Normal Modal Logics Containing KTB with some Finiteness onditions

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ABSTRACT. In this paper, we investigate normal modal logics over **KTB** with additional axioms which impose some finiteness conditions on Kripke frames for the logics. First we formulate the logics of reflexive and symmetric frames with finite *diameter*, and show some nice properties of such logics. We also prove of the existence of a continuum in the normal modal logics over $\mathbf{KTB} \oplus \Box^2 p \supset \Box^3 p$. Finally, we try to clarify the structure of the upper part of the lattice $Next(\mathbf{KTB})$.

1 Introduction

1.1 Background

The semantical study of propositional modal logics using Kripke type semantics has brought us a great success, especially to the investigation of the class of logics containing **K4**. This class of modal logics is characterized by the class of *transitive* frames. In order to analyze this class of logics, many algebraic and frame-theoretic techniques have been developed, and by using them, several general results have been established, which give us a perspective of the lattice of normal modal logics characterized by transitive frames (for example,[3],[6],[7],[17], [18]).

However, there is no effective way of clarifying the lattice structure of modal logics of non-transitive logics, and almost all techniques for logics of transitive frames are not valid, at least in their own form, for the logics of non-transitive frames. Therefore the study of logics of non-transitive frames has not yet grown to be so fruitful as that of logics of transitive frames.

Here we discuss the structure of the lattice of logics containing **KTB**, whose frames are reflexive and symmetric. Although the logic **KTB** has been known as *Brouwerian system*, there is very few results established about the structure of the lattice of extensions of **KTB**. In fact, various questions on the structure of this lattice remain open and untouched.

In this paper, we will focus on a little smaller class of the whole extensions of **KTB**, that is, a class of logics containing **KTB** that are characterized by

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modal algebras, or general frames with some *finiteness* conditions. These conditions include '*diameter*' of frames and a weakened form of transitivity. These can be expressed by modal formulas, and they correspond to some nice first order conditions on frames. To be the most important, they enable us to employ the standard technique of propositional modal logic.

The construction of this paper is as follows. The rest of this section below is devoted to preliminary facts. We introduce the syntax of our logics, algebraic and frame semantics of our logics, and we explain the correspondence between these two types of semantics.

In section 2, we characterize subdirectly irreducible members and simple members of KTB- Kripke algebras, and give an example of KTB- Kripke algebras that is subdirectly irreducible but not simple. Here it is seen that every KTB- Kripke frame that corresponds to a subdirectly irreducible KTB- algebra can be regarded as a *connected* undirected graph in a graph-theoretical sense.

We introduce the notion of '*diameter*' of a KTB frame, and investigate some nice properties of KTB-logics of frames with finite diameter in section 3. In particular, we explain the splitting technique here, and we show that this standard technique is applicable if the class of KTB-logics are characterized by KTB- frames of finite diameter. In fact, it is proved here that every finite algebra for a KTB-logic of frames with finite diameter splits the lattice above that KTB-logic.

In section 4, we show that there exists a continuum of normal modal logics over $Next(\mathbf{KTB} \oplus \Box^2 p \supset \Box^3 p)$. The author proved in [14] of the existence of a continuum of normal modal logics over $Next(\mathbf{KTB})$, where he used an embedding of a continuum of orthologics into $Next(\mathbf{KTB})$. Here we make use of the standard technique of modal logic, that is, splitting and p-morphism of frames. This result is in stark contract to a well known fact that there are only countable many normal modal logics over $\mathbf{KTB} \oplus \Box p \supset \Box^2 p$ $(= \mathbf{S5})$.

In section 5, we investigate the structure of the upper part of the lattice $Next(\mathbf{KTB})$, and in particular, we show that the logic determined by a frame of two reflexive points jointed with a symmetric relation is the third greatest logic of all Kripke complete KTB-logics.

In the last section, we list up some open questions about the KTB-logics and the structure of the lattice $Next(\mathbf{KTB})$.

1.2 Preliminaries

We will follow notions and nomenclature of modal logics from [5]. But here, we want to recall the basic terminology of syntax and semantics of modal logics that we will use in this paper.

First of all, we fix our language and introduce propositional modal logics. Our language consists of: (1) a denumerable set of propositional variables $\{p_0, p_1, \ldots\}$, (2) a propositional constant $\perp$, (3) conjunction $\wedge$, (4) negation $\neg$, (5) box $\Box$, and (6) a pair of parentheses (,). The connectives $\vee$ (disjunction), $\supset$ (implication), and $\Diamond$ (diamond) can be treated as auxiliary connectives (abbreviations). The set Φ of formulas on our language is defined as usual.

$\mathbf{L} \subseteq \Phi$ is a *normal modal logic* (a logic for short) if: (1) $\mathbf{L}$ contains the classical tautologies, (2) $\Box(p \wedge q) \supset (\Box p \wedge \Box q) \in \mathbf{L}$, and (3) $\mathbf{L}$ is closed under modus ponens, substitution, and the rule of necessitation. The smallest normal modal logic is denoted by $\mathbf{K}$. For a set of formulas Γ, the smallest normal modal logic that includes both $\mathbf{K}$ and Γ is denoted by $\mathbf{K} \oplus \Gamma$. For a normal modal logic $\mathbf{L}$, the class of normal extensions of $\mathbf{L}$ is denoted by $Next(\mathbf{L})$.

In this paper, we concentrate on the class $Next(\mathbf{KTB})$, where $\mathbf{T} := \Box p \supset p$ and $\mathbf{B} := p \supset \Box\Diamond p$, and $\mathbf{KTB} = \mathbf{K} \oplus \mathbf{T} \oplus \mathbf{B}$. Each member of $Next(\mathbf{KTB})$ is called a *KTB-logic* for short.

Next we review semantics for propositional modal logics. The first one is the algebraic semantics. A structure $\mathfrak{A} = \langle \mathsf{A}, \cap, \cup, -, I, 0, 1 \rangle$ is a *modal algebra* if: (1) $\langle \mathsf{A}, \cap, \cup, -0, 1 \rangle$ is a Boolean algebra, and (2) I is a unary operator that satisfies $(i) I(1) = 1$ and $(ii) I(a \cap b) = I(a) \cap I(b)$ for $a, b \in \mathsf{A}$. Formulas are interpreted in a modal algebra by a valuation in the standard way and it is also defined in the standard way that a formula A is *valid* in a modal algebra $\mathfrak{A}$ ($\mathfrak{A} \models A$). For a class $\mathcal{K}$ of modal algebras, $\mathbf{L}(\mathcal{K}) := \{A \in \Phi \mid \mathfrak{A} \models A$ for any $\mathfrak{A} \in \mathcal{K}\}$ is called the normal modal logic determined by $\mathcal{K}$. A modal algebra $\mathfrak{A}$ is a *KTB-algebra* if it satisfies that $I(x) \leq x$ and that $x \leq I(-I(-x))$ for any $x \in \mathsf{A}$.

Another semantical tools are called *frames*. A structure $\mathcal{F} = \langle W, R, P \rangle$ is a (*general*) *frame* if: (1) W is not an empty set, (2) R is a binary relation on W, and (3) $P \subseteq \mathcal{P}(W)$ contains $\emptyset, W$ and closed under $\cap, -, I_R$, where $I_R(X) := \{x \in W \mid \forall y \in W(xRy \text{ implies } y \in X)\}$ for $X \in \mathcal{P}(W)$. A frame $\langle W, R \rangle := \langle W, R, \mathcal{P}(W) \rangle$ is called a *Kripke frame*. The way how we interpret each formula in a frame is in the standard way. A class $\mathcal{C}$ of frames also determines a normal modal logic as: $\mathbf{L}(\mathcal{C}) := \{A \in \Phi \mid \mathfrak{A} \models A$ for any $\mathfrak{A} \in \mathcal{C}\}$. As is well known, the axioms $\mathbf{T}$ and $\mathbf{B}$ correspond to *reflexivity* and *symmetry* of Kripke frames respectively, that is, the following holds: for any Kripke frame $\mathcal{F}$, $\mathcal{F} \models \mathbf{T}$ iff $\mathcal{F} \models\!\!\models \forall x(xRx)$, and $\mathcal{F} \models \mathbf{B}$ iff $\mathcal{F} \models\!\!\models \forall x, y(xRy \text{ implies } yRx)$. Here, the symbol $\models\!\!\models$ represents the relation that the frame $\mathcal{F}$ is a model of the first order condition on the right hand side. A reflexive and symmetric general frame is called a *KTB-frame*.

