Decidability of *IF* Modal Logic of Perfect Recall

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ABSTRACT. IF Modal Logic of Perfect Recall is obtained from basic modal logic by allowing modal operators to be logically independent from certain syntactically superordinate dual operators. It is proven that the satisfiability and validity problems of this logic are decidable.

1 Plan of the Paper

The aim of this paper is to establish decidability of the satisfiability and validity problems of the logic (to be called 'IF Modal Logic of Perfect Recall') obtained from basic modal logic by allowing modal operators to be logically independent from certain syntactically superordinate dual operators. In *Section* 2, IF Modal Logic of Perfect Recall with k modality types (or, $\mathbf{IFML}^+[k]$) is defined in detail. Its semantics is defined in terms of evaluation games, which are simultaneously games of *imperfect information* and games of *perfect recall*. The former feature is used to implement modal operators' logical independence from syntactically superordinate modal operators. The latter feature follows from the syntactic restrictions regulating which operators may be marked as independent from which others. $\mathbf{IFML}^+[k]$ is a variant of the IF modal logic ($\mathbf{IFML}[k]$) of [16]. In *Section* 3 winning strategies of a player in evaluation games for $\mathbf{IFML}^+[k]$ are proven to have *history-free normal form*: if a player has a w.s., he has a w.s. whose value on a history only depends on the last 'known' position in the history. This result is shown not to be generalizable to $\mathbf{IFML}[k]$. In *Section* 4, $\mathbf{IFML}^+[k]$ is shown to have the strong finite model property, whence its satisfiablity problem is decidable. Our strategy for proving this result is hence the same as is typically used for proving the decidability of basic modal logic relative to the class of all modal structures (cf. [1], Th. 6.7). In *Section* 5 the validity and satisfiability problems of $\mathbf{IFML}^+[k]$ are observed not to be each others' duals. However, the validity problem of $\mathbf{IFML}^+[k]$ is proven to be decidable as well. *Section* 6 concludes the paper by pointing out related works, discussing the motivation behind IF modal logics, and mentioning open problems.

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2 IF Modal Logic of Perfect Recall

Syntax. Let a class **prop** $:= \{p_i : i < \omega\}$ of propositional atoms be given. For all $k < \omega$, we define recursively a class $\mathcal{L}[k]$ of strings. Every string φ in $\mathcal{L}[k]$ is simultaneously associated with its *set of indices*, $|\varphi|$. The class of $\mathbf{IFML}^+[k]$-formulas will then be introduced as a class of strings from $\mathcal{L}[k]$ meeting certain syntactic conditions; semantically, k will be the number of accessibility relations available in the models in terms of which the semantics of the logic $\mathbf{IFML}^+[k]$ will be defined.

- If $p_i \in \mathbf{prop}$, then $p_i, \neg p_i \in \mathcal{L}[k]$; and $|p_i| = |\neg p_i| = \{0\}$.
- If $\varphi, \psi \in \mathcal{L}[k]$ and $n = max(|\varphi| \cup |\psi|) + 1$, then $(\varphi \vee_n \psi) \in \mathcal{L}[k]$ and $(\varphi \wedge_n \psi) \in \mathcal{L}[k]$; and $|(\varphi \vee_n \psi)| = |(\varphi \wedge_n \psi)| = \{n\} \cup |\varphi| \cup |\psi|$.
- If $\varphi \in \mathcal{L}[k]$, $n = max|\varphi| + 1$, $i < k$, $W_n \subseteq]n, \infty[$, and $O \in \{\Box, \Diamond\}$, then $(O_{i,n}/W_n)\varphi \in \mathcal{L}[k]$; and $|(O_{i,n}/W_n)\varphi| = \{n\} \cup |\varphi|$.

We will write $\Diamond_{i,n}$ for $(\Diamond_{i,n}/\varnothing)$ and $\Box_{i,n}$ for $(\Box_{i,n}/\varnothing)$. Further, in strings of $\mathcal{L}[1]$, we allow writing $(\Diamond_n/W_n)$ for $(\Diamond_{0,n}/W_n)$ and $(\Box_n/W_n)$ for $(\Box_{0,n}/W_n)$.

