Utilitarian Deontic Logic

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ABSTRACT. This paper aims to examine Horty's proposal of utilitarian deontic logic [7]. It will focus on his dominance operators by way of simplified semantics. An axiomatization of the logic of the simplified semantics will be shown by the construction of a finite countermodel.

Analysis of the proof suggests possibilities of further investigations. The presented version of Horty's proposal does not make essential differences from standard deontic logic: the deontic operators are normal, and thus paradoxes of deontic logic occur. Non-monotonic agency operators should be considered.

1 Introduction

Horty [7], emphasizing game-theoretical motivations in the see-to-it-that (*stit*) theory of agency, proposes semantics of deontic operators with identifying duty as action according to dominant strategy. He introduces to stit semantics a utilitarian value assignment over branches, which represent possibilities at a moment, so as to define preference and dominance orderings among options at each moment. Defined with the dominance relations on choices, deontic operators are intended to label "actions at the dominant choice." He proposes several deontic modalities including operators according to properties of value assignment; "$\cdots$ holds in every dominant choice" and "$\cdots$ holds in every optimal choice," for example.

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It is natural to ask whether logic on the semantics is axiomatizable, and if yes, whether it is decidable. This paper answers yes to both questions.

Construction of a finite countermodel in this paper takes a simpler form than Horty's original semantics as argued in Section 3.1. It relies on the two related facts. First, the approach here follows Horty to take multi-S5 with an interactive axiom (independence of agents[1]) as agency operators, not operators argued in Belnap et al. [1]. The choice of agency operators makes the presentation considerably easier, while it does not incorporate essential properties of agency modalities required through conceptual analysis of the notion of real choices in the stit theory of agency.

Second, semantics in the paper is simplified than that in Horty's orignal presentation with branching time semantics.

Figure 1. Utilitarian frame for a single agent

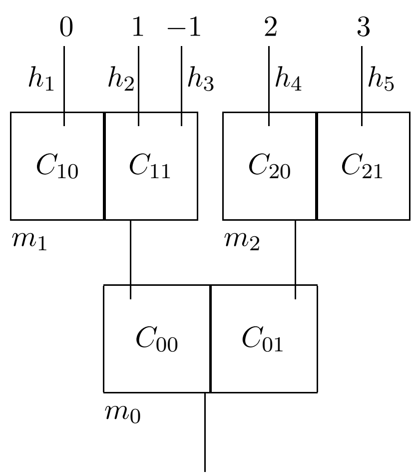

Figure 1 illustrates an utilitarian frame for a single agent in Horty's presentation. The tree structure represents a branching structure for the indeterministic time. It is open to the future, where each branch

[1]The corresponding semantic condition intuitively states that any combination of agents' choices can be realized.

(or *history*) represents a possible growth of the world. There are three moments where the agent can make a choice among two options C_{i0} and C_{i1} available at each m_i. A real number is assigned to each history in the structure to represent its utilitarian value, based on which the dominance ordering and the optimal ordering are defined. While readers should consult the following section for the exact definitions, it should also fit to a game-theoretical intuition that C_{01} is dominant over C_{00} at m_0 and C_{21} over C_{20} at m_2, while neither C_{01} nor C_{11} is dominant over each other at m_1.

Taking a language with only agency and deontic modalities whose truth conditions do not involve reference of any past or future moment, multi-S5 structures associated with a utilitarian value function preserve model and frame validity in branching time semantics as explained in Section 3.1.

2 Preliminary

2.1 Language

The language $\mathfrak{L}$ contains denumerably many propositional variables $p_0, p_1, \ldots$, denumerably many terms for agents $\alpha_0, \alpha_1, \ldots$, an identity symbol $=$, truth-functional operators $\neg$ and $\wedge$, and non-truth-functional operators $\Box$, $[\;\; cstit\text{:} \;\;]$ and $\odot$. Formulas are defined as usual, except $\alpha = \beta$, $\Box A$, $[\alpha\ cstit\text{: } A]$, $\odot[\alpha\ cstit\text{: } A]$ are formulas whenever α and β are agent terms for agents and A is a formula. Abbreviations such as $\rightarrow$, $\leftrightarrow$, $\vee$, $\top$, $\bot$ and $\Diamond$ are introduced as usual. The following abbreviations are also defined:

$$\alpha \neq \beta =_{df} \neg(\alpha = \beta),$$

$$\triangle \alpha A =_{df} [\alpha\ cstit\text{: } A],$$

$$\ominus \alpha A =_{df} \odot[\alpha\ cstit\text{: } A].$$

An additional abbreviation $\mathit{diff}(\beta_0, \ldots, \beta_n)$ (where $n > 0$) is defined as follows:

$$\mathit{diff}(\beta_0, \beta_1) =_{df} \beta_0 \neq \beta_1,$$

$$\mathit{diff}(\beta_0, \beta_1, \ldots, \beta_{n+1}) =_{df} \mathit{diff}(\beta_0, \ldots, \beta_n) \wedge (\textstyle\bigwedge_{0 \leqslant k \leqslant n} \beta_k \neq \beta_{n+1}).$$

A *stit formula* is a formula whose major operator is $\triangle\alpha$ for some α, and an *ought formula* is a formula whose major operator is $\ominus\alpha$ for some α.

2.2 Branching-time semantics

Horty [7] proposes a semantics for *stit* and dominant ought based on the branching semantics for indeterminist time proposed by Prior and Thomason (see [9], [11] and [12]) .

A tree structure (Branching-time structure, BT) is a pair $\mathfrak{F} = \langle \mathsf{T}, < \rangle$, where T is a nonempty set, whose members m, m' etc. are called *moments*; $<$ is a tree relation on T, i.e., a partial order on T being (1) linear to the past and (2) connected. A *history* in $\mathfrak{F}$ is a maximal linear subset of T. A history h is said to *go through a moment* m when $m \in h$. A *moment-history pair* is a pair of a moment m and a history which goes through m. H_m denotes the set of histories which go through m; $\mathsf{H}_{\mathfrak{F}}$ denotes the set of all the histories in $\mathfrak{F}$; and $Moment\text{-}History$ denotes the set of all moment-history pairs in $\mathfrak{F}$.

The stit analysis adds a set of agents and a choice function for each agents on a tree structure. A *stit frame* (Branching-time and agent-choice structure, $BT + AC$) is a quadruple $\mathfrak{F} = \langle \mathsf{T}, <, \mathsf{Agent}, \mathsf{choice} \rangle$ satisfying the following. Agent is an nonempty set of *agents*; choice is a function that assigns to each $a \in \mathsf{Agent}$ and $m \in T$ a partition choice_a^m of $\mathsf{H_m}$ satisfying the following conditions.

> **Independence of Agents**: For each function f that assigns to each pair of $a \in \mathsf{Agent}$ and $m \in \mathsf{T}$ a member of choice_a^m,
>
> $$\bigcap_{a \in \mathsf{Agent}} f(a) \neq \varnothing$$
>
> .
>
> In other words, let select^m be the set of all functions f on Agent such that $f(a) \in \mathsf{choice}_a^m$ for every $a \in \mathsf{Agent}$. For each proper subset A of Agent, let $\mathsf{State}_{\mathsf{A}}^m = \{\bigcap_{a \in \mathsf{Agent}-\mathsf{A}} f(a) \mid f \in \mathsf{select}^m\}$. Then, for any $m \in T$, $\varnothing \notin \mathsf{State}_{\varnothing}^m$.
>
> **No choice between undivided histories**: For any $h, h' \in$

$\mathsf{H}_{\mathfrak{F}}$, $m_0 \in h \cap h'$, and $\alpha \in \mathsf{Agent}$, if there is $m \in h \cap h'$ such that $m_0 < m$, $h \in \mathsf{choice}^{m_0}_{\alpha}$ if and only if $h' \in \mathsf{choice}^{m_0}_{\alpha}$.

