For_{\mathcal{L}}$ as follows:

1. $\overline{B} := B$, $\overline{\sim B} := \sim B$, for $B \in At_{\mathcal{L}}$.
2. $\overline{\sim\sim A} := \overline{A}$, for a formula A.
3. $\overline{A \diamond B} := \overline{A} \diamond \overline{B}$, where A and B are formulas and $\diamond \in \{\vee, \wedge, \rightarrow\}$.
4. For any formula A and $Q \in \{\forall, \exists\}$, $\overline{QxA} := Qx\overline{A}$.
5. For any formulas A and B, $\overline{\sim (A \vee B)} := \overline{\sim A} \wedge \overline{\sim B}$, $\overline{\sim (A \wedge B)} := \overline{\sim A} \vee \overline{\sim B}$, and $\overline{\sim (A \rightarrow B)} := \overline{A} \rightarrow \overline{\sim B}$.
6. For any formula A, $\overline{\sim \forall xA} := \exists x\overline{\sim A}$ and $\overline{\sim \exists xA} := \forall x\overline{\sim A}$.

PROPOSITION 8. *For any formula A, $\overline{A}$ is in negation normal form and* $\mathsf{QC} \vdash A \leftrightarrow \overline{A}$.

We now define the positive intuitionistic first-order language $\mathcal{L}'^{+}$ by deleting $\sim$ from $\mathcal{L}$ and adding for each predicate symbol P of $\mathcal{L}$ a new predicate symbol $P^{\sim}$ of the same arity.

DEFINITION 9. The transformation $(\cdot)^*$ of formulas from $For_{\mathcal{L}}$ into $For_{\mathcal{L}'^+}$ is defined as follows:

1. For an atomic $P(\overline{a})$, $(P(\overline{a}))^* := P(\overline{a})$ and $(\sim P(\overline{a}))^* := P^{\sim}(\overline{a})$.
2. $(A \diamond B)^* := A^* \diamond B^*$, where A and B are formulas in nnf and $\diamond \in \{\vee, \wedge, \rightarrow\}$.
3. $(QxA)^* := QxA^*$, where A is a formula in nnf and $Q \in \{\forall, \exists\}$.
4. $(A)^* := (\overline{A})^*$ for any formula A not in nnf.

PROPOSITION 10. *For any $A \in For_{\mathcal{L}}$,* $\mathsf{QC} \vdash A$ *iff* $\mathsf{QInt}^+ \vdash A^*$.

Proof. If $\mathsf{QC} \vdash A$, one can easily check by induction on the length of proofs that A^* is provable in QInt^+. If $\mathsf{QInt} \vdash A^*$ and $B_0, \ldots B_n = A^*$ is a proof, then $B'_0, \ldots B'_n = \overline{A}$, where B'_i is the result of replacing any subformula $P^{\sim}(\overline{a})$ by $\sim P(\overline{a})$, is a proof of the nnf of A in QC. By Proposition 8, we then have $\mathsf{QC} \vdash A$. ■

PROPOSITION 11. *For any $A \in For_{\mathcal{L}}$,* $\mathsf{QC} \vdash A$ *iff A is* QC*-valid.*

Proof. Let A_0 be a QC-valid formula. Assume that $\mathsf{QC} \not\vdash A_0$. Then $\mathsf{QInt}^+ \not\vdash (A_0)^*$ by Proposition 10. This means that there is a QInt^+-model $\mathcal{M} = \langle W, \leq, \Delta, D, v\rangle$ of the language $\mathcal{L}'^+$ and $t_0 \in W$ such that $\mathcal{M}, t_0 \not\models (A_0)^*$. We define a QC-model $\mathcal{M}' = \langle W, \leq, \Delta, D, v^+, v^-\rangle$ with the same W, $\leq$, and Δ. The assignment functions v^+ and v^- are defined as follows:

$$v^+(P(\overline{a})) := v(P(\overline{a})) \text{ and } v^-(P(\overline{a})) := v(P^{\sim}(\overline{a})).$$