For every string φ in $\mathcal{L}[k]$ there corresponds in an obvious way a *syntactic tree*, whose nodes consist of modal operators, propositional connectives and (negated) propositional atoms. For instance, the syntactic tree of the string $\Box_4(\Box_1 p_0 \wedge_3 ((\neg p_1 \wedge_1 p_2) \vee_2 (\Diamond_1/\{4\})p_3))$ is the one depicted in Figure 1:

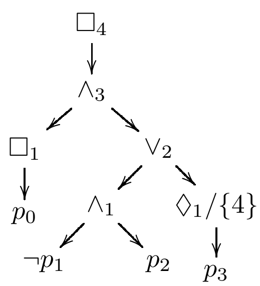

FIGURE 1

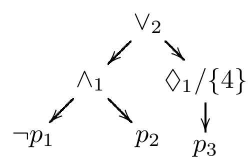

FIGURE 2

We call the ordering relation $(\prec_\varphi)$ of the syntactic tree of the string φ *(syntactic) superordination.* In Figure 1, $\Box_4$ for instance is superordinate to $\Diamond_1/\{4\}$; conversely $\Diamond_1/\{4\}$ is said to be *subordinate* to $\Box_4$. Whenever O is superordinate to O', there is a *path* from O to O', namely the sequence $\langle t_1, \ldots, t_n\rangle$, where $t_1 = O$, $t_n = O'$, and $t_i \prec_\varphi t_j$ iff $i < j$. For example, $(\wedge_3, \vee_2, \wedge_1, p_2)$ is a path from $\wedge_3$ to p_2 in the tree depicted in Figure 1. Operators appearing in a string and ordered according to the relation of superordinateness are said to be *nested.* If $(T, \prec_\varphi)$ is the syntactic tree of φ

and $t \in T$, the set $\{x : t = x \vee t \prec_\varphi x\}$ ordered by the relation $\prec_\varphi$ is a *full subtree* of $(T, \prec_\varphi)$. For instance the tree depicted in Figure 2 is a full subtree of the tree represented in Figure 1. The first index (i) of a modal operator $(O_{i,n}/W_n)$ appearing in string $\varphi \in \mathcal{L}[k]$ identifies a *modality type*, $i < k$; while the second (n) identifies the *occurrences* of nested modal operators.

The *length* of a string $\mathcal{L}[k]$, denoted $\sharp[\varphi]$, is the number of symbols appearing in φ. Hence if $p_i \in \mathbf{prop}$, then $\sharp[p_i] = 1 + d(i)$, where $d(i)$ is the number of digits of the numeral representing i in the binary system; $\sharp[(\varphi \vee \psi)] = \sharp[(\varphi \wedge \psi)] = \sharp[\varphi] + \sharp[\psi] + 3$; and $\sharp[(\Diamond_{i,n}/W_n)\varphi] = \sharp[(\Box_{i,n}/W_n)\varphi] = d(i) + d(n) + 4 + /W_n/ + \sharp[\varphi]$, where $/W_n/ := 1$, if $W_n = \varnothing$, and $/W_n/ := d(n_1) + \ldots + d(n_m) + m + 1$, if $W_n = \{n_1, \ldots, n_m\} \neq \varnothing$.

The set $Sub[\varphi]$ of *substrings* of a string φ of $\mathcal{L}[k]$ is defined recursively in a straightforward way: $Sub[p_i] = \{p_i\}$ and $Sub[\neg p_i] = \{\neg p_i\}$; for $\circ \in \{\vee_n, \wedge_n\}$: $Sub[(\varphi \circ \psi)] = \{(\varphi \circ \psi)\} \cup Sub[\varphi] \cup Sub[\psi]$; and for $O \in \{\Box, \Diamond\}$: $Sub[(O_{i,n}/W_n)\varphi] = \{(O_{i,n}/W_n)\varphi\} \cup Sub[\varphi]$. If $\psi \in Sub[\varphi]$ and an operator O appears in φ, we say that O is *superordinate to the substring* ψ, if O is superordinate to all nodes of the full subtree of $(T, \prec_\varphi)$ determined by ψ.