It is well established as *representation theory* that there exists a close relation between modal algebras and general frames. For any modal algebra $\mathfrak{A}$, we define $\mathfrak{A}_* = \langle W_{\mathfrak{A}}, R_{\mathfrak{A}}, P_{\mathfrak{A}} \rangle$ as follows: (1) $W_{\mathfrak{A}}$ is the set of all prime filters in A, (2) $R_{\mathfrak{A}}$ is the binary relation on $W_{\mathfrak{A}}$ defined as: for any $F, G \in W_{\mathfrak{A}}$, ${}_F R_{\mathfrak{A}G}$ iff for all $a \in \mathsf{A}$, $I(a) \in F$ implies $a \in G$, and (3) $P_{\mathfrak{A}} := \{\theta(a) | a \in \mathsf{A}\}$, where $\theta(a) := \{F \in W_{\mathfrak{A}} \mid a \in F\}$. Then, it is easily shown that $\mathfrak{A}_*$ is a general frame and that $\mathfrak{A}$ and $\mathfrak{A}_*$ validates the same set of formulas. Conversely, for any given general frame $\mathcal{F} = \langle W, R, P \rangle$, the corresponding modal algebra $\mathcal{F}^*$ can be defined as: $\mathcal{F}^* = \langle P, \cap, \cup, -, I_R, \emptyset, W \rangle$. Here, $\cap, \cup, -$ are the set theoretic operations, whereas I_R is the same one defined above. It is also easy to prove that both $\mathcal{F}$ and $\mathcal{F}^*$ validate the same set of formulas. Moreover, about two transformation $(\cdot)^*$ and $(\cdot)_*$, it is known that (1): $\mathfrak{A} \cong (\mathfrak{A}_*)^*$ for any modal algebra $\mathfrak{A}$ and that (2): $\mathcal{F} \cong (\mathcal{F}^*)_*$ iff the frame $\mathcal{F}$ is *descriptive*. We call an algebra $\mathfrak{A}$ a *Kripke algebra* if there exists a Kripke frame $\mathcal{F}$ such that $\mathfrak{A} \cong \mathcal{F}^*$.

2 Subdirectly irreducible and simple members of KTB-algebras

The first result in this paper is about subdirectly irreducible members, and simple members of KTB-algebras. An algebra $\mathfrak{A}$ is a *subdirect product* of an indexed family $\{\mathfrak{A}_i \mid i \in I\}$ of the same type if there exists a one-to-one homomorphism $f : \mathfrak{A} \to \prod_{i \in I} \mathfrak{A}_i$ such that for any $i \in I$, $\pi_i \circ f : \mathfrak{A} \to \mathfrak{A}_i$ is onto, where π_i is a projection map to i-th coordinate. Here the map f is called a *subdirect representation* of $\mathfrak{A}$. A non-trivial algebra $\mathfrak{A}$ is *subdirectly irreducible* (s.i. for short), if for any subdirect representation $f : \mathfrak{A} \to \prod_{i \in I} \mathfrak{A}_i$ of $\mathfrak{A}$, there exists $i \in I$ such that $\pi_i \circ f : \mathfrak{A} \to \mathfrak{A}_i$ is one-to-one.

The notion of subdirectly irreducible algebras is very important in universal algebra, since it is shown by G. Birkhoff ([1]) that every algebra is isomorphic to a subdirect product of subdirectly irreducible algebras of the same type. This famous theorem means that subdirectly irreducible algebras do form the building blocks of algebra. The notion is also important for logicians, because the following holds for logics determined by algebras.

PROPOSITION 1. *Let $\mathfrak{A}$ be a modal algebra that has a subdirect representation $f : \mathfrak{A} \to \prod_{i \in I} \mathfrak{A}_i$. Then $\mathbf{L}(\mathfrak{A}) = \bigcap_{i \in I} \mathbf{L}(\mathfrak{A}_i)$.* ■

Due to this proposition together with Birkhoff's theorem, when we consider a logic which is determined by a class of algebras, we have only to take into account the logics determined by a s.i. member of them, and focus on the intersection of such logics. The logics determined by subdirectly irreducible algebras do form the building blocks of algebraic logic. It is well known that a non-trivial $\mathfrak{A}$ is s.i. iff $\mathfrak{A}$ has the smallest non-trivial congruence. A non-trivial algebra $\mathfrak{A}$ is *simple* if the congruence lattice of $\mathfrak{A}$ consists of only two elements.

Subdirectly irreducible members, and simple members of modal algebras are well understood by the following characterization.

PROPOSITION 2. *Let $\mathfrak{A} = \langle \mathsf{A}, \cap, \cup, -, I, 0, 1\rangle$ be a modal algebra.*

(1) *$\mathfrak{A}$ is subdirectly irreducible if and only if there exists $d \in \mathsf{A}$ ($d \neq 1$) such that for any $a \in \mathsf{A}$ ($a \neq 1$), there exists $n \in \omega$, $a \cap I(a) \cap \cdots \cap I^n(a) \leq d$.*

(2) *$\mathfrak{A}$ is simple if and only if for any $a \in \mathsf{A}$ ($a \neq 1$), there exists $n \in \omega$, $a \cap I(a) \cap \cdots \cap I^n(a) = 0$.* ■

On the above proposition, (1) is by Rautenberg ([15]). The element d in (1) is called an *opremum.* (2) is shown by the fact that for any non-empty subset $X \subseteq \mathsf{A}$, the open filter F generated by X is given by: $F = \{a \in \mathsf{A} \mid \exists x_1, \ldots, x_k \in X, \text{ and } \exists n_1, \ldots, n_k \in \omega, I^{n_1}(x_1) \cap \cdots \cap I^{n_k}(x_k) \leq a \}$. Here, in a modal algebra $\mathfrak{A}$, an *open filter* F is a filter which satisfies that $a \in F$ implies $I(a) \in F$ for any $a \in \mathsf{A}$. Note that if $\mathfrak{A}$ is a KT-algebra, that is $\mathfrak{A} \models \Box p \supset p$, then the above conditions can be slightly simplified as follows: (1): $\mathfrak{A}$ is s.i. if and only if there exists $d \in \mathsf{A}$ ($d \neq 1$) such that for any $a \in \mathsf{A}$ ($a \neq 1$), there exists $n \in \omega$, $I^n(a) \leq d$. (2): $\mathfrak{A}$ is simple if and only if for any $a \in \mathsf{A}$ ($a \neq 1$), there exists $n \in \omega$, $I^n(a) = 0$.

Now we prove the following characterization of s.i. and simple members of KTB- Kripke algebras. Below, R^n is defined for $n \in \omega$ as: xR^0y iff $x = y$, and $xR^{n+1}y$ iff there is $z \in W$ such that xR^nz and zRy.