A *utilitarian stit frame* is a 5-tuple $\mathfrak{F} = \langle \mathsf{T}, <, \mathsf{Agent}, \mathsf{choice}, \mathsf{value} \rangle$, where $\langle \mathsf{T}, <, \mathsf{Agent}, \mathsf{choice} \rangle$ is a stit frame, and value is a function from $\mathsf{H}_{\mathfrak{F}}$ to the set of real numbers.

Let $\mathfrak{F} = \langle \mathsf{T}, <, \mathsf{Agent}, \mathsf{choice}, \mathsf{value} \rangle$ be a utilitarian stit frame. We will use K, K' etc. to range over subsets of $\mathsf{H}_{\mathfrak{F}}$. For each $a \in \mathsf{Agent}$, $m \in \mathsf{T}$, and $h \in \mathsf{H}_m$, we use $\mathsf{choice}^m_a(h)$ for the member of choice^m_a to which h belongs. For each $m \in \mathsf{T}$, $h_0, h_1 \in \mathsf{H_m}$, let $R^m_a h_0 h_1$ iff $\mathsf{choice}^m_a(h_0) = \mathsf{choice}^m_a(h_1)$.

2.3 States, preference, and dominance

The notion of *states* plays the main role in preference and dominance. Let $\Gamma \subseteq \mathsf{Agent}$ and $m \in \mathsf{T}$. Define $\mathsf{State}^m_\Gamma(h) = \bigcap_{a \notin \Gamma} \mathsf{choice}^m_a(h)$. Let $\mathsf{State}^m_\Gamma = \{\mathsf{State}^m_\Gamma(h) : h \in \mathsf{H}_m\}$.

For each $a \in \mathsf{Agent}$ and $m \in \mathsf{T}$, we will write State^m_a for $\mathsf{State}^m_{\{a\}}$. Let $K, K' \subseteq \mathsf{H_m}$. K' is *weakly preferred to* K, written $K \leq K'$, iff for each $h_0 \in K$ and each $h_1 \in K'$, $\mathsf{value}(h_0) \leq \mathsf{value}(h_1)$. K' is *strongly preferred to* K, written $K < K'$, iff $K \leq K'$ and not $K' \leq K$. Let $a \in \mathsf{Agent}$ and let $K, K' \in \mathsf{choice}^m_a$. K' *weakly dominates* K, written $K \preceq K'$, iff for each $S \in \mathsf{state}^m_a$, $K \cap S \leq K' \cap S$. K' *strongly dominates* K, written $K \prec K'$, iff $K \preceq K'$ and not $K' \preceq K$.

Horty [7] extablishes the following, which should aid the reader's intuition.

PROPOSITION 1. *Let* $\langle \mathsf{T}, <, \mathsf{Agent}, \mathsf{choice}, \mathsf{value} \rangle$ *be a utilitarian stit frame, let* $m \in \mathsf{T}$, $a \in \mathsf{Agent}$*, and let* State^m_a *be as defined above. Then the following hold.*

1. $X < Y$ *iff* $\mathsf{value}(h') \leq \mathsf{value}(h)$ *for each* $h' \in X$ *and* $h \in Y$*, and* $\mathsf{value}(h_2) < \mathsf{value}(h_1)$ *for some* $h_2 \in X$ *and* $h_1 \in Y$.

2. *The weak preference ordering* $\leq$ *is transitive.*

3. *The strong preference ordering* $<$ *is a strict partial ordering.*

4. $X < Y$ *and* $Y \leq Z$ *only if* $X < Z$.

5. *$X \prec Y$ iff $X \cap S \leq Y \cap S$ for every $S \in \mathsf{State}^m_a$, and $X \cap S < Y \cap S$ for some $S \in \mathsf{State}^m_a$.*

6. *The weak dominance ordering $\preceq$ is transitive.*

7. *The strong dominance ordering $\prec$ is a strict partial ordering.*

8. *$X \prec Y$ and $Y \preceq Z$ only if $X \prec Z$.*

2.4 Utilitarian stit models, truth conditions, and semantic notions

Let $\mathfrak{F} = \langle \mathsf{T}, <, \mathsf{Agent}, \mathsf{choice}, \mathsf{value}\rangle$ be a utilitarian stit frame. A *valuation τ on $\mathfrak{F}$* is a function that assigns each agent term a member of Agent and each atomic formula a subset of $Moment\text{-}History$. A *utilitarian stit model* is a pair $\mathfrak{M} = \langle \mathfrak{F}, \tau\rangle$ where $\mathfrak{F}$ is a utilitarian stit frame and τ is a valuation on $\mathfrak{F}$.

That A is *true in* $\mathfrak{M} = \langle \mathfrak{F}, \tau\rangle$ *at* m/h, written $\mathfrak{M}, m/h \vDash A$, is defined recursively as follows, where p is any atomic formula and α is any agent term.

$\mathfrak{M}, m/h \vDash p$	iff	$m/h \in \tau(p)$;
$\mathfrak{M}, m/h \vDash \alpha = \beta$	iff	$\alpha^\tau = \beta^\tau$;
$\mathfrak{M}, m/h \vDash \neg A$	iff	$\mathfrak{M}, m/h \nvDash A$ (i.e., not $\mathfrak{M}, m/h \vDash A$);
$\mathfrak{M}, m/h \vDash A \wedge B$	iff	$\mathfrak{M}, m/h \vDash A$ and $\mathfrak{M}, m/h \vDash B$;
$\mathfrak{M}, m/h \vDash \Box A$	iff	$\mathfrak{M}, m/h' \vDash A$ for every $h' \in H_m$;
$\mathfrak{M}, m/h \vDash \triangle\alpha A$	iff	$\mathfrak{M}, m/h' \vDash A$ for every $h' \in \mathsf{choice}^m_{\alpha^\tau}(h)$;
$\mathfrak{M}, m/h \vDash \ominus\alpha A$	iff	for each $K \in \mathsf{choice}^m_{\alpha^\tau}$ such that $K \nsubseteq \|A\|^\tau_m$, there is a $K' \in \mathsf{choice}^m_{\alpha^\tau}$ such that (i) $K \prec K'$, (ii) $K' \subseteq \|A\|^\tau_m$, and (iii) $K'' \subseteq \|A\|^\tau_m$ for every $K'' \in \mathsf{choice}^m_{\alpha^\tau}$ with $K' \preceq K''$.

where $\|A\|^\tau_m = \{h \in \mathsf{H}_m \mid \mathfrak{M}, m/h \vDash A\}$. Notions of validity and satisfiability are defined the same way as in modal logic. A is *valid in* $\mathfrak{M}$ if $\mathfrak{M}, m/h \vDash A$ for every $m/h \in Moment\text{-}History$, and A is *valid in* $\mathfrak{F}$ if A is valid in every model on $\mathfrak{F}$. A is *valid* if A is valid in every utilitarian stit frame. Φ is *satisfiable in* $\mathfrak{M}$ if for some m/h in $Moment\text{-}History$, $\mathfrak{M}, m/h \vDash A$ for every $A \in \Phi$, and Φ is *satisfiable* if Φ is satisfiable in a utilitarian stit model.

