On the Complexity of Fragments of Modal Logics

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ABSTRACT. We study and give a summary of the complexity of 15 basic normal monomodal logics under the restriction to the Horn fragment and/or bounded modal depth. As new results, we show that: a) the satisfiability problem of sets of Horn modal clauses with modal depth bounded by $k \geq 2$ in the modal logics $K4$ and $KD4$ is PSPACE-complete, in K is NP-complete; b) the satisfiability problem of modal formulas with modal depth bounded by 1 in $K4$, $KD4$, and $S4$ is NP-complete; c) the satisfiability problem of sets of Horn modal clauses with modal depth bounded by 1 in K, $K4$, $KD4$, and $S4$ is PTIME-complete.

We also study the complexity of the multimodal logics L_n under the mentioned restrictions, where L is one of the 15 basic monomodal logics. We show that, for $n \geq 2$: a) the satisfiability problem of sets of Horn modal clauses in $K5_n$, $KD5_n$, $K45_n$, and $KD45_n$ is PSPACE-complete; b) the satisfiability problem of sets of Horn modal clauses with modal depth bounded by $k \geq 2$ in K_n, KB_n, $K5_n$, $K45_n$, $KB5_n$ is NP-complete, and in KD_n, T_n, KDB_n, B_n, $KD5_n$, $KD45_n$, $S5_n$ is PTIME-complete.

1 Introduction

In the field of modal logics, a lot of works are devoted to monomodal logics that extend the modal logic K by some of the axioms D, T, B, 4, and 5. The reason is not that those logics are useful in practice, but because they are *basic* modal logics. Many useful multimodal logics, e.g. ones for reasoning about knowledge and belief, are also formed using the mentioned axioms and are extensions of some basic monomodal logics.

Advances in Modal Logic, Volume 5.

Decidability and complexity are important aspects of logics. In [15], Ladner proved that the complexity of the satisfiability problem in the modal logics K, T, B, and $S4$ is PSPACE-complete, and in $S5$ is NP-complete. This means that the satisfiability problem is NP-hard in all of those logics. In order to reduce the complexity to PTIME, one must focus on fragments of the considered logic. Such fragments are often specified by restrictions on the language. There are of course many kinds of restrictions, but the obtained fragments may be useful or not. The Horn fragment is very useful in logic programming, and in many logics it significantly reduces the complexity of the problem. For modal logics, the restriction of bounded modal depth is also acceptable, because in practice modal formulas often have small modal depth. We can also combine these two restrictions. Given an "acceptable" restriction and a modal logic, one may want to study the complexity of the satisfiability problem in the obtained fragment of the logic. The result may be positive (PTIME) or negative (NP-hard, PSPACE-hard, etc). Both of the cases are useful: the positive case is good for the fragment itself, while the negative case implies that every multimodal logic containing the fragment is hard at least as the fragment.

In this work, we study and give a summary of the complexity of the satisfiability problem in the basic normal monomodal logics (which are obtained from the logic K by adding an arbitrary combination of the axioms D, T, B, 4, and 5) under the restriction to the Horn fragment and/or bounded modal depth.

In [11], Halpern studied the effect of bounding modal depth on the complexity of modal logics and showed that the complexity of the satisfiability problem of formulas with modal depth bounded by $k \geq 2$ in K and T is NP-complete, and in $S4$ is PSPACE-complete. His arguments for K and T can also be applied for the logics KB, KDB, and B, to obtain the NP-completeness.

In [6], Fariñas del Cerro and Penttonen showed that the satisfiability problem of sets of Horn modal clauses in $S5$ is decidable in PTIME. In [4], Chen and Lin showed that the similar problem for a normal modal logic L being an extension of $K5$ (write $K5 \leq L$) is also decidable in PTIME. Chen and Lin also proved that for a normal

(in this column, $k \geq 2$)	K	KD, T KB, KDB, B	$K4$, $KD4$, $S4$	$K5$, $KD5$, $KB5$ $K45$, $KD45$, $S5$
no restrictions	PS-cp [15]	PS-cp [15]	PS-cp [15]	NP-cp [15]
$mdepth \leq k$	NP-cp [11]	NP-cp [11]	PS-cp [11] [$*$]	NP-cp [15]
$mdepth = 1$	NP-cp [11]	NP-cp [11]	NP-cp [$\star$]	NP-cp [15]
Horn	PS-cp [4]	PS-cp [4]	PS-cp [4]	PT-cp [6, 4]
Horn, $mdepth \leq k$	NP-cp [$\star$]	PT-cp [17]	PS-cp [4] [$*$]	PT-cp [6, 4]
Horn, $mdepth = 1$	PT-cp [$\star$]	PT-cp [17]	PT-cp [$\star$]	PT-cp [6, 4]

Table 1. The complexity of the satisfiability problem for modal logics

modal logic L such that $K \leq L \leq S4$ or $K \leq L \leq B$, the problem is PSPACE-hard. They also made a comment that the problem is still PSPACE-hard for $S4$ even when the modal depth is restricted to 2.

In [17], we showed that the complexity of the satisfiability problem of sets of Horn modal clauses with finitely bounded modal depth in KD, T, KB, KDB, and B is decidable in PTIME. These PTIME results can further be categorized as PTIME-complete, because the satisfiability problem of sets of Horn clauses in the classical propositional logic is PTIME-complete, as proved by Jones and Laaser [13].

In this work, we show that the satisfiability problem of sets of Horn modal clauses with modal depth bounded by $k \geq 2$ in the modal logics $K4$ and $KD4$ is PSPACE-complete, and in K is NP-complete. We also show that the satisfiability problem of modal formulas with modal depth bounded by 1 in $K4$, $KD4$, and $S4$ is NP-complete; the satisfiability problem of sets of Horn modal clauses with modal depth bounded by 1 in K, $K4$, $KD4$, and $S4$ is PTIME-complete.