THEOREM 3. *Let $\mathcal{F} = \langle W, R\rangle$ be a KTB- Kripke frame.*

(1) *$\mathcal{F}^*$ is subdirectly irreducible if and only if $\mathcal{F} \models \forall x, y, \exists n \in \omega\, (xR^ny)$.*

(2) *$\mathcal{F}^*$ is simple if and only if $\mathcal{F} \models \forall x, \exists n \in \omega, \forall y\, (xR^ny)$.*

Proof. (1): Suppose that $\mathcal{F}^*$ is s.i. Then, since $\mathcal{F}^*$ is a KT-algebra, we have that there exists $V \subsetneq W$ such that for any $U \subsetneq W$, for some $n \in \omega$, $I^n_R(U) \subseteq V$. Take arbitrary $x, y \in W$. Fix some element $w \in W - V$. First, put $U_0 := W - \{x\}$. Then by our condition, for this U_0, there exists $n \in \omega$ such that $I^n_R(U_0) \subseteq V$, which implies that $w \in -V \subseteq -I^n_R(U_0)$. Therefore we have in particular that wR^nx. Similarly we can show that wR^my for some $m \in \omega$. Thus we have $xR^{n+m}y$ because of symmetry of R. Conversely, suppose $\mathcal{F} \models \forall x, y, \exists n \in \omega(xR^ny)$. Here the following claim can be proved: for an arbitrary $x_0 \in W$, and for any $A \subset W$, there exists $n \in \omega$ such that

$I^n_R(A) \subseteq W - \{x_0\}$. In fact, for any $y \in W - A$, by our assumption there is some $n_0 \in \omega$ such that $x_0 R^{n_0} y$. Since $y \notin A$, this means $\{x_0\} \subseteq -I^{n_0}_R(A)$. Consider the open filter F in $\mathcal{F}^*$ that is generated from $W - \{x_0\}$. Then by the previous claim we can say that F is the smallest non-trivial open filter in $\mathcal{F}^*$, which corresponds to the smallest non-trivial congruence. Therefore $\mathcal{F}^*$ is subdirectly irreducible.
(2): Suppose that $\mathcal{F}$ is simple, which means that for any $U \subsetneq W$, there exists $n \in \omega$ such that $I^n_R(U) = \emptyset$. For any $x \in W$, put $U_0 := W - \{x\}$. Then for this U_0, there exists some $n_0 \in \omega$ such that $I^{n_0}_R(U_0) = \emptyset$. Therefore for any y, $y \in W = -I^n_R(U_0)$. Hence we have $xR^n y$. Conversely, suppose $\mathcal{F} \models \forall x, \exists n \in \omega, \forall y(xR^n y)$. Take an arbitrary non-trivial open filter F in $\mathcal{F}^*$. Then there exists $X \subsetneq W$ such that $X \in F$. Now, by our assumption, for any $x \in W - X$ there is some $n \in \omega$ such that for any $y \in W$ $xR^n y$. This means that $I^n_R(X) = \emptyset$. Since F is an open filter, $\emptyset \in F$, which implies that $F = \mathcal{P}(W)$. Therefore, there are only two open filters in $\mathcal{F}^*$. Hence $\mathcal{F}^*$ is simple. ■

In general, Kripke frames can be regarded as graphs, and reflexive and symmetry Kripke frames (KTB- Kripke frames) as undirected graphs. The above characterization shows that a KTB- Kripke frame which corresponds to a subdirectly irreducible algebra can be seen as a *connected* undirected graph. Here, the term *connected* is used in a graph-theoretical sense. As seen in an example below, any KTB- Kripke frame that corresponds to an s.i., but not simple algebra must be an infinite frame.

EXAMPLE 4. Let a Kripke frame $\mathcal{H} := \langle \omega, R \rangle$, where the relation R is defined as: mRn iff $|m-n| \leq 1$ for $m, n \in \omega$. Then, clearly this R is reflexive and symmetric, and so, $\mathcal{H}$ is a KTB- Kripke frame. By THEOREM 3, it is also easy to prove that $\mathcal{H}^*$ is s.i. but not simple.

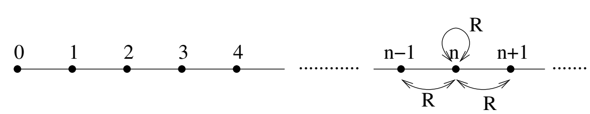

Figure 1. A frame $\mathcal{H}$ that corresponds to an s.i. but not simple algebra

To close this section, we make one more comment on the decomposition of a KTB- Kripke algebra into s.i. members. According to Blok's result ([2]), any Kripke algebra can be decomposed into s.i. Kripke algebras in a following way.

THEOREM 5. *Let $\mathcal{F} = \langle W, R \rangle$ be a Kripke frame. For each $x \in W$, define a Kripke frame $\mathcal{F}_x = \langle W_x, R_x \rangle$, where $W_x := \{a \in W \mid \exists n \in \omega \, xR^n a\}$, and $R_x = R \cap (W_x \times W_x)$. Then, for each $x \in W$, $\mathcal{F}_x^*$ is subdirectly irreducible, and $\mathcal{F}^*$ can be represented as a subdirect product of $\{\mathcal{F}_x^* \mid x \in W\}$, where the subdirect representation $f : \mathcal{F}^* \to \prod_{x \in W} \mathcal{F}_x^*$ is determined by $(f(X))_x := X \cap W_x$ for $X \in \mathcal{P}(W)$.* ■

For a KTB- Kripke frame $\mathcal{F} = \langle W, R \rangle$, define a binary relation $\approx$ on W as: $x \approx y$ if there exists $n \in \omega$ such that $xR^n y$. Then, because of symmetry of R, this $\approx$ turns out to be an equivalence relation, and so, $x \approx y$ iff $W_x = W_y$ for any $x, y \in W$. Therefore, in employing the above theorem to decompose a given KTB- Kripke algebra $\mathcal{F} = \langle W, R \rangle$, we factorize W by the relation $\approx$ first, and we collect one representative x_λ from every equivalence class W_λ, and then we can obtain our desirable subdirect representation of $\mathcal{F}$ from $\{{\mathcal{F}_{x_\lambda}}^* \mid x_\lambda \in W_\lambda\}$.

In this way, every KTB- Kripke algebra is decomposed into s.i. algebras. Therefore, when we consider a KTB-logic which is Kripke complete, we may assume that a class of KTB- Kripke frames, each member $\mathcal{F}$ of which satisfies $\mathcal{F} \models \forall x, y, \exists n \in \omega(xR^n y)$.

3 KTB-logics determined by frames with finite diameter

In this section, we will consider a finiteness condition on KTB- frames and KTB- logics determined by such frames. For transitive frames, a notion of '*depth*' of a frame is defined, by which locally tabular members of $Next(\mathbf{K4})$ can be characterized, that is, it is shown that a logic $\mathbf{L} \in Next(\mathbf{K4})$ is locally tabular if and only if it is characterized by a class of frames with finite depth ([11],[16]). However, in our case, the depth of a KTB- frame cannot be defined because of symmetry of our frame. Here we will introduce the notion of '*diameter*' of a frame, and present some nice properties of KTB logics characterized by a class of KTB frames with finite diameter.

Let $\mathcal{F} = \langle W, R, P \rangle$ be a general frame. A *path* (of length n) in $\mathcal{F}$ is a finite sequence $\{x_i\}_{i=0}^n \subseteq W$ of distinct points which satisfies $x_i R x_{i+1}$ for $i = 0, 1, \ldots, n-1$. The *diameter* of $\mathcal{F}$ is the maximum length of a path that $\mathcal{F}$ contains. It may happen that the diameter of a frame is infinite.

DEFINITION 6. For $n \geq 1$, a frame $\mathcal{F}$ is *n-bounded* if the diameter of $\mathcal{F}$ is at most $n-1$, specifically $\mathcal{F}$ is a model of the following first order sentence $\delta(n)$:

$$\delta(n) := \forall x_0, x_1, \ldots, x_n [x_0 R x_1 R x_2 \cdots x_{n-1} R x_n \text{ implies } \mho_{i \neq j}(x_i = x_j)]$$

$\mathcal{F}$ is *bounded* if it is n-bounded for some $n \geq 1$. ▮

In the above condition, $\mho$ is a logical connective of "big or". For each n, n-bounded Kripke frames can be characterized by a modal formula. For this purpose, first we define an exclusive sequence of formulas. An *exclusive sequence of formulas* is a sequence $\{F_i\}_{i=0}^{n}$ $(n \geq 1)$ of classical formulas (i.e. formulas on our language without $\Box, \Diamond$) such that (1) each F_i is satisfiable in one-point frame, or in the classical logic, and that (2) every pair of formulas F_i, F_j $(i \neq j)$, $F_i \wedge F_j$ is not satisfiable at any one point in a frame. Of course, we can easily construct an exclusive sequence of formulas for any $n \geq 1$. For example, if $n = 1$, then we can provide a sequence $\{F_i\}_{i=0}^{3}$, where $F_0 := p = 0 \wedge p = 1$, $F_1 := \neg p_0 \wedge p_1$, $F_2 := p_0 \wedge \neq p_1$, and $F_3 := \neg p_0 \wedge \neg p_1$. Of course, we can choose other combination of classical formulas to fulfill the conditions of an exclusive sequence. Now, we define a formula $\mathbf{D}_n$ by employing an exclusive sequence of formulas $\{F_i\}_{i=0}^{n}$.

$$\begin{aligned} \mathbf{D}_n &:= \neg\{F_0 \wedge \Diamond(F_1 \wedge \Diamond(\cdots \wedge \Diamond(F_{n+1} \wedge \Diamond F_n)\cdots))\} \\ &= F_0 \supset \Box\{F_1 \supset \Box(\cdots \supset \Box(F_{n-1} \supset \Box\neg F_n)\cdots)\} \end{aligned}$$

Then the following holds.