Using induction on the structure of a formula, we can show that

$$\mathcal{M}, t \models A^* \text{ if and only if } \mathcal{M}', t \models^+ A$$

for any $t \in W$ and formula A in nnf. Thus, $\mathcal{M}', t_0 \not\models^+ \overline{A_0}$ and $\mathcal{M}', t_0 \not\models^+ A_0$ by Proposition 8, which conflicts with the QC-validity of A_0. ■

5 The connexive modal logic CK

Let now $\mathcal{L}_{\Box,\Diamond} := \{\wedge, \vee, \to, \sim, \Box, \Diamond\}$. To obtain a connexive analogue of the smallest normal modal logic K, we consider the translation T_x from $For_{\mathcal{L}_{\Box,\Diamond}}$ into the first-order language $\mathcal{L}$ of QC:

$$\begin{array}{rcl}
p & \xmapsto{T_x} & P(x) \\
\sim p & \xmapsto{T_x} & \sim P(x) \\
A * B & \xmapsto{T_x} & T_x(A) * T_x(B),\ * \in \{\wedge, \vee, \to\} \\
\sim (A * B) & \xmapsto{T_x} & \sim (T_x(A) * T_x(B)),\ * \in \{\wedge, \vee, \to\} \\
\Box A & \xmapsto{T_x} & \forall y(R(x, y) \to T_y(A)) \\
\Diamond A & \xmapsto{T_x} & \exists y(R(x, y) \wedge T_y(A)) \\
\sim\Box A & \xmapsto{T_x} & \exists y(R(x, y) \wedge T_y(\sim A)) \\
\sim\Diamond A & \xmapsto{T_x} & \forall y(R(x, y) \to T_y(\sim A))
\end{array}$$

Consider the following axioms and rules.

$$\begin{array}{ll}
a6 & \Box A \wedge \Box B \to \Box(A \wedge B) \\
a7 & \Box(A \to A) \\
a8 & \Diamond(A \vee B) \to \Diamond A \vee \Diamond B \\
a9 & \Diamond(A \to B) \to (\Box A \to \Diamond B) \\
a10 & (\Diamond A \to \Box B) \to \Box(A \to B) \\
a11 & \sim\Box A \leftrightarrow \Diamond \sim A \\
a12 & \sim\Diamond A \leftrightarrow \Box \sim A \\
R_{\Box} & A \to B \,/\, \Box A \to \Box B \\
R_{\Diamond} & A \to B \,/\, \Diamond A \to \Diamond B
\end{array}$$

The connexive modal logic CK is defined as the deductive closure of axioms a1–a12 under the rules $R1$, $R_{\Box}$, and $R_{\Diamond}$. If Δ is a set of $\mathcal{L}_{\Box,\Diamond}$-formulas and A an $\mathcal{L}_{\Box,\Diamond}$-formula, the relation $\Delta \vdash A$ holds iff A belongs to the closure of $\mathsf{CK} \cup \Delta$ under $R1$. Axioms a9 and a10 ensure that $\Box$ and $\Diamond$ can be interpreted semantically with respect to a single accessibility relation R (instead of a pair of independent accessibility relations $R_\Box$ and $R_\Diamond$) (see [4], [13], [25]) and that CK can be faithfully embedded into QC.

A CK-frame is a triple $\mathcal{F} = \langle W, \leq, R\rangle$, where $\langle W, \leq\rangle$ is a C-frame and R is a binary relation on W such that:

1. $\leq$ is reflexive and transitive;
2. $\leq^{-1} \circ R \subseteq R\circ \leq^{-1}$;
3. $R\circ \leq\ \subseteq\ \leq \circ R$.