2.5 Classification of utilitarian stit frames

$\mathfrak{F} = \langle \mathsf{T}, <, \mathsf{Agent}, \mathsf{choice}, \mathsf{value} \rangle$ is *two-valued* if the range of value is $\{0, 1\}$. $\mathfrak{F}$ is *at-most-n-ary* (where $n > 0$) if $|\mathsf{choice}_a| \leqslant n$ for every $a \in \mathsf{Agent}$, and $\mathfrak{F}$ is a *finite-choice utilitarian stit frame* if choice_a^m is finite for every $a \in \mathsf{Agent}$. Counterpart notions on models are defined as usual.

Let $\mathfrak{F} = \langle \mathsf{T}, <, \mathsf{Agent}, \mathsf{choice}, \mathsf{value} \rangle$ be a utilitarian stit frame. For each $a \in \mathsf{Agent}$ and $m \in \mathsf{T}$, we define

$$\mathsf{optimal}_a^m = \{K \in \mathsf{choice}_a^m \mid \neg \exists K' (K' \in \mathsf{choice}_a \wedge K \prec K')\}.$$

$\mathfrak{F}$ is *optimal* at m if for each $a \in \mathsf{Agent}$, $K \in \mathsf{choice}_a^m - \mathsf{optimal}_a^m$ only if $K \prec K'$ for some $K' \in \mathsf{optimal}_a^m$. A utilitarian stit model $\mathfrak{M} = \langle \mathfrak{F}, \tau \rangle$ is *optimal* if $\mathfrak{F}$ is. It is easy to see that $\mathfrak{F}$ is optimal only if $\mathsf{optimal}_a^m \neq \varnothing$ for every $m \in \mathsf{T}$ and every $a \in \mathsf{Agent}$. It has been shown in Horty [7] that $\mathfrak{F}$ is a finite-choice utilitarian frame only if $\mathfrak{F}$ is optimal. It is also easy to verify the following.

PROPOSITION 2. *Let $\mathfrak{F} = \langle \mathsf{T}, <, \mathsf{Agent}, \mathsf{choice}, \mathsf{value} \rangle$ be any optimal utilitarian stit frame, and let $\mathfrak{M} = \langle \mathfrak{F}, \sigma \rangle$ be any model on $\mathfrak{F}$. Then for each m, h in $\mathfrak{F}$ with $m \in h$ and each formula $\ominus \alpha A$,*

(1) $\mathfrak{M}, m/h \vDash \ominus \alpha A$ *iff* $X \subseteq \|A\|_m^\sigma$ *for every* $X \in \mathsf{optimal}_{\alpha^\sigma}^m$.

Proof. Suppose on one hand that $\mathfrak{M}, m/h \vDash \ominus \alpha A$. Then, if $X \nsubseteq \|A\|_m^\sigma$ for any $X \in \mathsf{choice}_{\alpha^\sigma}^m$, we know by the truth-definition of $\ominus$ that there is an $Y \in \mathsf{choice}_{\alpha^\sigma}^m$ such that $X \prec Y$, and hence $X \notin \mathsf{optimal}_{\alpha^\sigma}^m$. It follows that $X \subseteq \|A\|_m^\sigma$ for every $X \in \mathsf{optimal}_{\alpha^\sigma}^m$.

Suppose on the other hand that $\mathfrak{M}, m/h \nvDash \ominus \alpha A$. Then by the truth-definition of $\ominus$, there is an $X \in \mathsf{choice}_{\alpha^\sigma}^m$ such that $X \nsubseteq \|A\|_m^\sigma$. Suppose for reductio that

(2) $X' \subseteq \|A\|_m^\sigma$ for every $X' \in \mathsf{optimal}_{\alpha^\sigma}^m$.

Then, since $\mathfrak{M}$ is optimal, there is an $Y \in \mathsf{optimal}_{\alpha^\sigma}^m$ such that $X \prec Y$, and then $Y \subseteq \|A\|_m^\sigma$. Applying the truth-definition of $\ominus$ again, there is a $Z \in \mathsf{choice}_{\alpha^\sigma}^m$ such that

(3) $Z \nsubseteq \|A\|_m^\sigma$

and $Y \preceq Z$. If $Z \notin \mathsf{optimal}^m_{\alpha^\sigma}$, there is an $X_0 \in \mathsf{Choice}^m_{\alpha^\sigma}$ such that $Z \prec X_0$, and then, since $Y \preceq Z$, $Y \prec X_0$ by Proposition 1 (8), and hence $Y \notin \mathsf{optimal}^m_{\alpha^\sigma}$, a contradiction. It follows that $Z \in \mathsf{optimal}^m_{\alpha^\sigma}$, and hence $Z \subseteq \|A\|^\sigma_m$ by (2), contrary to (3). From this reductio we conclude that $X' \nsubseteq \|A\|^\sigma_m$ for some $X' \in \mathsf{optimal}^m_{\alpha^\sigma}$. ■

2.6 Axiomatic systems

The logic L_0 takes as axioms all substitution instances of truth-functional tautologies and all formulas of the following forms, where in A8, $A(\alpha/\beta)$ is any formula obtained from A by replacing one or more occurrences of β with occurrences of α, and AIA is a scheme of schemes which holds for all $n > 0$:

A1 $\Box(A \to B) \to (\Box A \to \Box B)$, $\Box A \to A$, $\Diamond A \to \Box\Diamond A$

A2 $\triangle\alpha(A \to B) \to (\triangle\alpha A \to \triangle\alpha B)$, $\triangle\alpha A \to A$, $\neg\triangle\alpha A \to \triangle\alpha\neg\triangle\alpha A$

A3 $\ominus\alpha(A \to B) \to (\ominus\alpha A \to \ominus\alpha B)$

A4 $\Box A \to \triangle\alpha A \wedge \ominus\alpha A$

A5 $\ominus\alpha A \to \Diamond\triangle\alpha A$

A6 $\Box\ominus\alpha A \vee \Box\neg\ominus\alpha A$

A7 $\Box(\triangle\alpha A \to \triangle\alpha B) \to (\ominus\alpha A \to \ominus\alpha B)$

A8 $\alpha = \alpha$, $\alpha = \beta \to (A \to A(\alpha/\beta))$

AIA $\mathit{diff}(\beta_0, \ldots, \beta_n) \wedge (\bigwedge_{0\leqslant k\leqslant n} \Diamond\triangle\beta_k B_k) \to \Diamond(\bigwedge_{0\leqslant k\leqslant n} \triangle\beta_k B_k)$

and takes as rules of inference *modus ponens* and *necessitation*, i.e.,

RN from A to infer $\Box A$.

For each $n > 0$, let L_n be the logic obtained by adding APC_n below to L_0:

APC_n $\bigwedge_{0\leqslant k\leqslant n-1} \Diamond((\bigwedge_{0\leqslant i\leqslant k-1} \neg A_i) \wedge \triangle\alpha A_k) \to \bigvee_{0\leqslant k\leqslant n-1} A_k$

It has been shown in Horty [7] that for each agent term α, $\ominus\alpha$ is a normal modal operator. It is easy to show that A4–A8 are all valid in all utilitarian stit frames. Xu [15] shows that each instance of AIA is valid for all stit frames[2].

In the rest of this paper, when we use L without subscript, we presuppose that all axioms of L_0 are theorems of L, and *modus ponens* and RN hold in L. In particular, L could be any L_n with $n \geqslant 0$.

