Filtration via Bisimulation

VALENTIN SHEHTMAN

ABSTRACT. We develop a new version of the well-known filtration method in modal logic, allowing us to construct large countermodels and to solve some open problems on the finite model property for products of modal logics. This filtration is based on the bisimilarity relation between parts of the original model; it generalizes earlier versions of the filtration method introduced by E. Lemmon, K. Segerberg, D. Gabbay, and the author.

1 Introduction

The filtration method is the oldest and the most well-known method of finite model property proofs in modal logic. First developed in the 1960s by S. Kripke, E. Lemmon, K. Segerberg, and D. Gabbay, it was afterwards modified and successfully applied to different types of nonclassical logics.

However, in the field of many-dimensional modal logic, the traditional filtration method is not very popular in decidability proofs. Indeed, decidable many-dimensional logics may be of high complexity, but standard filtrations yield rather moderate upper complexity bounds. Other methods seem to be more successful here, like the method of quasimodels (or mosaics), which is widely used in the recent monograph [4].

Still, traditional filtrations seem so simple, that it is worth making them work for complex logics as well. In this paper we propose such a modification allowing us to simplify some earlier proofs and also to prove new fmp results — for example, for the logic $\mathbf{K} \times \mathbf{S4}$ and related ones. So we hope for further applications of this method.

Let us briefly describe the main idea. A filtration constructs a finite Kripke model, which is in some sense equivalent to a given infinite model. There are two main types of filtrations: selective filtrations (Kripke – Gabbay) and "epi-filtrations" (Lemmon – Segerberg). Selective filtrations extracting finite submodels, are not discussed here, details can be found in [3], for a more recent application cf. [10].

An epi-filtration identifies possible worlds by some "faithful" equivalence relation. In the simplest case [8], given a set of formulas Ψ, the equivalence $\equiv_\Psi$ is defined as the truth of the same formulas from Ψ. If Ψ is finite, we readily obtain a finite model, but the validity of the original logic may be lost. To avoid this, we can try to refine the equivalence relation.

The first modification of this kind was proposed in [2]: the new relation is $\equiv_\Phi$ for some $\Phi \supseteq \Psi$. Our modification is based on a more general construction from [9], [5]. We start from a finite set Ψ and a Kripke model $M = (W, R_1, ..., R_n, \theta)$. The worlds of M are identified by equivalence relation $\approx\ \subseteq\ \equiv_\Psi$ with finitely many equivalence classes. An appropriate choice of $\approx$ allows us to preserve the original logic in the filtrated frame.

In [5] this method is applied to the logic $\mathbf{S5} \times \mathbf{K}$. In this case every $R_1(x)$ is a cluster; we put $x \approx y$ iff $x \equiv_\Psi y$ and the same $\equiv_\Psi$-classes are presented in $R_1(x)$ and $R_1(y)$. But such a definition is no good if R_1 is not an equivalence (e.g. for the logic $\mathbf{K} \times \mathbf{K}$), because $x \approx y$ should somehow take the original R_1 into account. So we require that the corresponding generated R_1-submodels are *bisimilar*. To obtain finitely many $\approx$-classes, one should first restrict the depth of all these submodels. This step is crucial for the whole proof and it is not always possible. In fact, the method works quite well for intransitive R_1, but it may fail when all relations in M are transitive. For example, this happens for the logic $\mathbf{K4} \times \mathbf{K4}$, which lacks the fmp at all [7].

2 Basic definitions and facts

First let us recall some well-known material. Our terminology and notation mainly follow [5]. $\boldsymbol{\mathcal{L}}_n$ denotes the set of propositional formulas built from a countable set $PL = \{p_1, p_2, \dots\}$ of proposition letters, classical connectives $\to, \bot$, and modal connectives $\Box_1, \dots, \Box_n$. Let $\boldsymbol{\mathcal{L}}_n\lceil k$ be the set of all formulas in $\boldsymbol{\mathcal{L}}_n$ using only proposition letters from the set $PL\lceil k = \{p_1, p_2, \dots p_k\}$. *Closed formulas* do not contain proposition letters.

As usual, an *n-modal logic* is a set of $\mathcal{L}_n$-formulas containing the classical tautologies, the axiom $\Box(p_1 \to p_2) \to (\Box p_1 \to \Box p_2)$, closed under Substitution, Modus Ponens, and Necessitation $(A/\Box_i A)$, $1 \leq i \leq n$. For a set of $\mathcal{L}_n$-formulas Γ and an n-modal logic $\mathbf{\Lambda}$, the smallest n-modal logic containing

$(\mathbf{\Lambda} \cup \Gamma)$ is denoted by $\mathbf{\Lambda} + \Gamma$. $\mathbf{K}_n$ denotes the minimal n-modal logic, and

$$\mathbf{K}_{\pm n} := \mathbf{K}_{2n} + \{\Diamond_i \Box_{n+i} p \to p,\ \Diamond_{n+i} \Box_i p \to p \mid 1 \leq i \leq n\}$$

is the minimal n-temporal logic[1]; in the latter case $\Box_{n+i}$ is denoted by $\Box_i^{-1}$; similarly for $\Diamond$.

Recall that the *fusion* of two logics, n-modal $\mathbf{L}_1$ and m-modal $\mathbf{L}_2$ is $\mathbf{L}_1 * \mathbf{L}_2 := \mathbf{K}_{n+m} + \mathbf{L}_1 + \mathbf{L}_2^{+n}$, where $\mathbf{L}_2^{+n}$ is obtained from $\mathbf{L}_2$ by replacing every occurrence of any $\Box_j$ with $\Box_{j+n}$.

Kripke semantics is defined in a standard way. An *n-modal (Kripke) frame* is a tuple $F = (W, R_1, \dots, R_n)$, where $W \neq \varnothing$ is a set of possible worlds, $R_i \subseteq W \times W$ are accessibility relations. A *Kripke model* over F is a pair $M = (F, \theta)$, where $\theta : PL \longrightarrow 2^W$ is a *valuation.*

Valuations are extended to all formulas as usual:
$\theta(\bot) = \varnothing,\ \theta(A \to B) = (W - \theta(A)) \cup \theta(B)$,
$\theta(\Box_i A) = \{x \mid R_i(x) \subseteq \theta(A)\}$.

Similarly, a *k-restricted Kripke model* is $M = (F, \theta)$, where $\theta : PL\lceil k \longrightarrow 2^W$; in this case θ is extended to $\boldsymbol{\mathcal{L}}_n\lceil k$. A formula A is called *true at a world* w of M if $w \in \theta(A)$ (in another notation: $M, w \vDash A$). Since the truth value of a formula depends only on proposition letters occurring in it, we may fix k and assume that all Kripke models are k-restricted.

