Consistency Proofs for Systems of Multi-agent Only Knowing

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ABSTRACT. A new and natural multi-modal system of *only knowing* is proposed along with a sequent calculus. The main technical results are cut-elimination theorems and a proof of the consistency of the logic; the latter result follows from the subformula property of cut-free proofs. The system is extended to a multi-modal logic which allows the representation of belief states with confidence levels in a multi-agent context.

1 Introduction

Multi-agent belief logics can be viewed as systems designed for the reasoning about, and representation of, representations that agents use in reasoning about other agents' cognitive states. A multi-agent *only knowing* system has language constructs for representing upper and lower bounds of beliefs, which may be combined to express the exact content of an agent's belief state. The natural starting point for the design of such a system is to generalize the only knowing system of Levesque [8] to the multi-modal case. The tricky part is due to what I shall refer to as the $\Diamond$*-axiom* of Levesque's system: that $\Diamond\varphi$, "φ is logically possible", is an axiom *for each satisfiable, objective* φ ("objective" meaning that φ does not contain any modal operators). One route to generalization of this axiom goes through generalization of the semantical notion of satisfiability. This is what Halpern and Lakemeyer have attempted in a series of papers [7, 4, 5, 6], at the cost of coding the satisfiability relation into the system.

Another possibility is to express the generalized $\Diamond$-axiom syntactically. It is then natural to take the syntactical counterpart of Levesque's $\Diamond$-axiom as the starting point: $\vdash \Diamond\varphi$ if $\varphi \nvdash \bot$ (i.e. if φ is

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consistent), where φ is any purely Boolean formula. This seemingly circular pattern is in the single modality case completely innocent since the axiom is restricted to φs outside the modal part of the language; in particular it is easy to show that a purely Boolean φ is consistent in the system only if it is consistent in propositional logic. This constraint is clearly too restrictive in the multi-modal setting.

¿From the syntactical perspective, what seems to be appropriate is to generalize the $\Diamond$-axiom to the following: $\vdash \Diamond_k \varphi$ provided that $\varphi \nvdash \bot$, where k is an index individuating an agent, $\Diamond_k$ is a *conceivability* operator for agent k and φ is any formula in which every k-modality which occurs in it, occurs within the scope of a modality for another agent (Halpern and Lakemeyer call such a φ *k-objective*). This formulation arguably gives a cleaner and more transparent proof system than the systems studied by Halpern and Lakemeyer. Note that in the single agent case, the generalized $\Diamond$-axiom is identical to the single-agent formulation which we departed from.

However, the syntactically motivated $\Diamond$-axiom has a circular pattern, and an argument is needed in order to show that the circularity is not vicious. Note, incidentally, that consistency cannot easily be obtained by standard semantical methods. To prove soundness of the $\Diamond$-axiom one needs to recast the consistency condition into a corresponding semantical condition in terms of satisfiability and this, in turn, presupposes that completeness has already been established. The standard way of proving completeness is via maximal consistent sets, a construction which builds on consistency.

The scope of this paper is entirely syntactical. More precisely we define a multi-modal logic closed under a generalization of the syntactical form of the $\Diamond$-axiom. We introduce a sequent calculus for the proposed logic and prove cut-elimination theorems by constructive methods. A consequence of these results is that we obtain a restricted form of the subformula property: a formula φ is provable iff it has a proof using only subformulae of φ.

Modal contexts give rise to natural layers of representation in the syntax. A formula, say $\mathsf{B}_k \mathsf{B}_i \varphi$, expresses a property of agent k's representation of the beliefs of agent i (we regard agent k's representation as one distinct layer). This perspective is illuminating for the interpretation of the subformula property, since it means that the structural layers in the construction of formulae are mirrored in the proof theory. More precisely, statements about one representation layer can be proved only by means of statements about the same layer. It is immediate that every sequent calculus rule except the cut rule respects this property. Moreover, the cut-elimination results show that cut has no damaging effects. A consequence of this it that the circularity in the definition of the $\Diamond$-axiom is not vicious. Note that the introduction of a formula $\Diamond_k\varphi$ on the assumption that φ is consistent has no effect on the assumption itself as long as φ and $\Diamond_k\varphi$ belong to different representation layers. In particular, the consistency of the logic follows. Moreover, using the subformula property we show in section 3 that the system proposed in this paper is indeed equivalent to the system that Halpern and Lakemeyer claim is the correct multi-modal generalization of Levesque's system.

Since the logic for the single agent case is identical to Levesque's system, the paper also presents a sequent calculus for the propositional part of Levesque's logic. We shall also see that the cut-elimination results for the core system can be generalized to a system in which a preorder is imposed on the index set individuating the modalities. The motivation for the latter construct is to enable the representation of confidence levels for each agent; this is necessary for the representation of certain multi-modal defaults along the same lines as one can represent prioritized supernormal defaults in an extension of Levesque's system with confidence levels [9].

2 The multi-agent logic $\mathbf{L}_I$

2.1 Axiomatic system

The object language $\mathcal{L}_I$ contains a stock of propositional letters, the constants $\top$ and $\bot$, the Boolean connectives $\neg$, $\vee$, $\wedge$, $\supset$, and $\equiv$, and modal operators B_k and C_k for each k in a non-empty index set I. The intended interpretation is that the index set I represents the set of agents, B_k is a belief operator, and C_k a complementary *co-belief* operator for agent k. We define the symbol O_k by $\mathsf{O}_k\varphi = \mathsf{B}_k\varphi \wedge \mathsf{C}_k\neg\varphi$. Intuitively $\mathsf{O}_k\varphi$ means that φ is exactly what agent k believes; the construction is inspired by the "all I know" operator of Levesque [8].

Further abbreviations: b_k is $\neg\mathsf{B}_k\neg$, c_k is $\neg\mathsf{C}_k\neg$, $\Box_k\varphi$ is $\mathsf{B}_k\varphi \wedge \mathsf{C}_k\varphi$, the dual $\Diamond_k\varphi$ is $\mathsf{b}_k\varphi \vee \mathsf{c}_k\varphi$. The intended meaning of $\Box_k$ is logical necessity.

A formula is *purely Boolean* if it contains no occurrences of any modality, a *modal atom of modality* k if it is of the form $\mathsf{B}_k\psi$ or $\mathsf{C}_k\psi$ for a $k \in I$, *completely* k*-modalized* if it is a Boolean combination of modal atoms of modality k, and *free of modality* k if it is a Boolean combination of propositional letters and modal atoms not of modality k. φ is a *first-order formula* if, for each $k \in I$ and each subformula $\mathsf{B}_k\psi$ or $\mathsf{C}_k\psi$ in φ, ψ is free of modality k.

The *modal depth* $m(\varphi)$ of a formula φ expresses the nesting of alternating modalities in φ. Formally, the modal depth of a purely Boolean φ is 0. Otherwise, if φ is $\mathsf{B}_k\psi$ or $\mathsf{C}_k\psi$, let Ψ be the set of modal atoms which occur as subformulae in ψ. Then $m(\varphi)$ is the maximal number in $\{m(\chi)+1 \mid \chi \in \Psi \text{ and } \chi \text{ is not } k\text{-modalized}\} \cup \{m(\chi) \mid \chi \in \Psi \text{ and } \chi \text{ is } k\text{-modalized}\}$. Otherwise, the modal depth of φ is the maximal $m(\psi)$ for a subformula ψ of φ.