In Table 1, we summarize the complexity of the basic monomodal logics under the mentioned restrictions. There, *mdepth* stands for "modal depth"; PS-cp, NP-cp, and PT-cp respectively stand for PSPACE-complete, NP-complete, and PTIME-complete. The marks [$\star$] and [$*$] indicate the results of this work, where [$*$] involves with $K4$ and $KD4$.

As an extension to the preliminary version, we also study the complexity of the multimodal logics L_n under the mentioned restrictions, where L is one of the basic monomodal logics. Some results were established by Halpern and Moses [12] and Halpern [11]. Some of our results are:

- The satisfiability problem of sets of Horn modal clauses in $K5_n$, $KD5_n$, $K45_n$, and $KD45_n$ is PSPACE-complete.

- The satisfiability problem of sets of Horn modal clauses with modal depth bounded by $k \geq 2$ in K_n, KB_n, $K5_n$, $K45_n$, and $KB5_n$ is NP-complete, and in KD_n, T_n, KDB_n, B_n, $KD5_n$, $KD45_n$, and $S5_n$ is PTIME-complete.

This paper is structured as follows: In Section 2, we give preliminaries for monomodal logics. In Section 3, we present our results for monomodal logics. In Section 4, we discuss the complexity and give some results for multimodal logics. We conclude in Section 5.

2 Preliminaries

In this section we give preliminaries for monomodal logics. For abbreviation, we will ignore the prefix "mono" in this section and the next one.

2.1 Syntax and Semantics of Propositional Modal Logics

A modal formula, hereafter simply called a *formula*, is any finite sequence obtained by applying the following rules: any primitive proposition p_i is a formula, and if φ and ψ are formulas then so are $\neg\varphi$, $\varphi \wedge \psi$, $\varphi \vee \psi$, $\varphi \to \psi$, $\Box\varphi$, and $\Diamond\varphi$. We use letters p and q to denote primitive propositions, and Greek letters φ, ψ, ζ to denote formulas.

A *Kripke frame* is a triple $\langle W, \tau, R\rangle$, where W is a nonempty set of possible worlds, $\tau \in W$ is the actual world, and R is a binary relation on W, called the accessibility relation. If $R(w, u)$ holds then we say that the world u is accessible from the world w.

A *Kripke model* is a tuple $\langle W, \tau, R, h\rangle$, where $\langle W, \tau, R\rangle$ is a Kripke frame and h is a function mapping worlds to sets of primitive propositions. For $w \in W$, $h(w)$ is the set of primitive propositions which are "true" at w.

We call $\langle W, \tau, R, h\rangle$ a *flat model* if $W = \{\tau\}$ and $R = \emptyset$.

A *model graph* is a tuple $\langle W, \tau, R, H\rangle$, where $\langle W, \tau, R\rangle$ is a Kripke frame and H is a function mapping worlds to formula sets. We sometimes treat model graphs as models with H being restricted to the set of primitive propositions.

Given a Kripke model $M = \langle W, \tau, R, h\rangle$ and a world $w \in W$, the *satisfaction relation* $\models$ is defined as follows:

$M, w \models p$	iff	$p \in h(w)$;
$M, w \models \neg\varphi$	iff	$M, w \not\models \varphi$;
$M, w \models \varphi \wedge \psi$	iff	$M, w \models \varphi$ and $M, w \models \psi$;
$M, w \models \varphi \vee \psi$	iff	$M, w \models \varphi$ or $M, w \models \psi$;
$M, w \models \varphi \rightarrow \psi$	iff	$M, w \not\models \varphi$ or $M, w \models \psi$;
$M, w \models \Box\varphi$	iff	for all $v \in W$ s.t. $R(w, v)$, $M, v \models \varphi$;
$M, w \models \Diamond\varphi$	iff	there exists $v \in W$ s.t. $R(w, v)$ and $M, v \models \varphi$.

We say that φ *is satisfied at* w *in* M if $M, w \models \varphi$, and that φ *is satisfied in* M, write $M \models \varphi$ and call M a *model of* φ, if $M, \tau \models \varphi$.

The *size* of a finite Kripke model $\langle W, \tau, R, h\rangle$ is $|W| + |R| + \Sigma_{w\in W}$ $|h(w)|$. The *length* of a formula φ is the number of occurrences of connectives and primitive propositions in φ. The *modal depth* of a formula φ is the maximal nesting depth of modalities occurring in φ, e.g. $mdepth(p \wedge \Box(\Diamond q \vee \Diamond r)) = 2$.

The following lemma is well known and can be proved easily.

LEMMA 1. *Given a finite model M and a formula φ, the problem of checking whether $M \models \varphi$ is decidable in polynomial time (in the size of M and the length of φ).*

If as the class of admissible interpretations we take the class of all Kripke models (with no restrictions on the accessibility relations) then we obtain a normal modal logic which has a standard Hilbert-style axiomatization denoted by K. Other normal modal logics are obtained by adding to K certain axioms. The most popular axioms used for extending K are D, T, B, 4, and 5, whose schemata are listed in Table 2. These axioms respectively correspond to seriality, reflexiveness, symmetry, transitiveness, and euclideaness of the accessibility relation. A modal logic L is *serial* if it contains the axiom D.