By ordinary modal logic, we know that the following are derivable in each logic mentioned above:

DR1 $A \to B \ / \ \Box A \to \Box B$, $\triangle\alpha A \to \triangle\alpha B$, $\ominus\alpha A \to \ominus\alpha B$

DR2 $A \ / \ \triangle\alpha A$, $\ominus\alpha A$

T1 $\alpha = \beta \to \Box(\alpha = \beta)$

T2 $\alpha \neq \beta \to \Box(\alpha \neq \beta)$

T3 $\ominus\alpha A \leftrightarrow \ominus\alpha\triangle\alpha A$

T4 $\ominus\alpha A \to \neg\ominus\alpha\neg A$

3 Canonical modal frames

3.1 Modal utilitarian stit frames

The semantics presented in Section 2.2 is based on $BT + AC$ structures on p. 214. The formal language in this paper contains no operators whose interpretation involves temporal reference (such as

[2]Some technical discussions related to AIA and the postulate *independence of agents* can be found in Xu [15] and Xu [16], including examples showing that the postulate cannot be fully expressed in a language like ours. It is easy to see that our AIA can be replaced by the scheme $\mathit{diff}(\beta_0, \ldots, \beta_n) \wedge (\bigwedge_{0\leqslant k\leqslant n} \Diamond\triangle\beta_k B_k) \to \Diamond(\bigwedge_{0\leqslant k\leqslant n} B_k)$. Schemes that aim at expressing *independence of agents*, using *dstit* instead of *cstit*, with or without $\Box$, can also be found in Xu [15] and Xu [16] and that for each $n > 0$, APC_n is valid in a stit frame iff the frame is at-most-n-ary. Hence by our definition of utilitarian stit frames, each instance of AIA is valid in every utilitarian stit frame, and each APC_n is valid in every at-most-n-ary utilitarian stit frame. It follows that all axioms of L_0 are valid in all utilitarian stit frames, and all axioms of $\mathsf{L}_n = \mathsf{L}_0 + \mathsf{APC}_n$ with $n > 0$ are valid in at-most-n-ary utilitarian stit frames, and hence, since *modus ponens* and *necessitation* are obviously validity preserving, we have soundness theorems for all L_n with $n \geqslant 0$.

tense operators or achievement stit operators). Therefore, the temporal relation in utilitarian stit frames can be eliminated.

Let $\mathfrak{M} = \langle \mathsf{T}, <, \mathsf{Agent}, \mathsf{choice}, \mathsf{value}, \tau \rangle$ be an arbitrary utilitarian stit model and an arbitrary moment $m \in \mathcal{T}$. Then $\mathfrak{M}(m/h) = \langle \mathsf{H}_m, \mathsf{Agent}, \mathsf{choice}^m, \mathsf{value}^m, \tau^m \rangle$, where choice^m is the restriction of choice to m, value^m the restriction of value to m, and τ^m is the function which assigns $\{h \in \mathsf{H}_m : m/h \in \tau(p)\}$ to each propositional variable p. Counterpart notions of states, preference, and dominance on $\mathfrak{M}(m/h)$ are defined by taking a history instead of the corresponding moment-history pair. Then, by induction:

OBSERVATION 3. For each formula A of $\mathcal{L}$ and for each moment-history pair m/h, $\mathfrak{M}, m/h \models A$ iff $\mathfrak{M}(m/h), h \models A$.

Let us call $F = \langle W, Agent, \{R_a\}_{a \in Agent}, v, V \rangle$ a *modal utilitarian stit frame*, where $\langle W, Agent, \{R_a\}_{a \in Agent}, v \rangle$ be an arbitrary multi-S5 frame with $Agent$ being the (non-empty) index set for modalities, and V be a function from W to real numbers (a value assignment). A *modal utilitarian stit model* is $\langle F, \tau \rangle$ where τ is a truth-value assignment in the usual sense.

Given a modal utilitarian stit model $M = \langle W, Agent, R, v, \tau \rangle$, an equivalent utilitarian stit model can be easily constructed. That is, with $r \notin W$, $M_r = \langle W \cup \{r\}, Agent, \mathsf{choice}_r, v_r, \tau_r \rangle$ is a utilitarian stit model, where choice_r is a function which assigns to each member a of $Agent$ the partition generated from R_a.

Notions such as truth and validity on modal utilitarian stit frames and models are defined in the same way as in the branching-time (BT) version.

The construction in the next subsection aims a modal utilitarian stit model to satisfy a consistent formula of L.

3.2 Construction of a modal frame

Let Φ be any maximal L-consistent set. We use W_Φ to denote the set of all maximal L-consistent sets w such that $\{\Box A \mid \Box A \in \Phi\} \subseteq w$. It is easy to verify, by ordinary modal logic (applying A1, A8 and A4), that the following holds:

REMARK 4. Let Φ be any maximal L-consistent set. Then the following hold.

1. for each $w \in \mathsf{W}_\Phi$, $\{A \mid \Box A \in w\} = \{\Box A \mid \Box A \in w\} = \{\Box A \mid \Box A \in \Phi\}$;
2. for each $w \in \mathsf{W}_\Phi$, $\Box A \in w$ iff $A \in w'$ for every $w' \in \mathsf{W}_\Phi$;
3. for each α and β, $\alpha = \beta \in \Phi$ iff $\alpha = \beta \in w$ for every $w \in \mathsf{W}_\Phi$;
4. for each $w \in \mathsf{W}_\Phi$ and each α, $\{A \mid \Box A \in w\} \subseteq \{A \mid \triangle\alpha A \in w\}$.

For each set Ξ of formulas, we use $\mathsf{Sub}(\Xi)$ for the closure of Ξ under subformulas. A set Ξ of formulas is *suitable* if the following conditions hold:

1. $\Xi = \mathsf{Sub}(\Xi)$
2. α and β occur in Ξ only if $\alpha = \beta \in \Xi$,
3. $\ominus\beta B \in \Xi$ only if $\triangle\beta B \in \Xi$.

Note that each set of formulas has a smallest suitable extension. For each suitable set Ξ of formulas, we fix Ξ^+ for $\Xi \cup \{\neg A \mid A \in \Xi\}$.

Let Φ be any masimal L-consistent set, and let Ξ be any non-empty set of formulas. Ξ is *coherent with* Φ if for each A, and for each α and β with $\alpha = \beta \in \Phi$, $\triangle\alpha A \in \Xi$ iff $\triangle\beta A \in \Xi$, and $\ominus\alpha A \in \Xi$ iff $\ominus\beta A \in \Xi$.

Let Φ be any maximal L-consistent set, and let Ξ be any suitable set coherent with Φ. We define the "regular modal frame for L with respect to Φ and Ξ" as follows.

First, when Ξ contains no modal formula, consider the model $\langle\{\Phi\}, \{a\}, \{=\}, v_\Phi, V_\Phi\rangle$, where $v_\Phi(p) = \{\Phi\}$ if $p \in \Phi$ and $v_\Phi = \emptyset$ otherwise, and $V_\Phi(\Phi) = 1$. The argument in Section 4 will work for such a case.

Now, the following construction in this section assumes that Ξ contains modal formulas. Let $\mathsf{W} = \{w \cap \Xi^+ \mid w \in \mathsf{W}_\Phi\}$. We will use x, y etc. to range over members of W.

For each α and β occurring in Ξ, $\alpha \cong_{\Phi,\Xi} \beta$ iff $\alpha = \beta \in \Phi \cap \Xi^+$. Let us use $\langle\alpha\rangle_{\Phi,\Xi}$ for the $\cong_{\Phi,\Xi}$-equivalence class to which α belongs, and let Agent be the set of all $\cong_{\Phi,\Xi}$-equivalence classes of terms for agents. We will use $\mathbf{a}, \mathbf{b}$ etc. to range over members of Agent.

