On Notions of Completeness Weaker than Kripke Completeness

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ABSTRACT. We are going to show that the standard notion of Kripke completeness is the strongest one among many provably distinct algebraically motivated completeness properties, some of which seem to be of intrinsic interest. More specifically, we are going to investigate notions of completeness with respect to algebras which are either atomic, complete, completely additive or admit residuals (the last notion of completeness coincides with conservativity of minimal tense extensions); we will be also interested in combinations of these properties.

1 Motivation

It is known that Kripke frames correspond to complete, atomic and completely additive Boolean algebras with operators (BAO's). This fact became the basis of duality theory for Kripke frames, developed in the 1970's by Thomason [13], Goldblatt [4] and others. In this paper, we are going to investigate notions of completeness and consequence weaker than those associated with standard Kripke frames from an algebraic perspective. Our starting point is a simple question: can we still obtain incompleteness results if we drop at least one of the properties which hold in BAO's corresponding to Kripke frames? Is the phenomenon of Kripke incompleteness caused by any particular combination of these properties? It will be shown than in this way one obtains several provably distinct notions, whose mutual relationships seem to be of independent interest.

The structure of the present work is as follows. Section 2 introduces and systematizes those completeness notions and related consequence relations. Section 3 proves the existence of complete and completely additive BAO whose logic is inconsistent with respect to atomic BAO's;

that solves two open problems in the field (see Section 3 for details). Section 4 proves that non-finite tense logics of linear time are inconsistent with respect to complete BAO's; this is a strengthening of results obtained by Wolter [18] and Kowalski [6]. Finally, Section 5 discusses completeness with respect to BAO's which admit residuals, connection with van Benthem [14] and the notion of *weak second-order consequence* introduced there. The main new result of that section is an example of atomic and completely additive BAO whose logic is incomplete with respect to algebras admitting residuals.

The author hopes that his work provides new arguments for importance of algebraic methods and insights in modal logic.

2 Introduction

This section sums up some existing duality results and gives definitions crucial to what follows. We assume familiarity with basic (1) set-theoretical, (2) topological and, most importantly, (3) algebraic notions. Thus, the reader is supposed to be familiar with notions like: (1) cartesian product, powersets, cardinality of a set, (2) Euclidean topology (of the reals), open and closed intervals, regular open sets, (3) lattices, complete lattices, boolean algebras, products, homomorphic images, subalgebras, varieties, discriminator terms and so on. For arbitrary set X, let 2^X denote its powerset, X^* — the set of all finite sequences from X, X^2 — the cartesian product of X with itself and $[X]^2$ — the set of all dubletons (two-element sets) from X. $W - X$ denotes the set-theoretical difference; if the universe is understood from the context, we sometimes write $-X$. The converse of a relation R is denoted as R^{-1}.

DEFINITION 1 (Syntax). A *modal similarity type* is a finite set TYPE of unary operators; *the basic modal similarity type* is an arbitrary singleton. Formulas of propositional modal language are defined as

$$\varphi ::= \bot \mid \top \mid p \mid \neg\varphi \mid \psi \wedge \varphi \mid \psi \vee \varphi \mid \psi \to \varphi \mid \Diamond_\pi \varphi$$

for every $\pi \in$ TYPE. In addition, for every $\pi \in$ TYPE, we define the *dual operator* $\Box_\pi \varphi \leftrightharpoons \neg \Diamond_\pi \neg \varphi$. For the basic modal similarity type, we often drop the subscript.

A modal logic is any set of formulas closed under substitution, Modus Ponens, necessitation and containing all axioms of classical logic plus $\Box_\pi(p \to q) \to (\Box_\pi p \to \Box_\pi q)$ for every $\pi \in$ TYPE. Define also $\Diamond^+_\pi \varphi \leftrightharpoons \varphi \vee \Diamond_\pi \varphi$. $\Box^+_\pi$ is defined dually.

REMARK 2. As it may be seen from the above definition, we are interested only in *normal* modal logics. The definition entails also that the set of all logics in a given language is closed under arbitrary intersections. Thus, for every Γ, there exists the smallest logic containing Γ. We call it *the logic axiomatized by* Γ.

DEFINITION 3 (Frames). *A Kripke frame* is a structure $\langle W, \{R_\pi\}_{\pi \in \mathrm{TYPE}} \rangle$, where $W \neq \emptyset$ and $R_\pi \subseteq W \times W$ for every $\pi \in$ TYPE; for every $\pi \in$ TYPE and every $X \subseteq W$, let $\Diamond_\pi X \leftrightharpoons \{y \in W \mid \exists x \in X \; y R_\pi x\}$ and $\Box_\pi \leftrightharpoons -\Diamond_\pi - X$. *A (normal) neighbourhood frame* is a structure $\langle W, \{\mathcal{N}_\pi\}_{\pi \in \mathrm{TYPE}} \rangle$ where $W \neq \emptyset$ and every $\mathcal{N}_\pi$ is a function assigning a filter over W to every point. Recall that a filter is a non-empty family of elements of the powerset which is upward closed and closed under finite intersections.

DEFINITION 4 (BAO's). *A boolean algebra with (unary) operators* (BAO) is a structure $\langle \mathfrak{A}, \to, \wedge, \vee, \neg, \{\Diamond_\pi\}_{\pi \in \mathrm{TYPE}}, \top, \bot \rangle$ s.t. $\langle \mathfrak{A}, \to, \wedge, \vee, \neg, \top, \bot \rangle$ is a boolean algebra and $\Diamond_\pi \bot = \bot$, $\Diamond_\pi(x \vee y) = \Diamond_\pi x \vee \Diamond_\pi y$ hold for every $\pi \in$ TYPE and every $x, y \in \mathfrak{A}$. *A trivial algebra* is any algebra whose universe is a singleton.

Thus, if not stated otherwise, we will make systematic confusion between (1) an algebra and its carrier set, (2) syntactic connectives (or derived terms) and corresponding algebraic operations; at least, if the underlying algebra is clear from the context. Definition of satisfaction and validity in frames and algebras are standard; the latter is done, as usual, by identifying modal formulas with algebraic terms. Just as we introduced the notion of the logic axiomatized by Γ, we may introduce the notion of the variety corresponding to Γ; it is the smallest variety s.t. all formulas from Γ are valid in all algebras from the variety. We write $\Gamma \vDash \varphi$ if φ is valid in the variety axiomatized by Γ. It is well-known that $\Gamma \vDash \varphi$ iff φ belongs to the logic axiomatized by Γ.

DEFINITION 5 (Basic properties).

- $\mathfrak{A}$ is a $\mathcal{C}$-BAO if it is non-trivial and lattice-complete, i.e, closed under arbitrary joins and meets.

- $\mathfrak{A}$ is a $\mathcal{A}$-BAO if it is non-trivial and atomic, i.e., below every element distinct from the bottom there is an *atom* — smallest element which is not equal to the bottom itself. The set of all atoms will be denoted by $At\mathfrak{A}$.