In this work, we consider all of the 15 basic modal logics that are obtained from K by adding an arbitrary combination of the above axioms, namely K, KD, T, KB, KDB, B, $K4$, $KD4$, $S4$, $K5$, $KD5$, $K45$, $KD45$, $KB5$, $S5$. The names of these logics often consist of K and the names of the added axioms, e.g. KDB is the logic which extends K with the axioms D and B. The special cases are T, B,

Axiom	Schema	Corresponding Condition on R
D	$\Box\varphi \to \Diamond\varphi$	$\forall w\ \exists u\ R(w,u)$
T	$\Box\varphi \to \varphi$	$\forall w\ R(w,w)$
B	$\varphi \to \Box\Diamond\varphi$	$\forall w,u\ R(w,u) \to R(u,w)$
4	$\Box\varphi \to \Box\Box\varphi$	$\forall w,u,v\ R(w,u) \land R(u,v) \to R(w,v)$
5	$\Diamond\varphi \to \Box\Diamond\varphi$	$\forall w,u,v\ R(w,u) \land R(w,v) \to R(u,v)$

Table 2. Modal logics and frame restriction

$S4$, and $S5$, which stand for KT, KTB, $KT4$, and $KT5$, respectively. For a further reading about modal logics, see, e.g., [2, 3].

We refer to the properties of the accessibility relation of a modal logic L as the *L-frame restrictions*. We call a model M an *L-model* if the accessibility relation of M satisfies all L-frame restrictions. We say that φ is *L-satisfiable* if there exists an L-model of φ. A formula is *L-valid* if it is satisfied in every L-model. We write $\varphi \models_L \psi$ to denote that ψ is satisfied in every L-model of φ.

2.2 Modal Horn Formulas and Positive Modal Logic Programs

We call formulas of the form p or $\neg p$, where p is a primitive proposition, *classical literals* and use letters a, b, c to denote them. We call formulas of the form p, $\Box p$, or $\Diamond p$ *atoms* and use letters A, B, C to denote them.

A *clause* is a formula of the form $\Box^s(A_1 \lor \ldots \lor A_n \lor \neg B_1 \lor \ldots \lor \neg B_m)$, where $s, m, n \geq 0$. The sequence $\Box^s$ is called the *modal context* of the clause[1]. If $s = 0$ then the clause is called a *simple clause*. Note that the modal depth of a clause is not greater than the length of its modal context plus 1.

A formula set is sometimes considered as the conjunction of its formulas, in particular when we are talking about length, modal depth, or satisfiability.

A formula is in *negative normal form* if it does not contain the connective $\to$, and the connective $\neg$ can occur only immediately before a primitive proposition. Every formula can be transformed to

[1] Assume that the modal context of $\Box^s\Box p$ is $\Box^{s+1}$.

the equivalent negative normal form in the usual way. A formula is called *negative* if in its negative normal form every primitive proposition is prefixed by negation. A formula is called *non-negative* if it is not negative, and *positive* if its negation is a negative formula.

A formula φ is a *Horn formula* if it is of one of the following forms:

- a primitive proposition or a negative formula,
- $\Box\psi$, $\Diamond\psi$, or $\psi \wedge \zeta$, where ψ and ζ are Horn formulas,
- $\psi \rightarrow \zeta$, where ψ is a positive formula and ζ is a Horn formula,
- a disjunction of a negative formula and a Horn formula.

A clause is called a *Horn clause* if it is a Horn formula.

Our definitions of Horn clauses/formulas are different than the one of Chen and Lin [4]. A Horn clause by our definition is also a Horn clause by the definition of Chen and Lin, and the latter is a Horn formula by our definition, but not vice versa. These definitions, however, are equivalent. As stated by Lemma 2 given below, every Horn formula φ can be translated to a set X of Horn clauses such that for any normal modal logic L, φ is L-satisfiable iff X is L-satisfiable.

A *positive propositional modal logic program* is a finite set of rules of the following form: $\Box^s(B_1 \wedge \ldots \wedge B_k \rightarrow A)$, where $s \geq 0$, $k \geq 0$, and $A, B_1, \ldots, B_k$ are atoms of the form p, $\Box p$, or $\Diamond p$, where p is a primitive proposition.

Formula sets X and Y are said to be *equisatisfiable* in a logic L (or L-equisatisfiable) iff (X is L-satisfiable iff Y is L-satisfiable).

LEMMA 2. *For any formula set X, there exists a clause set Y s.t.:*

- *X and Y are equisatisfiable in any normal modal logic.*
- *If X is a set of Horn formulas, then Y is a set of Horn clauses.*
- *The modal depth of Y is equal to the modal depth of X, and the length of Y is of quadratic order in the length of X.*

Moreover, if X is a set of Horn formulas and Y is divided into P and Q such that P contains only non-negative clauses and Q contains only negative clauses, then P can be treated as a positive program, and X

is L-satisfiable iff $P \not\models_L \neg Q$, where L is any normal modal logic. The translation from X to Y is computable in polynomial time.

The proof for the case when X is a set of Horn formulas can be found in [17]. The proof for the other case is similar. The translation technique is based on replacing a complicated formula by a fresh primitive proposition and "defining" that primitive proposition by the formula. For example, $\Box^s(\Diamond\varphi \vee \psi)$, where $s \geq 0$ and φ is not a primitive proposition, is replaced by $\Box^s(\Diamond p \vee \psi)$ and $\Box^{s+1}(\neg p \vee \varphi)$, where p is a fresh primitive proposition.

2.3 Ordering Kripke Models

Let $M = \langle W, \tau, R, h\rangle$ and $N = \langle W', \tau', R', h'\rangle$ be Kripke models. We say that M is *less than or equal to N w.r.t. $r \subseteq W \times W'$*, and write $M \leq N$ w.r.t. r, if the following conditions hold:

1. $r(\tau, \tau')$
2. $\forall x, x', y \;\; R(x, y) \wedge r(x, x') \rightarrow \exists y' \; R'(x', y') \wedge r(y, y')$
3. $\forall x, x', y' \;\; R'(x', y') \wedge r(x, x') \rightarrow \exists y \; R(x, y) \wedge r(y, y')$
4. $\forall x, x' \;\; r(x, x') \rightarrow (h(x) \subseteq h'(x'))$.