A formula A is *valid in a frame F* (notation: $F \vDash A$) if it is true at every world of every Kripke model over F. A set of formulas Γ is *valid* in F (notation: $F \vDash \Gamma$) if every $A \in \Gamma$ is valid. In the latter case we also say that F is a Γ-*frame.* A logic $\mathbf{\Lambda}$ is *determined* by a class of frames $\mathcal{C}$ if $\mathbf{\Lambda}$ is the set of all formulas valid in all frames from $\mathcal{C}$.

[1] $\mathbf{K}_{\pm 1}$ is the well known logic $\mathbf{K.t}$.

DEFINITION 1. Let $F = (W, R_1, \dots, R_n)$ be a frame, $u, v \in W$, $m \geq 1$. A *path of length* m from u to v is a sequence $(u_0, j_0, u_1, \dots, j_{m-1}, u_m)$ such that $u = u_0,\ v = u_m$ and for all $i < m$, $u_i R_{j_i} u_{i+1}$. A singleton sequence (u) is the *path of length 0* (from u to u). Recall that the *subframe of* F *generated by* u (notation: F^u) is the restriction of F to the set of all v such that there exists a path from u to v; similarly a *generated Kripke submodel* M^u is defined.

DEFINITION 2. A *tree* with root u is a frame F such that $F = F^u$ and for every $v \in F$ there exists a unique path from u to v. The length of this path is called the *height* of v and denoted by $h(v)$. The height of F ($h(F)$) is the maximal $h(v)$ (if it exists), or ∞ otherwise.

DEFINITION 3. For a $2n$-modal tree $G = (W, S_1, \dots, S_{2n})$, the frame $F = (W, R_1, \dots, R_n, R_1^{-1}, \dots, R_n^{-1})$, where $R_i = S_i \cup S_{n+i}^{-1}$, is called the *$n$-temporal tree (with the pattern G)*. The height function in F is then defined as the height function in G.

Speaking informally, a temporal tree is a modal tree, in which some of the arrows are inverted.

DEFINITION 4. A 1-tree $(W, \sqsubset)$ is called *standard* if its worlds are (some) finite sequences of natural numbers, its root is the void sequence λ and $\alpha \sqsubset \beta$ iff β is obtained by adding a single element at the end of α. An n-tree $(W, R_1, \dots, R_n)$ is called *standard* if the 1-tree $(W, R_1 \cup \dots \cup R_n)$ is standard. An n-temporal tree is called *standard* if its pattern is standard.

DEFINITION 5. Let $M = (W, R_1, \dots, R_n, \theta)$ be an n-modal Kripke model, Ψ a set of n-modal formulas closed under subformulas. For $x \in W$ let $\Psi(x) := \{A \in \Psi \mid M, x \vDash A\}$. Two worlds $x, y \in W$ are called Ψ-*equivalent in* M (notation: $(M, x) \equiv_\Psi (M, y)$, or just $x \equiv_\Psi y$) if $\Psi(x) = \Psi(y)$.

DEFINITION 6. (cf. [5]) Under the assumptions of Definition 5, let $\approx$ be an equivalence relation on W. Let $x^\sim$ denote the $\approx$-class of x. A Kripke model $M' = (W', R'_1, ..., R'_n, \theta')$ is called a *filtration of* M *through* $\Psi, \approx$ if for any $x, y \in W$, for any formula A, $1 \leq i \leq n$:

(f1) $\approx \subseteq \equiv_\Psi$;

(f2) $W' = W/\approx$;

(f3) $xR_iy \quad\Longrightarrow\quad x^{\sim}R'_iy^{\sim}$;

(f4) $x^{\sim}R'_iy^{\sim}$ & $M,x \vDash \Box_iA$ & $\Box_iA \in \Psi \Longrightarrow M,y \vDash A$;

(f5) if $q \in \Psi \cap PL$, then $M,x \vDash q \Longleftrightarrow M',x^{\sim} \vDash q$.[2]

LEMMA 7. *(Filtration Lemma). Let M' be a filtration of M through $\Psi, \approx$. Then for any $x \in W$, for any $A \in \Psi$*
$M,x \vDash A$ iff $M',x^{\sim} \vDash A$.

Proof. Standard, by induction on the length of A, cf. [5], [9]. In the case $A = \Box_iB$ use (f3) for 'if' and (f4) for 'only if'. ■

LEMMA 8. *Let $M = (W, R_1, \ldots, R_n, \theta)$, Ψ be the same as in Definition 5, $\approx$ an equivalence relation on W such that $\approx \subseteq \equiv_\Psi$; and let $W' = W/\approx$. Then the model $\underline{M} = (W', \underline{R}_1, \ldots, \underline{R}_n, \theta')$ such that*

- *for any $x,y \in W$, $x^{\sim}\underline{R}_iy^{\sim}$ iff $\exists x_1 \in x^{\sim}\exists y_1 \in y^{\sim}\ x_1R_iy_1$;*
- *for any $q \in \Psi \cap PL$, $\quad \theta'(q) = \{x^{\sim} \mid M,x \vDash q\}$*

*is a filtration of M through $\Psi, \approx$ (*the least filtration*).*

Proof. Also standard; cf. [5], [9]. (f5) follows from the definition of θ'. To check (f4), assume $x \vDash \Box_iA$, $\Box_iA \in \Psi$, $x^{\sim}R'_iy^{\sim}$. Then $x_1R_iy_1$ for some $x_1 \approx x$, $y_1 \approx y$, and thus $x_1 \vDash \Box_iA$, $y_1 \vDash A$, and hence $y \vDash A$. ■

DEFINITION 9. Let J be a finite set of positive integers. A binary relation R is called *J-quasitransitive* if $R^{j+1} \subseteq R$ for any $j \in J$.

LEMMA 10. *For any binary relation R, the smallest J-quasitransitive relation containing R (the J-*quasitransitive closure*) is $R^+ := \bigcup_{h \in H} R^{h+1}$, where H is the additive closure of $J \cup \{0\}$ in ω.*

Proof. Assume that $R \subseteq S$ and S is J-quasitransitive. Let $H_0 = \{h \mid R^{h+1} \subseteq S\}$, then obviously, $0 \in H_0$. Next, assume $h \in H_0$, $j \in J$.

[2] [5] contains misprints in this item.

Then $(h+j) \in H_0$; in fact, $R^{h+j+1} = R^{h+1} \circ R^j \subseteq S \circ R^j \subseteq S^{j+1} \subseteq S$. Therefore $H \subseteq H_0$, which readily implies $R^+ \subseteq S$.

It remains to show that R^+ is J-quasitransitive. So let us check that $(R^+)^{j+1} \subseteq R^+$ for $j \in J$. Assume $x(R^+)^{j+1}y$; then $x(R^{h_1+1} \circ \dots \circ R^{h_{j+1}+1})y$ for some $h_1, \dots, h_{j+1} \in H$. But this means $xR^{h_1+\dots+h_{j+1}+j+1}y$, while $(h_1 + \dots + h_{j+1} + j) \in H$ (remember that $J \subseteq H$ and H is additive closed). Hence xR^+y. ■

Remark. R^+ is a particular case of Horn closure, cf. [5].