- $\mathfrak{A}$ is a $\mathcal{V}$-BAO if it is non-trivial and completely additive, i.e., for every $\pi \in$ TYPE and any family of elements $X \subseteq \mathfrak{A}$ s.t. whenever the join $\bigvee\limits_{x \in X} x$ exists, the join $\bigvee\limits_{x \in X} \Diamond_\pi x$ exists as well and is equal to $\Diamond_\pi \bigvee\limits_{x \in X} x$.

- $\mathfrak{A}$ is a $\mathcal{T}$-BAO if it is non-trivial and admits residuals, i.e., for any $\pi \in$ TYPE there exists a function $\mathbf{h}_\pi$ s.t. for each $x, y \in \mathfrak{A}$, $\Diamond_\pi x \leq y$ iff $x \leq \mathbf{h}_\pi y$.

The reader is asked to observe that we don't require residuals to be term-definable. This is the difference between our $\mathcal{T}$-BAO's and *residuated* BAO's of Jipsen [5]. Also, we have the following well-known

FACT 6. $\Diamond_\pi$ has a residual $\mathbf{h}_\pi$ iff f has a *conjugate* $\mathbf{p}_\pi$, i.e, for $x, y \in \mathfrak{A}$, $\Diamond_\pi x \wedge y = \bot$ iff $x \wedge \mathbf{p}_\pi y = \bot$. Both operations are then related to each other by equation $\mathbf{h}_\pi x = \neg \mathbf{p}_\pi \neg x$, i.e., $\mathbf{h}_\pi$ is the dual of $\mathbf{p}_\pi$.

DEFINITION 7 (Complex properties). Let PROPERTIES $\leftrightharpoons \{\mathcal{C}, \mathcal{A}, \mathcal{V}, \mathcal{T}\}$. For any $\mathcal{X}^* \in$ PROPERTIES*, we say that $\mathfrak{A}$ is a $\mathcal{X}^*$-BAO if $\mathfrak{A}$ is a $\mathcal{X}$-BAO for any $\mathcal{X}$ appearing in $\mathcal{X}^*$. E.g., a $\mathcal{AV}$-BAO is atomic and completely additive.

Some of these notions coincide. For example, we have the following

LEMMA 8. *Every $\mathcal{T}$-BAO is a $\mathcal{V}$-BAO. For $\mathcal{C}$-BAO's the converse also holds: $\mathfrak{A}$ is a $\mathcal{CV}$-BAO iff it is a $\mathcal{CT}$-BAO.*

Proof. The first statement is well-known. The second statement is justified by the observation that in a $\mathcal{CV}$-BAO $\mathfrak{A}$ for every $\pi \in$ TYPE

Dualities

	Kripke frames	neighbourhood frames
frames $\rightarrow$ algebras	$\mathcal{CAV}$-BAO: powerset algebra of W with operators $\Diamond X$ as in Def. 3 (*the complex algebra*)	$\mathcal{CA}$-BAO: powerset algebra of W with operators $\Diamond X \leftrightharpoons -\{x \in W \mid -X \in \mathcal{N}x\}$
algebras $\rightarrow$ frames	for a $\mathcal{CAV}$-BAO $\mathfrak{A}$: $\langle At\mathfrak{A}, R\rangle$ aRb iff $a \leq \Diamond b$ (*the atom structure*)	for a $\mathcal{CA}$-BAO $\mathfrak{A}$: $\langle At\mathfrak{A}, \mathcal{N}\rangle$ $\mathcal{N} : At\mathfrak{A} \ni a \longrightarrow \{X \subseteq At\mathfrak{A} \mid a \leq \Box \bigvee X\}$
morphisms	p-morphisms $\eta : \langle W, R\rangle \longrightarrow \langle V, S\rangle$ $\eta x S y$ iff $\exists z\ (xRz \& \eta z = y)$	neighbourhood morphisms $\eta : \langle W, \mathcal{M}\rangle \longrightarrow \langle V, \mathcal{N}\rangle$ $\eta^{-1}B \in \mathcal{M}a$ iff $B \in \mathcal{N}\eta a$
dual categories (by contravariant functors)	$\mathcal{CAV}$-BAO's with complete morphisms	$\mathcal{CA}$-BAO's with complete morphisms

Table 1. Summary of basic information. The part concerning neighbourhood frames based on Došen [3] . For simplicity, we work in the basic similarity type.

we may define the conjugate of $\Diamond_\pi$ as

$$\mathbf{p}_\pi a \leftrightharpoons \bigwedge\{x \mid a \leqslant \Box_\pi x\},$$

where $\Box_\pi$, as everywhere in this work, denotes the dual of $\Diamond_\pi$. $\dashv$

It is known that Kripke frames correspond to $\mathcal{CAV}$-BAO's and neighbourhood frames correspond to $\mathcal{CA}$-BAO. See Table 1 for the summary of known dualities.

REMARK 9. Lemma 8 implies that, in particular, $\mathcal{CAV}$-BAO's are also $\mathcal{CAT}$-BAO's. Thus, by Table 1 we may treat Kripke frames as $\mathcal{CAT}$-BAO's in disguise. The reason why we may choose to work with $\mathcal{T}$-BAO's instead of $\mathcal{V}$-BAO's is that the former are much more tractable. In particular, the property of being a $\mathcal{T}$-BAO in a finite type is expressible by a Σ^1_1-sentence (*there exists a function such that ...*) and hence preserved by ultraproducts. In general, however, it is not preserved by subalgebras, hence even the universal class generated by a given class of such BAO's contains algebras which are not $\mathcal{T}$-BAO's. Fortunately, the situation changes when residuals are term-definable, as it is often the case. In such a situation, all algebras in the variety must be $\mathcal{T}$-BAO's. Nothing like this can happen with $\mathcal{A}$-BAO's or $\mathcal{C}$-BAO's; non-degenerate varieties always contain a BAO which is neither complete nor atomic. Further discussion is postponed till Section 5.

DEFINITION 10 (Completeness notions). For $\mathcal{X}^* \in$ PROPERTIES*, Γ — arbitrary set of modal formulas in a fixed similarity type, φ — a formula in the same language, let

$$\Gamma \vDash_{\mathcal{X}^*} \varphi \quad \leftrightharpoons \quad \varphi \text{ holds in all } \mathcal{X}^*\text{-BAO's in the variety of BAO's corresponding to } \Gamma.$$

Thus, a logic axiomatized by Γ is $\mathcal{X}^*$-*incomplete* iff for some φ, $\Gamma \vDash_{\mathcal{X}^*} \varphi$ and yet $\Gamma \nvDash \varphi$.

A particularly interesting notion arising in this way is

DEFINITION 11 ($\mathcal{X}^*$-inconsistency). A logic axiomatized by Γ is $\mathcal{X}^*$-inconsistent if it is consistent, i.e., distinct from the set of all formulas in a given similarity type and yet $\Gamma \vDash_{\mathcal{X}^*} \bot$, i.e., there is no $\mathcal{X}^*$-BAO in the corresponding variety.