The logic L_I is defined by means of the axioms and inference rules below. We write $\vdash \varphi$ if φ is theorem and $\varphi_1, \ldots, \varphi_n \vdash \psi$ if $(\varphi_1 \wedge \cdots \wedge \varphi_n) \supset \psi$ is a theorem; the provability symbol '$\vdash$' is used also for extensions of L_I and shall always be understood relative to the system at hand. A formula φ is *consistent* if $\varphi \nvdash \bot$.

Let us say that a *tautology* is any substitution instance of a formula valid in classical propositional logic such as $\Box_k\varphi \supset \Box_k\varphi$. L_I is defined as the least set that contains all tautologies, is closed under all instances of the rules

$$\frac{\varphi}{\Box_k\varphi}\ \mathrm{RN} \qquad\qquad \frac{\varphi \quad \varphi \supset \psi}{\psi}\ \mathrm{MP}$$

and contains all instances of the following schemata for each $k \in I$:

K_B: $\mathsf{B}_k(\varphi \supset \psi) \supset (\mathsf{B}_k\varphi \supset \mathsf{B}_k\psi)$
K_C: $\mathsf{C}_k(\varphi \supset \psi) \supset (\mathsf{C}_k\varphi \supset \mathsf{C}_k\psi)$
$B_\Box$: $\mathsf{B}_k\varphi \supset \Box_k\mathsf{B}_k\varphi$
$C_\Box$: $\mathsf{C}_k\varphi \supset \Box_k\mathsf{C}_k\varphi$
$\overline{B}_\Box$: $\neg\mathsf{B}_k\varphi \supset \Box_k\neg\mathsf{B}_k\varphi$
$\overline{C}_\Box$: $\neg\mathsf{C}_k\varphi \supset \Box_k\neg\mathsf{C}_k\varphi$

$\Diamond$: $\quad \Diamond_k\varphi$ provided $\varphi \nvdash \bot$, φ free of modality k

The core system without the $\Diamond$-axiom is called L'_I. It is easy to show that B_k and C_k are both K45 modalities. The axioms can be viewed as modality reduction rules; the main properties are captured in the following two lemmata, both of which are proved without the $\Diamond$-axiom and hence hold in L'_I.

LEMMA 1. *Let κ be completely k-modalized and φ be any formula. Then $\vdash \mathsf{B}_k(\kappa \vee \varphi) \equiv (\kappa \vee \mathsf{B}_k\varphi)$ and $\vdash \mathsf{C}_k(\kappa \vee \varphi) \equiv (\kappa \vee \mathsf{C}_k\varphi)$. Hence $\vdash \mathsf{B}_k\kappa \equiv (\kappa \vee \mathsf{B}_k\bot)$ and $\vdash \mathsf{C}_k\kappa \equiv (\kappa \vee \mathsf{C}_k\bot)$.*

Proof. We first show, by induction on θ, that $\vdash \theta \supset \Box_k\theta$ for each completely k-modalized θ. The base case in which θ is $\mathsf{B}_k\psi$ follows from axiom $B_\Box$; likewise for the other base cases (using $C_\Box$, $\overline{B}_\Box$, or $\overline{C}_\Box$). The induction step follows from the induction hypothesis and simple modal reasoning. To establish the first equivalence we use instances of the K_B axiom, that $\vdash \kappa \supset \mathsf{B}_k\kappa$ and that $\vdash \neg\kappa \supset \mathsf{B}_k\neg\kappa$. The second equivalence is proved similarly. The latter statements follow from the former since $\vdash \kappa \equiv (\kappa \vee \bot)$. ■

LEMMA 2. *Any formula is provably equivalent to a first-order formula with the same modal depth.*

Proof. A modal atom $\mathsf{B}_k\varphi$ is equivalent to a formula of the form $\mathsf{B}_k(\psi_1 \wedge \cdots \wedge \psi_m)$ where each ψ_i is a disjunction of a formula of modality k and a formula free of modality k (either disjunct may be absent). If φ is not free of modality k, we can distribute B_k over the conjunctions and use Lemma 1 and the induction hypothesis on each $\mathsf{B}_k\psi_j$. ■

A *k-block* is a conjunction of formulae of the form $\mathsf{B}_k\psi$ or $\mathsf{C}_k\psi$, possibly negated. An I-block is a conjunction of k-blocks (for zero or more $k \in I$) and possibly also a purely Boolean formula.

LEMMA 3. *Let φ be L_I-consistent. Then there is an I-block ψ such that ψ is L_I-consistent, $\psi \supset \varphi$ is provable in L'_I and $m(\psi) = m(\varphi)$.*

Proof. By Lemma 2 φ has a first-order equivalent with the same modal depth; this formula can be put on disjunctive normal form such that each disjunct is an I-block. If there is a proof of $\bot$ from each disjunct, simple propositional reasoning yields that $\varphi \vdash \bot$. Hence, if φ is L_I-consistent, one of the disjuncts ψ in its DNF equivalent must also be consistent; ψ clearly implies φ. ■

If φ is free of k, the $\Diamond$-axiom entails that $\Box_k\varphi$ is provable only if φ is provable; otherwise the property holds by the axioms. Hence, by RN, $\vdash \Box_k\varphi$ iff $\vdash \varphi$. The logic of $\Box_k$ is hence S5. The next lemma shows that the $\Diamond$-axiom makes B_k and C_k complementary modalities.

LEMMA 4. *Let φ and ψ be formulae free of modality k such that $\{\neg\varphi, \neg\psi\}$ is L_I-consistent. Then $\vdash \neg(\mathsf{B}_k\varphi \wedge \mathsf{C}_k\psi)$.*

Proof. By assumption, $\Diamond_k(\varphi \vee \psi)$ is an instance of the $\Diamond$-axiom. The result follows from this and the theorems $\mathsf{B}_k\varphi \supset \mathsf{B}_k(\varphi \vee \psi)$ and $\mathsf{C}_k\psi \supset \mathsf{C}_k(\varphi \vee \psi)$. ■

Structural rules

$$\frac{\Gamma \Rightarrow \Delta}{\Gamma, \varphi \Rightarrow \Delta}\ \text{L}\textsc{t} \qquad \frac{\Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \varphi, \Delta}\ \text{R}\textsc{t}$$

$$\frac{\Gamma, \varphi, \varphi \Rightarrow \Delta}{\Gamma, \varphi \Rightarrow \Delta}\ \text{L}\textsc{c} \qquad \frac{\Gamma \Rightarrow \varphi, \varphi, \Delta}{\Gamma \Rightarrow \varphi, \Delta}\ \text{R}\textsc{c}$$

Identity rules

$$\frac{}{\varphi \Rightarrow \varphi}\ \text{I}\textsc{d} \qquad \frac{\Gamma_1 \Rightarrow \varphi, \Delta_1 \quad \Gamma_2, \varphi \Rightarrow \Delta_2}{\Gamma_1, \Gamma_2 \Rightarrow \Delta_1, \Delta_2}\ \text{C}\textsc{ut}$$