The first three conditions state that r is a bisimulation of the frames of M and N. Intuitively, $r(x, x')$ states that the world x is less than or equal to x'.

We say that a model M is *less than or equal*[2] *to N*, and write $M \leq N$, if $M \leq N$ w.r.t. some r. This relation is a pre-order [17]. Also see [17] for the proof of the following lemma.

LEMMA 3. *Suppose that $M \leq N$. Then $M \models \varphi$ implies $N \models \varphi$ for every positive formula φ.*

Let P be a positive program in a normal modal logic L. We say that M is a *least L-model* of P if M is an L-model of P and M is less than or equal to every L-model of P. Observe that if P is a positive program in a normal modal logic L, and M is a least L-model of P, then for any positive formula φ, $M \models \varphi$ iff $P \models_L \varphi$.

[2] This kind of "equality" is induced by the pre-order $\leq$. By Lemma 3, if $M \leq N$ and $N \leq M$ then for every positive formula φ, $M \models \varphi$ iff $N \models \varphi$.

A model M is called the *least flat model* of a positive program P if it is a flat model of P and is less than or equal to any flat model of P. In [17], we showed that any positive modal logic program that has some flat model has the least flat model, which can be constructed in polynomial time and has polynomial size.

3 New Results for Monomodal Logics

We first consider the complexity of the satisfiability problem of sets of Horn formulas with modal depth bounded by $k \geq 2$ in the logics $K4$, $KD4$, and $S4$.

If X and Y are formula sets then we write $X; Y$ to denote the union of them. We write $X; \varphi$ for $X; \{\varphi\}$. We need the two following auxiliary lemmas. The first one is used to reduce lengths of modal contexts of clauses.

LEMMA 4. *In the following, let p and q be new primitive propositions (i.e. p and q occur only at the indicated positions) and φ a simple clause. Then the following pairs of formula sets are equisatisfiable in any normal modal logic that is an extension of $K4$.*

(1)	$X; \Box^2\varphi$	*and*	$X; \Box^2 p; \Box(\neg p \vee \varphi)$	
(2)	$X; \Box^{2k}\varphi$	*and*	$X; \Box^k q; \Box(\neg q \vee \Box^k\varphi)$	*where* $k \geq 2$
(3)	$X; \Box^{2k+1}\varphi$	*and*	$X; \Box^{k+1} q; \Box(\neg q \vee \Box^k\varphi)$	$k \geq 1$
(4)	$X; \Box(a \vee \Box^{2k}\varphi)$	*and*	$X; \Box(a \vee \Box^k q); \Box(\neg q \vee \Box^k\varphi)$	$k \geq 1$
(5)	$X; \Box(a \vee \Box^{2k+1}\varphi)$	*and*	$X; \Box(a \vee \Box^{k+1} q); \Box(\neg q \vee \Box^k\varphi)$	$k \geq 0$

Proof. $\rightarrow$) Choose one of the pairs. Suppose that the LHS set is satisfied in a model $M = \langle W, \tau, R, h\rangle$. Let $M' = \langle W, \tau, R, h'\rangle$ with $x \in h'(u)$ iff $x \in h(u)$ for $x \neq p$ and $x \neq q$, $p \in h'(u)$ iff $M, u \vDash \varphi$, and $q \in h'(u)$ iff $M, u \vDash \Box^k\varphi$, where p and q are the new primitive propositions. It is easily seen that the RHS set is satisfied in M'.

$\leftarrow$) Choose one of the pairs. We show that the RHS formula set implies the LHS set in any modal logic that is an extension of $K4$.

The assertion holds for the pair (1) because that the formulas $\Box(\neg p \vee \varphi) \rightarrow \Box^2(\neg p \vee \varphi)$ and $\Box^2 p \wedge \Box^2(\neg p \vee \varphi) \rightarrow \Box^2\varphi$ are $K4$-valid.

The assertion holds for the pair (2) because that the formulas $\Box(\neg q \vee \Box^k\varphi) \to \Box^k(\neg q \vee \Box^k\varphi)$ and $\Box^k q \wedge \Box^k(\neg q \vee \Box^k\varphi) \to \Box^{2k}\varphi$ are $K4$-valid.

The assertion holds for the pair (4) because the following formulas are $K4$-valid: $\Box(\neg q \vee \Box^k\varphi) \to \Box^{k+1}(\neg q \vee \Box^k\varphi)$ and

$$\Box(a \vee \Box^k q) \wedge \Box^{k+1}(\neg q \vee \Box^k\varphi) \to \Box(a \vee \Box^{2k}\varphi)$$

Analogously, the assertion holds for the pairs (3) and (5). ■

LEMMA 5. *Let L be a normal modal logic that is an extension of $K4$. Every formula set X can be translated to an L-equisatisfiable set Y of clauses with modal depth bounded by 2. Furthermore, if X is a set of Horn formulas then Y is a set of Horn clauses. The translation can be done in polynomial time and the length of Y is bounded by a polynomial in the length of X.*

Proof. By Lemma 2, we can translate X in polynomial time to a clause set Z such that: X and Z are L-equisatisfiable; if X is a set of Horn formulas then Z is a set of Horn clauses; the modal depth of Z is equal to the modal depth of X, and the length of Z is of quadratic order in the length of X.