For each $\mathbf{a} \in \mathsf{Agent}$, let $R_{\mathbf{a}}$ be the relation on W such that for each $x, y \in \mathsf{W}$, $R_{\mathbf{a}}xy$ iff $\{\triangle\alpha A \in x \mid \alpha \in \mathbf{a}\} = \{\triangle\alpha A \in y \mid \alpha \in \mathbf{a}\}$. Obviously, each $R_{\mathbf{a}}$ is an equivalence relation. We use choice to denote the function on Agent that assigns each $\mathbf{a} \in \mathsf{Agent}$ the set $\mathsf{choice}_{\mathbf{a}}$ of all $R_{\mathbf{a}}$-equivalence classes, and use $[x]_{\mathbf{a}}$ for the $R_{\mathbf{a}}$-equivalence class to which x belongs, i.e., $[x]_{\mathbf{a}} = \mathsf{choice}_{\mathbf{a}}(x)$.

Finally, $\langle \mathsf{W}, \mathsf{Agent}, \mathsf{choice} \rangle$ is the *regular modal frame for* L *with respect to* Φ *and* Ξ if W, Agent and choice are as specified above. Let, for each $\mathbf{a} \in \mathsf{Agent}$, $\Sigma_{\mathbf{a}} = \{\triangle\alpha A \mid \ominus\alpha A \in \Phi \cap \Xi^{+} \wedge \alpha \in \mathbf{a}\}$, and let $\Sigma = \bigcup_{\mathbf{a} \in \mathsf{Agent}} \Sigma_{\mathbf{a}}$. It is easy to see that $\Sigma = \{\triangle\alpha A \mid \ominus\alpha A \in \Phi \cap \Xi^{+}\}$. The *canonical modal frame for* L *with respect to* Φ *and* Ξ is $\langle \mathsf{W}, \mathsf{Agent}, \mathsf{choice}, \mathsf{value} \rangle$, where $\langle \mathsf{W}, \mathsf{Agent}, \mathsf{choice} \rangle$ is the regular modal frame for L with respect to Φ and Ξ, and value is the function on W such that for each $x \in \mathsf{W}$, $\mathsf{value}(x) = 1$ if $\Sigma \subseteq x$, and $\mathsf{value}(x) = 0$ otherwise. When Φ and Ξ are given explicit in the context, we will drop the references to Φ or Ξ as much as we can, and thus use $\langle \alpha \rangle$ for $\langle \alpha \rangle_{\Phi,\Xi}$, and $[x]_{\langle \alpha \rangle}$ for $[x]_{\langle \alpha \rangle_{\Phi,\Xi}}$ etc. From now on, when we speak of "the regular (canonical) modal frame for L with respect to Φ and Ξ", we presuppose that Φ is a maximal L-consistent set and Ξ is a suitable set of formulas coherent with Φ. We say that $\mathfrak{F}$ is a *regular* (*canonical*) *modal frame for* L if it is a regular (canonical) modal frame for L with respect to some Φ and Ξ.

We want to consider both the compactness and the finite model property at the same time, and thus, consider the ordinary canonical frames and their filtrations at the same time—this way we can avoid going over similar arguments twice. It is easy to see that when $\langle \mathsf{W}, \mathsf{Agent}, \mathsf{choice}, \mathsf{value} \rangle$ is the canonical modal frame for L with respect to Φ and Ξ, if Ξ is the set of all formulas, $\mathsf{W} = \mathsf{W}_{\Phi}$, and thus we have a subframe (generated by Φ) of the ordinary canonical frame; and if Ξ is ("logically") finite, we have a filtration.

The following remark is an immediate consequence of $\mathsf{A8}$, our definition of coherence and ordinary modal logic:

REMARK 5. Let $\langle \mathsf{W}, \mathsf{Agent}, \mathsf{choice}, \mathsf{value} \rangle$ be the regular modal frame for L with respect to Φ and Ξ. Then the following hold.

1. for each $x \in \mathsf{W}$ and each $A \in \Xi$, $A \in x$ or $\neg A \in x$;
2. for each $x, y \in \mathsf{W}$, $\{\triangle\alpha A \mid \triangle\alpha A \in x\} = \{\triangle\alpha A \mid \triangle\alpha A \in y\}$ only if $R_{\langle\alpha\rangle}xy$;
3. for each $x \in \mathsf{W}$, $\{\triangle\alpha A \mid \ominus\alpha A \in \Phi \cap \Xi^{+}\} \subseteq x$ only if $\Sigma_{\langle\alpha\rangle} \subseteq x$.

The lemma below handles the desired properties of $\Box$ and $\triangle$, on the basis of which we can start dealing with *independence of agents* and the condition for the operator of dominant ought.

LEMMA 6. *Let* $\langle \mathsf{W}, \mathsf{Agent}, \mathsf{choice}\rangle$ *be the regular modal frame for* L *with respect to* Φ *and* Ξ*, and let* $x \in \mathsf{W}$*. Then the following hold.*

1. *for each* $\Box A \in \Xi$*,* $\Box A \in x$ *iff* $A \in y$ *for every* $y \in \mathsf{W}$*;*
2. *for each* α *and* β *occurring in* Ξ*,* $\alpha = \beta \in x$ *iff* $\langle\alpha\rangle = \langle\beta\rangle$*;*
3. *for each* $\triangle\alpha A \in \Xi$*,* $\triangle\alpha A \in x$ *iff* $A \in y$ *for every* $y \in [x]_{\langle\alpha\rangle}$*.*

The following lemma shows that for each L, all regular modal frames satisfy the required version of *independence of agents*.

LEMMA 7. *Let* $\langle \mathsf{W}, \mathsf{Agent}, \mathsf{choice}\rangle$ *be the regular modal frame for* L *with respect to* Φ *and* Ξ*. Then for each function* f *on* Agent *such that* $f(\mathbf{a}) \in \mathsf{choice}_{\mathbf{a}}$ *for every* $\mathbf{a} \in \mathsf{Agent}$*,* $\bigcap_{\mathbf{a}\in\mathsf{Agent}} f(\mathbf{a}) \neq \varnothing$*.*

The theorem below follows immediately from our definition and Lemma 7.

THEOREM 8. *Each regular modal frame for* L *is a stit frame, and each canonical modal frame for* L *is a two-valued utilitarian stit frame.*

The lemma below gives us a property all regular modal frames have, which will be used to deal with the semantic condition for the dominant ought operators.

LEMMA 9. *Let* $\langle \mathsf{W}, \mathsf{Agent}, \mathsf{choice}\rangle$ *be the regular modal frame for* L *with respect to* Φ *and* Ξ*. Then for each* $\ominus\alpha A \in \Xi$ *and each* $x \in \mathsf{W}$*,* $\ominus\alpha A \in x$ *iff* $\triangle\alpha A \in y$ *for every* $y \in \mathsf{W}$ *such that* $\Sigma_{\langle\alpha\rangle} \subseteq y$*.*

From now on, for each regular modal frame $\mathfrak{F} = \langle \mathsf{W}, \mathsf{Agent}, \mathsf{choice}\rangle$ for L, let $\mathsf{D}_{\mathfrak{F}}$ stand for the set $\{y \in \mathsf{W} \mid \Sigma \subseteq y\}$. For a canonical modal frame $\mathfrak{F} = \langle \mathsf{W}, \mathsf{Agent}, \mathsf{choice}, \mathsf{value} \rangle$ for L, let $\mathsf{D}_{\mathfrak{F}} = \mathsf{D}_{\mathfrak{F}'}$, where $\mathfrak{F}' = \langle \mathsf{W}, \mathsf{Agent}, \mathsf{choice} \rangle$. Note that for each canonical modal frame $\mathfrak{F}$ for L, $\mathsf{D}_{\mathfrak{F}} = \{y \in \mathsf{W} \mid \mathsf{value}(y) = 1\}$ by definition.