Logical rules for the connectives

$$\frac{\Gamma \Rightarrow \varphi, \Delta}{\Gamma\neg\varphi \Rightarrow \Delta}\ \text{L}\neg \qquad \frac{\Gamma, \varphi \Rightarrow \Delta}{\Gamma \Rightarrow \neg\varphi, \Delta}\ \text{R}\neg$$

$$\frac{\Gamma, \varphi_i \Rightarrow \Delta}{\Gamma, \varphi_1 \wedge \varphi_2 \Rightarrow \Delta}\ \text{L}\wedge_i\ (i = 1, 2) \qquad \frac{\Gamma_1 \Rightarrow \varphi, \Delta_1 \quad \Gamma_2 \Rightarrow \psi, \Delta_2}{\Gamma_1, \Gamma_2 \Rightarrow \varphi \wedge \psi, \Delta_1, \Delta_2}\ \text{R}\wedge$$

$$\frac{\Gamma_1, \varphi \Rightarrow \Delta_1 \quad \Gamma_2, \psi \Rightarrow \Delta_2}{\Gamma_1, \Gamma_2, \varphi \vee \psi \Rightarrow \Delta_1, \Delta_2}\ \text{L}\vee \qquad \frac{\Gamma \Rightarrow \varphi_i, \Delta}{\Gamma \Rightarrow \varphi_1 \vee \varphi_2, \Delta}\ \text{R}\vee_i\ (i = 1, 2)$$

$$\frac{\Gamma_1 \Rightarrow \varphi, \Delta_1 \quad \Gamma_2, \psi \Rightarrow \Delta_2}{\Gamma_1, \Gamma_2, \varphi \supset \psi \Rightarrow \Delta_1, \Delta_2}\ \mathrm{L}{\supset} \qquad \frac{\Gamma, \varphi \Rightarrow \psi, \Delta}{\Gamma \Rightarrow \varphi \supset \psi, \Delta}\ \mathrm{R}{\supset}$$

$$\frac{\Gamma_1, \varphi, \psi \Rightarrow \Delta_1 \quad \Gamma_2 \Rightarrow \varphi, \psi, \Delta_2}{\Gamma_1, \Gamma_2, \varphi \equiv \psi \Rightarrow \Delta_1, \Delta_2}\ \mathrm{L}{\equiv} \quad \frac{\Gamma_1, \varphi \Rightarrow \psi, \Delta_1 \quad \Gamma_2, \psi \Rightarrow \varphi, \Delta_2}{\Gamma_1, \Gamma_2 \Rightarrow \varphi \equiv \psi, \Delta_1, \Delta_2}\ \mathrm{R}{\equiv}$$

Logical rules for the modalities

$$\frac{\Theta^k, \Gamma \Rightarrow \Delta^k, \varphi}{\Theta^k, \Gamma^{\mathsf{B}_k} \Rightarrow \Delta^k, \mathsf{B}_k\varphi}\ \mathrm{LRB}_k \qquad \text{provided } \Delta^k, \varphi \text{ is non-empty}$$

$$\frac{\Theta^k, \Gamma \Rightarrow \Delta^k, \varphi}{\Theta^k, \Gamma^{\mathsf{C}_k} \Rightarrow \Delta^k, \mathsf{C}_k\varphi}\ \mathrm{LRC}_k \qquad \text{provided } \Delta^k, \varphi \text{ is non-empty}$$

$$\frac{}{\mathsf{B}_k\varphi, \mathsf{C}_k\psi \Rightarrow}\ \textsc{Conv} \qquad \text{provided } \{\neg\varphi, \neg\psi\} \text{ is } \mathrm{L}_I\text{-consistent}$$

The φ may be absent in both LRB_k and LRC_k. In Conv φ and ψ are both free of modality k.

Figure 1. Sequent calculus rules for L_I.

2.2 Sequent calculus

A *sequent* is an expression of the form $\Gamma \Rightarrow \Delta$, where the *antecedent* Γ and the *succedent* Δ are finite multisets of formulae, i.e. sets which may contain several occurrences of the same formula. We shall use usual conventions like Γ, φ for $\Gamma \cup \{\varphi\}$. Intuitively a sequent $\Gamma \Rightarrow \Delta$ corresponds to the formula $(\wedge\Gamma) \supset (\vee\Delta)$, which we in the following shall denote $\Gamma \rightarrow \Delta$; $(\wedge\Gamma)$ is $\top$ when $\Gamma = \emptyset$ and $(\vee\Delta)$ is $\bot$ when $\Delta = \emptyset$.

The rules of the sequent calculus for L_I are given in Figure 1. In the definition of the rules Γ^{B_k} is the set $\{\mathsf{B}_k\varphi \mid \varphi \in \Gamma\}$, Γ^{C_k} is defined correspondingly. When a set Γ is superscripted with k, like Γ^k, it contains only modal atoms of modality k.

Proofs are trees regulated by the rules whose leaves are axioms. Note that an axiom is a zero-premiss rule. Two inferences are *successive* if the conclusion of the first is a premiss of the next. In derivation figures a series of 0 or more successive structural inferences is indicated with a double line. The *height* $h(\pi)$ of a proof π is defined as the number of successive inferences (including structural inferences) on the longest branch in π. If π is an axiom, $h(\pi) = 1$.

THEOREM 5 (Correctness). $\Gamma \Rightarrow \Delta$ *is provable in the sequent calculus iff* $\vdash \Gamma \rightarrow \Delta$.

Proof. "If" is proved by routine induction on the axiomatic derivation. The axioms are provable in the sequent calculus; in particular is $\Rightarrow \Diamond_k\varphi$ provable if $\Diamond_k\varphi$ is an instance of the $\Diamond$-axiom. RN is replaced by applications of LRB_k and LRC_k, while MP is replaced by Cut. The other direction is proved by induction on a proof π of $\Gamma \Rightarrow \Delta$. All the rules for the Boolean connectives are standard. We consider the three modality rules.

Case 1: the last inference in π is

$$\frac{\Theta^k, \Gamma \Rightarrow \Delta^k, \varphi}{\Theta^k, \Gamma^{\mathsf{B}_k} \Rightarrow \Delta^k, \mathsf{B}_k\varphi}\ \mathrm{LRB}_k$$

The rule can be used with φ absent, provided that Δ is non-empty. By induction hypothesis, $\vdash \Theta^k, \Gamma \rightarrow \Delta^k, \varphi$. By RN and modal logic, $\vdash \mathsf{B}_k(\Theta^k, \Gamma \rightarrow \Delta^k, \varphi)$. Let $\Theta^k = \{\theta_1, \ldots, \theta_m\}$ and $\Delta^k = \{\delta_1, \ldots, \delta_n\}$. There are two cases, depending on whether φ is present in the succedent of the premiss or not. Assume first that φ is present. It follows from Lemma 1 and standard modal logic that (i) $\vdash \mathsf{B}_k(\theta_1 \wedge \cdots \wedge \theta_m) \equiv (\theta_1 \wedge \cdots \wedge \theta_m) \vee \mathsf{B}_k\bot$, (ii) $\vdash \mathsf{B}_k(\delta_1 \vee \cdots \vee \delta_n \vee \varphi) \equiv (\delta_1 \vee \cdots \vee \delta_n \vee \mathsf{B}_k\varphi)$. By (i), (ii) and principles of normal modal logics it follows that $\vdash (\theta_1 \wedge \cdots \wedge \theta_m) \vee \mathsf{B}_k\bot, \Gamma^{\mathsf{B}_k} \rightarrow \Delta, \mathsf{B}_k\varphi$, from which we conclude that $\vdash \Theta^k, \Gamma^{\mathsf{B}_k} \rightarrow \Delta^k, \mathsf{B}_k\varphi$. If φ is absent, we can reason as if φ is $\bot$. We then get $\vdash (\theta_1 \wedge \cdots \wedge \theta_m) \vee \mathsf{B}_k\bot, \Gamma^{\mathsf{B}_k} \rightarrow \Delta^k, \mathsf{B}_k\bot$. Since Δ^k is non-empty, $\vdash \mathsf{B}_k\bot \rightarrow \Delta^k$. Hence $\vdash \Theta^k, \Gamma^{\mathsf{B}_k} \rightarrow \Delta^k$.