LEMMA 7. *Let $\mathcal{F} = \langle W, R\rangle$ be a Kripke frame and $n \geq 1$. Then $\mathcal{F} \models \mathbf{D}_n$ if and only if $\mathcal{F} \models \delta(n)$*

Proof. Suppose $\mathcal{F} \not\models \delta(n)$. Then there exists a sequence of distinct points $x_0, x_1, \ldots, x_n \in W$ such that $x_0 R x_1 R \cdots R x_n$. Since $\{F_i\}_{i=0}^{n}$ is exclusive, we can define a valuation V on $\mathcal{F}$ as: for each i, $x_i \models F_i$. Then the formula $\neg\mathbf{D}_n$ is satisfiable at x_0 in $\mathcal{F}$, which means $(\mathcal{F}, V) \not\models_{x_0} \mathbf{D}_n$. Thus we have $\mathcal{F} \not\models \mathbf{D}_n$. Conversely, suppose $\mathcal{F} \models \delta(n)$, and consider any valuation V on $\mathcal{F}$. Take arbitrary points $x_0, \ldots, x_n \in W$ such that $x_0 \models F_0$, $x_{i-1} R x_i$ and $x_i \models F_i$ for $1 \leq i \leq n-1$, and $x_{n-1} R x_n$. Then, because of $\delta(n)$, there exists at least one pair (x_i, x_j) such that $x_i = x_j$. But since $\{F_i\}_{i=0}^{n}$ is exclusive, we can see that for some i $(0 \leq i \leq n-1)$, $x_i = x_n$. Therefore, $x_i = x_n \models F_i$, which means that $x_n \not\models F_n$. Thus, we have $(\mathcal{F}, V) \models_{x_0} \mathbf{D}_n$, and so $\mathcal{F} \models \mathbf{D}_n$ ■

By this lemma, it is obvious that the logic $\mathbf{K} \oplus \mathbf{D}_n$ does not depend on what its exclusive sequence of formulas $\{F_i\}_{i=0}^{n}$ really looks like. To be more important, this lemma also says that for $n \geq 1$, the logic $\mathbf{K} \oplus \mathbf{D}_n$ is elementary. This fact implies that these logics have the following nice property. Below, a logic $\mathbf{L}$ is *d-persistent* if for every descriptive frame $\mathcal{F} = \langle W, R, P\rangle$, $\mathcal{F} \models \mathbf{L}$ implies $\kappa\mathcal{F} \models \mathbf{L}$, where $\kappa\mathcal{F} = \langle W, R\rangle$ (underlying Kripke frame of $\mathcal{F}$).

THEOREM 8. *For every $n \geq 1$,*

(1) $\mathbf{K} \oplus \mathbf{D}_n$ *is d-persistent, in particular canonical.*

(2) $\mathbf{K} \oplus \mathbf{D}_n$ *is Kripke complete.*

(3) $\mathbf{K} \oplus \mathbf{D}_n$ *has the finite model property.* ∎

For the proof of (3) above, we can appeal to the filtration method. Since the axioms **T** and **B** will not cause any trouble along the proof, we can apply the same argument to the logic $\mathbf{KTB} \oplus \mathbf{D}_n$ to show the following theorem on it.

THEOREM 9. *For every $n \geq 1$,*

(1) $\mathbf{KTB} \oplus \mathbf{D}_n$ *is d-persistent, in particular canonical.*

(2) $\mathbf{KTB} \oplus \mathbf{D}_n$ *is Kripke complete.*

(3) $\mathbf{KTB} \oplus \mathbf{D}_n$ *has the finite model property.* ∎

Note that it is easily seen that every bounded KTB- Kripke frame corresponds to a simple algebra, and that we can build an infinite KTB- Kripke frame which gives us a simple algebra. It is still unknown whether the notion of diameter of a frame can characterize the locally tabular members of $Next(\mathbf{KTB})$.

4 Splittings in $Next(\mathbf{KTB})$

The notion of splitting was first introduced in lattice theory, and was used for classifying some subvarieties of the variety of lattices. In modal logic, this technique was used for investigating the structure of the lattice $Next(\mathbf{L})$ for some particular logics **L**. Specifically, by splitting technique, the problem of the degree of Kripke incompleteness was solved by Blok ([2]) for the lattices $Next(\mathbf{K})$, $Next(\mathbf{KT})$, and $Next(\mathbf{KD})$, where the axiom **D** is $\Box p \supset \Diamond p$.
It has not yet become clear whether there is any valuable splitting in the lattice $Next(\mathbf{KTB})$. Instead of the whole $Next(\mathbf{KTB})$, we investigate splittings of $Next(\mathbf{KTB} \oplus \mathbf{D}_n)$ and splittings of $Next(\mathbf{KTB} \oplus \Box^n p \supset \Box^{(n+1)} p)$ for $n \in \omega$. In consequence of the latter analysis, we show that there exists a continuum of normal modal logics over $\mathbf{KTB} \oplus \Box^2 p \supset \Box^3 p$.

4.1 Splittings in $Next(\mathbf{KTBD_n})$ and in $Next(\mathbf{KTB} \oplus \Box^n p \supset \Box^{(n+1)} p)$

First of all, we recall the notion of *splitting* of a lattice, that is often used for investigating the lattice of modal logics.

DEFINITION 10 (Splitting). Let $\mathcal{L} = \langle L, \wedge, \vee, 0, 1\rangle$ be a complete lattice and $a \in L$. Then *a splits* $\mathcal{L}$ if there exists $b \in L$ such that for any $x \in L$, either $x \leq a$ or $b \leq x$, but not both. Such a pair (a, b) is called a *splitting pair* of the lattice $\mathcal{L}$. In this case the element b is a *splitting* of $\mathcal{L}$.

When we think about splittings of a lattice of logics, if a logic $\mathbf{L}(\mathfrak{A})$ (or $\mathbf{L}(\mathcal{F})$) splits the lattice, then we say that the algebra $\mathfrak{A}$ (or the frame $\mathcal{F}$) splits it. Next we explain a general splitting theorem by Rautenberg ([15]).

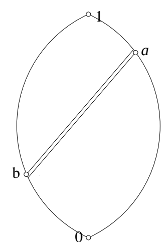

Figure 2. A splitting of a complete lattice $\mathcal{L}$

Consider the following formula: $t^m := (p \wedge \Box p \wedge \cdots \wedge \Box^m p) \supset \Box^{(m+1)} p$ for $m \in \omega$. This axiom corresponds to the following first order condition on Kripke frames. For a Kripke frame $\mathcal{F} = \langle W, R\rangle$,

$$\mathcal{F} \models t^m \quad \text{if and only if} \quad \mathcal{F} \models \forall x, y[xR^{(m+1)}y \text{ implies } \mho_{i=0}^{m}(xR^i y)]$$

The condition on the right means that if it is possible to go from x to y by $m + 1$ steps, then that is possible by at most m steps. This condition is a weaker form of usual transitivity and is called *m-transitivity*.