We refer to the pairs of equisatisfiable formula sets given in Lemma 4 as translation rules (with left to right direction of application). We then apply[3] these translation rules to Z. We apply the rule (1) only when the modal depth of φ is 1, and the rule (5) only when $k \geq 1$, or $k = 0$ and φ is not a classical literal. We apply the rules until no more changes can be made to the set. Let Y be the resulting set. Observe that the modal depth of Y is bounded by 2.

Observe also that each of the applications decreases the modal depth of some formula of the set by a half (with an inaccuracy up to 2) and increases the length of the set by a constant number (of symbols). Hence there exists a constant h such that we can decrease the modal depth of the set by a half (with an inaccuracy up to 2) while the length of the set increases not more than h times. Hence the process terminates in polynomial time. It is easily seen that the length of Y is bounded by a polynomial in the size of Z, and Y is a set of Horn clauses if so is Z.

[3]Each application of a rule is done for the whole formula set but not a fragment.

Hence, the translation from X to Y (via Z) is done in polynomial time, the length of Y is bounded by a polynomial in the length of X, and Y is a set of Horn clauses if X is a set of Horn formulas. ∎

As a consequence we have the following result:

THEOREM 6. *The complexity of the satisfiability problem of sets of Horn formulas with modal depth bounded by $k \geq 2$ in the logics K4, KD4, and S4 is PSPACE-complete.*

This theorem follows from the above lemma and the reason that the similar problem without bounding modal depth is PSPACE-complete [4]. The assertion for $S4$ has been previously proved by Chen and Lin [4].

By this theorem, the complexity of the satisfiability problem of formula sets (without the Horn restriction) with modal depth bounded by $k \geq 2$ in $K4$, $KD4$, and $S4$ is PSPACE-complete (the upper bound follows from [15]).

THEOREM 7. *The complexity of the satisfiability problem of sets of Horn formulas with modal depth bounded by $k \geq 2$ in the logic K is NP-complete.*

Proof. The upper bound follows from Halpern [11]. For the lower bound, we use a reduction from the 3SAT problem, which is known to be NP-hard. The 3SAT problem is to check satisfiability of a clause set $X = \{C_1, \ldots, C_n\}$, where $C_i = c_{i1} \vee c_{i2} \vee c_{i3}$ and c_{i1}, c_{i2}, c_{i3} are classical literals. Given such a set X, we construct in polynomial time a set Y of Horn formulas with modal depth bounded by 2 such that X is satisfiable iff Y is K-satisfiable.

Let t and f be new propositions, which informally stand for "true" and "false". The presence of the formula $\Box f$ (resp. $\Diamond t$) at a world w informally says that there are no worlds (resp. there is some world) accessible from w. Let Y be the set consisting of the formulas

$$\begin{gathered}\Diamond p_i, \Diamond q_i, \neg\Diamond(p_i \wedge q_i), \neg\Diamond^2 f, \Box^2 t,\\ \Diamond(p_i \wedge \Box f) \wedge \Diamond(q_i \wedge \Box f) \rightarrow c_{i1},\\ \Diamond(p_i \wedge \Diamond t) \rightarrow c_{i2},\\ \Diamond(q_i \wedge \Diamond t) \rightarrow c_{i3},\end{gathered}$$

for $1 \leq i \leq n$, and p_i and q_i are new propositions. Denote the set Y also by $\pi_{3SAT}(X)$. Note that Y contains only Horn formulas with modal depth bounded by 2.

Suppose that X is satisfied by a variable assignment V. We show that Y is K-satisfiable. Let $M = \langle W, \tau, R, h \rangle$ be a model defined as: $W = \{\tau, w_{1p}, w_{1q}, \ldots, w_{np}, w_{nq}, u\}$, $h(\tau) = \{p \mid V(p)\}$, $h(u) = \{t\}$, and for $1 \leq i \leq n$, $h(w_{ip}) = \{p_i\}$ and $h(w_{iq}) = \{q_i\}$, and
$R = \{(\tau, w_{ip}), (\tau, w_{iq}) \mid 1 \leq i \leq n\} \cup \{(w_{ip}, u) \mid 1 \leq i \leq n \text{ and } V(c_{i2})\}$
$\cup \{(w_{iq}, u) \mid 1 \leq i \leq n \text{ and } V(c_{i3})\}$.

It is easy to verify that $M \vDash Y$. Therefore Y is K-satisfiable.

Now suppose that Y is K-satisfiable. We show that X is satisfiable. Let M be a model of Y. Let w_{ip}, w_{iq} be worlds accessible from τ such that $M, w_{ip} \vDash p_i$ and $M, w_{iq} \vDash q_i$, for $1 \leq i \leq n$. If there exists a world accessible from w_{ip}, then $M, \tau \vDash \Diamond(p_i \wedge \Diamond t)$, and hence $M, \tau \vDash c_{i2}$. Similarly, if there exists a world accessible from w_{iq}, then $M, \tau \vDash c_{i3}$. If there are no worlds accessible from w_{ip} or w_{iq}, then $M, \tau \vDash \Diamond(p_i \wedge \Box f) \wedge \Diamond(q_i \wedge \Box f)$, and hence $M, \tau \vDash c_{i1}$. Consequently, $M, \tau \vDash C_i$, for $1 \leq i \leq n$. Hence $M, \tau \vDash X$, and X is satisfiable. ■

In the remainder of this section, we study the satisfiability problem of modal formulas with modal depth bounded by 1. The problem is NP-complete, and for the Horn fragment it is PTIME-complete, for all of the monomodal logics considered in this work. Some parts of these results immediately follow from known ones. We complete the picture by the two following theorems.