LEMMA 10. *Let* $\langle \mathsf{W}, \mathsf{Agent}, \mathsf{choice} \rangle$ *be the regular modal frame for* L *with respect to* Φ *and* Ξ. *Then for each* $\ominus\alpha A \in \Xi$ *and each* $x \in \mathsf{W}$, $\ominus\alpha A \in x$ *iff* $\triangle\alpha A \in y$ *for every* $y \in \mathsf{D}_{\mathfrak{F}}$.

Applying Proposition 10, Lemma 9 and $\mathsf{T4}$, one can easily verify the following.

COROLLARY 11. *Let* $\langle \mathsf{W}, \mathsf{Agent}, \mathsf{choice} \rangle$ *be the regular modal frame for* L *with respect to* Φ *and* Ξ. *Then* $\mathsf{D}_{\mathfrak{F}} \neq \varnothing$.

Because the canonical modal frame $\mathfrak{F} = \langle \mathsf{W}, \mathsf{Agent}, \mathsf{choice}, \mathsf{value} \rangle$ for L with respect to Φ and Ξ is a utilitarian stit frame, $\mathsf{optimal}_a$ is defined for every $a \in \mathsf{Agent}$ (recall that $\mathsf{optimal}_a = \{K \in \mathsf{choice}_a \mid \neg\exists K'(K' \in \mathsf{choice}_a \wedge K \prec K')\}$). Our next goal is to show that each canonical modal frame is optimal, but first we need to show the following key lemma.

LEMMA 12. *Let* $\mathfrak{F} = \langle \mathsf{W}, \mathsf{Agent}, \mathsf{choice}, \mathsf{value} \rangle$ *be the canonical modal frame for* L *with respect to* Φ *and* Ξ. *Then for each* $x \in \mathsf{W}$ *and each* $\mathbf{a} \in \mathsf{Agent}$, $\Sigma_{\mathbf{a}} \subseteq x$ *iff* $[x]_{\mathbf{a}} \in \mathsf{optimal}_{\mathbf{a}}$.

Proof. Let $x \in \mathsf{W}$ and $\mathbf{a} \in \mathsf{Agent}$. First suppose that $\Sigma_{\mathbf{a}} \subseteq x$. We show below that $[x]_{\mathbf{a}} \in \mathsf{optimal}_{\mathbf{a}}$, i.e., $[x]_{\mathbf{a}} \prec K$ for no $K \in \mathsf{choice}_{\mathbf{a}}$. To that end, we observe that by definitions of $\Sigma_{\mathbf{a}}$ and $R_{\mathbf{a}}$,

(4) $\Sigma_{\mathbf{a}} \subseteq y$ for every $y \in [x]_{\mathbf{a}}$.

Consider any $K \in \mathsf{choice}_{\mathbf{a}}$. We first show that for each $S \in \mathsf{state}_{\mathbf{a}}$,

(5) $S \cap K \leq S \cap [x]_{\mathbf{a}}$.

By definition of value, it is sufficient to show that

if $\mathsf{value}(y) = 1$ for some $y \in S \cap K$, $\mathsf{value}(z) = 1$ for all $z \in S \cap [x]_{\mathbf{a}}$.

Let $S \in \mathsf{state}_{\mathbf{a}}$, $K \in \mathsf{choice}_{\mathbf{a}}$ and $x_0 \in S \cap K$ with $\mathsf{value}(x_0) = 1$. We know by definition of $\mathsf{state}_{\mathbf{a}}$ that $S = \bigcap_{\mathbf{b} \in \mathsf{Agent} - \{\mathbf{a}\}} [x_0]_{\mathbf{b}}$. For each $\mathbf{b} \in \mathsf{Agent} - \{\mathbf{a}\}$, since $\mathsf{value}(x_0) = 1$, $\Sigma_{\mathbf{b}} \subseteq \Sigma \subseteq x_0$, and then by definitions of $\Sigma_{\mathbf{b}}$ and $R_{\mathbf{b}}$, $\Sigma_{\mathbf{b}} \subseteq y$ for every $y \in [x_0]_{\mathbf{b}}$. It follows from (4) that $\Sigma \subseteq z$ for every $z \in [x]_{\mathbf{a}} \cap (\bigcap_{\mathbf{b} \in \mathsf{Agent}-\{\mathbf{a}\}} [x_0]_{\mathbf{b}})$, and hence $\mathsf{value}(z) = 1$ for every $z \in S \cap [x]_{\mathbf{a}}$. It follows that (5) holds. Now if $[x]_{\mathbf{a}} \prec K$, $K \not\preceq [x]_{\mathbf{a}}$ by definition, i.e., there is an $S' \in \mathsf{state}_{\mathbf{a}}$ such that $S' \cap K \not\leq S' \cap [x]_{\mathbf{a}}$, contrary to (5). Hence $[x]_{\mathbf{a}} \not\prec K$. It follows that $[x]_{\mathbf{a}} \in \mathsf{optimal}_{\mathbf{a}}$.

Next suppose that $\Sigma_{\mathbf{a}} \not\subseteq x$. Then there are $\alpha \in \mathbf{a}$ and $\triangle\alpha A \in \Sigma_{\mathbf{a}}$ such that $\triangle\alpha A \notin x$. By definition of $R_{\mathbf{a}}$, $\triangle\alpha A \notin y$ for each $y \in [x]_{\mathbf{a}}$, and thus $\Sigma_{\mathbf{a}} \not\subseteq y$ for each $y \in [x]_{\mathbf{a}}$, and hence

(6) $[x]_{\mathbf{a}} \cap \mathsf{D}_{\mathfrak{F}} = \varnothing$.

There is, by Corollary 11, a $z \in \mathsf{D}_{\mathfrak{F}}$. Since $\Sigma \subseteq z$, an argument similar to the one given above will show that for each $\mathbf{b} \in \mathsf{Agent}$, $\Sigma_{\mathbf{b}} \subseteq y$ for every $y \in [z]_{\mathbf{b}}$. Letting $S_z = \bigcap_{\mathbf{b} \in \mathsf{Agent}-\{\mathbf{a}\}} [z]_{\mathbf{b}}$, we know that $S_z \in \mathsf{state}_{\mathbf{a}}$ and $z \in S_z \cap [z]_{\mathbf{a}}$. Since $\mathsf{value}(z) = 1$ and $S_z \cap [x]_{\mathbf{a}} \neq \varnothing$ (by Theorem 8), $S_z \cap [z]_{\mathbf{a}} \not\leq S_z \cap [x]_{\mathbf{a}}$ by (6), and hence $[z]_{\mathbf{a}} \not\preceq [x]_{\mathbf{a}}$. It follows, also from (6), that for each $S \in \mathsf{state}_{\mathbf{a}}$, $S \cap [x]_{\mathbf{a}} \leq S \cap [z]_{\mathbf{a}}$, i.e., $[x]_{\mathbf{a}} \preceq [z]_{\mathbf{a}}$. Hence $[x]_{\mathbf{a}} \prec [z]_{\mathbf{a}}$, and therefore $[x]_{\mathbf{a}} \notin \mathsf{optimal}_{\mathbf{a}}$. ■

THEOREM 13. *Each canonical modal frame for* L *is optimal.*

Proof. Let $\mathbf{a} \in \mathsf{Agent}$, and let $K \in \mathsf{choice}_{\mathbf{a}} - \mathsf{optimal}_{\mathbf{a}}$. We show as follows that there is a $K' \in \mathsf{optimal}_{\mathbf{a}}$ such that $K \prec K'$, which is sufficient. Let $x \in K$. We know that $K = [x]_{\mathbf{a}}$ and, by Lemma 12, that $\Sigma_{\mathbf{a}} \not\subseteq x$. An argument similar to that in Lemma 12 shows that there is a $z \in \mathsf{D}_{\mathfrak{F}}$ such that $[x]_{\mathbf{a}} \prec [z]_{\mathbf{a}}$. Since $\Sigma_{\mathbf{a}} \subseteq \Sigma \subseteq z$, $[z]_{\mathbf{a}} \in \mathsf{optimal}_{\mathbf{a}}$ by Lemma 12. ■

For each $n > 0$, we know that the characteristic axiom of L_n, i.e., APC_n, corresponds to the semantic condition that the stit frames are at-most-n-ary. The following lemma shows that all regular (canonical) modal frames for L_n are at-most-n-ary.