LEMMA 11. *(cf. [5])* [3] *Let M, Ψ be the same as in Definition 5, and suppose that R_i is J_i-quasitransitive for $1 \le i \le n$. Also let*

$$\Phi \supseteq \Psi \cup \{\Box_i^j C \mid 1 \le i \le n,\ J_i \neq \varnothing,\ 1 \le j \le \max(J_i),\ \Box_i C \in \Psi\},$$

be a set of formulas closed under subformulas, and let $\underline{M} := (W', \underline{R}_1, \dots, \underline{R}_n, \theta')$ be the least filtration of M through $\Phi, \approx$. If $R'_i = (\underline{R}_i)^+$ is the J_i-quasitransitive closure of $\underline{R}_i$, then $M' := (W', R'_1, \dots, R'_n, \theta')$, is a filtration of M through $\Psi, \approx$. Moreover, if R_i is reflexive (respectively, symmetric), then R'_i also is.

Proof. Obviously, $\approx \subseteq \equiv_\Phi \subseteq \equiv_\Psi$. Since (f3), (f5) hold for $\underline{M}$, they also hold for M' (note that $\underline{R}_i \subseteq R'_i$). So let us check (f4) for R'_i, provided $J_i \neq \varnothing$. To simplify notation, we drop the subscript i.

Assume $x^\sim R' y^\sim$, $x \vDash \Box A$, $\Box A \in \Psi$. Let H be the additive closure of $J \cup \{0\}$, and let us show that for any $h \in H$

(∗) $x^\sim \underline{R}^h y^\sim$ implies $y \vDash \Box A$.

One can argue by induction on the number r of summands in the representation $h = j_1 + \dots + j_r$, with $j_1, \dots, j_r \in J$, and it suffices to prove that

(∗∗) $y \vDash \Box A$ & $j \in J$ & $y^\sim \underline{R}^j z^\sim$ implies $z \vDash \Box A$.

So assume $y \vDash \Box A$, $y^\sim \underline{R}^j z^\sim$. Then we have $y \approx y_0 R z_1 \approx y_1 R \dots y_{j-1} R z_j \approx z$, and thus $y_0 \vDash \Box A$ (since $\approx \subseteq \equiv_\Psi$ and $\Box A \in \Psi$),

$y_0 \vDash \Box^{j+1} A$ (since R is J-quasitransitive), $z_0 \vDash \Box^j A$, $y_1 \vDash \Box^j A$ (since $\approx \subseteq \equiv_\Phi$, $\Box^j A \in \Phi$), $z_1 \vDash \Box^{j-1} A$, $y_2 \vDash \Box^{j-1} A$ (since $\Box^{j-1} A \in \Phi$), …, $y_{j-1} \vDash \Box^2 A$, $z_j \vDash \Box A$, and finally $z \vDash \Box A$. This proves (**).

[3] But again there are misprints in the definition of Φ in [5], p. 123.

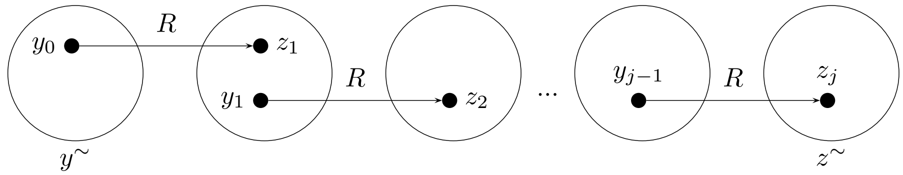

Figure 1.

Now, given $x \vDash \Box A$, $\Box A \in \Psi$, $x^\sim R' y^\sim$, we have $x^\sim \underline{R}^h z^\sim \underline{R} y^\sim$ for some z and some $h \in H$. By (*), we obtain $z \vDash \Box A$, and since (f4) holds for $\underline{R}$, this implies $y \vDash A$.

If R_i is reflexive (respectively, symmetric), then obviously, $\underline{R}_i$ is reflexive (respectively, symmetric), and thus $R'_i = \bigcup\limits_{h \in H} \underline{R}_i^{h+1}$ is reflexive (symmetric) as well. ∎

3 Filtration via bisimulation

DEFINITION 12. A *bisimulation* between Kripke models $M = (W, R_1, ..., R_n, \theta)$, $N = (V, S_1, ..., S_n, \eta)$ is a relation $E \subseteq W \times V$ with the following properties:

- $pr_1(E) = W$;
- $pr_2(E) = V$;
- $E \circ S_i \subseteq R_i \circ E$ for $1 \leq i \leq n$;
- $R_i^{-1} \circ E \subseteq E \circ S_i^{-1}$ for $1 \leq i \leq n$;
- if xEy, then for any $q \in PL\lceil k$, $M, x \vDash q$ iff $N, y \vDash q$.

$E : M, x \asymp N, y$ denotes that E is a bisimulation between M and N such that xEy. Two worlds $x \in M$, $y \in N$ are called *bisimilar* (notation: $M, x \asymp N, y$) if there exists a bisimulation $E : M, x \asymp N, y$.

If a bisimulation E is a function, it is a *p-morphism* from M onto N. In this case the condition $R_i^{-1} \circ E \subseteq E \circ S_i^{-1}$ is the *monotonicity*: $xR_iy \Longrightarrow E(x)S_iE(y)$, and $E \circ S_i \subseteq R_i \circ E$ is the *lift property*: $E(x)S_iz \Longrightarrow \exists y\ (xR_iy\ \&\ E(y) = z)$.

It follows easily from the definition that bisimilarity is an equivalence relation. The following Bisimulation Lemma is well-known:

LEMMA 13. *Bisimulations preserve truth values of formulas, i.e.,*

$M, x \asymp N, y$ *implies* $M, x \vDash A$ *iff* $N, y \vDash A$ *for any formula* $A \in \mathcal{L}_n \lceil k$.

Proof. Cf. [1], Theorem 2.20. ■

DEFINITION 14. Let $M = (W, R_1, \ldots, R_n, S_1, \ldots, S_m, \theta)$ be a Kripke model, Ψ a set of formulas closed under subformulas. For $x \in W$ the model $x\!\uparrow := (W, R_1, \ldots, R_n, R_1^{-1}, \ldots, R_n^{-1}, \theta)^x$ (cf. Definition 1) is called the *n-trace* of x. Put $x \approx y$ iff there exists $E : x\!\uparrow, x \asymp y\!\uparrow, y$ such that $E \subseteq \equiv_\Psi$ (*bisimilarity modulo* Ψ *with respect to* $R_1, \ldots, R_n$).