A CK-model is a structure $\mathcal{M} = \langle W, \leq, R, v^+, v^-\rangle$, where $\langle W, \leq, R\rangle$ is a CK-frame and v^+ and v^- are valuation functions from $At_\mathcal{L}$ into $\langle W, \leq\rangle^+$. The relations $\mathcal{M}, t \models^+ A$ and $\mathcal{M}, t \models^- A$ are inductively defined as for C, except that in addition we have:

$$\begin{array}{lll} \mathcal{M}, t \models^+ \Box A & \text{iff} & \forall u \geq t\, \forall v(uRv \text{ implies } \mathcal{M}, v \models^+ A) \\ \mathcal{M}, t \models^- \Box A & \text{iff} & \exists u(tRu \text{ and } \mathcal{M}, u \models^- A) \\[4pt] \mathcal{M}, t \models^+ \Diamond A & \text{iff} & \exists u(tRu \text{ and } \mathcal{M}, u \models^+ A) \\ \mathcal{M}, t \models^- \Diamond A & \text{iff} & \forall u \geq t\, \forall v(uRv \text{ implies } \mathcal{M}, v \models^- A) \end{array}$$

A discussion of the support of truth conditions for the modal operators can be found in [24]. Support of truth and support of falsity are persistent with respect to the relation $\leq$ of possible expansion of information states. A formula A is valid in a model $\mathcal{M} = \langle W, \leq, R, v^+, v^-\rangle$ ($\mathcal{M} \models A$) if $\mathcal{M}, t \models^+ A$ for all $t \in W$; A is said to be valid on a frame $\mathcal{F}$ ($\mathcal{F} \models A$) if A is valid in any model based on $\mathcal{F}$.

PROPOSITION 12. *For any $\mathcal{L}$-formula A, $\mathsf{CK} \vdash A$ iff A is valid on every CK-frame.*

The proof is basically the completeness proof for the constructive modal logic FSK^d, see [13]. A canonical model is defined and shown to satisfy the above conditions 1.–3. on its underlying frame.

DEFINITION 13. We define the transformation $\overline{(\cdot)}$ on $For_{\mathcal{L}_{\Box,\Diamond}}$ as in Definition 7 except that the clause for quantified formulas is replaced by the following:

6. For any $\mathcal{L}_{\Box,\Diamond}$-formula A, $\overline{\sim \Box A} := \Diamond\overline{\sim A}$ and $\overline{\sim \Diamond A} := \Box\overline{\sim A}$.

PROPOSITION 14. *For any $\mathcal{L}_{\Box,\Diamond}$-formula A, $\overline{A}$ is in negation normal form and $\mathsf{CK} \vdash A \leftrightarrow \overline{A}$.*

This negation normal form theorem allows CK to be rewritten as FS^+, the positive fragment of Fischer Servi's intuitionistic modal logic FS, see [3], [4], [5], [24].

PROPOSITION 15. *The translation T_x is a faithful embedding of CK into QC.*

Proof. Observe that the translation of $\mathcal{L}_{\Box,\Diamond}$-formulas in negation normal form into the language $\mathcal{L}_{\Box,\Diamond}{'^+}$ that maps every propositional variable p to itself, sends every negated propositional variable $\sim p$ to a fresh propositional variable p', and commutes with the positive connectives is a faithful embedding of CK into FS^+. This is enough to establish the claim, since FS^+ is faithfully embedded by the standard translation, coinciding on negation-free formulas with T_x, into $\mathsf{QInt}^+ = \mathsf{QC}^+$, positive QK. ■

Since FS is decidable, we obtain the following:

COROLLARY 16. *The connexive modal logic CK is decidable.*

PROPOSITION 17. *The logic CK satisfies the disjunction property and the constructible falsity property.*

PROPOSITION 18. *The set $\{A \mid \mathsf{CK} \vdash A\}$ is closed under the rule $A \leftrightarrow B$ / $C(A) \leftrightarrow C(B)$.*