THEOREM 8. *The complexity of the satisfiability problem of formulas with modal depth bounded by 1 in the logics K4, KD4, and S4 is NP-complete.*

Proof. The lower bound NP-hard follows from the fact that the satisfiability problem in the classical propositional logic is NP-complete. For the upper bound, let L be one of the logics $K4$, $KD4$, $S4$, and let X be any L-satisfiable formula set with modal depth bounded by 1.

It can be proved that X has an L-model $M = \langle W, \tau, R, h \rangle$ such that for any u and v different to τ, if $R(\tau, u)$ and $R(u, v)$ hold, then $u = v$. In fact, if $M' = \langle W, \tau, R', h \rangle$ is an L-model of X, then by deleting edges (u, v) with $u \neq \tau$ from R' and adding edges (u, u) for $u \neq \tau$ to the frame, we obtain such a mentioned L-model M of X.

An L-model M of X with the mentioned frame restriction can be nondeterministically constructed in polynomial time by building an L-model graph for X (see, e.g., [21, 10, 18] for the technique). Therefore the satisfiability problem of formulas with modal depth bounded by 1 in $K4$, $KD4$, and $S4$ belongs to the NP class. ∎

THEOREM 9. *The complexity of the satisfiability problem of sets of Horn formulas with modal depth bounded by 1 in K, K4, KD4, and S4 is PTIME-complete.*

Proof. The lower bound PTIME-hard follows from that the complexity of the satisfiability problem of sets of Horn formulas in the classical propositional logic is PTIME-complete (Jones and Laaser [13]).

By the result of [17], every positive modal logic program with modal depth bounded by 1 has the least $KD4$-model and the least $S4$-model, which can be constructed in polynomial time and have polynomial size. Consequently, by Lemmas 2 and 1, the problem of checking satisfiability of sets of Horn formulas with modal depth bounded by 1 in $KD4$ and $S4$ is decidable in PTIME.

It remains to show that the similar problem for the logics K and $K4$ is decidable in PTIME. Let L denote K or $K4$, and P be any positive modal logic program with modal depth bounded by 1. Let $M = \langle W, \tau, R, H \rangle$ be the model graph constructed as follows.

1. Let $W = \{\tau, \rho\}$, $R = \{(\tau, \rho)\}$, $H(\tau) = P$, $H(\rho) = \emptyset$.
2. For every $w \in W$, and every $\varphi \in H(w)$,
   (a) Case $\varphi = (B_1 \wedge \ldots \wedge B_k \rightarrow A)$: if $M, w \models B_i$ for all $1 \leq i \leq k$, then add A to $H(w)$;
   (b) Case $\varphi = \Box\psi$: add ψ to every world u accessible from w;
   (c) Case $\varphi = \Diamond p$: if $M, w \not\models p$ then add a new world u with content $\{p\}$ to W and connect w to u (i.e. let $W = W \cup \{u\}$, $H(u) = \{p\}$, $R = R \cup \{(w, u)\}$).
3. While some change occurred, repeat step 2.

Observe that, for any w and u, $R(w, u)$ holds only when $w = \tau$ (since the modal depth is bounded by 1). Hence, the above algorithm terminates in polynomial time. It can be shown by induction on the structure of φ that for any $w \in W$ and any $\varphi \in H(w)$, $M, w \vDash \varphi$. Hence M is a K-model of P. By the mentioned property of R, M is also a $K4$-model of P.

If $N = \langle W', \tau', R', h' \rangle$ is a model of P such that $R' \neq \emptyset$ and for any x, y, $R'(x, y)$ holds only when $x = \tau$, then $M \leq N$. This claim can be proved by showing that it is an invariant of the loop of the above algorithm that there exists a relation $r \subseteq W \times W'$ such that the following assertions hold:

$r(\tau, \tau')$
$\forall x \;\; R(\tau, x) \rightarrow \exists x' \; R'(\tau', x') \wedge r(x, x')$
$\forall x' \; R'(\tau', x') \rightarrow \exists x \; R(\tau, x) \wedge r(x, x')$
$\forall x, x' \; \forall \varphi \in H(x) \;\; r(x, x') \rightarrow N, x' \vDash \varphi$

Such relations r can be built as follows: After the execution of step 1, let $r = \{(\tau, \tau')\} \cup \{(\rho, w') \mid R'(\tau', w')\}$, and after each execution of step 2c, let $r = r \cup \{(u, u') \mid R'(\tau', u') \text{ and } p \in h'(u')\}$.

If P has a flat model, then let M' be the least flat model of P, else let $M' = M$. Both M and M' can be constructed in polynomial time and have size bounded by a polynomial in the size of P.

We claim that for any positive formula φ with modal depth bounded by 1, $P \nvDash_L \varphi$ iff $M \nvDash \varphi$ or $M' \nvDash \varphi$. The "if" part clearly holds. For the "only if" part, suppose that $P \nvDash_L \varphi$, where φ is a positive formula with modal depth bounded by 1. It follows that there exists an L-model N of P such that $N \nvDash \varphi$. Let $N_{|1}$ be the model obtained from N by deleting all edges not starting from τ. We have $N_{|1} \vDash P$ and $N_{|1} \nvDash \varphi$, because the modal depths of P and φ are bounded by 1. If $N_{|1}$ is a flat model, then M' is the least flat model of P, and hence $M' \nvDash \varphi$. Otherwise, $M \leq N_{|1}$, and hence $M \nvDash \varphi$.

By Lemmas 2 and 1, we conclude that checking satisfiability of sets of Horn formulas with modal depth bounded by 1 in K and $K4$ is decidable in PTIME. ■

4 On the Complexity of Multimodal Logics

A language for multimodal logics uses n pairs of modal operators $\Box_i$ and $\Diamond_i$, for $1 \leq i \leq n$, where n is a fixed number *greater than* 1.