REMARK 12. $\mathcal{X}^*$-inconsistency implies $\mathcal{X}^*$-incompleteness. For any $\mathcal{Y}^* \supseteq \mathcal{X}^*$, $\mathcal{X}^*$-inconsistency implies $\mathcal{Y}^*$-inconsistency and $\mathcal{X}^*$-incompleteness implies $\mathcal{Y}^*$-incompleteness. Thus, the construction of Thomason [12] may be viewed as the first example of a $\mathcal{CAT}$-inconsistent logic, or even a $\mathcal{CA}$-inconsistent one, as for tense logics there is no difference between those two notions; see also Section 5.

REMARK 13 (The basic similarity type). It is not possible to produce any example of $\mathcal{CAT}$-inconsistent logic in the basic modal similarity type. By Makinson's theorem, any nontrivial variety of BAO's with one unary operator contains at least one two-element algebra (cf., e.g., [2, theorem 8.67]). Nevertheless, the method of reduction invented by Thomason and developed by Kracht and Wolter (cf., e.g., [7, chapter 6]) shows how to transfer examples of $\mathcal{X}^*$-inconsistent logics into $\mathcal{X}^*$-incomplete ones in the basic similarity type.

Relationships between those notions of completeness (or consequence) are as shown by Figure 1 below. For $\vDash_{\mathcal{X}^*} \subseteq \vDash_{\mathcal{Y}^*}$, solid lines denote established proper inclusion $\vDash_{\mathcal{X}^*} \subsetneq \vDash_{\mathcal{Y}^*}$, i.e. the existence of $\mathcal{X}^*$-complete logics which are $\mathcal{Y}^*$-incomplete.

$$\begin{tikzcd}
 & \models \arrow[dl, dash] \arrow[d, dash] \arrow[dr, dash] & \\
\models_{\mathcal{A}} \arrow[d, dash] \arrow[dr, dash] & \models_{\mathcal{C}} \arrow[dl, dash] \arrow[dr, dash] & \models_{\mathcal{T}} \arrow[dl, dash] \arrow[d, dash] \\
\models_{\mathcal{CA}} \arrow[dr, dash] & \models_{\mathcal{AT}} \arrow[d, dash] & \models_{\mathcal{CT}} \arrow[dl, dash] \\
 & \models_{\mathcal{CAT}} &
\end{tikzcd}$$

Figure 1. Proper inclusions between various algebraic consequence relations. This is not necessarily the Hasse diagram — neither joins nor meets have to be faithful.

As the drawing suggests, one may show that all inclusions are proper. The most economical way to achieve this goal is to prove three facts:

$$\models_{\mathcal{A}} \;\not\subseteq\; \models_{\mathcal{CT}}, \quad \models_{\mathcal{C}} \;\not\subseteq\; \models_{\mathcal{AT}}, \quad \models_{\mathcal{T}} \;\not\subseteq\; \models_{\mathcal{CA}} .$$

The first one will be proven in Section 3, the second one in Section 4, the last one will be discussed in Section 5. The reader certainly noted that $\mathcal{V}$-completeness is missing from the picture. Some reasons for that were already given by Remark 9. Further discussion and an updated diagram will be given in Section 5.

REMARK 14. The consequence relation symbol introduced above should not be confused with $\mathfrak{A} \models \varphi$ ($\mathfrak{A} \models_{\mathfrak{V}} \varphi$), which means, as usual, that the equation corresponding to φ holds in $\mathfrak{A}$ (under valuation $\mathfrak{V}$).

In forthcoming sections, it will be often useful to represent BAO's as *general frames*.

DEFINITION 15. *A general frame* is a pair $\langle \mathfrak{F}, \mathfrak{A} \rangle$ s.t. $\mathfrak{F}$ is a Kripke frame and $\mathfrak{A}$ is a subalgebra of the full complex algebra of $\mathfrak{F}$ (see Table 1 for the definition). An algebra thus represented will be called *a complex algebra*.

3 Varieties with no atomic algebras

Let us start with the proof that $\vDash_{\mathcal{A}} \quad \not\subseteq \quad \vDash_{\mathcal{CT}}$. We generalize here a result of Venema [16], who shows that there are $\mathcal{A}$-inconsistent logics. We are going to prove that there is $\mathcal{CV}$-BAO (hence a $\mathcal{CT}$-BAO) whose logic is $\mathcal{A}$-inconsistent. That solves at least two open problems in the field. The first one was posed to the present author by V. Shehtman: according to him, the question whether every $\mathcal{C}$-complete logic is neighbourhood complete (i.e., whether $\vDash_{\mathcal{CA}} \quad = \quad \vDash_{\mathcal{C}}$) was a folklore since V. Rybakov and L. Maksimova's work on the subject in the 1970's. Another one was posed by Y. Venema himself: can one produce a variety with no atomic members using a $\mathcal{T}$-BAO?

The original construction of Venema [16] is unsuited for the goal we have in mind. It is based on a particular representation of the atomless countable boolean algebra. Hence, lattice-incompleteness and lack of complete additivity seem irreparable. Nevertheless, we try to exhibit existing analogies between that construction and ours by the choice of notation and terminology. On the other hand, the earlier construction gives an example of an *atomless* variety, whereas ours — just a variety with no atomic members. It is enough for our purposes, though.

NOTATION 16. Let $\mathbf{I}$ be the unit interval $(0,1)$, $\mathbf{P} = [\mathbf{I} \cup \{0,1\}]^2$, $\mathfrak{P}$ — the boolean algebra whose universe is $2^{\mathbf{P}}$ and the operations coincide with standard set-theoretical ones, $\mathfrak{O}$ — the boolean algebra of regular open sets (i.e., interiors of closed sets) from $\mathbf{I}$. Note that $\vee$ and $\neg$ in $\mathfrak{O}$ differ from the set-theoretical operations. It is well-known that both $\mathfrak{P}$ and $\mathfrak{O}$ are complete lattices. For $0 \leq a < b \leq 1$, the open interval determined by a and b will be denoted as (a,b), whereas $\langle a,b \rangle$ will be simply an ordered pair.

THEOREM 17. *There exists a $\mathcal{CV}$-BAO $\mathfrak{A}$ s.t. the variety generated by $\mathfrak{A}$ — i.e., $HSP(\mathfrak{A})$ – contains no atomic members.*

Proof. We will proceed via a series of easily verifiable claims.

CLAIM 18. $B \in \mathfrak{O}$ only if there exists at most countable set J s.t. $B = \bigcup_{j \in J} (a_j, b_j)$, where for each $j \in J$, $0 \leq a_j < b_j \leq 1$ and for each $j \neq k$, either $b_j < a_k$ or $b_k < a_j$. The set $\{\{a_j, b_j\} \mid j \in J\}$ will be called *the canonical representation of* B and denoted as Z_B.

NOTATION 19. For any $z \in \mathbf{P}$, $z_< \leftrightharpoons inf(z)$, $z_> \leftrightharpoons sup(z)$ and $\llcorner z \lrcorner \leftrightharpoons (z_<, z_>)$.

Our main object of interest will be a BAO whose underlying boolean reduct is $\mathfrak{A}_- \leftrightharpoons \mathfrak{P} \times \mathfrak{O}$. As this is a product of two complete boolean algebras, we get

CLAIM 20. $\mathfrak{A}_-$ is a complete boolean algebra.