Note hat T_x differs from the standard translation ST_x from $For_{\mathcal{L}_{\Box,\Diamond}}$ into $For_{\mathcal{L}}$:

$$\begin{array}{rcl}
p & \stackrel{ST_x}{\longmapsto} & P(x) \\
\sim A & \stackrel{ST_x}{\longmapsto} & \sim ST_x(A) \\
A * B & \stackrel{ST_x}{\longmapsto} & ST_x(A) * ST_x(B),\ * \in \{\wedge, \vee, \rightarrow\} \\
\Box A & \stackrel{ST_x}{\longmapsto} & \forall y(R(x,y) \rightarrow ST_y(A)) \\
\Diamond A & \stackrel{ST_x}{\longmapsto} & \exists y(R(x,y) \wedge ST_y(A))
\end{array}$$

Whereas T_x guarantees the syntactic duality between $\Box$ and $\Diamond$, the standard translation of $\sim \Box A$ is equivalent with $\exists y(R(x,y) \rightarrow \sim ST_y(A))$, which is not equivalent with the standard translation of $\Diamond \sim A$. Also $\sim \Diamond A$ and $\Box \sim A$ are not equivalent under ST_x. The standard translation of $\sim \Diamond A$ yields a formula equivalent with $\forall y(\sim R(x,y) \vee \sim ST_y(A))$. The support of falsity clauses for modal formulas induced by ST_x are:

$$\begin{array}{lll}
\mathcal{M}, t \models^- \Diamond A & \text{iff} & \forall s \geq t\ \forall u(sR^{\sim}u \text{ or } \mathcal{M}, u \models^- A) \\
\mathcal{M}, t \models^- \Box A & \text{iff} & \exists s\, \forall u \geq s(tRs \text{ implies } \mathcal{M}, u \models^- A)
\end{array}$$

where $R^{\sim}$ is an impossibility relation. The support of falsity clause for formulas $\Diamond A$ is plausible: if for every expansion s of information state t, every state u is such that either u is impossible relative to s or u supports the falsity of A, then t supports the falsity of $\Diamond A$. The support of falsity clause for $\Box A$, however, appears to be much more problematic. To support the falsity of $\Box A$, it is enough for a state t that there exists one inaccessible state, which is not very convincing.

6 Adding intuitionistic negation

Axiom a5 calls for some explanation. Whereas the direction from right to left can be justified by rejecting the view that if A implies B and A is inconsistent, A implies any formula, in particular B, the direction from left to right seems rather strong. If the verification conditions of implications are dynamic, then a5 indicates that the falsification conditions of implications are dynamic as well. The falsity of $(A \to B)$ thus implies that if A is true, B is *false*. Yet, one might wonder why it is not required that the falsity of $(A \to B)$ implies that if if A is true, B is *not true*. This cannot be expressed in a language with just one negation $\sim$ expressing falsity instead of absence of truth. (classically at the state of evaluation or intuitionistically at all related states). If one adds to C the further axiom $\sim A \to (A \to B)$ to obtain a connexive variant of Nelson's three-valued logic $\mathsf{N3}$, intuitionistic negation $\neg$ is definable by setting: $\neg A := A \to\sim A$. Then a5 could be replaced by

$$\text{a5}' \quad \sim (A \to B) \leftrightarrow (A \to \neg B).$$

The resulting system satisfies AT, AT', BT, and BT', because $A \to \neg \sim A$ and $\sim A \to \neg A$ are theorems. For BT, for example, we have:

1.	$A \to B$	assumption
2.	$B \to \neg \sim B$	theorem
3.	$A \to \neg \sim B$	1., 2., transitivity of $\to$
4.	$(A \to \neg \sim B) \to \sim (A \to \sim B)$	a5$'$
5.	$\sim (A \to \sim B)$	3., 4., R1
6.	$(A \to B) \to \sim (A \to \sim B)$	1., 5., deduction theorem

This logic, however, is the trivial system. Like C it contains contradictions, which together with the ex falso axiom trivialize.