LEMMA 14. *Let* $n > 0$*, and let* $\langle \mathsf{W}, \mathsf{Agent}, \mathsf{choice} \rangle$ *be the regular modal frame for* L_n *with respect to* Φ *and* Ξ*. Then for each* $\mathbf{a} \in \mathsf{Agent}$, $|\mathsf{choice}_{\mathbf{a}}| \leqslant n$.

Proof. Let $\mathbf{a} \in \mathsf{Agent}$. Suppose for reductio that $|\mathsf{choice}_{\mathbf{a}}| > n$. Then there are $U_0, \ldots, U_n \in \mathsf{choice}_{\mathbf{a}}$ such that $U_0, \ldots, U_n$ are all different. Select $\alpha \in \mathbf{a}$, select $x_0 \in U_0, \ldots, x_n \in U_n$, and select $w_0, \ldots, w_n \in \mathsf{W}_\Phi$ such that $x_k = w_k \cap \Xi^+$ for every k with $0 \leqslant k \leqslant n$. Consider any i and k such that $1 \leqslant k \leqslant n$, $0 \leqslant i \leqslant n$ and $i \neq k$. By definition of $R_{\mathbf{a}}$, there is an A such that either $\neg\triangle\alpha A \in x_k$ and $\triangle\alpha A \in x_i$, or $\triangle\alpha A \in x_k$ and $\neg\triangle\alpha A \in x_i$; and then by $\mathsf{A2}$, either $\triangle\alpha\neg\triangle\alpha A \in w_k$ and $\neg\triangle\alpha\neg\triangle\alpha A \in w_i$, or $\triangle\alpha A \in w_k$ and $\neg\triangle\alpha A \in w_i$; and hence there is an A' such that $\triangle\alpha A' \in w_k$ and $\neg\triangle\alpha A' \in w_i$. It follows that for each k with $1 \leqslant k \leqslant n$, there are $\triangle\alpha A_{k,0}, \ldots, \triangle\alpha A_{k,n} \in w_k$ such that $\neg\triangle\alpha A_{k,i} \in w_i$ for every i with $0 \leqslant i \leqslant n$ and $i \neq k$. For each k with $1 \leqslant k \leqslant n$, letting $B_k = \triangle\alpha A_{k,0} \wedge \ldots \wedge \triangle\alpha A_{k,n}$, it is easy to see by ordinary modal logic (applying $\mathsf{A2}$, RN and $\mathsf{A4}$) that $\triangle\alpha B_k \in w_k$ and $\neg B_k \in w_i$ for every i with $0 \leqslant i \leqslant n$ and $i \neq k$. It follows that $\triangle\alpha B_1 \in w_1$, $\neg B_1 \wedge \triangle\alpha B_2 \in w_2, \ldots, \neg B_1 \wedge \ldots \wedge \neg B_{n-1} \wedge \triangle\alpha B_n \in w_n$, and then by Remark 4 (2), $\Diamond\triangle\alpha B_1 \in w_0$, $\Diamond(\neg B_1 \wedge \triangle\alpha B_2) \in w_0, \ldots, \Diamond(\neg B_1 \wedge \ldots \wedge \neg B_{n-1} \wedge \triangle\alpha B_n) \in w_0$, and hence by APC_n, $B_1 \vee \ldots \vee B_n \in w_0$. But, since $\neg B_i \in w_0$ for every i with $0 \leqslant i \leqslant n$, $\neg(B_1 \vee \ldots \vee B_n) \in w_0$, contrary to the assumption of L-consistency on w_0. We conclude from this reductio that $|\mathsf{choice}_{\mathbf{a}}| \leqslant n$. ■

4 Main results

Let $\mathfrak{F} = \langle \mathsf{W}, \mathsf{Agent}, \mathsf{choice}, \mathsf{value}\rangle$ be the canonical frame for L with respect to Φ and Ξ. $\mathfrak{M} = \langle \mathfrak{F}, \tau\rangle$ is the *canonical model for* L *with respect to* Φ *and* Ξ if τ is the valuation on $\mathfrak{F}$ such that for each agent term α, $\alpha^\tau = \langle\alpha\rangle$, and for each atomic formula $p \in \Xi$, $\tau(p) = \{x \in \mathsf{W} \mid p \in x\}$.

THEOREM 15. *Let* $\mathfrak{M} = \langle \mathfrak{F}, \tau\rangle$ *be the canonical model for* L *with respect to* Φ *and* Ξ*, where* $\mathfrak{F} = \langle \mathsf{W}, \mathsf{Agent}, \mathsf{choice}, \mathsf{value}\rangle$*. Then for each* $A \in \Xi$ *and for each* $x \in \mathsf{W}$*,* $\mathfrak{M}, x \vDash A$ *iff* $A \in x$*.*

Proof. By a routine induction we can show that for each $A \in \Xi$ and $x \in \mathsf{W}$,

(7) $\mathfrak{M}, x \vDash A$ iff $A \in x$.

It is straightforward that (7) holds when A is an atomic formula, or $\neg B$ or $B \wedge C$. We know by Lemma 6 (1)–(3). that (7) holds when

A is $\alpha = \beta$, $\Box B$ or $\triangle\alpha B$. Now let A be $\ominus\alpha B \in \Xi$. Consider any $x \in \mathsf{W}$. By Lemma 9, $\ominus\alpha B \in x$ iff $\triangle\alpha B \in y$ for every $y \in \mathsf{W}$ such that $\Sigma_{\langle\alpha\rangle} \subseteq y$, and then by Lemma 12,

$$(8) \quad \ominus\alpha B \in x \quad \text{iff} \quad \triangle\alpha B \in y \text{ for every } y \in \mathsf{W} \text{ such that } [y]_{\langle\alpha\rangle} \in \mathsf{optimal}_{\langle\alpha\rangle}.$$

Consider any $y \in \mathsf{W}$ such that $[y]_{\langle\alpha\rangle} \in \mathsf{optimal}_{\langle\alpha\rangle}$. We know by Lemma 6 (3) that $\triangle\alpha B \in y$ iff $B \in z$ for every $z \in [y]_{\langle\alpha\rangle}$, and then by induction hypothesis, $\triangle\alpha B \in y$ iff $[y]_{\langle\alpha\rangle} \subseteq \|B\|^{\tau}$. It follows from (8) that $\ominus\alpha B \in x$ iff $[y]_{\langle\alpha\rangle} \subseteq \|B\|^{\tau}$ for every $y \in \mathsf{W}$ such that $[y]_{\langle\alpha\rangle} \in \mathsf{optimal}_{\langle\alpha\rangle}$. Since each $K \in \mathsf{optimal}_{\langle\alpha\rangle}$ is $[y]_{\langle\alpha\rangle}$ for some $y \in \mathsf{W}$, we know that $\ominus\alpha B \in x$ iff $K \subseteq \|B\|^{\tau}$ for every $K \in \mathsf{optimal}_{\langle\alpha\rangle}$. Hence by Theorem 13 and the definition of $\mathsf{optimal}$ on p. 217, $\ominus\alpha B \in x$ iff $\mathfrak{M}, x \vDash \ominus\alpha B$. ■

THEOREM 16. L_0 *is complete and compact.*

Proof. For each L_0-consistent set Ψ of formulas, let Φ be any maximal L_0-consistent set including Ψ, let Ξ be the set of all formulas, and let $\mathfrak{M} = \langle \mathsf{W}, \mathsf{Agent}, \mathsf{choice}, \mathsf{value}, \tau\rangle$ be the canonical model for L_0 with respect to Φ and Ξ. It is easy to see that $\Phi \in \mathsf{W}$. Then by Theorem 15, $\mathfrak{M}, \Phi \vDash \Psi$. ■

By Theorem 13, each canonical modal frame for L_0 is an optimal utilitarian stit frame, and hence the only formulas valid in all optimal utilitarian stit frames are theorems of L_0. It follows that the theorem below holds, which shows a certain limitation of the language.