Note that $x\!\uparrow$ is $2n$-modal; the use of R_i^{-1} is essential for Lemma 15 below. It follows that $\approx$ is an equivalence relation. In fact, reflexivity and symmetry are obvious. For transitivity, note that $E_1 : x\!\uparrow, x \asymp y\!\uparrow, y$ and $E_2 : y\!\uparrow, y \asymp z\!\uparrow, z$ imply $(E_1 \circ E_2) : x\!\uparrow, x \asymp z\!\uparrow, z$; this is checked in a straightforward way: $E_1 \circ E_2 \circ R_i = E_1 \circ R_i \circ E_2 = R_i \circ E_1 \circ E_2$, and the same for R_i^{-1}; we also have $E_1 \circ E_2 \subseteq (\equiv_\Psi) \circ (\equiv_\Psi) = (\equiv_\Psi)$. By definition, $\approx \subseteq \equiv_\Psi$, and thus we can consider filtrations based on $\approx$.

LEMMA 15. *Under the conditions of Definition 14, for any* $i \leq n$, $\approx \circ R_i = R_i \circ \approx$.

Proof. First let us show that $\approx \circ R_i \subseteq R_i \circ \approx$. Assume $x \approx y R_i z$. Then x, y are equivalent modulo Ψ and there exists $E : x\!\uparrow, x \asymp y\!\uparrow, y$ such that $E \subseteq \equiv_\Psi$. So $x(E \circ R_i)z$, and thus $x(R_i \circ E)z$, i.e., $x R_i u E z$ for some u. Since $u\!\uparrow = x\!\uparrow$ and $y\!\uparrow = z\!\uparrow$, the same E yields $u \approx z$. In a similar way it follows that $\approx \circ R_i^{-1} \subseteq R_i^{-1} \circ \approx$, or $R_i \circ \approx \subseteq \approx \circ R_i$. ■

Hence we obtain filtrations preserving some commutation properties.

LEMMA 16. *Let* $\approx$ *be the bisimilarity modulo* Ψ *as in Definition 14. Then for the least filtration of* M *through* $\Psi, \approx$ *we have*

(1) $S_j \circ R_i \subseteq R_i \circ S_j \Longrightarrow \underline{S}_j \circ \underline{R}_i \subseteq \underline{R}_i \circ \underline{S}_j$;

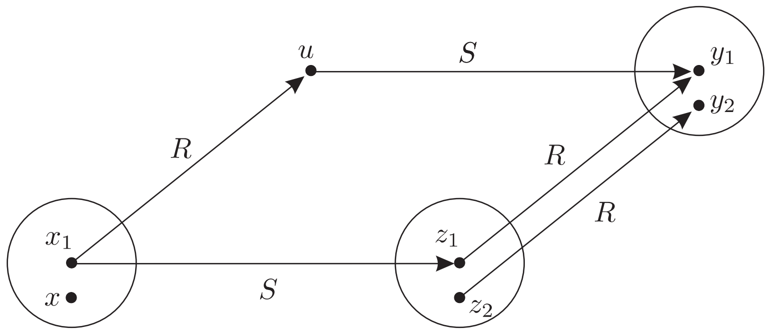

Figure 2.

(2) $R_i \circ S_j \subseteq S_j \circ R_i \Longrightarrow \underline{R}_i \circ \underline{S}_j \subseteq \underline{S}_j \circ \underline{R}_i$;

(3) $R_i^{-1} \circ S_j \subseteq S_j \circ R_i^{-1} \Longrightarrow \underline{R}_i^{-1} \circ \underline{S}_j \subseteq \underline{S}_j \circ \underline{R}_i^{-1}$.

Proof. We prove only (1); the proofs of (2),(3) are similar. To simplify the notation, we drop the subscripts i, j. Assume $S \circ R \subseteq R \circ S$, $x^{\sim}(\underline{S} \circ \underline{R})y^{\sim}$. Then for some z, $x^{\sim}\underline{S}z^{\sim}\underline{R}y^{\sim}$, and hence by definition in Lemma 8, we obtain $x_1 \approx x$, $z_1 \approx z_2 \approx z$, $y_2 \approx y$ such that $x_1 S z_1$, $z_2 R y_2$. So $z_1(\approx \circ R)y_2$, and thus by Lemma 15, there exists y_1 such that $z_1 R y_1$, $y_1 \approx y_2 \approx y$.

Next, $S \circ R \subseteq R \circ S$ implies $x_1(R \circ S)y_1$. So there exists u such that $x_1 R u S y_1$, and thus $x^{\sim}\underline{S}u^{\sim}\underline{R}y^{\sim}$. ■

LEMMA 17. *Let M, Ψ be the same as in Definition 14. Assume that R_i is J_i-quasitransitive, $1 \leq i \leq n$. Construct the set Φ as in Lemma 11. Let $\approx$ be the bisimilarity modulo Φ, and according to Lemma 11, consider the filtration $M' = (W', R'_1, \ldots, R'_n, S'_1, \ldots, S'_m)$ of M through $\Psi, \approx$ such that*

$R'_i = \underline{R}_i^+$, the J_i-quasitransitive closure of $\underline{R}_i$, and $S'_i = \underline{S}_i$.

Then the following holds:

(1) $S_k \circ R_i \subseteq R_i \circ S_k \Longrightarrow S'_k \circ R'_i \subseteq R'_i \circ S'_k$;

(2) $R_i \circ S_k \subseteq S_k \circ R_i \Longrightarrow R'_i \circ S'_k \subseteq S'_k \circ R'_i$;

(3) $R_i^{-1} \circ S_k \subseteq S_k \circ R_i^{-1} \Longrightarrow {R'}_i^{-1} \circ {S'}_k \subseteq S'_k \circ {R'}_i^{-1}$.

Proof. Follows easily from the previous Lemma. By Lemma 10, R'_i can be presented as $\bigcup\limits_{h \in H_i} \underline{R}_i^{h+1}$, where H_i is the additive closure of $\{0\} \cup J_i$. By Lemma 16 (1), $S_k \circ R_i \subseteq R_i \circ S_k$ implies $\underline{S}_k \circ \underline{R}_i \subseteq \underline{R}_i \circ \underline{S}_k$, and hence by induction, $\underline{S}_k \circ \underline{R}_i^{j+1} \subseteq \underline{R}_i^{j+1} \circ \underline{S}_k$, which implies $\underline{S}_k \circ R'_i \subseteq R'_i \circ \underline{S}_k$. Similarly, for the claims (2), (3). ■

4 Main results on finite model property

Let us now recall the definitions of products and relativised products.

DEFINITION 18. The *product* of Kripke frames $F = (W, R_1, \ldots, R_n)$, $G = (V, S_1, \ldots, S_m)$ is the frame

$$F \times G = (W \times V, R_{11}, \ldots, R_{n1}, S_{12}, \ldots, S_{m2})$$

such that

$$(x, y)R_{i1}(x', y') \Leftrightarrow xR_i x' \ \&\ y = y';$$
$$(x, y)S_{j2}(x', y') \Leftrightarrow x = x' \ \&\ yS_j y'.$$

A *relativised product* of F and G is an arbitrary subframe of $F \times G$.