Now we are going to define five unary operations on the universe of our algebra. Subscripts will be chosen so as to emphasize analogies with Venema [16] construction.

First, we are going to define four auxiliary mappings:

$$
\begin{aligned}
\eta_\lhd : \mathfrak{O} \ni B \quad &\longrightarrow \quad \{z \in \mathbf{P} \mid \exists y \in B \ y \in \llcorner z \lrcorner\} \in \mathfrak{P},\\
\eta_\rhd : \mathfrak{P} \ni A \quad &\longrightarrow \quad \bigvee_{a \in A}{}^{\mathfrak{O}} \llcorner a \lrcorner \in \mathfrak{O},\\
\eta_> : \mathfrak{P} \ni A \quad &\longrightarrow \quad \{z \in \mathbf{P} \mid \exists z^* \in A \ \llcorner z^* \lrcorner \supsetneq \llcorner z \lrcorner\} \in \mathfrak{P},\\
\eta_L : \mathfrak{P} \ni A \quad &\longrightarrow \quad \{z \in \mathbf{P} \mid \exists z^* \in A \ z_< = z^*_<, z_> = z^*_< + \\
&\qquad\quad (z^*_> - z^*_<)/2\} \in \mathfrak{P}.
\end{aligned}
$$

Through this section, we set TYPE $\leftrightharpoons \{\lhd, \rhd, >, L, \mathbf{E}\}$.

DEFINITION 21. Define $\mathfrak{A} \leftrightharpoons \langle \mathfrak{A}_-, \{\Diamond_\pi\}_{\pi \in \mathrm{TYPE}} \rangle$, where $\Diamond_\lhd \langle A, B \rangle \leftrightharpoons \langle \eta_\lhd B, \emptyset \rangle$, $\Diamond_\rhd \langle A, B \rangle \leftrightharpoons \langle \emptyset, \eta_\rhd A \rangle$, $\Diamond_> \langle A, B \rangle \leftrightharpoons \langle \eta_> A, \emptyset \rangle$, $\Diamond_L \langle A, B \rangle \leftrightharpoons \langle \eta_L A, \emptyset \rangle$, and $\Diamond_\mathbf{E}$ will be *the unary boolean discriminator function* on $\mathfrak{A}_-$ (Jipsen [5]); i.e., $\Diamond_\mathbf{E} x = \bot$ iff $x = \bot$ and $\Diamond_\mathbf{E} x = \top$ otherwise.

We want to prove that these operations distribute over arbitrary joins. It is obvious for the unary discriminator, $\Diamond_>$ and $\Diamond_L$ as both $\Diamond_>$ and $\Diamond_L$ are easily seen to be operators given by certain relations on a Kripke frame whose universe is $\mathbf{P}$. Assume now we have a family of pairs $\{\langle A_j, B_j \rangle\}_{j \in J} \subseteq \mathfrak{A}$. Then

$$\bigvee_{j \in J} \Diamond_\lhd \langle A_j, B_j \rangle = \bigvee_{j \in J} \langle \eta_\lhd B_j, \emptyset \rangle = \langle \bigcup_{j \in J} \eta_\lhd B_j, \emptyset \rangle$$

and, similarly,

$$\bigvee_{j\in J} \Diamond_{\rhd}\langle A_j, B_j\rangle = \bigvee_{j\in J}\langle\emptyset, \eta_{\rhd}A_j\rangle = \langle\emptyset, \bigvee_{j\in J}\eta_{\rhd}A_j\rangle.$$

Hence, it is enough to establish

(a) $\bigcup\limits_{j\in J}\eta_{\lhd}B_j = \eta_{\lhd}\bigvee\limits_{j\in J}B_j$

and

(b) $\bigvee\limits_{j\in J}\eta_{\rhd}A_j = \eta_{\rhd}\bigcup\limits_{j\in J}A_j$.

Observe that in both cases, the $\subseteq$-direction is trivial. To establish the converse for a, assume there exists $i \in \mathbf{P}$ s.t.

$$(\exists y \in \bigvee_{j\in J} B_j \; y \in \llcorner i \lrcorner) \;\&\; (\forall j \in J \;\forall y \in B_j \; y \notin \llcorner i \lrcorner).$$

But then for each $j \in J$, $B_j \subseteq \neg\llcorner i\lrcorner$, hence $B_j = Int(B_j) \subseteq Int(\neg\llcorner i\lrcorner) \in \mathfrak{O}$. Thus, we get $\bigvee\limits_{j\in J} B_j \subseteq Int(\neg\llcorner i\lrcorner) \subseteq \neg\llcorner i\lrcorner$, a contradiction.

To establish

$$\bigvee_{a\in\bigcup\limits_{j\in J}A_j} \llcorner a\lrcorner \subseteq \bigvee_{j\in J}\bigvee_{a\in A_j}\llcorner a\lrcorner,$$

take any X s.t. for each $j \in J$, $\bigvee\limits_{a\in A_j}\llcorner a\lrcorner \subseteq X$. Obviously, for each $a \in \bigcup\limits_{j\in J} A_j$ there exists $j_a \in J$ s.t. $a \in A_{j_a}$. Hence for each $a \in \bigcup\limits_{j\in J} A_j$, $\llcorner a\lrcorner \subseteq \bigvee\limits_{a\in A_{j_a}}\llcorner a\lrcorner \subseteq X$ and thus we get $\bigvee\limits_{a\in\bigcup_{j\in J}A_j}\llcorner a\lrcorner \subseteq X$, which gives us b. In fact, this a sort of law of infinite associativity which holds in every complete lattice; cf., e.g., [9]. Thus. we got the desired

CLAIM 22. $\Diamond_{\lhd}$, $\Diamond_{\rhd}$, $\Diamond_{>}$, $\Diamond_L$ and $\Diamond_{\mathbf{E}}$ are completely additive operators.

Define $c \leftrightharpoons \Diamond_{\rhd}\top$ and $Fx \leftrightharpoons \Diamond_{\rhd}\Box_L(\Box_{\lhd}x \wedge \neg\Diamond_{>}\Box_{\lhd}x)$.

CLAIM 23. In $\mathfrak{A}$, c is equal to $\langle\emptyset, \mathbf{I}\rangle$. Hence, an element of $\mathfrak{A}$ is below c iff it is of the form $\langle\emptyset, B\rangle$ for some $B \in \mathfrak{O}$.