THEOREM 17. *There is no set* Θ *of formulas such that for each utilitarian stit frame* $\mathfrak{F}$*,* $\mathfrak{F}$ *is optimal iff all members of* Θ *are valid in* $\mathfrak{F}$*; and if* L *is the set of all formulas valid in all optimal utilitarian stit frames,* $\mathsf{L} = \mathsf{L}_0$ *(identifying* L_0 *with the set of all its theorems).*

Similarly, because canonical modal frames for L_0 are all two-valued, the following holds.

THEOREM 18. *There is no set* Θ *of formulas such that for each utilitarian stit frame* $\mathfrak{F}$*,* $\mathfrak{F}$ *is two-valued iff all members of* Θ *are valid in* $\mathfrak{F}$*; and if* L *is the set of all formulas valid in all two-valued* *utilitarian stit frames,* $\mathsf{L} = \mathsf{L}_0$ *(identifying* L_0 *with the set of all its theorems).*

The theorem below follows from Theorem 15 and Lemma 14, whose proof is similar to that of Theorem 16, with an application of Lemma 14.

THEOREM 19. *Let* $n > 0$. *Then the following hold:*

1. L_n *is complete; and*
2. *for each set* Θ *of formulas, if every finite subset of* Θ *has an at-most-n-ary utilitarian stit model, so does* Θ.

In the following, we prove the decidability of all L_n with $n \geqslant 0$ by way of the finite model property. For each L-consistent formula A, let Φ be a maximal L-consistent set containing A, and let

$\Pi_0 = \{\alpha = \beta \mid \alpha \text{ and } \beta \text{ occur in } A\}$,

$\Pi_1 = \{\triangle\beta B, \ominus\beta B \mid \ominus\alpha B \in \mathsf{Sub}(\{A\}),\ \alpha = \beta \in \Phi,\ \beta \text{ occurs in } A\}$,

$\Pi_2 = \{\triangle\beta B \mid \triangle\alpha B \in \mathsf{Sub}(\{A\}), \alpha = \beta \in \Phi,\ \beta \text{ occurs in } A\}$, and

$\Xi_A^\Phi = \mathsf{Sub}(\{A\}) \cup \Pi_0 \cup \Pi_1 \cup \Pi_2$.

It is easy to verify that for each A, Ξ_A^Φ is finite, suitable, and coherent with Φ.

THEOREM 20. *For each* $n \geqslant 0$, L_n *has the finite model property, and hence is decidable.*

Proof. Let $n \geqslant 0$ and let A be any L_n-consistent formula. Select a maximal L_n-consistent set Φ containing A, let $\Xi = \Xi_A^\Phi$. Then, let $\mathfrak{M} = \langle \mathsf{W}, \mathsf{Agent}, \mathsf{choice}, \mathsf{value}, \tau \rangle$ be the canonical model for L_n with respect to Φ and Ξ. Since Ξ is finite, so is W. Because $A \in \Phi \cap \Xi^+ \in \mathsf{W}$, we know by Theorem 15 that $\mathfrak{M}, x \vDash A$, where $x = \Phi \cap \Xi^+$. When $n > 0$, we apply Lemma 14 to conclude that $\mathfrak{M}$ is at-most-n-ary. ∎

5 Conclusion

There are several classes of frames proposed by Horty and Thomason as argued in Section 2.5. Some logics of them share the same axiomatic system L_0. In particular, both the logic of the class of all optimal frames and that of the class of all two-valued frames coincide with L_0. It implies that, despite of Horty's introduction of the real-number valued value assignment, the resulting logic fails to be essentially different from standard deontic logic.

The considerations suggest that Horty's proposal needs subtler investigations than presented here. In fact, his original presentation with branching time semantics allows to take non-monotonic agency modalities instead of multi-S5. For example, the *deliberative stit* (dstit) operators $[\ \ ]$ induces the dstit-based dominant ought operator $\oplus\alpha$ can be defined as follows: $[\alpha]A =_{df} \triangle\alpha A \wedge \neg\Box A$, and $\oplus\alpha A =_{df} \ominus\alpha A \wedge \neg\Box A$. $[\ \ ]$ can be taken primitive, and let $\triangle\alpha A =_{df} [\alpha]A \vee \Box A$, $\oplus\alpha A =_{df} \odot[\alpha]A$ and $\ominus\alpha A =_{df} \oplus\alpha A \vee \Box A$. Different choices of agency operators would lead to different logics, which are to be examined.

BIBLIOGRAPHY

[1] Nuel Belnap, Michael Perloff, and Ming Xu. *Facing the Future: Agents and Choices in Our Indeterminist World*. Oxford Univerity Press, Oxford, 2001.

[2] Dagfinn Føllesdal and Risto Hilpinen. Deontic logic: an introduction. In Hilpinen (ed.) *Deontic Logic: introductory and systematic readings*, 1–35. D. Reidel, Dordrecht, 1971.

[3] Dov Gabbay and F. Guenthner, editors. *Handbook of Philosophical Logic*, volume 2. D. Reidel Publishing Company, Dordrecht, 1984.

[4] Dov M. Gabbay and Hans J. Ohlbach, editors. *Temporal Logic, First International Conference, ICTL'94, Bonn, Germany, Proceedings*, volume 827. Springer-Verlag, 1994.

[5] Peter Geach. What happened in deontic logic? *Philosophia*, 11:1–11, 1982.

[6] Risto Hilpinen, editor. *New Studies in Deontic Logic*. D. Reidel Publishing Company, Dordrecht, 1981.

[7] John F. Horty. *Agency and Deontic Logic*. Oxford University Press, Oxford, 2000.

[8] John F. Horty and Nuel Belnap The Deliberative *Stit*: a study of action, omission, ability, and obligation *Journal of Philosophical Logic*, 24:583–644, 1995.

[9] Arthur Prior. *Past, Present and Future*. Oxford Univerity Press, Oxford, 1967.

[10] Krister Segerberg. Getting started: beginnings in the logic of action. *Studia Logica*, 51:347–378, 1992.

[11] Richmond H. Thomason. Indeterminist time and truth-value gaps. *Theoria*, 36:264–281, 1970.

[12] Richmond H. Thomason. Combination of tense and modality. In Gabbay and Guenthner [3], pages 135–165.

[13] Georg Henrik von Wright. *Norm and Action*. Routledge and Kegan Paul, London, 1963.

[14] Georg Henrik von Wright. Problems and prospects of deontic logic: A survey. In Evandro Agazzi (ed.) *Modern Logic: A Survey*, pages 399–423. D. Reidel, Dordrecht, 1980.
[15] Ming Xu. Decidability of deliberative *stit* theories with multiple agents. In Gabbay and Ohlbach [4], pages 332–348.
[16] Ming Xu. Axioms for Deliberative *Stit*. *Journal of Philosophical Logic*, 27:505–552, 1998.

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