7 Systems of consequential implication

Aristotle's and Boethius' Theses express, as it seems, some pre-theoretical intuitions about meaning relations between negation and implication. But is not clear that a language must contain only one negation operation and only one implication. In the previous section we have encountered a system with two negations, and the language of systems of *consequential implication* comprises two implication connectives together with one negation, see [14], [15], [16], [17]. In [18], the notion of a normal system of analytic consequential implication is defined. The smallest such normal consequential logic that satisfies AT is called CI. Alternatively, CI can be characterized as the smallest normal system that satisfies Weak Boethius' Thesis:

$$(A \to B) \supset \neg(A \to \neg B),$$

where $\to$ is consequential implication, $\supset$ is material implication, and $\neg$ is classical negation.

Pizzi and Williamson show that CI can be faithfully embedded into the normal modal logic KD, and vice versa. Analytic consequential implication is interpreted according to the following translation function φ:

$$\varphi(A \to B) = \Box(\varphi A \supset \varphi B) \wedge (\Box\varphi B \supset \Box\varphi A) \wedge (\Diamond\varphi B \supset \Diamond\varphi A)$$

As Pizzi and Williamson [18, p. 571] point out, their investigation is a "contribution to the modal treatment of logics intermediate between logics of consequential implication and connexive logics." They emphasize a difficulty of regarding consequential implication as a genuine implication connective by showing that in any normal system of consequential logic that admits modus ponens for consequential implication and contains BT, the following formulas are provable:

$$(a)\ \ (A \to B) \equiv (B \to A), \quad (b)\ \ (A \to B) \equiv \neg(A \to \neg B),$$

where $\equiv$ is classical equivalence. Since $(A \to B) \leftrightarrow\, \sim (A \to \sim B)$ is a theorem of C, the more problematic fact, from the point of view of the present paper, is the provability of (a). Pizzi and Williamson also show that in any normal system of consequential logic that contains BT, the formula $(A \to B) \equiv (A \equiv B)$ is provable, if $(A \to B) \supset (A \supset B)$ is provable, in other words, consequential implication collapses into classical equivalence if $(A \to B) \supset (A \supset B)$ is provable.

8 Subconnexive logic and the logic of scientific research

In the literature, there is a discussion about whether some system of constructive logic may be regarded as the logic of scientific research. According to Grzegorczyk [6], intuitionistic logic may be viewed as the logic of scientific research, understood positivistically as the logic of verification. However, intuitionistic falsity, as expressed by intuitionistic negation, is still a rather weak notion of falsity. The intuitionistic negation of a formula A is verified at an information state u iff A is *not true* at all possible extensions of u. Gurevich [7] criticizes this weak notion of falsity and suggests taking direct falsification seriously by admitting falsification of atomic formulas on the spot. These considerations lead him to Nelson's three-valued logic $\mathsf{N3}$ with strong, constructive negation. The system $\mathsf{N3}$ is a logic of scientific research in which both truth and falsity are understood positivistically in the sense that both truth and falsity are persistent with respect to the relation of possible growth of information states. Moreover, truth and falsity are mutually exclusive: there is no information state u and no formula A such that both A and its strong negation are true at u. In [26] is has been pointed out that Gurevich's emphasis on falsification may be seen to have a precursor in Popper's idea of falsification in the natural sciences, see [19], [20].