THEOREM 11 (Rautenberg). *Let* $\mathbf{L}_0 \in Next(\mathbf{K} \oplus t^m)$ $(m \in \omega)$. *Let* $\mathfrak{A}$ *be a finite, subdirectly irreducible modal algebra such that* $\mathfrak{A} \models \mathbf{L}_0$. *Then* $\mathfrak{A}$ *splits* $Next(\mathbf{L}_0)$. ■

Proof of this theorem proceeds in the following. For each element $a \in \mathsf{A}$, prepare propositional variable p_a, and let δ be the conjunction of all the following formulas. For any $a, b \in \mathsf{A}$,

$$p_a \wedge p_b \leftrightarrow p_{a \cap b}, \qquad p_a \vee p_b \leftrightarrow p_{a \cup b}, \qquad \neg p_a \leftrightarrow p_{-a}, \qquad \Box p_a \leftrightarrow p_{Ia}$$

Here $\alpha \leftrightarrow \beta := (\alpha \supset \beta) \wedge (\beta \supset \alpha)$. Put $\kappa_{\mathfrak{A}} := (\delta \wedge \Box\delta \wedge \cdots \wedge \Box^m \delta) \supset p_d$, where d is an opremum of $\mathfrak{A}$. This $\kappa_{\mathfrak{A}}$ is called the *characteristic formula* for $\mathfrak{A}$. Then it can be verified that for any algebra $\mathfrak{B}$ that validates t^m, the following three conditions are equivalent.

(1) $\mathfrak{A} \in SH(\mathfrak{B})$. (This means that $\mathfrak{A}$ is a subalgebra of a homomorphic image of $\mathfrak{B}$.)

(2) $\mathbf{L}(\mathfrak{B}) \subseteq \mathbf{L}(\mathfrak{A})$.

(3) $\kappa_{\mathfrak{A}} \notin \mathbf{L}(\mathfrak{B})$.

Now let $\mathbf{L}^\sharp := \mathbf{L}_0 \oplus \kappa_{\mathfrak{A}}$. For any $\mathbf{L} \in Next(\mathbf{L}_0)$, there is a modal algebra $\mathfrak{C}$ such that $\mathbf{L} = \mathbf{L}(\mathfrak{C})$ and $\mathfrak{C} \models t^m$. Then by the above equivalence, $\mathbf{L} = \mathbf{L}(\mathfrak{C}) \subseteq \mathbf{L}(\mathfrak{A})$ if and only if $\kappa_{\mathfrak{A}} \notin \mathbf{L}(\mathfrak{C})$ if and only if $\mathbf{L}_0 \oplus \kappa_{\mathfrak{A}} \not\subseteq \mathbf{L}(\mathfrak{C})$. Hence $(\mathbf{L}(\mathfrak{A}), \mathbf{L}^\sharp)$ is a splitting pair.

It is rather difficult to find splittings in $Next(\mathbf{KTB})$. The reason seems that the variety of KTB-algebras does not have the *EDPC* (equationally definable principal congruence). So we discuss here splittings in $Next(\mathbf{KTB} \oplus \mathbf{D}_n)$ and in $Next(\mathbf{KTB} \oplus \Box^n p \supset \Box^{(n+1)} p)$. By the first order characterization of the formula $\mathbf{D}_n$, it is easy to show that for a modal algebra $\mathfrak{A}$ and for $n \in \omega$, if $\mathfrak{A} \models \mathbf{D}_n$ then $\mathfrak{A} \models t^n$. Therefore in considering splittings in $Next(\mathbf{KTB} \oplus \mathbf{D}_n)$, we can make use of Rautenberg's theorem. Because every finite KTB- algebra is a Kripke algebra, and so a finite KTB- algebra for $\mathbf{D}_n$ is simple by THEOREM 3, the following holds.

THEOREM 12. *Let $\mathfrak{A}$ be an arbitrary finite KTB-algebra such that $\mathfrak{A} \models \mathbf{D}_n$ $(n \in \omega)$. Then $\mathfrak{A}$ splits $Next(\mathbf{KTBD}_n)$.* ■

Here, the characteristic formula $\kappa_{\mathfrak{A}}$ for $\mathfrak{A}$ is $\Box^n \delta \supset p_0$, because we have the axiom $\mathbf{T}$ and in this case an opremum is 0. The obtained splitting pair is $(\mathbf{L}(\mathfrak{A}), \mathbf{KTBD}_n \oplus \kappa_{\mathfrak{A}})$

It is easily proved that for any finite KTB- algebra $\mathfrak{A}$, if $\mathfrak{A}_* \models \mathbf{D}_n$ for some $n \in \omega$, then $\mathfrak{A}_* \models \mathbf{D}_m$ for any $m (m > n)$. Therefore, THEOREM 12 says that any finite KTB-algebra splits the lattice $Next(\mathbf{KTBD}_\ell)$ for every sufficiently large $\ell \in \omega$. From the opposite point of view, this means that if there exists a KTB-logic that prevents some KTB-algebra from being a splitting algebra of $Next(\mathbf{KTB})$, it must be the logic determined by infinite s.i. algebras such as $\mathcal{H}^*$ in EXAMPLE 4, and that logic must be located in the bottom part of the lattice $Next(\mathbf{KTB})$.

Now we consider another splittings. Let $\mathbf{Tra}(n, n+1) := \Box^n p \supset \Box^{(n+1)} p$. Apparently this formula is a generalization of transitivity, indeed the following first order characterization holds. For a Kripke frame $\mathcal{F} = \langle W, R \rangle$,

$$\mathcal{F} \models \mathbf{Tra}(n, n+1) \text{ if and only if } \mathcal{F} \models \forall x, y (x R^{(n+1)} y \text{ implies } x R^n y)$$

The following can be proved by the similar argument as in the previous theorem.

THEOREM 13. *Let $\mathfrak{A}$ be an arbitrary finite KTB-algebra such that $\mathfrak{A} \models \mathbf{Tra}(n, n+1)$ $(n \in \omega)$. Then $\mathfrak{A}$ splits $Next(\mathbf{KTB} \oplus \mathbf{Tra}(n, n+1))$.* ■

Here, again the characteristic formula $\kappa_{\mathfrak{A}}$ for $\mathfrak{A}$ is $\Box^n\delta \supset p_0$. We will use this theorem for $n = 2$ in the next part.

4.2 The existence of a continuum of logics over KTB $\oplus\ \Box^2 p \supset \Box^3 p$

First, we introduce the tool of *p-morphism* between two general frames.

DEFINITION 14. Let $\mathcal{F} = \langle W, R, P\rangle$ and $\mathcal{G} = \langle U, S, Q\rangle$ be general frames. A map $f : W \to U$ is a *p-morphism* from $\mathcal{F}$ to $\mathcal{G}$, if f satisfies the following:

(pi) f is onto.

(pii) For all $x, y \in W$, xRy implies ${}_{f(x)}S_{f(y)}$

(piii) For all $x \in W$ and for all $a \in U$, if ${}_{f(x)}S_a$, then there exists $y \in W$ such that xRy and $f(y) = a$.

(piv) For all $X \in Q$, $f^{-1}(X) \in P$.

There is a close connection between the fact that there is a p-morphism from one frame to another and the fact that one an algebra is a subalgebra of another. Specifically the following holds.

PROPOSITION 15. *Let $\mathcal{F} = \langle W, R, P\rangle$ and $\mathcal{G} = \langle U, S, Q\rangle$ be general frames, and suppose that there is a p-morphism from $\mathcal{F}$ to $\mathcal{G}$. Then $\mathcal{G}^*$ is a subalgebra of $\mathcal{F}^*$.* ■

We have also the converse proposition as follows.

PROPOSITION 16. *Let $\mathfrak{A}, \mathfrak{B}$ be modal algebras, and suppose that $\mathfrak{B}$ is a subalgebra of $\mathfrak{A}$. Then there exists a p-morphism from $\mathfrak{A}_*$ to $\mathfrak{B}_*$.* ■

We will use the following corollary derived from the above two propositions in the sequel.