Case 2: the last inference in π is LRC_k. Same as case 1.

Case 3: the last inference in π is Conv. Apply Lemma 4. ■

2.3 Cut-elimination theorems

The proof of the Cut-elimination theorem is inspired by Girard [3] and Shvarts [10]. As usual for such arguments, the proof breaks down into a list of cases, each of which is simple to handle. In our proof of the cut-elimination theorems we follow Gentzen's construction in the original paper [2] and replace the CUT rule by the MIX:

$$\frac{\Gamma_1 \Rightarrow (\varphi)^m, \Delta_1 \quad \Gamma_2, (\varphi)^n \Rightarrow \Delta_2}{\Gamma_1, \Gamma_2 \Rightarrow \Delta_1, \Delta_2} \text{ MIX}$$

where $\varphi \notin \Delta_1$, $\varphi \notin \Gamma_2$, and $(\varphi)^m$ denotes m occurrences of φ. φ is called the *mix formula* of the inference. It is easy to show that a CUT can be accomplished by a MIX and structural inferences, and *vice versa*.

DEFINITION 6. The *degree* $d(\chi)$ of a formula χ is inductively defined by

(i) $d(p) = 1$, p a propositional letter,

(ii) $d(\neg\varphi) = d(\mathsf{B}_k\varphi) = d(\mathsf{C}_k\varphi) = d(\varphi) + 1$,

(iii) $d(\varphi \circ \psi) = \max\{d(\varphi), d(\psi)\} + 1$; $\circ$ a binary connective.

The degree of a MIX (CUT) inference is the degree of its mix (cut) formula. The degree $d(\pi)$ of a proof π is the maximum $d(\varphi)$ such that φ is the mix formula of a MIX inference in π *and such that the* φ *in the right premiss of the* MIX *is not introduced by a* CONV *inference.* Note that a mix formula introduced by a CONV inference in the right premiss of a MIX does not contribute to the degree of a proof.

LEMMA 7 (Principal lemma). *Let* $\Sigma - \varphi$ *be* Σ *with every occurrence of* φ *removed. Assume that* π_1 *is a proof of* $\Sigma_1 \Rightarrow \Phi_1$ *and* π_2 *is a proof of* $\Sigma_2 \Rightarrow \Phi_2$ *such that* $d(\pi_1) < d(\varphi)$ *and* $d(\pi_2) < d(\varphi)$. *Then there is a proof* π *of* $\Sigma_1, \Sigma_2 - \varphi \Rightarrow \Phi_1 - \varphi, \Phi_2$ *such that* $d(\pi) < d(\varphi)$.

Proof. Note first that if $\varphi \notin \Phi_1$, $\Phi_1 = \Phi_1 - \varphi$; π can be constructed using thinnings:

$$\frac{\begin{array}{c}\pi_1 \vdots \\ \Sigma_1 \Rightarrow \Phi_1 - \varphi\end{array}}{\Sigma_1, \Sigma_2 - \varphi \Rightarrow \Phi_1 - \varphi, \Phi_2}$$

In particular this is the case when π_1 is an identity axiom $\psi \Rightarrow \psi$ where $\psi \neq \varphi$. In the same way we use thinnings when $\varphi \notin \Sigma_2$. We assume in the following that $\varphi \in \Phi_1$ and $\varphi \in \Sigma_2$.

Assume that the last rule r_1 of π_1 has premisses $\Sigma'_i \Rightarrow \Phi'_i$ proved by π'_i and that the last rule r_2 of π_2 has premisses $\Sigma''_j \Rightarrow \Phi''_j$ proved by π''_j. The lemma is proved by induction on $h(\pi_1) + h(\pi_2)$. There are several cases to consider, but the only interesting one is really the case in which r_1 and r_2 are logical inferences such that φ is the principal formula of r_1 in the succedent of the conclusion and φ is the principal formula of r_2 in the antecedent of the conclusion. We consider the subcase where $\varphi = \mathsf{B}_k\psi$; the case when φ is $\mathsf{C}_k\psi$ is symmetrical, and all the other cases are standard. π_1 must be as follows:

$$\dfrac{\overset{\pi'_1 \vdots}{\Theta_1^k, \Gamma_1 \Rightarrow \Delta_1^k, (\mathsf{B}_k\psi)^m, \psi}}{\Theta_1^k, \Gamma_1^{\mathsf{B}_k} \Rightarrow \Delta_1^k, (\mathsf{B}_k\psi)^m, \mathsf{B}_k\psi}\ r_1$$

Case 1: r_2 is LRB$_k$. π_2 is then

$$\dfrac{\overset{\pi''_1 \vdots}{\Theta_2^k, \Gamma_2, (\psi)^j, (\mathsf{B}_k\psi)^n \Rightarrow \Delta_2^k, \chi}}{\Theta_2^k, \Gamma_2^{\mathsf{B}_k}, (\mathsf{B}_k\psi)^j, (\mathsf{B}_k\psi)^n \Rightarrow \Delta_2^k, \mathsf{B}_k\chi}\ r_2$$

Apply the induction hypothesis to π'_1 and π_2; call the resulting proof π'. Then, apply the induction hypothesis to π''_1 and π_1; this gives a proof π''. The two proofs are then merged with a MIX of degree less than $d(\varphi)$ (the MIX is the uppermost inference):

$$\dfrac{\dfrac{\dfrac{\overset{\pi' \vdots}{\Theta_1^k, \Gamma_1, \Theta_2^k, \Gamma_2^{\mathsf{B}_k} \Rightarrow \Delta_1^k, \Delta_2^k, \mathsf{B}_k\chi, \psi} \qquad \overset{\pi'' \vdots}{\Theta_1^k, \Gamma_1^{\mathsf{B}_k}, \Theta_2^k, \Gamma_2, (\psi)^j \Rightarrow \Delta_1^k, \Delta_2^k, \chi}}{\Theta_1^k, \Theta_1^k, \Theta_2^k, \Theta_2^k, \Gamma_1, \Gamma_2, \Gamma_1^{\mathsf{B}_k}, \Gamma_2^{\mathsf{B}_k} \Rightarrow \Delta_1^k, \Delta_1^k, \Delta_2^k, \Delta_2^k, \mathsf{B}_k\chi, \chi}}{\Theta_1^k, \Theta_2^k, \Gamma_1, \Gamma_2, \Gamma_1^{\mathsf{B}_k}, \Gamma_2^{\mathsf{B}_k} \Rightarrow \Delta_1^k, \Delta_2^k, \mathsf{B}_k\chi, \chi}}{\dfrac{\Theta_1^k, \Theta_2^k, \Gamma_1^{\mathsf{B}_k}, \Gamma_1^{\mathsf{B}_k}, \Gamma_2^{\mathsf{B}_k}, \Gamma_2^{\mathsf{B}_k} \Rightarrow \Delta_1^k, \Delta_2^k, \mathsf{B}_k\chi, \mathsf{B}_k\chi}{\Theta_1^k, \Theta_2^k, \Gamma_1^{\mathsf{B}_k}, \Gamma_2^{\mathsf{B}_k} \Rightarrow \Delta_1^k, \Delta_2^k, \mathsf{B}_k\chi}}\ \mathrm{LRB}_k$$