CLAIM 24. For any $B \in \mathfrak{O}$,

$$\Box_{\lhd}\langle\emptyset, B\rangle = \langle\{z \in \mathbf{P} \mid \llcorner z \lrcorner \subseteq B\}, \mathbf{I}\rangle,$$

$$\Box_{\lhd}\langle\emptyset, B\rangle \wedge \neg\Diamond_{>}\Box_{\lhd}\langle\emptyset, B\rangle = \langle Z_B, \mathbf{I}\rangle,$$

$$F\langle\emptyset, B\rangle = \langle\emptyset, \bigcup_{z \in Z_B} (z_<, z_< + (z_> - z_<)/2)\rangle.$$

This gives us the following

CLAIM 25. There exists a constant term c and an unary term Fx in language determined by TYPE s.t. $\mathfrak{A} \models c > \bot$ and $\mathfrak{A} \models \forall x\ (\bot < x \leq c \rightarrow \bot < Fx < x)$. As $\mathfrak{A}$ is a discriminator algebra, i.e., $\Diamond_{\mathbf{E}}$ behaves like universal modality, those two facts may be reformulated as follows: $\mathfrak{A} \models \Diamond_{\mathbf{E}} c = \top$ and $\mathfrak{A} \models \Diamond_{\mathbf{E}} x \wedge \Box_{\mathbf{E}}(x \rightarrow c) \ \leq\ \Diamond_{\mathbf{E}} Fx \wedge \Diamond_{\mathbf{E}}(x \wedge \neg Fx) \wedge \Box_{\mathbf{E}}(Fx \rightarrow x)$.

The above claim implies Theorem 17. For similar arguments cf. Venema [16] or an earlier, undebugged attempt of such a construction by Kracht and Kowalski [8] . $\dashv$

4 Varieties with no complete algebras

This section establishes $\vDash_{\mathcal{C}} \ \not\subseteq\ \vDash_{\mathcal{AT}}$. Examples of $\mathcal{C}$-incomplete logics (above the unimodal logic **K4** and higher) have been already provided in a previous work of the author [11]. Nevertheless, we are now going to obtain a stronger result ($\mathcal{C}$-inconsistency) by means of surprisingly natural examples. We are going to work in the similarity type of tense logic in this section, and hence we set TYPE $\risingdotseq \{<, >\}$. In the 90's, Kowalski [6] and Wolter [18] proved independently that there are exactly countably many maximal consistent extensions of **Lin** — the tense logic of all linear time flows, i.e., frames which are transitive and connected ($\forall x \forall y (xRy \vee yRx \vee x = y)$)). We are going to show that every maximal **Lin** logic which is not *tabular*, i.e., which is not determined by a finite frame, is $\mathfrak{C}$-inconsistent. As far as the author is aware, this is a new result: up to now, it was known only that such logics do not have any Kripke frames.

DEFINITION 26. *A head-and-tail logic* is the tense logic determined by a general frame $\mathfrak{F}_{\mathbf{a}} \leftrightharpoons \langle W_{\mathbf{a}}, \{R^{\mathbf{a}}_{\pi}\}_{\pi \in \mathrm{TYPE}}, \mathfrak{A}_{\mathbf{a}} \rangle$, where

- $\mathbf{a} = \langle \kappa, r, \lambda \rangle$, $\kappa, \lambda \in \omega + 1$, $r \in \{0, 1\}$. If $\omega \in \{\kappa, \lambda\}$, then $r = 0$.
- $W_{\mathbf{a}} \leftrightharpoons \{n_{<} \mid n \in \kappa\} \cup \{o_{*} \mid o \in r\} \cup \{m_{>} \mid m \in \lambda\}$,
- $R^{\mathbf{a}}_{<} \leftrightharpoons \{\langle n_{<}, m_{>} \rangle \mid n \in \kappa, m \in \lambda\} \cup \{\langle n_{<}, m_{<} \rangle \mid n, m \in \kappa, n < m\} \cup \{\langle n_{>}, m_{>} \rangle \mid n, m \in \lambda, n > m\} \cup \{\langle n_{<}, o_{*} \rangle, \langle o_{*}, o_{*} \rangle, \langle o_{*}, m_{>} \rangle \mid n \in \kappa, m \in \lambda, o \in r\}$. $R^{\mathbf{a}}_{>} \leftrightharpoons R^{\mathbf{a}-1}_{<}$.
- $\mathfrak{A}_{\mathbf{a}}$ is the algebra of finite and co-finite subsets over $W_{\mathbf{a}}$.

Note that to make the definition more condensed, we followed von Neumann convention of representing ordinals; hence, e.g., $1 = \{0\}$. Finite irreflexive chain of length m may thus be represented as $\mathfrak{F}_{\langle k,0,l \rangle}$, for any k, l s.t. $k + l = m$.

FACT 27. On every $\mathfrak{A}_{\mathbf{a}}$, the following term (*universal modality*) defines the unary boolean discriminator:

$$\mathbf{E}x \leftrightharpoons \Diamond_{>} x \wedge x \wedge \Diamond_{<} x.$$

THEOREM 28. *A tense logic containing* **Lin** *is maximal consistent iff it is a head-and-tail logic.*

Proof. Cf. Kowalski [6] or Wolter [18]. ⊣

Thus, we will use the names *head-and-tail logic* and *maximal* **Lin** *logic* interchangeably. Those determined by a frame of the form $\mathfrak{F}_{\langle \kappa, r, \lambda \rangle}$ with either κ or λ (or both) equal to ω are called *infinite*. Those which are not infinite are called *tabular* (or *finite*).

¿From now on in this section, to fix the discourse we concentrate on head-and-tail logics with $\kappa = \omega$, but proofs are instantly adaptable for those with finite κ and $\lambda = \omega$ — it is enough to exchange modalities. As in the infinite case r is always equal to 0, instead of notation $\mathcal{F}_{\langle \omega, r, \lambda \rangle}$, we write $\mathfrak{F}_{\omega, \lambda}$. Define a sequence of variable-free formulas which will serve as names for points: $\underline{i} \leftrightharpoons \Diamond^{i}_{>} \top \wedge \Box^{i+1}_{>} \bot$ $(n \in \omega)$.

LEMMA 29. *The following variable-free formulas are true in every* $\mathfrak{F}_{\omega, \lambda}$*:*

(c) $\mathbf{E}\underline{i} \quad (i \in \omega)$,

(d) $\underline{i} \to \lozenge_{<}\underline{i+n} \quad (i \in \omega, n > 0)$.

In addition,

(e) $\underline{i} \to \neg\underline{j}$

is a theorem of **Lin** *for every* $i \neq j$.

Proof. The easy observation that *numeral* $\underline{i}$ is true exactly in one point (namely i) and Fact 27 entail statement c. To prove d it is enough to observe in addition that in the future of every i there is a point verifying $\underline{j}$ for every $j > i$. Statement e follows from the fact that the right conjunct of $\underline{i}$ is a negation of the left conjunct of $\underline{i+1}$ and that $\Box_{<}p \to \Box^2_{<}p$ is a theorem of **Lin**; the rest is proven by simple induction. ⊣

LEMMA 30. *Every complete (or even ω-complete)* BAO *verifying all variable-free formulas true in $\mathfrak{F}_{\omega,\lambda}$ refutes* the weak Grzegorczyk axiom *or* the law of weak foundation, *i.e., the formula*

$$\mathbf{wf} \leftrightharpoons \neg(p \ \wedge \ \Box^{+}_{<}(p \ \to \ \lozenge_{<}(\neg p \ \wedge \ \lozenge_{<}p))).$$

Proof. Take any complete BAO $\mathfrak{B}$ verifying all variable-free formulas true in $\mathfrak{F}_{\omega,\lambda}$. Define two sequences of variable-free formulas and corresponding elements of $\mathfrak{B}$: $a_n \leftrightharpoons \underline{2n}$ and $b_n \leftrightharpoons \underline{2n+1}$ $(n \in \omega)$.