DEFINITION 19. (cf. [5]) A *quasitranstitive (QT) formula* is one of the following kinds:

- $\Box_i p \to \Box_i^j p$ $(j \geq 0)$;
- $\Diamond_i \Box_i p \to p$.

A *QTC-logic* is a modal logic axiomatized by a finite set containing QT-formulas and maybe also closed formulas. A *$QTC_{\pm}$-logic* has the form $\mathbf{K}_{\pm m} + \mathbf{L}_0$, where $\mathbf{L}_0$ is a QTC-logic.

Remark. The above two kinds of QT-formulas correspond to the following conditions on frames: $R_i^j \subseteq R_i$, $R_i = R_i^{-1}$. A more general type, *pseudotransitive (PT)* formulas, was also considered in [5]: $\triangledown_1 \Box_j p \to \Delta_2 p$, where $\triangledown_1$ is a sequence of diamonds, Δ_2 is a sequence of boxes; such a formula corresponds to the frame condition

$(R_{\Delta_1})^{-1}R_{\Delta_2} \subseteq R_j$, where R_{Δ_i} are the corresponding compositions of basic relations in a frame. A *PTC-logic* is axiomatized by PT-formulas plus maybe, closed formulas. From [5] it is known that any two PTC-logics are product-matching, but the question if every PTC-logic has the fmp, is open[4]. This explains why we deal with a smaller class of QTC-logics in this paper.

DEFINITION 20. A *weak product* of an n-modal logic $\mathbf{L}_1$ and an m-modal logic $\mathbf{L}_2$ is obtained from their fusion $\mathbf{L}_1 * \mathbf{L}_2$ by adding some commutation axioms of the forms
$\Diamond_i\Box_{n+j}p \to \Box_{n+j}\Diamond_i p,\ \Box_i\Box_{n+j}p \to \Box_{n+j}\Box_i p,\ \Box_{n+j}\Box_i p \to \Box_i\Box_{n+j}p,$
for $1 \leq i \leq n$, $1 \leq j \leq m$.

Recall that the corresponding frame conditions are:

$$R_i^{-1}\circ R_{n+j} \subseteq R_{n+j}\circ R_i^{-1},\ R_{n+j}\circ R_i \subseteq R_i\circ R_{n+j},\ R_i\circ R_{n+j} \subseteq R_{n+j}\circ R_i.$$

THEOREM 21. *Let $\mathbf{\Lambda}$ be a weak product of $\mathbf{K}_n$ and a $QTC_{\pm}$-logic $\mathbf{L}_2$. Then $\mathbf{\Lambda}$ is determined by some class of relativized products $G \subseteq F_1 \times F_2$, where F_1 is an n-tree, $F_2 \vDash \mathbf{L}_2$.*

Proof. By an appropriate p-morphism construction similar to [4, Section 9.1]. First note that $\mathbf{\Lambda}$ is elementary and complete by Sahlqvist's theorem. Assume that $A \not\in \mathbf{\Lambda}$, then there exists a rooted countable frame $F = (W, R_1, R_2) \vDash \mathbf{\Lambda}$ refuting A. It follows that F is a p-morphic image of some relativized product $G \subseteq F_1 \times F_2$, where F_1 is a standard (infinite) tree, F_2 is the Horn closure (corresponding to the QT-axioms of $\mathbf{L}_2$) of a standard tree. G is selected from their product by a game-theoretic argument (see below), so that it validates the commutation axioms from Definition 20. Since QT-formulas correspond to universal first order conditions, their validity is preserved for subframes. The validity of closed formulas is reflected by p-morphisms, so we obtain that $G \vDash \mathbf{\Lambda}$.

Let us describe the corresponding game for the case $n = 1$, $\mathbf{L}_2 = \mathbf{K}$. We may assume that the relations R_1, R_2 in F are non-empty — otherwise the claim is trivial. Let T_ω be the standard (intransitive

[4]The simplest unclear cases are $\mathbf{K} + \Diamond_1\Box_1 p \to \Box_2 p$ and $\mathbf{K} + \Box_1 p \to \Box_2\Box_1\Box_2 p$.

irreflexive) countable tree consisting of all finite sequences in ω, and let S_1, S_2 be the basic relations in the square $T_\omega \times T_\omega$.

An *arrow* in F is a triple (x, j, y), such that $x, y \in W,\ j \in \{1, 2\}$, and xR_jy.

A *network over* F is a function $h : N \to W$ such that

- $(\lambda, \lambda) \in N$, where λ is the empty sequence.
- $N \subseteq T_\omega \times T_\omega$ is a finite connected subset (i.e., in N the root (λ, λ) is connected by a path with every point).
- h is monotonic: $\forall a, b \in N\ (aS_ib \Rightarrow h(a)R_ih(b))$.

If h, g are two networks, we write: $h \subseteq g$ to denote that g prolongs h.

The *rectification game over* F *of length* ω (notation: $RG_\omega(F)$) is a game between two players, $\forall$ and $\exists$, who build a countable increasing sequence of networks: $h_0 \subseteq h_1 \subseteq \ldots \subseteq h_i \subseteq \ldots$, where $h_i : N_i \longrightarrow W$, according to the rules:

- $N_0 = \{(\lambda, \lambda)\};\ h_0(\lambda, \lambda) = u_0$ (the root of F).
- $\forall$ starts the game, and $\forall$ and $\exists$ make their moves in turn.
- h_i is built from h_{i-1} at the i-th move of $\exists$.
- $\forall$ is allowed to make the i-th move of one of the two types:
  - (i) choose a 'lift enquiry': a quadruple (a, x, j, y), where $a \in N_{i-1},\ x = h_{i-1}(a)$, and (x, j, y) is an arrow in F;
  - (ii) choose a 'commutation enquiry', which is a pair of arrows in N_{i-1}: if the axiom $\Diamond_1\Box_2 p \to \Box_2\Diamond_1 p$ is present in $\mathbf{\Lambda}$, this is $((a, 1, b), (a, 2, c))$; if the axiom $\Box_1\Box_2 p \to \Box_2\Box_1 p$ is in $\mathbf{\Lambda}$, this is $((a, 2, b), (b, 1, c))$, and similarly for the axiom $\Box_2\Box_1 p \to \Box_1\Box_2 p$.
- $\exists$ is allowed to respond to the moves of $\forall$ as follows:

(i) in a response to a lift enquiry (a, x, j, y) — to build a network $h_i : N_i \longrightarrow W$ such that $h_{i-1} \subseteq h_i$, and for some $b \in N_i,\ aS_jb, h_i(b) = y$, (i.e. h_i lifts the arrow (x, j, y) to (a, j, b));

(ii) in a response to a commutation enqury — to build a network $h_i : N_i \longrightarrow F$ extending h_{i-1} such that N_i contains the missing element. That is, in the case of the enquiry $((a, 1, b), (a, 2, c))$, N_i should contain d such that bS_2d and cS_1d; in the case of the enquiry $((a, 2, b), (b, 1, c))$, N_i should contain d such that aS_1d and dS_2c etc.