Case 2: π_2 is

$$\cfrac{}{\mathsf{B}_k\psi, \mathsf{C}_k\chi \Rightarrow}\ \textsc{Conv}$$

The induction hypothesis gives a proof π' of $\Theta_1^k, \mathsf{C}_k\chi, \Gamma_1 \Rightarrow \Delta_1^k, \psi$. π is then

$$\cfrac{\cfrac{\cfrac{\cfrac{\pi' \vdots}{\Theta_1^k, \mathsf{C}_k\chi, \Gamma_1 \Rightarrow \Delta_1^k, \psi}}{\Theta_1^k, \mathsf{C}_k\chi, \Gamma_1^{\mathsf{B}_k} \Rightarrow \Delta_1^k, \mathsf{B}_k\psi} \quad \cfrac{}{\mathsf{B}_k\psi, \mathsf{C}_k\chi \Rightarrow}\ \textsc{Conv}}{\Theta_1^k, \mathsf{C}_k\chi, \mathsf{C}_k\chi, \Gamma_1^{\mathsf{B}_k} \Rightarrow \Delta_1^k}\ \textsc{Cut}}{\Theta_1^k, \mathsf{C}_k\chi, \Gamma_1^{\mathsf{B}_k} \Rightarrow \Delta_1^k}$$

At this point a Cut is introduced in the proof. By definition of degree, this Cut inference does not increase the degree of the proof. ■

THEOREM 8 (Basic Cut-elimination). *Let $\Gamma \Rightarrow \Delta$ be L_I-provable. Then there is a proof of $\Gamma \Rightarrow \Delta$ which contains a* Cut *inference only if the cut formula in the right premiss is introduced by* Conv.

Proof. First we can prove the following result by induction on π (the induction hypothesis uses the Principal lemma 7): Let π prove $\Gamma \Rightarrow \Delta$, and assume $d(\pi) > 0$. Then there is a proof π' of $\Gamma \Rightarrow \Delta$ such that $d(\pi') < d(\pi)$. By repeated application of this result we can transform any proof into a proof with the same end-sequent, and with degree 0. ■

COROLLARY 9. *For each formula φ there is a first-order formula ψ such that the sequent $\Rightarrow \varphi \equiv \psi$ has a cut-free proof.*

Proof. We know from the proof of Lemma 2 that the equivalence can be proved without reference to the $\Diamond$-axiom. There is thus a sequent calculus proof of it which does not use Conv. Hence, if the proof contains instances of Mix, they can all be eliminated. ■

LEMMA 10. *Let L be a logic containing classical logic. Assume that $\{\neg\varphi, \neg\psi\}$ is L-consistent and that $\theta_1, \ldots, \theta_n \Rightarrow \varphi$ is provable. Then there is a θ_i, $1 \leq i \leq n$, such that $\{\neg\theta_i, \neg\psi\}$ is L-consistent.*

Proof. Assume the contrary, i.e. that $\neg\psi \Rightarrow \theta_j$ is provable for each $j \leq n$. Since we have a proof of $\theta_1, \ldots, \theta_n \Rightarrow \varphi$, n applications of CUT give a proof of $\neg\psi \Rightarrow \varphi$, contradicting the consistency of $\{\neg\varphi, \neg\psi\}$. ∎

THEOREM 11 (Cut-elimination for normal form input). *Let* $\Gamma \Rightarrow \Delta$ *be* L_I*-provable and assume that all formulae in* Γ, Δ *are first-order. Then the sequent has a cut-free proof.*

Proof. By Theorem 8 there is a proof of $\Gamma \Rightarrow \Delta$ with degree 0. This means that any right CUT formula is introduced by CONV. Consider the last inference in the proof π.

$$\dfrac{\begin{array}{c}\pi'\vdots\\ \Theta \Rightarrow \Psi, \mathsf{B}_k\varphi\end{array} \qquad \overline{\mathsf{B}_k\varphi, \mathsf{C}_k\psi \Rightarrow}}{\Theta, \mathsf{C}_k\psi \Rightarrow \Psi}\ \text{CUT}$$

We may without loss of generality assume that Θ and Ψ contain only first-order modal atoms of modality k. The idea is to replace π' by another proof π'' and derive $\Theta, \mathsf{C}_k\psi \Rightarrow \Psi$ by thinnings from the end-sequent of π''. There are two cases. If the succedent occurrence of $\mathsf{B}_k\varphi$ in the end-sequent of π' is not traceable to an axiom (i.e. it is never used to close a branch), we can simply remove it from all sequents in π' in which it occurs. This will give us a proof π'' of $\Theta \Rightarrow \Psi$, from which $\Theta, \mathsf{C}_k\psi \Rightarrow \Psi$ follows by thinning. Otherwise, we must use LRB_k on $\mathsf{B}_k\varphi$. Assume for simplicity that this is the lowermost logical inference in π'; the argument for the general case is essentially the same. By this assumption the premiss of the inference must be of the form $\Gamma^k, \theta_1, \ldots, \theta_n \Rightarrow \Delta^k, \varphi$, where $\mathsf{B}_k\theta_1, \ldots, \mathsf{B}_k\theta_n$ are in Θ. Since $\theta_1, \ldots, \theta_n, \varphi$ are all first-order they will be lost in a further inverse application of a LR modality rule. Hence $\theta_1, \ldots, \theta_n \Rightarrow \varphi$ must be provable. Moreover, by definition of the CONV rule $\{\neg\varphi, \neg\psi\}$ is L_I-consistent and free of modality k. By Lemma 10 there is a θ_j such that $\{\neg\theta_j, \neg\psi\}$ is L_I-consistent. Thus the following is an axiom:

$$\overline{\mathsf{B}_k\theta_j, \mathsf{C}_k\psi \Rightarrow}\ \text{CONV}$$

The axiom now serves the function as π''. Since $\mathsf{B}_k\theta_j \in \Theta$, we see that $\Theta, \mathsf{C}_k\psi \Rightarrow \Psi$ follows from $\mathsf{B}_k\theta_j, \mathsf{C}_k\psi \Rightarrow$ by thinnings. By repeating this procedure, we can remove every instance of CUT. ■

2.4 Consistency results for $\mathbf{L}_I$

L_I is a *consistent* logic if there are formulae in the language which are not theorems of L_I; equivalently, if $\perp$ is not derivable. We shall now see why the cut-elimination results entail that L_I is consistent.

Let us define the *rank* of an instance $\Diamond_k\varphi$ of the $\Diamond$-axiom as the modal depth of φ. The rank of an instance of the CONV-rule with conclusion $\mathsf{B}_k\varphi, \mathsf{C}_k\psi \Rightarrow$ is the maximum of $m(\varphi)$ and $m(\psi)$.