COROLLARY 17. *Let $\mathcal{F}, \mathcal{G}$ be finite Kripke frames such that both $\mathcal{F}^*, \mathcal{G}^*$ are simple. Then $\mathbf{L}(\mathcal{F}) \subseteq \mathbf{L}(\mathcal{G})$ if and only if there exists a p-morphism from $\mathcal{F}$ to $\mathcal{G}$.*

Proof. For an algebra $\mathfrak{A}$, let $V(\mathfrak{A})$ denote the variety generated by $\mathfrak{A}$. For a class $\mathcal{C}$ of algebras, let $(\mathcal{C})_{SI}$ denote the class of all subdirectly irreducible members in $\mathcal{C}$, and $P_u(\mathcal{C})$ the class of *ultraproducts* of members in $\mathcal{C}$. Now $\mathbf{L}(\mathcal{F}) \subseteq \mathbf{L}(\mathcal{G})$ iff $\mathbf{L}(\mathcal{F}^*) \subseteq \mathbf{L}(\mathcal{G}^*)$ iff $V(\mathcal{G}^*) \subseteq V(\mathcal{F}^*)$ iff $\mathcal{G}^* \in (V(\mathcal{F}^*))_{SI} \subseteq HSP_u(\mathcal{F}^*)$ by Jónsson's lemma ([9]). Further, by the congruence extension property of modal algebras, finiteness and simplicity of $\mathcal{F}^*$, this is equivalent to $\mathcal{G}^* \in SH(\mathcal{F}^*) = S(\mathcal{F}^*)$. Hence, by the above propositions, this means equivalently that there exists a p-morphism from $(\mathcal{F}^*)_*$ to $(\mathcal{G}^*)_*$. These are isomorphic to $\mathcal{F}$ and $\mathcal{G}$ respectively. ■

Now we will prove the existence of a continuum of logics in $Next(\mathbf{KTB} \oplus \Box^2 p \supset \Box^3 p)$. Roughly speaking, we will construct an infinite series of finite frames $\{\mathcal{F}_n\}_{n\in\omega}$ and an infinite series of formulas $\{A_m\}_{m\in\omega}$ such that (1): for every $n \in \omega$, $\mathcal{F}_n \not\models A_n$ and (2): for every $m \in \omega (m \neq n)$, $\mathcal{F}_m \not\models A_n$. We employ the following frames.

DEFINITION 18 (Wheel frame). For $n \in \omega, n \geq 5$, the *wheel frame* $\mathcal{W}_n = \langle W, R\rangle$ of degree n consists of the following underlying set and binary relation: $W = \mathrm{rim}(W) \cup \{h\}$, where $\mathrm{rim}(W) := \{0, 1, \ldots, n-1\}$ and $h \notin \mathrm{rim}(W)$. An element in $\mathrm{rim}(W)$ is called a *rim element*, whereas the element h the *hub element*. The relation R is defined as: $R := \{(x, y) \in (\mathrm{rim}(W))^2 \mid |x - y| \leqq 1(\mathrm{mod}\, n)\} \cup \{(h, h)\} \cup \{(h, x), (x, h) \mid x \in \mathrm{rim}(W)\}$.

Note that $|x - y|$ means the absolute value of the difference, and for example $|(n-1) - 0| = |n-1| \equiv 1(\mathrm{mod}\, n)$. The following is the wheel frame $\mathcal{W}_7$. It is trivial that for $n \geq 5$, $\mathcal{W}_n \models \mathbf{B}, \mathbf{T}$ and $\mathcal{W}_n \models \mathbf{Tra}(2, 3)$.

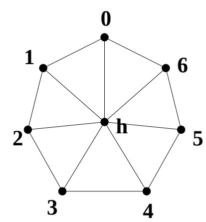

Figure 3. The wheel frame $\mathcal{W}_7$

The following proposition can be also verified.

PROPOSITION 19. *For $m > n \geq 5$, $\mathbf{L}(\mathcal{W}_n) \not\subseteq \mathbf{L}(\mathcal{W}_m)$.*

Proof. Because $m > n$, $\mathcal{W}_m \not\models \mathbf{D}_m$, but $\mathcal{W}_n \models \mathbf{D}_m$. Thus we have $\mathbf{D}_m \in \mathbf{L}(\mathcal{W}_n)$, but $\mathbf{D}_m \notin \mathbf{L}(\mathcal{W}_m)$. ■

The following lemma is crucial to our result in this section.

LEMMA 20. *For $m \geq n \geq 5$, suppose there is a p-morphism from $\mathcal{W}_m$ to $\mathcal{W}_n$. Then m is divisible by n.*

Proof. Let $\mathcal{W}_m := \langle W, R\rangle$, where $W = \mathrm{rim}\,(W) \cup \{h\} = \{0, 1, \ldots m - 1, h\}$ and $\mathcal{W}_n := \langle U, S\rangle$, where $U = \mathrm{rim}\,(U) \cup \{\underline{h}\} = \{\underline{0}, \underline{1}, \ldots, \underline{n-1}, \underline{h}\}$. This means that we distinguish elements in $\mathcal{W}_n$ from elements in $\mathcal{W}_m$ by underlining. Let $f : W \to U$ be the p-morphism from $\mathcal{W}_M$ to $\mathcal{W}_n$. For the relation R, let $R(a) := \{x \in W \mid aRx\}$ for $a \in W$. The same notation is used

for the relation S. We consider hub elements and rim elements separately.
(i) On hub elements

We claim that $f(h) = \underline{h}$ and $f^{-1}(\underline{h}) = \{h\}$. Because, suppose there is $\underline{a} \in$ rim (U) such that $f(h) = \underline{h}$. For hub element h, we have hRx for all $x \in W$, so ${}_{f(h)}S_{f(x)}$ in $\mathcal{W}_n$ for all $x \in W$. But for the rim element $\underline{a} \in U$, card $(S(\underline{a})) = 4$. If $f(z) \in S(\underline{a})$ for all $z \in W$, then this contradicts to the condition that f is onto because $n \geq 5$. Thus we have $f(h) = \underline{h}$. Moreover, suppose there exists $b \in$ rim (W) such that $b \in f^{-1}(\underline{h})$. Then since card$(R(b)) = 4$ and $n \geq 5$, we can pick up an element $c \in W$ such that ${}_{f(b)}S_{f(c)}$ but there is no $z \in W$ that satisfies bRz and $f(z) = f(c)$. This is also a contradiction to (piv) of p-morphism. Thus we have $f^{-1}(\underline{h}) = \{h\}$.
(ii) On rim elements

By (i) above, we have that for any $x \in$ rim (W), $f(x) \in$ rim (U), and that for any $\underline{x} \in$ rim (U), $f^{-1}(\underline{x}) \subseteq$ rim (W). We claim here that for any distinct elements $x, y, z \in\in$ rim (W) such that $xRyRz$, $f(x), f(y), f(z)$ are all distinct and ${}_{f(x)}S_{f(y)}S_{f(z)}$ holds. Now we go on to case analysis.

(Case A): $f(x) = f(y) = \underline{a}$.
Let $\underline{b}, \underline{c} \in$ rim (U) be elements that are next to $\underline{a}$, that is, $\underline{b}S\underline{a}S\underline{c}$. Since $n \geq 5$, $\underline{b}, \underline{a}, \underline{c}$ can be all distinct. Then, by ${}_{f(y)}S_{\underline{b}}$ and $f(x) = \underline{a} \neq \underline{b}$, $z \in f^{-1}(\underline{b})$ holds. Also, by ${}_{f(y)}S_{\underline{c}}$ and $f(x) = \underline{a} \neq \underline{c}$, $z \in f^{-1}(\underline{c})$ holds. This is a contradiction. Therefore the case $f(x) = f(y)$ never happens. Similarly, the case $f(y) = f(z)$ never happens.

(Case B): $f(x) = f(z) = \underline{a}$.
Let $\underline{b}, \underline{c} \in$ rim(U) be elements that are next to $\underline{a}$, that is, $\underline{b}S\underline{a}S\underline{c}$. Of course, $\underline{b}, \underline{a}, \underline{c}$ can be all distinct

(subcase B-1): $f(y) = \underline{a}$.
We have ${}_{\underline{b}}S_{f(y)}$, but $f(x) = f(z) = \underline{a}$, which means that $x, z \notin f^{-1}(\underline{b})$. This contradicts to (piv) of conditions for p-morphism f.