We assume that the player $\exists$ wins in every infinite play of the game; she loses if at some stage she cannot respond to a move of $\forall$.

A *winning strategy* for $\exists$ in the game is defined as usual; this is a function of the already constructed network and the last move of $\forall$, giving the response of $\exists$.

LEMMA 22. *If the player $\exists$ has a winning strategy in $RG_\omega(F)$, then F is a p-morhic image of a relativized product of two trees validating the appropriate commutation axioms.*

Proof.

Assume that $\exists$ has a winning strategy. Let us find a strategy for $\forall$, allowing $\exists$ to construct a required p-morphism.

Let $Ar(F)$ be the set of all arrows in F; then the elements of the countable set $\Pi = T_\omega \times Ar(F)$ and all commutation enquiries in $T_\omega \times T_\omega$ can be put into a sequence $\pi_1, \ldots, \pi_n, \ldots$ Now we choose the following strategy for $\forall$:

Every move of $\forall$ is the first occurence of a lift enquiry or a commutation enquiry in the sequence π, which is an allowed move and which has not been used in the previous moves; if it does not exist, this is a repetition of his previous move.

Obviously, $\forall$ can make the first move, since by our assumption, $R_1(u_0) \cup R_2(u_0) \neq \varnothing$.

If $\exists$ uses her winning strategy in response to these moves of $\forall$, they can build a sequence of networks $h_0 \subseteq h_1 \subseteq \ldots$.

Let $h = \bigcup\limits_{i=0}^{\infty} h_i,\ N_i = \mathrm{dom}(h_i),\ N = \mathrm{dom}(h);$ then $N = \bigcup\limits_{i=0}^{\infty} N_i$.

We claim that h is a required p-morphism. Obviously, h is monotonic, since every h_i is a network. By the same reason, N is a connected subset of $\Phi_1 \times \Phi_2$, where the sets $\Phi_1 := pr_1(N),\ \Phi_2 := pr_2(N)$ are standard trees. Let us show that the commutation axioms hold in N. In fact, suppose the contrary. Then there exists a commutation enquiry in N, say, $\pi_n = (a, 1, b, 2, c)$, but there does not exist $d \in N$ such that aS_2dS_1c. Take i such that $a, b, c \in N_i$. Then the chosen strategy for $\forall$ suggests that π_n should be his even move with a number $k \leq i + n$. In fact, after the i-th move is made, the only reason to postpone the move π_n is the existence of an allowed commutation enquiry with a number less than n. But all these enquiries should be exhausted by the $(i + n)$-th move.

In a similar way, let us check the lift property for h. Assume that $a \in N_{k-1},\ h(a) = xR_jy$.

Then the lift enquiry (a, x, j, y) is an allowed move of $\forall$ with a number $\geq k$. This enquiry occurs in the sequence π as some π_n, and according to the strategy of $\forall$, this must be his move with a number $l \leq k + n$, because this enquiry, after the k-th move is made, can be postponed only in favour of π_i with $i < n$. The response of $\exists$ is a network $h_l : N_l \longrightarrow F$ lifting (x, j, y) to (a, j, b). So $h(b) = y,\ aR_jb$. ∎

LEMMA 23. *$\exists$ has a winning strategy in $RG_\omega(F)$.*

Proof. Consider the i-th move of $\forall$, which is a lift enquiry (a, x, k, y), and assume that $k = 2$. (If $k = 1$, the argument is the same.)

If $a = (\alpha, \beta)$ then we take n such that $b = (\alpha, \beta n) \notin N_{i-1}$ (it exists because N_{i-1} is finite), and put $N_i = N_{i-1} \cup \{b\}, h_i(b) = y$. It is clear that h_i is monotonic and N_i is connected; so h_i is a correct response.

Suppose the i-th move of $\forall$ is a commutation enquiry $(a, 1, b, 2, d)$. Then $\Box_2\Box_1 p \to \Box_1\Box_2 p \in \mathbf{\Lambda}$, and thus $R_1 \circ R_2 \subseteq R_2 \circ R_1$ holds in F. It is clear that there exists a unique $c \in T_\omega \times T_\omega$ such that aS_2cS_1d. If $c \in N_{i-1}$, the response of $\exists$ will be $h_i = h_{i-1}$. Otherwise, let $h_{i-1}(a) = x,\ h_{i-1}(b) = y,\ h_{i-1}(d) = z$. Since h_{i-1} is monotonic, we have xR_1yR_2z, and due to the commutation property of F, $x(R_2 \circ R_1)z$, i.e., there exists t such that xR_2tR_1y. Then the response of $\exists$ will be $h_i = h_{i-1} \cup \{(c, t)\}$.

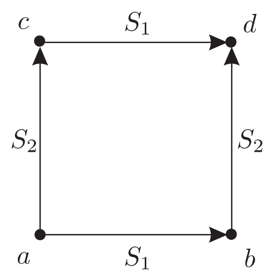

Figure 3.

For other types of commutation enquiries the argument is quite similar. ■

An analogous construction can be applied to $\mathbf{L}_2 = \mathbf{K}_n$. If $\mathbf{L}_2$ also has axioms $\Box_i p \to \Box_i^j p$ or $\Diamond_i \Box_i p \to p$, they are valid in F, and we should take the corresponding Horn closures G^+, F_2 of G, Φ_2. Then h is a p-morphism from G^+ onto F, cf. [5], Proposition 7.9.

■

LEMMA 24. *There exist finitely many equivalence classes with respect to the relation $M, x \asymp N, y$ for n-tree models M, N of fixed finite height with roots x, y.*

Proof. Since in trees bisimilar paths are of equal length, $M, x \asymp N, y$ implies $h(M) = h(N)$. Now the argument is by induction on $h(M)$.

If $h(M) = 0$, M, N are singletons, and $x \asymp y$ iff the same proposition letters are true at x and y. So in this case there exist 2^k $\asymp$-classes (recall that k is the fixed number of proposition letters).

Assume that $h(M) = l$ and there exist r $\asymp$-classes for n-tree models of height $< l$. We claim that $M, x \asymp N, y$ iff the same proposition letters are true at x and y, and also $\forall i \leq n$ $(\{z_{\asymp} \mid xR_i z\} = \{t_{\asymp} \mid yS_i t\})$, where R_i, S_i are the relations in M, N respectively, and $z_{\asymp}$ denotes the $\asymp$-class of M^z, z (or N^z, z).