LEMMA 12. *Let φ be L_I-provable. Then there is an L_I-proof π of φ such that the rank of every $\Diamond$-axiom in π is less than $m(\varphi)$.*

Proof. It is immediate from Corollary 9 and Theorem 11 that there is a sequent calculus proof of $\Rightarrow \varphi$ such that the rank of every of its CONV-inferences is less than $m(\varphi)$. By inspecting the proof of Theorem 5, which constructs an axiomatic proof from a sequent calculus proof, we see that each CONV-inference is mapped to a $\Diamond$-axiom with the same rank, and no other instances of the $\Diamond$-axiom are introduced. ■

THEOREM 13. *L_I is consistent.*

Proof. It is clear from Theorem 8 that the core system L'_I is consistent since there are obviously sequents without a cut-free proof in the calculus without the CONV-rule. Since the modal depth of $\perp$ is 0, it is immediate from Lemma 12 that there is no proof in L_I of it either. ■

In fact we can prove a stronger property, namely that the $\Diamond$-axiom is well-defined. To prove this, we must prove that for any instance $\Diamond_k\varphi$ of the $\Diamond$-axiom it is sound to establish the unprovability of $\neg\varphi$ in the side condition of the axiom without any reference to this instance of the $\Diamond$-axiom. However, should this not be the case, there must be a proof of $\neg\varphi$ with an instance of the $\Diamond$-axiom with a rank at least equal to $m(\neg\varphi)$, contradicting Lemma 12. Hence, there can be no circularity in proofs due to the $\Diamond$-axiom.

3 Equivalence with Halpern and Lakemeyer's logic

In an attempt to define a correct multi-modal generalization of the single-agent $\Diamond$-axiom of Levesque, Halpern and Lakemeyer [6] enrich the object language with two dual operators *Val* and *Sat*. Let us call their language $\mathcal{L}_{HL}$; note that $\mathcal{L}_{HL}$ is an extension of $\mathcal{L}_I$.

Recall that the core system L'_I is the logic over $\mathcal{L}_I$ defined in section 2.1 without the $\Diamond$-axiom. The system L_{HL} by Halpern and Lakemeyer is defined over $\mathcal{L}_{HL}$ as the weakest system which is closed under the axioms and rules of L'_I, the axioms

V1 $\mathit{Val}(\varphi \supset \psi) \supset (\mathit{Val}(\varphi) \supset \mathit{Val}(\psi))$

V2 $\mathit{Sat}(\varphi)$, provided φ is a satisfiable purely Boolean formula

V3 $(\mathit{Sat}(\alpha \wedge \beta_1) \wedge \cdots \wedge \mathit{Sat}(\alpha \wedge \beta_m) \wedge$
$\mathit{Sat}(\gamma \wedge \delta_1) \wedge \cdots \wedge \mathit{Sat}(\gamma \wedge \delta_n) \wedge \mathit{Val}(\alpha \vee \gamma)) \supset$
$\mathit{Sat}(\mathsf{B}_k\alpha \wedge \mathsf{b}_k\beta_1 \wedge \cdots \wedge \mathsf{b}_k\beta_m \wedge \mathsf{C}_k\gamma \wedge \mathsf{c}_k\delta_1 \wedge \cdots \wedge \mathsf{c}_k\delta_n)$,
provided α, γ and each β_i, δ_j are free of modality k

V4 $(\mathit{Sat}(\alpha) \wedge \mathit{Sat}(\beta)) \supset \mathit{Sat}(\alpha \wedge \beta)$, if α is free of modality k and β is completely k-modalized

SAT $\mathit{Sat}(\varphi) \supset \Diamond_k\varphi$, provided φ is free of modality k

and the following rule of inference:

$$\frac{\varphi}{\mathit{Val}(\varphi)}\ \mathrm{VAL}$$

The deducibility relation for L_{HL} is denoted $\vdash_{HL}$. By VAL and V1, *Val* is a normal modal operator; *Val* can in fact be shown to be S5. Moreover, *Val* and *Sat* correspond to provability and consistency in the intended way.

PROPOSITION 14. *Let φ be a formula in $\mathcal{L}_{HL}$. If $\vdash_{HL} \varphi$, then $\vdash_{HL}$ $\mathit{Val}(\varphi)$. If $\nvdash_{HL} \varphi$, then $\vdash_{HL} \neg\mathit{Val}(\varphi)$.*

Proof. Consult Proposition 5.1 of [6]. ■

THEOREM 15. *Let φ be a formula in $\mathcal{L}_{HL}$. Then φ is a theorem of L_I if and only if it is a theorem of L_{HL}.*

Proof. We prove both directions of the theorem simultaneously by induction on the modal depth of φ. For the left-to-right direction, assume that φ is L_I-probable. By Lemma 12 there is an L_I-proof π of φ such that the modal depth of every $\Diamond$-axiom in π is less that $m(\varphi)$. By induction on the length of π we can now construct an L_{HL}-proof of φ. The only axiom in L_I which is not in $L_{HL}$is the $\Diamond$-axiom. If φ is purely Boolean, there are no such occurrences and we are done; this situation occurs in the base case of the outer induction. However, if $\Diamond_k\chi$ is an instance of this axiom in π we know that $m(\chi) < m(\varphi)$, that χ is free of modality k and that χ is L_I-consistent. By the induction hypothesis there is an L_{HL}-proof of $Sat(\chi)$. By axiom SAT we conclude that $\Diamond_k\chi$ is L_{HL}-provable. There is hence a proof of φ in L_{HL}.

For the right-to-left direction, it suffices, by Proposition 14, to prove that $\varphi \nvdash \bot$ entails that $\vdash_{HL} Sat(\varphi)$. So assume that φ is L_I-consistent. If φ is purely Boolean, the result follows from axiom V2 and the fact that $\varphi \nvdash \bot$ iff φ is propositionally consistent. In the inductive step we use Lemma 3 to conclude that there is an I-block ψ with $m(\psi) = m(\varphi)$ such that ψ is L_I-consistent and $\psi \supset \varphi$ is provable in the core system L'_I. ψ is the conjunction of a purely Boolean ψ_0 and k-blocks ψ_k for zero or more $k \in I$. Since ψ is L_I-consistent, each k-block ψ_k is L_I-consistent. We show that $\vdash_{HL} Sat(\psi_k)$ for each of these. If $m(\psi_k) < m(\psi)$, this follows by the induction hypothesis. Otherwise, ψ_k is of the form

$$\mathsf{B}_k\alpha \wedge \mathsf{b}_k\beta_1 \wedge \cdots \wedge \mathsf{b}_k\beta_m \wedge \mathsf{C}_k\gamma \wedge \mathsf{c}_k\delta_1 \wedge \cdots \wedge \mathsf{c}_k\delta_n,$$

where $\alpha, \beta_1, \ldots, \beta_m, \gamma, \delta_1, \ldots, \delta_n$ are formulae free of modality k. $\alpha\wedge\beta_i$ is then L_I-consistent for every $i \leq m$. For suppose not. Then $\alpha \vdash \neg\beta_i$, and so $\psi_k \vdash \mathsf{B}_k\neg\beta_i \wedge \mathsf{b}_k\beta_i$, contradicting the consistency of ψ_k. By the same argument, $\gamma \wedge \delta_j$ is consistent for every $j \leq n$. Furthermore, $\vdash \alpha \vee \gamma$. For if not, then, by Lemma 4, $\vdash \neg(\mathsf{B}_k\alpha \wedge \mathsf{C}_k\gamma)$, contradicting the consistency of ψ_k. By the induction hypothesis, $\vdash_{HL} Sat(\alpha \wedge \beta_i)$, $\vdash_{HL} Sat(\gamma \wedge \delta_j)$ and $\vdash_{HL} \alpha \vee \gamma$. By VAL, the last theorem entails that $\vdash_{HL} Val(\alpha \vee \gamma)$. By axiom V3,

$$\vdash_{HL} Sat(\mathsf{B}_k\alpha \wedge \mathsf{b}_k\beta_1 \wedge \cdots \wedge \mathsf{b}_k\beta_m \wedge \mathsf{C}_k\gamma \wedge \mathsf{c}_k\delta_1 \wedge \cdots \wedge \mathsf{c}_k\delta_n)$$

i.e. $\vdash_{HL} Sat(\psi_k)$. Repeated application of axiom V4 gives $\vdash_{HL} Sat(\psi)$.