Formulas in multimodal logics are formed in the usual way. Interpretations used for multimodal logics are usually Kripke models with n accessibility relations (one for each of the pairs $\Box_i$ and $\Diamond_i$, $1 \leq i \leq n$). The satisfaction relation is also defined in the usual way.

Multimodal logics can be formed by combining modal logics. The combination of modal logics has been intensively studied in the last decade (see, e.g., [1, 7, 14, 9, 16, 5, 22, 19, 8]). A simple way to combine modal logics is to make their fusion and we can consider fusions of variants (by renaming modal operators) of the same logic. Given a monomodal logic L, the fusion L_n is the multimodal logic axiomatized by the axioms of the classical propositional logic, the modus ponens rule, the modal axioms and modal rules of L with $\Box$ and $\Diamond$ replaced respectively by $\Box_i$ and $\Diamond_i$, for each $1 \leq i \leq n$. Note that there are no interaction axioms between different kinds of modal operators in L_n.

In this section, we discuss the complexity of the multimodal logics L_n under the restriction to the Horn fragment and/or bounded modal depth, where L is one of the 15 basic monomodal logics. We show how Table 1 changes when each logic L is replaced by L_n.

In [12], Halpern and Moses showed that the satisfiability problem in the multimodal logics K_n, T_n, $S4_n$, $KD45_n$, and $S5_n$ is PSPACE-complete. Halpern in [11] claimed that the PSPACE-complete complexity also holds for $K45_n$, as its proof does not differ much from the proof for $KD45_n$.

The complexity PSPACE-complete also holds for KD_n, $K4_n$, $KD4_n$, $K5_n$, and $KD5_n$. The reasons are as follows:

- Nondeterministic PSPACE algorithms for checking satisfiability in KD_n, $K4_n$, $KD4_n$, $K5_n$, and $KD5_n$ can be developed, e.g., in a similar way as for K_n, $S4_n$, and $KD45_n$ in [12]. Hence we have the upper bound PSPACE.

- If $L_n \in \{KD_n, K4_n, KD4_n\}$, then the lower bound for L_n follows from the lower bound for L (PSPACE-hard). The lower bound PSPACE-hard for $K5_n$ and $KD5_n$ will be shown later in this section.

As a corollary (of the upper bound), every PSPACE-completeness result in Table 1 for $L \in \{K, KD, T, K4, KD4, S4\}$ also holds for L_n. Note that Theorem 6 is useful here for $K4_n$ and $KD4_n$.

We guess that the satisfiability problem in the multimodal logics KB_n, KDB_n, B_n, and $KB5_n$ is also PSPACE-complete. For KB_n, KDB_n, and B_n, it suffices to show the upper bound. If our prediction is true, then the satisfiability problem of sets of Horn clauses in KB_n, KDB_n, and B_n is also PSPACE-complete.

In [11], Halpern showed that the satisfiability problem of formulas with modal depth bounded by $k \geq 2$ in K_n, T_n, $K45_n$, $KD45_n$, and $S5_n$ is NP-complete. (The lower bound NP-hard follows from the lower bound of the monomodal case, the upper bound NP can be seen not difficultly.) Using similar argumentations, one can claim that the assertion also holds for KD_n, KB_n, KDB_n, B_n, $K5_n$, $KD5_n$, $KB5_n$.

Next, we claim that the satisfiability problem of formulas with modal depth bounded by 1 in L_n is NP-complete for L being any one the 15 basic monomodal logics. The only point that needs justification is the case of $K4_n$, $KD4_n$, and $S4_n$. For this case, use similar argumentations as the proof of Theorem 8.

We now consider the restriction to the Horn fragment.

Let Horn formulas be defined similarly as in the case of monomodal logics. A *clause* is a formula of the form $\Delta(A_1 \vee \ldots \vee A_h \vee \neg B_1 \vee \ldots \vee \neg B_k)$, where Δ is a sequence of universal modal operators, A_i and B_j are atoms of the form p, $\Box_t p$, or $\Diamond_t p$. Let Horn clauses and positive logic programs be defined similarly as in the case of monomodal logics. It can be seen that Lemma 2 still holds for normal multimodal logics.

In the following, we show that the satisfiability problem of sets of Horn clauses in $K5_n$, $KD5_n$, $K45_n$, and $KD45_n$ is PSPACE-complete.

Let X be a set of clauses in the language of monomodal logics. Let $\pi_{bi}(X)$ be the set of clauses obtained from X as follows: modal operators at odd modal nesting depths are subscripted by 1, and modal operators at even modal nesting depths are subscripted by 2. For example, the clause $\Box\Box\Box(p \vee \Box q \vee \Diamond r)$ is replaced by $\Box_1\Box_2\Box_1(p \vee \Box_2 q \vee \Diamond_2 r)$. It is clear that if X is a set of Horn clauses then $\pi_{bi}(X)$ is also a set of Horn clauses.

LEMMA 10. *Let X be a set of Horn clauses in the language of monomodal logics, L_n be either $K5_n$ or $K45_n$, and LD_n be either $KD5_n$ or $KD45_n$. Then X is K-satisfiable iff $\pi_{bi}(X)$ is L_n-satisfiable; and X is KD-satisfiable iff $\pi_{bi}(X)$ is LD_n-satisfiable.*

The proof of this lemma is not included due to the lack of space.

THEOREM 11. *The satisfiability problem of sets of Horn clauses in $K5_n$, $KD5_n$, $K45_n$, and $KD45_n$ is PSPACE-complete.*

Proof. The upper bound PSPACE has been justified earlier (for the case without restrictions). The lower bound PSPACE-hard follows from the above lemma and the facts that the satisfiability problem of sets of Horn clauses in K and KD is PSPACE-complete [4] and $\pi_{bi}(X)$ can be obtained from X in linear time and has a linear size (in the size of X). ■

COROLLARY 12. *The satisfiability problem (without restrictions) in $K5_n$ and $KD5_n$ is PSPACE-complete.*

We now consider the combination of the restriction to the Horn fragment and the restriction to bounded modal depth.