In fact, the direction 'only if' is easy. To check 'if', assume that the truth values of proposition letters coincide in x and y and $\{z_{\asymp} \mid z \in R_i(x)\} = \{t_{\asymp} \mid t \in S_i(y)\}$ for any $i \leq n$. Then there

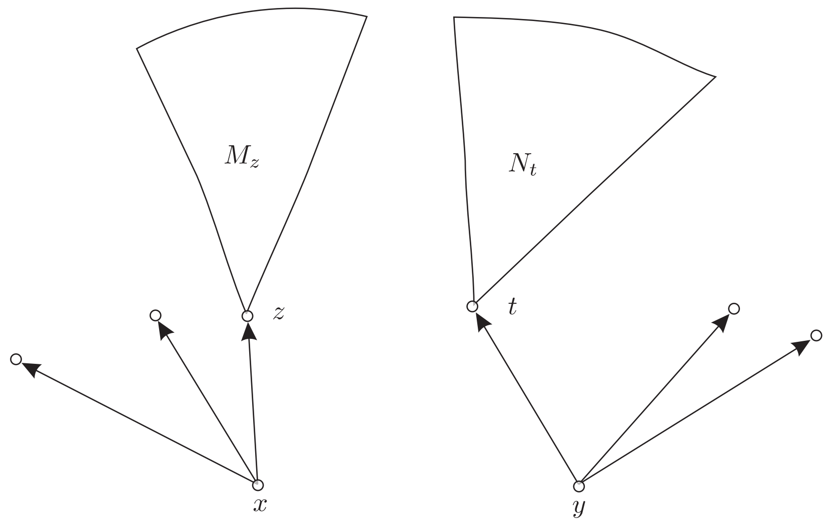

Figure 4.

exist bisimulations $E_{izt} : M^z, z \asymp N^t, t$, with $(z,t) \in V_i$, for some set $V_i \subseteq R_i(x) \times S_i(y)$ such that $pr_1(V_i) = R_i(x),\ pr_2(V_i) = S_i(y)$.

Then

$$E := \{(x,y)\} \cup \bigcup \{E_{izt} \mid 1 \leq i \leq n,\ (z,t) \in V_i\}$$

is a bisimulation between M, x and N, y. In fact, E_{izt} works properly between M^z and N^t. Since $pr_2(V_i) = S_i(y)$, we also have
$\forall z \in R_i(x)\ \exists t \in S_i(y)\ tEz$, and

similarly, every $t \in S_i(y)$ corresponds to some $z \in R_i(x)$, and finally, E sends x, the unique predecessor of z, to y, the unique predecessor of t.

Thus the $\asymp$-class of M, x is uniquely determined by the set $\{p_i \mid M, x \vDash p_i\}$ and $\{z_{\asymp} \mid z \in R_i(x)\}$ (for $1 \leq i \leq n$), and so there exist at most $(2^k \cdot (2^r)^n)$ $\asymp$-classes of M, x with $h(M) = l$. ■

LEMMA 25. *Let M, $\approx$ be the same as in Definition 14 and assume that every n-trace $x\!\uparrow$ is an n-tree of height $\leq l$ for some fixed l. Then the set $W/\approx$ is finite.*

Proof. In fact, $x \approx y$ only if $x\uparrow, x \asymp y\uparrow, y$, only if $x\uparrow, u \asymp y\uparrow, v$, where u, v are the roots of $x\uparrow$, $y\uparrow$. So the number of $\approx$-classes is finite, by Lemma 24. ■

THEOREM 26. *Every logic* $\mathbf{\Lambda}$ *from Theorem 21 has the fmp.*

Proof. By Theorem 21, every $A \notin \mathbf{\Lambda}$ is refuted in a model M over a $\mathbf{\Lambda}$-frame $F \subseteq F_1 \times F_2$, where F_1 is an n-tree; moreover, we may assume that $M, (x_0, y_0) \not\models A$, where x_0 is the root of F_1, y_0 is the root of F_2.

Let $d_1(B)$ denote the modal depth of a formula B with respect to $\Box_1, \ldots, \Box_n$ (the maximal number of nested modalities of this type). Assume that $d_1(A) = r$. For $x \in F$ let $h_1(x)$ be the height of its first coordinate $pr_1(x)$ in F_1.

Let M^- (respectively, F^-) be the restriction of M (respectively, F) to the set $\{x \mid h_1(x) \leq r\}$. Then for any $v \in F^-$, for any formula B:

$$(1) \quad \text{if } h_1(v) + d_1(B) \leq r, \text{ then } M, v \models B \iff M^-, v \models B.$$

This is proved by induction on $d_1(B)$ (cf. Lemma 9.11 in [5]). In fact, if $d_1(B) = 0$, the claim is obvious. If $d_1(B) > 0$, the only non-trivial case is when $B = \Box_i C$. So assume that (1) is proved for C, $M, v \models B$. Then $M, w \models C$ for any $w \in R_{i1}(v)$. Since $vR_{i1}w$ implies $h_1(w) = h_1(v) - 1$ and $d_1(C) = d_1(B) - 1$, we obtain $d_1(C) + h_1(w) = d_1(B) + h_1(v) \leq r$, and thus by induction hypothesis, $M^-, w \models C$. Thus $M^-, w \models B$.

If $B = \Box_{n+j} C$, then $M, v \models B$ means $\forall w \in S_{j2}(v)\ M, w \models C$. But $vS_{j2}w$ implies $h_1(v) = h_1(w)$, and thus $d_1(C) + h_1(w) < d_1(B) + h_1(v) \leq r$. Hence $M^-, w \models C$, by induction hypothesis, and thus $M^-, w \models B$.

The converse ($M^-, v \models B \Longrightarrow M, v \models B$) is proved in the same way.

Since $h_1(x_0, y_0) = 0$, from (1) we obtain:

$$(2) \quad M^-, (x_0, y_0) \not\models A.$$

We also have:

$$(3) \quad F^- \models \mathbf{\Lambda}.$$

In fact, for every $x \in F^-$ the subframes generated by x along the second coordinate in F and in F^- are the same.

Thus $F^- \vDash \mathbf{K}_n * \mathbf{L}_2$.

It is also clear that F^- inherits the commutation properties of F. For example, assume that $R_{i1} \circ S_{j2} \subseteq S_{j2} \circ R_{i1}$ holds in F. Now if $(x_1, y_1)R_{i1}(x_2, y_1)S_{j2}(x_2, y_2)$ in F^-, then $h_1(x_1, y_2) = h_1(x_1, y_1) \leq r$, and so $(x_1, y_2) \in F^-$; thus $(x_1, y_1)(S_{j2} \circ R_{i1})(x_2, y_2)$ holds in F^-.

Therefore we can take the filtration of M^- as in Lemma 17. The resulting model M' is finite, due to Lemma 25. ■

The previous Theorem can be generalized to the temporal case.