Since $\psi \supset \varphi$ is provable in L'_I, $\vdash_{HL} \psi \supset \varphi$. Use VAL and V1 to conclude that $\vdash_{HL} Sat(\varphi)$. ■

4 Including a preorder on I

We will in this section address an interesting extension of the multi-modal logic by imposing further structure to the modalities, and show that the main results in this paper still hold.

Assume that there is a preorder $\leq$ (reflexive and transitive, not necessarily anti-symmetric) on I. Let E be the transitive, symmetric closure of $\leq$; if iEk, we say that i and k are *E-equivalent*. The equivalence class of k modulo E is denoted $\overline{k}$. Extending the terminology used previously we say that a modal atom of a modality E-equivalent to k is a *modal $\overline{k}$-atom*; the notions of formulae free of $\overline{k}$-modalities and completely $\overline{k}$-modalized formulae are defined analogously. By a first-order formula we now understand a formula in which the ψ in each subformula of the form $\mathsf{B}_k\psi$ or $\mathsf{C}_k\psi$ is free of $\overline{k}$-modalities, for any $k \in I$.

The idea is to use the equivalence classes of I modulo E to represent agents. The indices in an equivalence class then denote different confidence levels for the given agent. Hence, if k_1 and k_2 are distinct members of I, they denote two different confidence levels of beliefs; moreover, these two confidence levels pertain to the same agent if and only if k_1 and k_2 are E-equivalent.

The logic we address in this section extends L_I in two ways. First, we introduce a modality $\Box_{\overline{k}}$ for each $k \in I$, add the axiom

$$\Box_{\overline{k}}\varphi \equiv \Box_i\varphi \ \text{ if } iEk$$

and change the occurrence of $\Box_k$ to $\Box_{\overline{k}}$ in the definition of RN and the axioms $B_\Box$, CN, $\overline{B}_\Box$, and $\overline{C}_\Box$. Second, we change the $\Diamond$-axiom and add two persistence axioms to the system:

P_B: $\mathsf{B}_i\varphi \supset \mathsf{B}_k\varphi,\ i \leq k$
P_C: $\mathsf{C}_k\varphi \supset \mathsf{C}_i\varphi,\ i \leq k$
$\Diamond$: $\Diamond_{\overline{k}}\varphi$ provided $\varphi \nvdash \bot$, φ free of modality $\overline{k}$

The generalization of the two modality reduction lemmata 1 and 2 are easily seen to hold.

The rules for the extended sequent calculus are as in the previous sections, with the following two modifications. First, each part of the sequent in a modality rule in Figure 1 which is superscripted with a k is now superscripted with $\overline{k}$; for instance is Θ^k in LRB_k now changed to $\Theta^{\overline{k}}$ to indicate that Θ contains only modal $\overline{k}$-atoms. Second, the calculus contains in addition the following rules provided $i \leq k$:

$$\frac{\Theta^{\overline{k}}, \mathsf{B}_k\varphi \Rightarrow \Delta^{\overline{k}}}{\Theta^{\overline{k}}, \mathsf{B}_i\varphi \Rightarrow \Delta^{\overline{k}}}\ \mathrm{PB}_i \qquad\qquad \frac{\Theta^{\overline{k}}, \mathsf{C}_i\varphi \Rightarrow \Delta^{\overline{k}}}{\Theta^{\overline{k}}, \mathsf{C}_k\varphi \Rightarrow \Delta^{\overline{k}}}\ \mathrm{PC}_k$$

Correctness of the calculus is proved by extending the proof of Theorem 5 and is straightforward.

To prove cut-elimination, we must change the definition of the degree of a formula. According to Definition 6, $d(\mathsf{B}_k\varphi) = d(\mathsf{B}_i\varphi)$. This is as it should if k and i are unrelated by $\leq$; if, however, $i < k$, we need to state that the degree of $\mathsf{B}_k\varphi$ is less than the degree of $\mathsf{B}_i\varphi$. Hence we change d to d', where $d'(\chi) = d(\chi)$ if χ is not a modal atom, otherwise $d'(\mathsf{B}_k\varphi) = d(\mathsf{B}_k\varphi) + \epsilon(k)$, $\epsilon(k) < 1$. ϵ is a function satisfying $\epsilon(k) < \epsilon(i)$ iff $i < k$, and can, since $\leq$ is a preorder, easily be defined.

The two new rules cause some new cases to be checked in the proof of the Principal lemma. We consider the new subcase to the case that we have addressed for the other system. Assume r_1 and r_2 are logical inferences such that φ is the principal formula of r_1 in the succedent of the conclusion and φ is the principal formula of r_2 in the antecedent of the conclusion. Let φ be $\mathsf{B}_i\psi$ and π_1 end with an instance r_1 of LRB_i (below to the left).

$$\frac{\begin{array}{c}\pi_1'\vdots\\ \Theta_1^{\overline{k}}, \Gamma_1 \Rightarrow \Delta_1^{\overline{k}}, (\mathsf{B}_i\psi)^m, \psi\end{array}}{\Theta_1^{\overline{k}}, \Gamma_1^{\mathsf{B}_i} \Rightarrow \Delta_1^{\overline{k}}, (\mathsf{B}_i\psi)^m, \mathsf{B}_i\psi}\ r_1 \quad \frac{\begin{array}{c}\pi_1''\vdots\\ \Theta_2^{\overline{k}}, \mathsf{B}_k\psi, (\mathsf{B}_i\psi)^n \Rightarrow \Delta_2^{\overline{k}}\end{array}}{\Theta_2^{\overline{k}}, \mathsf{B}_i\psi, (\mathsf{B}_i\psi)^n \Rightarrow \Delta_2^{\overline{k}}}\ r_2$$

We must check the case that π_2 ends with a PB_i, depicted above to the right. Apply the induction hypothesis to π_1' and π_2, giving π', and to π_1 and π_1'', giving π''. In the proof π below the application of LRB_i in π_1 has been changed to an instance of LRB_k. The instance of PB_i

in π_2 has been changed to many applications of PB_i (indicated by double lines in the proof):