In $\mathsf{N3}$ and $\mathsf{N4}$, positive and negative information are treated on a par, by distinguishing a support of truth relation $\models^+$ and a support of falsity relation $\models^-$. Whereas the persistence with respect to $\models^-$ is compatible with Popper's notion of falsification, persistence of positive information is incompatible with Popper's notion of confirmation. According to Popper, empirical hypotheses may be falsified by observation, measurement, and suitable experiments. This means falsification on the spot, justifying, in Popper's view, a dismissal of the statement under consideration henceforth. In contrast to this, confirmation is only preliminary confirmation. It may happen that a future state falsifies the hypothesis in question, so that confirmation at the given state does not establish the hypothesis once and for all. It does not help to consider the logic resulting from $\mathsf{N3}$ by giving up the persistence condition for support of truth of atomic formulas. In this subconstructive logic, *falsification*, too, is no longer persistent. Consider simple law-like hypotheses like $\forall x(S(x) \to B(x))$. If strong negation embodies the notion of falsification, then $\forall x(S(x) \to B(x))$ is falsified at a state u iff for some element a from the individual domain of u, $(S(a) \to B(a))$ is falsified at u iff $S(a)$ is verified at u and $B(a)$ is falsified at u. Although falsification is falsification on the spot, falsification in *Nelson's systems* involves *persistent* verification. But if persistence of verification is given up, so that verification may be regarded as confirmation in Popper's sense, then, if for some a, $S(a)$ is confirmed and $B(a)$ is falsified, falsification at all future states is not guaranteed. This problem of failure of persistence of negative information, of course, arises for the falsification of other kinds of compound formulas, too.[5]

[5] In [23], it has been suggested to consider a certain system of implication-free logic as Popper's logic of scientific research. In this system, called falsificationist logic, the

The problem of identifying a formal system that captures Popper's conception of the logic of scientific research looses much of its attraction, when it is recognized that Popper's model of scientific inquiry is an outdated and abandoned model. It is well-know that Popper's much celebrated distinction between confirmation and falsification of scientific theories has been vigorously criticized by many philosopher's of science like, for example, Kuhn [9] and Lakatos [10]. Since virtually every empirical theory is confronted with some kind of anomalies, Popper's thesis that anomalies falsify a theory once and for all would have the highly undesirable consequence that there is no non-falsified empirical theory. There simply is nothing like persistent direct falsification in the empirical sciences. Moreover, and maybe even more importantly, there is little reason to assume that the development of empirical or non-empirical scientific theories is adequately described by a system of propositional or first-order *logic* instead of a suitable first- or higher-order *theory*.

Whether or not the system QC or the sub-connexive logic obtained from QC by giving up both persistence of support of truth and persistence of support of falsity of atomic information are logics of scientific research, these systems provide an interesting model of *information processing*. If the informational interpretation of constructive logics (see [26] and references therein) is taken seriously as a model of information processing, then it is conspicuous that although support of truth and support of falsity in Nelson's logics are treated on a par, there still is an asymmetry between verification and falsification: the former may be dynamic, the latter is static. The support of falsity conditions of implications in C are one way of overcoming this asymmetry.[6] Interestingly, this modification of the classical falsity conditions of implications gives rise to a system of connexive logic. As a system of constructive logic with *strong* negation, QC may also play a role in investigations of contrariety, see [27], [28]. More specifically, CK may

evaluation clauses for negation are as follows (notation adjusted):

$$u \models^+ \sim A \quad \text{iff} \quad (\exists w \geq u)\, w \models^- A; \quad u \models^- \sim A \quad \text{iff} \quad (\forall w \geq u)\, w \models^+ A.$$

Moreover, the verification of atomic information is taken to be backward-persistent: if $u \models^+ p$ and $u \geq w$, then $w \models^+ p$.

[6] Another suggestion for rendering the evaluation of negated dynamic implications dynamic has been made independently of the present paper in [8], where N. Kamide defines a sequent calculus CLS, in which the following equivalence is provable (notation adjusted):

$$\sim (A \to B) \leftrightarrow (\sim A \to \sim B).$$

Kamide points out that "the substitution theorem with respect to $\leftrightarrow$ does not hold". The treatment of negated implications is motivated by the idea of treating $\sim$ as a modal operator satisfying the K axiom schema, leading to the left-to-right direction of the displayed formula.

play a role in investigations of necessity versus unnecessity and possibility versus impossibility.

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