By statement e of Lemma 29, any a and b intersect at $\bot$ and it follows that $\bigvee_{n\in\omega} a_n \leqslant \neg \bigvee_{n\in\omega} b_n$ and $\bigvee_{n\in\omega} b_n \leqslant \neg \bigvee_{n\in\omega} a_n$. By claim d, $a_n \leqslant \lozenge_{<}b_n \leqslant \lozenge_{<} \bigvee_{n\in\omega} b_n$ and it follows that $\bigvee_{n\in\omega} a_n \leqslant \lozenge_{<} \bigvee_{n\in\omega} b_n$; in a similar way, one proves $\bigvee_{n\in\omega} b_n \leqslant \lozenge_{<} \bigvee_{n\in\omega} a_n$. Hence, if we set $\mathfrak{V}(p) \leftrightharpoons \bigvee_{n\in\omega} a_n$, $\mathfrak{V}(\mathbf{wf}) = \mathfrak{V}(\neg p)$, as the value of the subformula of $\mathbf{wf}$ preceded by $\Box^{+}_{<}$ is $\top$. By statement c of Lemma 29, $\mathfrak{V}(p) \neq \bot$, hence $\mathfrak{V}(\neg p) \neq \top$. ⊣

THEOREM 31. *There is no complete* BAO *in the variety corresponding to an infinite maximal* **Lin** *logic.*

Proof. By Lemma 30 it is enough to prove that the law of weak foundation (both for $\Diamond_>$ and $\Diamond_<$) cannot be refuted in any frame of the form $\mathfrak{F}_{\omega,\lambda}$. Assume there is a point y_0 where formula **wf** is refuted under some valuation $\mathfrak{V}'$. In the future of this point, there must a point y_1 belonging to $\mathfrak{V}'(\neg p)$; in the future of y_1 there must be a point y_2 in $\mathfrak{V}'(p)$ and proceeding in this way we can prove there is an infinite set with infinite complement in $\mathfrak{A}_{\omega,\lambda}$, namely $\mathfrak{V}'(p)$, which is a contradiction. Hence, the law of weak foundation belongs to every maximal consistent **Lin** logic and yet it is refuted in any complete algebra verifying all variable-free formulas true in a given infinite maximal **Lin** logic. $\dashv$

REMARK 32. One may prove that all head-and-tail logics are decidable and their satisfiability problem is NP-complete. Thus, $\mathcal{C}$-inconsistency does not imply a high level of computational complexity. Moreover, all of them with the exception of the one determined by $\mathfrak{F}_{\omega,\omega}$ are finitely axiomatizable. These results follow from Wolter [18] and a forthcoming paper Litak, Wolter [10]. By use of methods from those papers one may actually prove that all extensions of **Lin** are $\mathcal{AT}$-complete. Thus, the situation in the lattice of all extensions of **Lin** is rather peculiar: although incompleteness is a common phenomenon for logics from that lattice, the sole perpetrator of all problems is lattice-completeness of BAO's corresponding to Kripke frames. Section 3 showed that in general it is not true even for logics with conjugated operators and universal modality. Proof of this result would require too much techniques outside the scope of the present work. Therefore, it will be provided in a separate paper together with a more in-depth analysis of incompleteness problems for tense logics of linear time.

5 Varieties which do not admit residuals

This section deals with $\mathcal{T}$-incompleteness and $\mathcal{AV}$-completeness. Let us formulate the following representation theorem, which, albeit simple, will be of some importance later on.

THEOREM 33. *An algebra is a $\mathcal{AV}$-BAO iff it can be represented as a complex algebra containing all singletons of the underlying frame.*

An algebra is a $\mathcal{AT}$-BAO *iff it can be represented as a reduct of a complex algebra containing all singletons of a frame in which every relation has its converse — i.e., a tense frame.*

Proof. Follows from Venema [15, Theorem 5.1]. The second part may be proven analogously. $\dashv$

OBSERVATION 34. A term P_π defines the conjugate of an operator $\Diamond_\pi$ on all algebras in the variety $\mathfrak{V}$ iff $\mathfrak{A} \models p \to \neg P_\pi \neg \Diamond_\pi p$ and $\mathfrak{V} \models p \to \Box_\pi P_\pi p$ hold for all $\mathfrak{A} \in \mathfrak{V}$. Hence, the most straightforward method to obtain a $\mathcal{T}$-complete extension of a given logic is to add for $\pi \in$ TYPE a new operator P_π and stipulate the above equalities. This procedure is known as forming *minimal tense extensions*. Thus we arrive at the most important reason for independent interest in $\mathcal{T}$-completeness; it boils down to conservativity of minimal tense extensions, as the latter means that logic is complete with respect to a class of algebras which are reducts of $\mathcal{T}$-BAO's.

For the time being, we are going to work in the basic modal similarity type. It is already known that logics with non-conservative minimal tense extensions indeed do exist; in fact, Zakharyaschev et al. [19, Theorem 124] generalizes the Blok Incompleteness Theorem for degrees of $\mathcal{T}$-incompleteness. For the purpose of this work, it is enough to observe that

LEMMA 35. $\Box\Diamond\top \to \Box(\Box(\Box p \to p) \to p) \vDash_{\mathcal{T}} \Diamond\Box\bot \vee \Box\bot$.

Proof. Assume $\mathfrak{A} \not\models \Diamond\Box\bot \vee \Box\bot = \top$ and $\Diamond$ has a conjugate $\mathbf{p}_\Diamond$ (not necessarily term-definable). It means that $a \leftrightharpoons \Box\Diamond\top \wedge \Diamond\top$ is distinct from the bottom element. Define $b \leftrightharpoons \mathbf{p}_\Diamond a$. $b \leq \Diamond\top$ because of the fact that $a \leq \Box\Diamond\top$. Our goal is to show that

(f) $a \leq \Diamond b$

and

(g) $b \leq \Diamond b$.

As, by assumption, $a \leq \Box\Diamond\top$, it will follow that

(h) $\bot < a \leq \Box\Diamond\top \wedge \Diamond(\Box(\Diamond b \vee \neg b) \wedge b)$, which implies our theorem. h follows from g and f, because g implies that $b \wedge \Box\neg b = \bot$ and hence $b = \Box(\Diamond b \vee \neg b) \wedge b$. f follows from the fact that for any $x \leq \Diamond\top$, $x \leq \Diamond\mathbf{p}_\Diamond x$. As $b \leq \Diamond\top$ as well and for every x, $x \leq \Box\mathbf{p}_\Diamond x$, we get that $a \leq \Box\Diamond b$, which gives us g. ⊣

Nevertheless, Wolter [17, Section 4.6] defined a $\mathcal{CA}$-BAO which is readily seen to separate $\Diamond\Box\bot\vee\Box\bot$ from $\Box\Diamond\top \to \Box(\Box(\Box p \to p) \to p)$. This completes the proof that $\vDash_{\mathcal{T}} \quad \not\subseteq \quad \vDash_{\mathcal{CA}}$ and that all inclusions in Figure 1 are proper. By adding the unary discriminator to the similarity type of the algebra defined by Wolter and following an argument analogous to the proof of Lemma 35, one may actually prove a $\mathcal{T}$-inconsistency theorem:

FACT 36. There exists a consistent bimodal logic that is $\mathcal{T}$-inconsistent and $\mathcal{CA}$-complete.