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\overset{\pi'\vdots}{\Theta_1^{\overline{k}}, \Theta_2^{\overline{k}}, \Gamma_1 \Rightarrow \Delta_1^{\overline{k}}, \Delta_2^{\overline{k}}, \psi}}{\Theta_1^{\overline{k}}, \Theta_2^{\overline{k}}, \Gamma_1^{\mathsf{B}_k} \Rightarrow \Delta_1^{\overline{k}}, \Delta_2^{\overline{k}}, \mathsf{B}_k\psi}\,\mathrm{LRB}_k \qquad \overset{\pi''\vdots}{\Theta_1^{\overline{k}}, \Theta_2^{\overline{k}}, \Gamma_1^{\mathsf{B}_i}, \mathsf{B}_k\psi \Rightarrow \Delta_1^{\overline{k}}, \Delta_2^{\overline{k}}}}{\Theta_1^{\overline{k}}, \Theta_1^{\overline{k}}, \Theta_2^{\overline{k}}, \Theta_2^{\overline{k}}, \Gamma_1^{\mathsf{B}_k}, \Gamma_1^{\mathsf{B}_i} \Rightarrow \Delta_1^{\overline{k}}, \Delta_1^{\overline{k}}, \Delta_2^{\overline{k}}, \Delta_2^{\overline{k}}}\,\textsc{Mix}}{\Theta_1^{\overline{k}}, \Theta_2^{\overline{k}}, \Gamma_1^{\mathsf{B}_k}, \Gamma_1^{\mathsf{B}_i} \Rightarrow \Delta_1^{\overline{k}}, \Delta_2^{\overline{k}}}}{\Theta_1^{\overline{k}}, \Theta_2^{\overline{k}}, \Gamma_1^{\mathsf{B}_i}, \Gamma_1^{\mathsf{B}_i} \Rightarrow \Delta_1^{\overline{k}}, \Delta_2^{\overline{k}}}\,\mathrm{PB}_i}{\Theta_1^{\overline{k}}, \Theta_2^{\overline{k}}, \Gamma_1^{\mathsf{B}_i} \Rightarrow \Delta_1^{\overline{k}}, \Delta_2^{\overline{k}}}
$$

π contains a Mix on $\mathsf{B}_k\psi$. Note that $d'(\pi) < d'(\varphi)$, even though $d(\pi) = d(\varphi)$. This is why we must use d' as a measure of degree instead of d in the proof of the Principal lemma. The case in which C_k is principal in r_1 is symmetrical. Consistency of the system follows from the cut-elimination results as for L_I.

THEOREM 16. *In the sequent calculus for the extended version of L_I, a sequent is provable iff it has a proof which contains a* Cut *only if the cut formula in the right premiss is introduced by* Conv. *If every formula in the sequent is first-order the sequent is provable iff it has a cut-free proof.*

5 Application to Gricean implicatures

The system presented in section 4 is capable of representing an interesting class of Gricean implicatures covered by Gazdard in [1]. The nature of implicatures is multi-modal; they result from the audience's reflections about what an utterer meant by an utterance put forth in a particular conversational context. The representation of implicatures must have a mechanism which reflects the priority mechanism that governs Gricean implicatures.

To give an idea of the representation, let us address a simple example. Assume that U has said "p or q" and that the context of utterance is represented by the formula κ; the context typically represents what has been said in the conversation in addition to $p \vee q$. Since the representation of the context is constrained by the language of L_I, contextual information like common knowledge is excluded; this puts limitations to the formalization. If we neglect the priority mechanism we can use L_I (i.e. the system of section 2; let the index set be $\{A, U\}$ for audience and utterer.

There are two kinds of implicatures relevant for this example, both generated on the basis of the Gricean Maxim of Quantity. Scalar implicatures assume that the strength of expressions in some cases can be ordered on a linguistic scale. In particular, conjunction is stronger than disjunction. The idea is that if U says $p \vee q$ and the context does not entail that U meant $p \wedge q$, then U implicates $\neg(p \wedge q)$, i.e. that the disjunction should be taken exclusively. The mechanism behind this pragmatical inference can be formalized by

$$\mathrm{sc}(A) = \neg \mathsf{B}_U \neg (p \wedge q) \supset \mathsf{B}_A \neg \mathsf{B}_U \neg (p \wedge q).$$

The formula has the form of a supernormal default in a multi-modal language and expresses that if $p \wedge q$ it is compatible with the utterer's beliefs, then the audience will know about this. Conversely, if A does not explicitly know that U said $p \wedge q$, then A believes that U does not mean that $p \wedge q$. Using the sequent calculus it is easy to prove that if $\kappa, \mathsf{B}_U(p \vee q) \nvdash \neg \mathsf{B}_U \neg (p \wedge q)$, then $\vdash \mathsf{O}_A(\kappa \wedge \mathsf{B}_U(p \vee q) \wedge \mathrm{sc}(A)) \equiv \mathsf{O}_A(\kappa \wedge \mathsf{B}_U(p \vee q) \wedge \mathsf{B}_U \neg (p \wedge q))$, otherwise $\vdash \mathsf{O}_A(\kappa \wedge \mathsf{B}_U(p \vee q) \wedge \mathrm{sc}(A)) \equiv \mathsf{O}_A(\kappa \wedge \mathsf{B}_U(p \vee q))$.

Another type of implicature arises from implications like $\mathsf{B}_U p \supset \mathsf{B}_U(p \vee q)$. If U says $p \vee q$ and the context does not entail that U knows which disjunct is true, then A believes that U does not know this (this is called a clausal implicature). If we let $\iota(\varphi)$ be the formula $\mathsf{B}_U \varphi \supset \mathsf{B}_A \mathsf{B}_U \varphi$, there are four clausal implicatures in this example: $\iota(p) \wedge \iota(\neg p) \wedge \iota(q) \wedge \iota(\neg q)$. Let us call this formula $\mathrm{cl}(A)$. We can now, e.g., prove that if $\kappa, \mathsf{B}_U(p \vee q), \neg \mathsf{B}_U p, \neg \mathsf{B}_U \neg p, \neg \mathsf{B}_U q, \neg \mathsf{B}_U \neg q \nvdash \bot$, then $\vdash \mathsf{O}_A(\kappa \wedge \mathsf{B}_U(p \vee q) \wedge \mathrm{cl}(A)) \equiv \mathsf{O}_A(\kappa \wedge \mathsf{B}_U(p \vee q) \wedge \neg \mathsf{B}_U p \wedge \neg \mathsf{B}_U \neg p \wedge \neg \mathsf{B}_U q \wedge \neg \mathsf{B}_U \neg q)$, as expected.

In the general case there may, however, be a conflict between clausal and scalar implicatures, in which case the clausal ones should be given priority [1]. In the modal language we can capture this by having two modalities for the audience, representing two confidence levels: one for the beliefs after clausal implicatures have been accounted for and one for beliefs after the scalar implicatures have been processed. We can implement this in the language introduced in section 4 by taking $I = \{A_1, A_2, U\}$ and have only one preference constraint, namely $A_1 < A_2$. The representation $\mathbf{O}_{A_1}(\kappa \wedge \mathsf{B}_U(p \vee q) \wedge \mathrm{cl}(A_1)) \wedge \mathbf{O}_{A_2}(\kappa \wedge \mathsf{B}_U(p \vee q) \wedge \mathrm{cl}(A_1) \wedge \mathrm{sc}(A_2))$ will then serve as a formalization with the desired properties. Details of this encoding will be closer addressed in a subsequent paper.

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