(subcase B-2): $f(y) = \underline{b}$.
Let $\underline{d} \in$ rim (U) be the element next to $\underline{b}$ but not $\underline{a}$. Then we have ${}_{\underline{d}}S_{f(y)}$, but $x, z \notin f^{-1}(\underline{d})$. This also contradicts to (piv) of conditions for p-morphism f.

(subcase B-3): $f(y) = \underline{c}$.
Similar treatment can be applied to show that this case is impossible.

Our case analysis is completed and the claim has been justified. Now we will show that m is divisible by n. If $m = n$, then we have nothing else to prove. If $m > n$, then there exists $\underline{t} \in$ rim (U), and $x, y \in$ rim $(W)(x \neq y)$ such that $x, y \in f^{-1}(\underline{t})$. Let $\underline{s}, \underline{u} \in$ rim (U) be elements that are next to

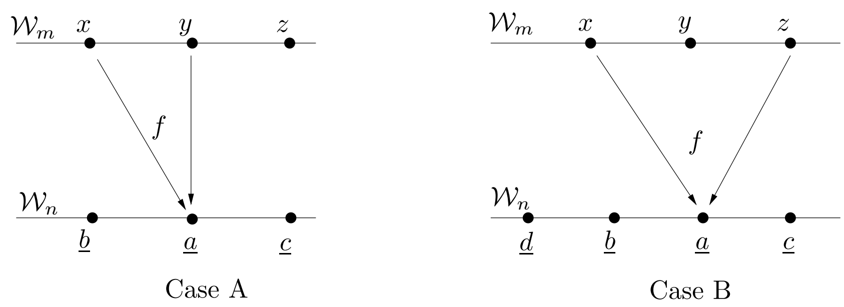

Figure 4. Case A and Case B

$\underline{t}$, that is, $\underline{s}S\underline{t}S\underline{u}$. Here $\underline{s}, \underline{t}, \underline{u}$ are all distinct. Then by our claim that we have shown above, we can find distinct elements $x', x'', y', y'' \in \text{rim}\,(W)$ such that $x', y' \in f^{-1}(\underline{s})$, $x'', y'' \in f^{-1}(\underline{u})$, $x'RxRx''$, and $y'RyRy''$ hold. Put $k := \max_{\underline{x} \in \text{rim}\,(U)} \text{card}\,(f^{-1}(\underline{x}))$. By the observation just above, if $\text{card}\,(f^{-1}(\underline{a})) = k$ for some $\underline{x} \in \text{rim}\,(U)$, then for elements $\underline{b}, \underline{c}$ that are next to $\underline{a}$, $\text{card}\,(f^{-1}(\underline{b})) = \text{card}\,(f^{-1}(\underline{c})) = k$. Thus we can state that $\text{card}\,(f^{-1}(\underline{z})) = k$ for all $\underline{z} \in \text{rim}\,(U)$. Hence we have that $m = k \cdot n$. ■

Now we are in a position to prove our main result in this part. Let $Prim := \{n \in \omega \mid n \text{ is prime }, \ m \geq 5\}$. The following theorem holds.

THEOREM 21. *For any $p \in Prim$, there exists a formula A_p such that:*

(1) $\mathcal{W}_p \not\models A_p$.

(2) $\mathcal{W}_q \models A_p$ *for all* $q \in Prim$ $(p \neq q)$

Proof. As already mentioned in this section, $\mathcal{W}_p^*$ is a finite simple KTB-algebra and $\mathcal{W}_p^* \models \mathbf{Tra}(2,3)$. Then by THEOREM 11, $(\mathbf{L}(\mathcal{W}_p^*), \mathbf{KTB} \oplus \mathbf{Tra}(2,3) \oplus A_p)$ is a splitting pair of $Next(\mathbf{KTB} \oplus \mathbf{Tra}(2,3))$, where $A_p := \Box^2\delta \supset p_0$ is the characteristic formula for $\mathcal{W}_p^*$ defined just below THEOREM 11. Denote $\mathbf{L}^\sharp := \mathbf{KTB} \oplus \mathbf{Tra}(2,3) \oplus A_p$. Because of this splitting pair, we have in particular, $A_p \not\in (\mathbf{L}(\mathcal{W}_p^*)$, which means that $\mathcal{W}_p \not\models A_p$.
Furthermore, take any $q \in Prim$ $(q \neq p)$. If $q > p$, then, since q is not divisible by p, there does never exist any p-morphism from $\mathcal{W}_q$ to $\mathcal{W}_p$ by LEMMA 20. This means by COROLLARY 17, that $\mathbf{L}(\mathcal{W}_q^*) \not\subseteq \mathbf{L}(\mathcal{W}_p^*)$, thus $\mathbf{L}^\sharp \subseteq \mathbf{L}(\mathcal{W}_p^*)$ by splitting. Therefore we have $\mathcal{W}_q \models A_p$. If $p < q$, then by PROPOSITION 19, $\mathbf{L}(\mathcal{W}_q^*) \not\subseteq \mathbf{L}(\mathcal{W}_p^*)$, thus also by splitting, $\mathbf{L}^\sharp \subseteq \mathbf{L}(\mathcal{W}_p^*)$. Therefore $\mathcal{W}_q \models A_p$. ■

COROLLARY 22. *There exists a continuum of normal modal logics over* $\mathbf{KTB} \oplus \mathbf{Tra}(2,3)$. ■

This proof is one of the quite standard ways to show an existence of a continuum of modal logics, by employing a series of finite algebras and using the splitting technique effectively. Aside from the fact that it turned out to be possible to apply such a standard tool of modal logic to KTB-logics, there is the following significance of this result.

The author showed already the existence of a continuum in $Next(\mathbf{KTB})$ in [14], where he first presented a continuum of orthologics , and then used an embedding from orthologics to KTB-logics. In that proof, he used a class of ortholattices to show the continuum of orthologics, where such ortholattices are not orthomodular ([13]). Therefore, the resulting continuum of KTB-logics is located not very near from the top of the lattice $Next(\mathbf{KTB})$. The continuum we obtained in this study, instead, is in $Next(\mathbf{KTB} \oplus \mathbf{Tra}(2,3))$, which means that this is located in quite upper part of $Next(\mathbf{KTB})$.

It is well known that there exists only countably many normal modal logics in $Next(\mathbf{S5})$, and that $\mathbf{S5}$ is pretabular. In context of this paper, it is helpful to rewrite $\mathbf{S5} = \mathbf{KTB} \oplus \mathbf{Tra}(1,2)$. Then we recognize that there lies a boundary of countability or uncountability of a class of normal modal logics between $\mathbf{KTB} \oplus \mathbf{Tra}(1,2)$ and $\mathbf{KTB} \oplus \mathbf{Tra}(2,3)$.

5 The upper part of the lattice $Next(\mathbf{KTB})$

As we mentioned above, the logic $\mathbf{L}(\bullet)$ is the second greatest element in $Next(\mathbf{KTB})$. Then what kind of logics follow $\mathbf{L}(\bullet)$? A plausible candidate is the logic $\mathbf{L}(\bullet\!-\!\bullet)$, where . $\bullet\!-\!\bullet$ is a frame of two reflexive points jointed with symmetric relation. In fact, by Jónsson's lemma ([9]) we can show that above it are only $\mathbf{L}(\bullet)$ and the inconsistent logic. Moreover, it can be proved by the following argument that the logic $\mathbf{L}(\bullet\!-\!\bullet)$ is the third greatest of all Kripke complete logics in $Next(\mathbf{KTB})$.

We denote the frame $\bullet\!-\!\bullet$ by $\mathcal{G}_2$, that is, $\mathcal{G}_2 = \langle V, S\rangle$, where $V := \{0,1\}$ and $S := \{(0,0),(1,1),(0,1),(1,0)\}$. Then, we can show the following.