Observe that we may prove also

LEMMA 37. $\Box\Diamond\top \to \Box(\Box(\Box p \to p) \to p) \vDash_{\mathcal{AV}} \Diamond\Box\bot \vee \Box\bot$

Proof. This time, a is defined as an arbitrary atom below $\Box\Diamond\top\wedge\Diamond\top$ in an $\mathcal{AV}$-BAO $\mathfrak{D}$. As $a \leq \Diamond \bigvee\{x \mid x \in At\mathfrak{D}\}$ and $\mathfrak{D}$ is completely additive, there must exist an atom b s.t. $a \leq \Diamond b$ (we use here the fact that a is an atom). If $b \leq \Diamond b$, then we have both f and g and that gives us h in a manner similar to the proof of lemma 35. If g does not hold, then, as b is an atom, $b \leq \Box\neg b$ and that gives us h anyhow. ⊣

REMARK 38. We have proven above that $\vDash_{\mathcal{T}} \cap \vDash_{\mathcal{AV}} \quad \not\subseteq \quad \vDash_{\mathcal{CA}}$. Unfortunately, it is not at all clear how we could strengthen the above observations to a proof that $\vDash_{\mathcal{V}} \quad \not\subseteq \quad \vDash_{\mathcal{CA}}$ or at least to a proof of existence of a $\mathcal{V}$-incomplete logic. Atomicity in the proof of Lemma 37 or existence of residuals in the proof of Lemma 35 are used in a crucial way. This shows once again that complete additivity itself is not a very tractable property.

Van Benthem [14] formulated a monadic *weak second-order logic* complete with respect to Henkin models closed under first-order definability. Closure under first-order definability means that for any formula ψ without second-order quantifiers, any sequence of admissible subsets $\bar{X}$ and any sequence of elements $\bar{w}$, the set of elements x satisfying $\psi(x, \bar{w}, \bar{X})$ is admissible. Such a model is always a general frame, as $\Diamond X$ is definable in the standard way. This motivates definition of yet another property of BAO's and associated notion of consequence/completeness:

DEFINITION 39. $\mathfrak{A}$ is a $\mathcal{E}$-BAO if it can be depicted as a complex algebra containing all first-order definable subsets of some general frame. $\Gamma \vDash_{\mathcal{E}} \varphi$ and the notion of $\mathcal{E}$-completeness are defined in an analogous way to other properties.

THEOREM 40. *Every $\mathcal{E}$-BAO is a $\mathcal{AT}$-BAO.*

Proof. All singletons are definable by identity formulas with one parameter. $\mathbf{p}_{\Diamond}X$ may be defined as $\exists y(yRx \& y \in X)$. It is enough now to apply Theorem 33. $\dashv$

OBSERVATION 41. We have thus a following sequence of inclusions:

(i) $\vDash_{\mathcal{AV}} \;\subseteq\; \vDash_{\mathcal{AT}} \;\subseteq\; \vDash_{\mathcal{E}} \;\subsetneq\; \vDash_{\mathcal{CAT}}$.

As $\vDash_{\mathcal{AT}} \;\not\subseteq\; \vDash_{\mathcal{CA}}$, the notion of weak second-order consequence relation is too strong for analysis of consequence relation over neighbourhood frames. Nevertheless, the question now arises whether all of the inclusions in i are proper. Van Benthem [14] showed that

$$\Box\Diamond\top \to \Box(\Box(\Box p \to p) \to p) \vDash_{\mathcal{E}} \Diamond\Box\bot \vee \Box\bot.$$

By inequality i, this observation is weaker than either of Lemmas 35 and 37. It was, however, an open question whether Lemma 37 is an actual strengthening of van Benthem's result. Now we are going to show that it is indeed the case. Actually, the result we are going to prove is stronger: $\vDash_{\mathcal{T}} \;\not\subseteq\; \vDash_{\mathcal{AV}}$. This is the main new result of this section. Set TYPE $\leftrightharpoons \{<, >\}$.

THEOREM 42. *There exists a $\mathcal{AV}$-algebra which generates a $\mathcal{T}$-incomplete logic.*

Proof. Consider a frame $\mathfrak{G} \coloneqq \langle W, \{R_\pi\}_{\pi \in \mathrm{TYPE}}, \mathfrak{A}\rangle$. $W \coloneqq \{\infty\} \cup \{a_n\}_{n\in\omega}$, $R_< \coloneqq \{\langle a_i, a_j\rangle | i < j\} \cup \{\langle \infty, a_{2i}\rangle | i \in \omega\}$, $R_> \coloneqq \{\langle a_i, a_j\rangle | i > j\}$ (i.e., $R_> = R_<^{-1} \cap (W - \{\infty\})^2$), $\mathfrak{A}$ is the algebra of finite and cofinite subsets of W. Observe that — just like in case of frames from Section 4 — all points are definable by variable-free formulas. For our purposes, we need only the following

CLAIM 43. Define $\underline{1} \coloneqq \Diamond_> \top \wedge \Box^2_> \bot$, $\underline{\infty} \coloneqq \Diamond_< \Box_> \bot$. Then $\underline{\infty}^{\mathfrak{A}} = \{\infty\}$ and $\underline{1}^{\mathfrak{A}} = \{a_1\}$.

LEMMA 44. *Let* $\blacksquare\varphi \coloneqq \neg\varphi \wedge \Box_< \varphi$. *The following formulas hold in* $\mathfrak{G}$*:*

$$\underline{\infty} \to \Diamond^2_< \underline{1} \wedge \Box_< \neg \underline{1}, \tag{j}$$
$$\underline{\infty} \wedge \Diamond_< \Box_< x \wedge \Diamond^2_< \neg p \to \Diamond_< \blacksquare p \vee \Diamond^2_< \blacksquare p, \tag{k}$$
$$\Diamond_< \blacksquare p \to \Diamond_< (p \wedge \Box_< p \wedge \Diamond^2_> \blacksquare p \wedge \Box^3_> \neg \blacksquare p), \tag{l}$$
$$\underline{\infty} \wedge \neg \Diamond_< \blacksquare p \to \Box_< \neg (p \wedge \Box_< p \wedge \Diamond^2_> \blacksquare p \wedge \Box^3_> \neg \blacksquare p), \tag{m}$$
$$\underline{\infty} \wedge \Box_< p \to \Box_< \Diamond_< p, \tag{n}$$
$$\Box_< \Diamond_< p \to \Diamond_< \Box_< p. \tag{o}$$

Conjunction of these formulas will be denoted as Γ.