THEOREM 13. *The satisfiability problem of sets of Horn clauses with modal depth bounded by $k \geq 2$ in the multimodal logics K_n, KB_n, $K5_n$, $K45_n$, $KB5_n$ is NP-complete, and in KD_n, T_n, KDB_n, B_n, $KD5_n$, $KD45_n$, $S5_n$ is PTIME-complete.*

The two groups of modal logics mentioned in this theorem differ at the aspect that logics in the first group are non-serial, while logics in the second group are serial. For $L \in \{KB, K5, K45, KB5\}$, the complexity jumps from PTIME-complete for L to NP-complete for L_n because that L is *almost serial*[4] while L_n does not have such a similar property.

Sketch of the proof Consider the case of K_n, KB_n, $K5_n$, $K45_n$, $KB5_n$. The essential point here is the lower bound NP-hard. Let X

[4]A frame $\langle W, \tau, R\rangle$ is *connected* if W contains only worlds reachable directly or indirectly from τ via R. A monomodal logic L is *almost serial* if every connected L-frame $\langle W, \tau, R\rangle$ with $W \neq \{\tau\}$ is serial (i.e. $\forall x \exists y\, R(x, y)$).

be a set of clauses of the form $c_1 \vee c_2 \vee c_3$, where c_1, c_2, c_3 are classical literals. We use the translation π_{3SAT} as in the proof of Theorem 7. Consider the set $\pi_{bi}(\pi_{3SAT}(X))$. We claim that for $L_n \in \{K_n, KB_n, K5_n, K45_n, KB5_n\}$, X is satisfiable iff $\pi_{bi}(\pi_{3SAT}(X))$ is L_n-satisfiable. The proof of this is more or less the same as the proof of Theorem 7. Hence the 3SAT problem is reducible to the satisfiability problem of sets of Horn clauses with modal depth bounded by $k \geq 2$ in the multimodal logics K_n, KB_n, $K5_n$, $K45_n$, $KB5_n$. Therefore the latter problem is NP-hard.

For the case of $KD_n, T_n, KDB_n, B_n, KD5_n, KD45_n, S5_n$, the essential point is the upper bound PTIME. The proof of this is similar to the proof given in [17] of that the problem of checking satisfiability of sets of Horn clauses with modal depth bounded by $k \geq 2$ in KD, T, KDB, and B is in PTIME. The key of the proof is that if P is a positive logic program (in the language of multimodal logics) consisting of clauses whose modal depths are bounded by some constant k, then a least[5] L_n-model of P, for $L_n \in \{KD_n, T_n, KDB_n, B_n, KD5_n, KD45_n, S5_n\}$, can be constructed in polynomial time, and has a polynomial size if $L_n \in \{KD_n, T_n, KD5_n, KD45_n, S5_n\}$, or can be encoded in polynomial space if $L_n \in \{KDB_n, B_n\}$ (similarly as in the case of KDB and B [17]). ■

Analogously as for the proof of Theorem 9, one can show that the complexity of the satisfiability problem of sets of Horn clauses with modal depth bounded by 1 in K_n, KB_n, $K4_n$, $KD4_n$, $S4_n$, $K5_n$, $K45_n$, and $KB5_n$ is PTIME-complete. The main change is that, for K_n, KB_n, $K4_n$, $K5_n$, $K45_n$, $KB5_n$ we need to use 2^n models instead of 2 models as in the proof of Theorem 9. More precisely, for each set $I \subseteq \{1, \ldots, n\}$, consider the case when $\exists x\, R_i(\tau, x)$ holds iff $i \in I$ and construct a "minimal" model of the considered program for that case. We conclude that the satisfiability problem of sets of Horn clauses with modal depth bounded by 1 in L_n is PTIME-complete for every L being one the 15 basic monomodal logics.

In summary, there are open problems on the complexity of the satisfiability problem in KB_n, KDB_n, B_n, $KB5_n$ and the satisfia-

[5] An ordering of Kripke models in multimodal logics is defined similarly as for the monomodal case. See [20] for details.

bility problem of sets of Horn modal clauses in $KB5_n$ and $S5_n$. It is probable that these problems are PSPACE-complete. Under this assumption, Table 1 changes as follows when every logic L in that table is replaced by L_n:

- The satisfiability problem in $K5_n$, $KD5_n$, $K45_n$, $KD45_n$, $KB5_n$, $S5_n$ is PSPACE-complete for both of the cases: without restrictions or with the restriction to the Horn fragment.

- The satisfiability problem of sets of Horn clauses with modal depth bounded by $k \geq 2$ in the multimodal logics KB_n, $K5_n$, $K45_n$, and $KB5_n$ is NP-complete.

5 Conclusions

We have summarized the complexity of the satisfiability problem in all of the 15 basic normal monomodal logics under the restriction to the Horn fragment and/or bounded modal depth. To fulfill the complexity table, we have given some new results. Our Theorems 6 and 7 show that the modal logics K, $K4$, and $KD4$ are hard even under the mentioned restrictions. The restriction of modal depth to 1 is quite tight and the corresponding fragments are rather useless. However, our results for that case are still interesting from the theoretical point of view.

We have also discussed and given some results on the complexity of the multimodal logics L_n under the mentioned restrictions, where L is one of the 15 basic monomodal logics. There remain some open problems.

Acknowledgements

The author would like to thank professor Andrzej Szałas and the anonymous reviewers for helpful comments.

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