LEMMA 23. *Let* $\mathcal{F} = \langle W, R\rangle$ *be a KTB- Kripke frame, where* $\mathcal{F}^*$ *is subdirectly irreducible and* $|W| \geq 2$. *Then* $\mathbf{L}(\mathcal{F}) \subseteq \mathbf{L}(\mathcal{G}_2)$

Proof. Since $\mathcal{F}^*$ is s.i., we have $\mathcal{F} \models \forall x, y \exists n \in \omega(xR^n y)$ by THEOREM 2. We define a partition $\{W^{(i)}\}_{i=0}^{\infty}$ of W in the following: For $i = 0$, pick up an arbitrary $x_0 \in W$ and $W^{(0)} := \{x_0\}$. For $i = n + 1$, $W^{(n+1)} := \{y \in W \mid xRy \text{ for some } x \in W^{(n)}\} - \bigcup_{i=0}^{n} W^{(i)}$. Then by this construction, $|W| \geq 2$, and our first order condition for $\mathcal{F}$, we have $W^{(0)} \neq \emptyset$, $W^{(1)} \neq \emptyset$, and $\bigcup_{i=0}^{\infty} W^{(i)} = W$. Here a map $f : W \to \{0,1\}$ is defined as: for $a \in W$,

$$f(a) = \begin{cases} 0 & (\text{if } a \in W^{(2k)} \text{ for some } k \in \omega) \\ 1 & (\text{if } a \in W^{(2k+1)} \text{ for some } k \in \omega) \end{cases}$$

Then about this f, first, because $W^{(0)} \neq \emptyset$ and $W^{(1)} \neq \emptyset$, we can say that f is onto. Suppose aRb for $a, b \in W$. Then by our construction of $W^{(i)}$, we can see that one of the following cases occurs, that is, (1): for some m, $a \in W^{(m)}$ and $b \in W^{(m+1)}$, (2): for some m, $a \in W^{(m+1)}$ and $b \in W^{(m)}$, (3): for some m, $a, b \in W^{(m)}$. In case (1) and (2), ${}_{f(a)}S_{f(b)}$ holds due to the definition of f. In (3), ${}_{f(a)}S_{f(b)}$ also holds since S is reflexive. Last, suppose ${}_{f(a)}Sz$ for $a \in W$ and $z \in \{0,1\}$. Then, because $W^{(0)}, W^{(1)} \neq \emptyset$ and R is reflexive, it is obvious that there exists $b \in W$ such that aRb and $f(b) = z$. Thus we have checked that this f is a p-morphism from $\mathcal{F}$ to $\mathcal{G}_2$, and so, by COROLLARY 17 , $\mathbf{L}(\mathcal{F}) \subseteq \mathbf{L}(\mathcal{G}_2)$ holds. ■

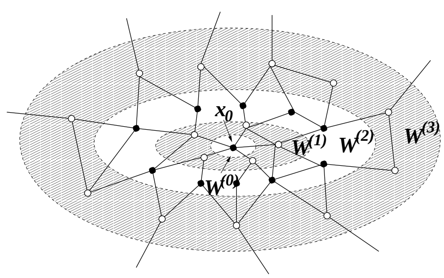

Figure 5. The partition of $\mathcal{F}$

From a graph theoretical point of view, this lemma can be reformulated into the following way. Namely, every countable, undirected, and connected graph (here *connected* means the graph theoretical sense), whose number of nodes are greater than one, can be colored in two colors, say, black and white, in such a way that any black node can see at least a white node, and vice a versa. This fact suggests that an existence of a p-morphism from a KTB- Kripke frame $\mathcal{F}$ to a KTB- Kripke frame $\mathcal{G}$ has something to do with the possibility of coloring the undirected graph $\mathcal{F}$ in some *coloring pattern* determined by the frame $\mathcal{G}$. By the above lemma, we have the following.

THEOREM 24. *The KTB-logic* $\mathbf{L}(\bullet\!\!-\!\!\bullet)$ *is the third greatest logics of all Kripke complete KTB-logics.* ■

In fact, by using Jónsson's lemma for some finite frames, we can draw a picture of the upper part of the lattice $Next(\mathbf{KTB})$. The figure below shows how the lattice structure of the upper part of $Next(\mathbf{KTB})$, which consists of connected frames of, at most, four reflexive points looks like. Each of ten circles represents the logic determined by the Kripke frame in it. In Blok's paper ([2]), the following result is established on the lattice

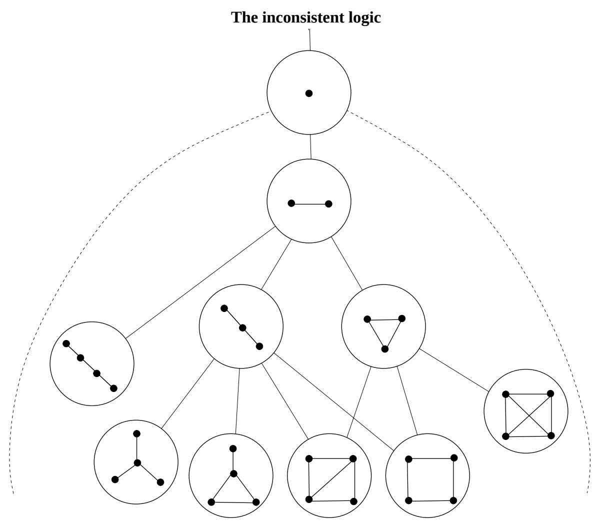

Figure 6. The upper part of $Next(\mathbf{KTB})$

$Next(\mathbf{KT})$.

THEOREM 25. *For any consistent logic* $\mathbf{L} \in Next(\mathbf{KT})$*, there exists a continuum of logics in* $Next(\mathbf{KT})$ *that are covered by* $\mathbf{L}$. ■

Of course this theorem can be applied to the logic $\mathbf{L}(\bullet)$ to show that there are uncountably many predecessors of this logic in $Next(\mathbf{KT})$. When the axiom $\mathbf{B}$ comes in their sight, it is not clear how many logics can survive of all predecessors of $\mathbf{L}(\bullet)$ in $Next(\mathbf{KT})$. But there might be a possibility that THEOREM 25 can be transformed, in some way, into the fact about the class $Next(\mathbf{KTB})$ and that the fact will show there is no third greatest logic in $Next(\mathbf{KTB})$.

6 Remarks and questions

The author cannot help saying that the study of the lattice $Next(\mathbf{KTB})$ here is very far from complete at present, because this paper treats only a very restricted class of the whole KTB-logics. But the approach in this study of assuming some conditions in order to make the object logics tractable seems to be reasonable, in the face of the situation that there is very few known facts about this class of logics.

The logics in $Next(\mathbf{KTB})$ are extremely difficult to investigate by nature, which is mentioned in several texts and papers in modal logics (for instance,[5],[8],[10]). In fact, there are only few facts known about this class of logics at present. Namely,

(1) There exists a continuum in $Next(\mathbf{KTB})$ ([14]).

(2) The cardinality of pretabular members in $Next(\mathbf{KTB})$ is at least countably infinite ([12]).

(3) The universal frame of rank 2 for $\mathbf{KTB} \oplus \Box^2 p \supset \Box^3 p$ is infinite ([4]).

To finish off this paper, we will list up some open questions about the KTB-logics and the structure of the lattice $Next(\mathbf{KTB})$. We expect that many researchers are getting interested in this area of modal logics, and they will find answers to the following questions. Moreover, we hope that the investigation of this class of logics will lead to a discovery of some new techniques and new perspectives for modal logics of broader classes rather than logics of transitive frames.

(Q1): Is the logic $\mathbf{L}(\bullet\!\!-\!\!\bullet)$ the third greatest element of $Next(\mathbf{KTB})$?

(Q2): What is the degree of incompleteness of $\mathbf{L}(\bullet)$ with respect to $\mathbf{KTB}$?

(Q3): Is there any KTB-logic that is finitely axiomatizable but not Kripke complete?

(Q4): Is there any KTB-logic that is finitely axiomatizable but not decidable?

(Q5): Is there any KTB-logic that is decidable but without the finite model property? ([8])

(Q6): Is it possible to show that for any $n \in \omega$, every normal modal logic over $\mathbf{KTBD_n}$ has the finite model property?

(Q7): How are locally tabular members in $Next(\mathbf{KTB})$ characterized? Can the diameter be the index of characterizing locally tabular members in $Next(\mathbf{KTB})$?

(Q8): Does there exist a continuum of pretabular members in $Next(\mathbf{KTB})$? ([12])

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