Proof. Statement j follows directly from the definition of the frame and Claim 43. For k, assume $\{\infty\} \leq \mathfrak{V}(\Diamond_< \Box_< p \wedge \Diamond^2_< \neg p)$. It means that for some i, $\{a_i\} \leq \mathfrak{V}(\Box p)$ and yet $\mathfrak{V}(\neg p)$ is nonempty. Thus, there must exist a maximal point a_j in $\mathfrak{V}(\neg p)$ and ∞ $R_<$-sees a_j in one or two steps. For statement l, assume xRa_i and $\{a_i\} \leq \mathfrak{V}(\blacksquare p)$. But then $xR_< a_{i+2}$ and a_{i+2} is the one and only point $\mathfrak{V}$-satisfying the formula arising from the successor by erasing the initial $\Diamond_<$. Similar reasoning establishes m. Statement n is straightforward. For o, assume $\{w\} \leq \mathfrak{V}(\Box_< \Diamond_< p)$. It means that for every i, there exists $j > i$ s.t. $\{a_j\} \leq \mathfrak{V}(p)$, hence $\mathfrak{V}(p)$ is infinite. But then the complement of $\mathfrak{V}(p)$ must be finite and for some i, $a_i \leq \mathfrak{V}(\Box_< p)$. ⊣

LEMMA 45. $\Gamma \vDash_{\mathcal{T}} \neg \underline{\infty}$.

Proof. Assume that for some $\mathcal{T}$-BAO $\mathfrak{A}$ $\mathfrak{A} \not\vDash \underline{\infty} = \bot$ and there exists a conjugate $\mathbf{p}_<$ of $\Diamond_<$. We will show that $\underline{\infty} \leq \Box_< \Diamond_< \mathbf{p}_< \underline{\infty} \wedge$

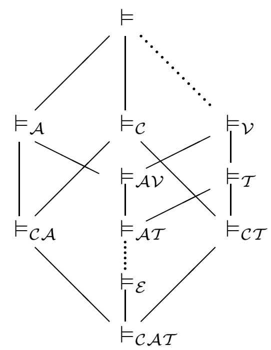

Figure 2. Dotted lines denote inclusions in whose case it is unclear whether they are proper or not.

$\Box_{<}\Diamond_{<}\neg\mathbf{p}_{<}\underline{\infty}$, thus contradicting the fact that $\mathfrak{A}$ validates McKinsey Axiom (statement o). That $\underline{\infty} \leq \Box_{<}\Diamond_{<}\mathbf{p}_{<}\underline{\infty}$ follows from n of the previous lemma. Assume now $\underline{\infty} \wedge \Diamond_{<}\Box_{<}\mathbf{p}_{<}\underline{\infty} \neq \bot$. By j, $\underline{\infty} \leq \Diamond^2_{<}\underline{1} \leq \Diamond^2_{<}\neg\mathbf{p}_{<}\underline{\infty}$. Thus, $\underline{\infty} \wedge \Diamond_{<}\Box_{<}\mathbf{p}_{<}\underline{\infty} \wedge \Diamond^2_{<}\neg\mathbf{p}_{<}\underline{\infty} \neq \bot$. By k, it means that

(p) $\Diamond_{<}\blacksquare\mathbf{p}_{<}\underline{\infty} \neq \bot$.

Define

$$c \leftrightharpoons \mathbf{p}_{<}\underline{\infty} \wedge \Box_{<}\mathbf{p}_{<}\underline{\infty} \wedge \Diamond^2_{>}\blacksquare\mathbf{p}_{<}\underline{\infty} \wedge \Box^3_{>}\neg\blacksquare\mathbf{p}_{<}\underline{\infty}.$$

By definition, $c \leq \mathbf{p}_{<}\underline{\infty}$. On the other hand, m and the fact that

$$\underline{\infty} \leq \Box_{<}\mathbf{p}_{<}\underline{\infty} \leq \Box_{<}(\mathbf{p}_{<}\underline{\infty} \vee \Diamond_{<}\neg\mathbf{p}_{<}\underline{\infty}) = \Box_{<}\neg\blacksquare\mathbf{p}_{<}\infty$$

imply $\mathbf{p}_{<}\underline{\infty} \leq \neg c$. Thus, $c = \bot$ but this contradicts l and p. $\dashv$

As $\mathfrak{A} \not\models \neg\underline{\infty}$, Theorem 42 follows. $\dashv$

Let us finish then with Figure 2 — a refined version of Figure 1. It is less symmetric, but perhaps more thought-provoking.

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BIBLIOGRAPHY

[1] P. Blackburn, M. de Rijke and Y. Venema, **Modal Logic**, Cambridge Tracts in Theoretical Computer Science 53, 2001.

[2] A.V. Chagrov and M.V. Zakharyaschev, **Modal Logic**, Clarendon Press, Oxford, 1997.

[3] K. Došen, *Duality between modal algebras and neighbourhood frames*, **Studia Logica**, 48:219–234, 1989.

[4] R.I. Goldblatt, **Mathematics of modality**, CSLI Lecture Notes 43, CSLI Publications, 1993.

[5] P. Jipsen, *Discriminator varieties of Boolean algebras with operators*, **Algebraic Logic**, Banach Center Publications vol. 28, Warsaw 1993.

[6] T. Kowalski, *Varieties of tense algebras*, **Reports on Mathematical Logic**, 32 (1998), pp. 53–95.

[7] M. Kracht, **Tools and Techniques in Modal Logic**, Studies in Logic and the Foundations of Mathematics vol. 142, Elsevier, 1999.

[8] M. Kracht, T. Kowalski, *Atomic incompleteness or how to kill one bird with two stones*, **Bulletin of the Section of Logic**, 30:71–78, 2001.

[9] K. Kuratowski, A. Mostowski, **Set Theory : with An Introduction to Descriptive Set Theory**, Studies in logic and the foundations of mathematics v. 86, Elsevier, 1976.

[10] T. Litak, F. Wolter, *All finitely axiomatizable tense logics of linear time flows are coNP-complete*, to appear in **Studia Logica**.

[11] T. Litak, *Modal incompleteness revisited*, **Studia Logica**, 76:329–342, 2004.

[12] S.K. Thomason, *Semantic analysis of tense logics*, **Journal of Symbolic Logic**, 37:150–158, 1972.

[13] S.K. Thomason, *Categories of frames for modal logic*, **Journal of Symbolic Logic**, 40:439–442, 1975.

[14] J.F.A.K. van Benthem, *Syntactic aspects of modal incompleteness theorems*, **Theoria**, 45:63–77, 1979.

[15] Y. Venema, *Atom structures*, **Advances in Modal Logic I**, California, CSLI, 1998, pp. 63–72.

[16] Y. Venema, *Atomless varieties*, **Journal of Symbolic Logic**, 68:607–614, 2003.

[17] F. Wolter, **Lattices of Modal Logics**, PhD Thesis, 1993.

[18] F. Wolter, *Tense logic without tense operators*, **Mathematical Logic Quarterly**, 42 (1996), 145–171.

[19] M. Zakharyaschev, F. Wolter and A. Chagrov, *Advanced modal logic*, [in:] **Handbook of Philosophical Logic, 2nd Edition**, vol. 3, Kluwer, 2001, pp. 83–266.

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