THEOREM 27. *Let* $\mathbf{\Lambda}$ *be a weak product of* $\mathbf{K}_{\pm n}$ *and a* $QTC_{\pm}$*-logic* $\mathbf{L}_2$*. Then* $\mathbf{\Lambda}$ *has the fmp.*

Proof. (Sketch.) The idea of the proof is the same as above, but n-trees are replaced with temporal n-trees. Note that now we do not need Church – Rosser axioms $\Diamond_i \Box_{n+j} p \rightarrow \Box_{n+j} \Diamond_i p$ — they can be replaced with $\Box_{n+j} \Box_i^{-1} p \rightarrow \Box_i^{-1} \Box_{n+j} p$. The proof of Theorem 21 is easily modified for this case. For example, if $\mathbf{\Lambda} = \mathbf{K}_{\pm 1} \times \mathbf{K}_{\pm 1}$, we take the standard temporal tree $T_\omega^\pm$ and construct $G \subseteq T_\omega^\pm \times T_\omega^\pm$ by a rectification game, in which commutation enquiries may be of the form $(a, i, b), (b, j, c))$, where $i = \pm 1,\ j = \pm 2$ or $j = \pm 1,\ i = \pm 2$.

The proof of the analogue of Lemma 24 is slightly more delicate, because now a path can use the same arrow in both directions many times. So we first make a reduction of a given temporal tree model M (with a chosen root x) as follows. We can identify two arrows (x, i, y) and (x, i, z) (or (y, i, x) and (z, i, x)) if there exists a bisimulation of M associating x with x and y with z. So we successively identify arrows in this way as long as possible. The reduced tree is bisimilar to the original one. Every point (except x) has a unique *father* (the predecessor in the unique path from x to that point); non-terminal points may also have R_i- or R_i^{-1}-*sons*. Now similar to the proof of Lemma 24, we have the criterion of bisimilarity for two reduced n-temporal trees of the same height: $M, x \asymp N, y$ iff the same proposition letters are true in M, x and N, y and the set of $\asymp$-classes of M_z, z and N_z, z are the same for R_i-sons of x and y, and also for R_i^{-1}-sons of x and y. Here M_z denotes the subtree of M with root

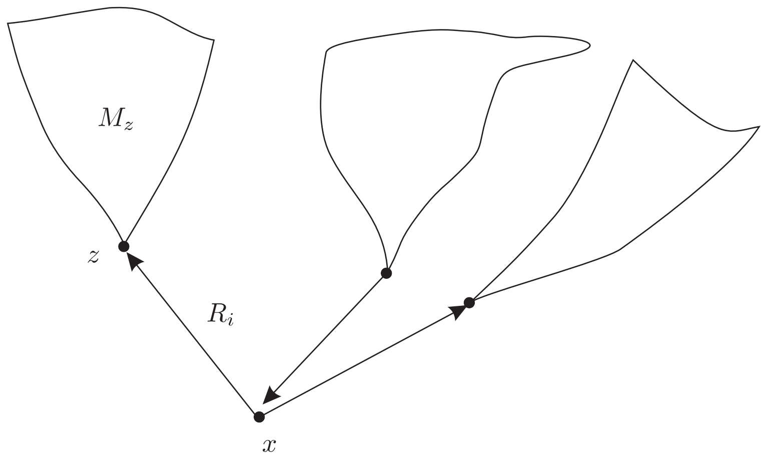

Figure 5.

z obtained from M by eliminating all points connected with z via x (i.e., those meeting x on on the shortest path to z).

■

Hence we obtain the fmp for some product logics:

COROLLARY 28. *The logics* $\mathbf{K}_{\pm n} \times \mathbf{K}_{\pm m}$, $\mathbf{K}_{\pm n} \times \mathbf{S5}_m$, $\mathbf{K}_{\pm n} \times \mathbf{K4}_{\pm m}$, $\mathbf{K}_{\pm n} \times \mathbf{S4}_{\pm m}$ *have the fmp.*

Remarks. 1. This implies the fmp of the logic $\mathbf{K.t}^2$ (which has been an open question) and therefore, of $\mathbf{K} \times \mathbf{K.t}$. The proof of the latter result given in [6], is much more complicated.

2. We also obtain the fmp for $\mathbf{K} \times \mathbf{S5}_m$, $m > 1$. Note that the earlier proof given in [5], contained a gap noticed by A. Kurucz [4], Section 5.3. Apparently, the methods of this paper can be extended to the products $\mathbf{S5}_m \times \mathbf{L}_2$, where $\mathbf{L}_2$ is a QTC-logic.

3. This also implies a positive answer to the question about the fmp of $\mathbf{K} \times \mathbf{K4}$ put in [4], p.339. However the question, whether the more powerful (but decidable) logic $\mathbf{K} \times \mathbf{PDL}$ has the fmp, remains open; our methods are not directly transferable to this case.

Acknowledgements. I would like to thank the anonymous referees for very useful comments on the paper. The work on this paper was partly supported by Russian Foundation for Basic Research, projects No. 02-01-22003, 01-01-00493, 02-01-01041, and by Jumelage (twinship) program CNRS-IUM.

BIBLIOGRAPHY

[1] P. Blackburn, M. de Rijke, Y. Venema. Modal logic. Cambridge University Press, 2001.
[2] D. Gabbay. General filtration method for modal logic. Journal of Philosophical Logic, v. 1 (1972), pp. 29-34,.
[3] D. Gabbay, I. Hodkinson, M. Reynolds. Temporal logic, v. 1. Oxford University Press, 1994.
[4] D. Gabbay, A. Kurucz, F. Wolter, M. Zakharyaschev. Many-dimensional modal logics. Theory and applications. Elsevier, 2003.
[5] D. Gabbay, V. Shehtman. Products of modal logics, I. Logic Journal of the IGPL, v.6 (1998), pp. 73-146.
[6] D. Gabbay, V. Shehtman. Products of modal logics. III. Products of modal and temporal logics. Studia Logica, v. 72 (2002), No. 2, p. 157-183.
[7] D. Gabelaia, A. Kurucz, M. Zakharyaschev. Products of 'transitive' modal logics without the (abstract) finite model property. In: AIML-2004, Advances in Modal Logic. Department of Computer Science University of Manchester, Technical Report Series, UMCS-04-91, pp. 104-115.
[8] K.Segerberg. Decidability of S4.1. Theoria, v.34 (1968), p. 7-20.
[9] V.Shehtman. A logic with progressive tenses. Diamonds and Defaults: Studies in Pure and Applied Intensional Logic. (Ed. by M. De Rijke). Kluwer Academic Publishers, 1993, p. 255-285.
[10] V. Shehtman, I. Shapirovsky. Chronological future modality in Minkowski spacetime. Advances in Modal Logic, v. 4, pp. 437-459. King's College Publications, 2003.

Institute for Information Transmission Problems, B. Karetny 19, 101447, Moscow, Russia

Department of Mathematics, Moscow State University

CNRS Laboratory Poncelet, B. Vlasyevsky 11, 119002, Moscow, Russia

shehtman@lpcs.math.msu.su