The class of formulas of *IF Modal Logic of Perfect Recall with k modality types* (or the class of $\mathbf{IFML}^+[k]$-formulas), is defined as the class of strings $\varphi \in \mathcal{L}[k]$ meeting the following conditions:

(1) *Closedness:* If $O \in \{\Box, \Diamond\}$ and $(O_{i,n}/W_n)$ appears in φ, then there appear $(O_{i_1,n_1}/W_{n_1}), \ldots, (O_{i_m,n_m}/W_{n_m})$ in φ s.t. $W_n = \{n_1, \ldots, n_m\}$ and $(O_{i_1,n_1}/W_{n_1}) \prec_\varphi \ldots \prec_\varphi (O_{i_m,n_m}/W_{n_m}) \prec_\varphi (O_{i,n}/W_n)$.

(2.i) *Diamonds independent only from boxes:* If $(\Diamond_{i,n}/W_n)$ appears in φ, $j \in W_n$ and $(O_{r,j}/W_j) \prec_\varphi (\Diamond_{i,n}/W_n)$, then $O_{r,j} = \Box_{r,j}$.

(2.ii) *Boxes independent only from diamonds:* If $(\Box_{i,n}/W_n)$ appears in φ, $j \in W_n$, and $(O_{r,j}/W_j) \prec_\varphi (\Box_{i,n}/W_n)$, then $O_{r,j} = \Diamond_{r,j}$.

(3) *Minimal requirement on inherited dependencies:*

(i) If $(\Box_{i,n}/W_n) \prec_\varphi \vee_m \prec_\varphi (\Diamond_{r,j}/W_j)$, and there is no operator $O \in \{\Box, \Diamond\}$ with $(\Box_{i,n}/W_n) \prec_\varphi O \prec_\varphi \vee_m$, then $n \notin W_j$.

(ii) If $(\Diamond_{i,n}/W_n) \prec_\varphi \wedge_m \prec_\varphi (\Box_{r,j}/W_j)$, and there is no operator $O \in \{\Box, \Diamond\}$ with $(\Diamond_{i,n}/W_n) \prec_\varphi O \prec_\varphi \wedge_m$, then $n \notin W_j$.

Observe that hence the *formulas* of $\mathbf{IFML}^+[k]$ are a certain kind of *strings* of $\mathcal{L}[k]$, namely the strings satisfying the conditions (1), (2) and (3). The set of *subformulas* of an $\mathbf{IFML}^+[k]$-formula φ is defined to be $Sub[\varphi]$, the set of substrings of φ. So in general subformulas of $\mathbf{IFML}^+[k]$-formulas are *not* themselves *formulas* of $\mathbf{IFML}^+[k]$, but merely strings of $\mathcal{L}[k]$.

Condition (1) rules out formulas with operators $(O_{i,n}/W_n)$, where some indices in W_n do not refer to any operators superordinate to $(O_{i,n}/W_n)$. Notice that the more subordinate operators an operator has, the greater its index; and that due to conjunctions and disjunctions, there may well be several operators with the same index. The semantic correlate of the restrictions (2.i,ii) will be that in the evaluation games for $\mathbf{IFML}^+[k]$, players recall their own past moves; the restrictions are motivated by the fact that while it certainly makes sense to speak of an operator being logically independent from its dual, it is not evident that there is such a properly logical phenomenon as the independence of a diamond, say, from another diamond.

Condition (3) imposes a specific restriction on possible independencies between dual modal operators. The semantic counterpart of condition (3.i) will be this: if the relevant player of an evaluation game knows, when making a move for a disjunction, the opponent's choice corresponding to a box − which in particular is the closest modal operator superordinate to the disjunction − then he will know the opponent's choice for the box also always later in the appropriate play of the game. This condition has an air of arbitrariness about it. The natural condition would be obtained by extending independence indications to disjunctions and conjunctions (allowing writing e.g. $\vee_2/\{5\}$ and $\wedge_1/\{9, 11\}$), and requiring that the independencies of a formula do not violate the principle "a player never forgets a move he has once known." The formulation and study of such a more general syntax is left to another occasion; we content ourselves here with (3), which is the weakest condition that together with (1) and (2) results in a logic with the property that winning strategies in the associated evaluation games can be assumed to be 'history-free' (cf. Theorem 6 and Example 11). If a game is said to be of *perfect recall* whenever a player always recalls his own past moves, and does not forget what he has known, the evaluation games of $\mathbf{IFML}^+[k]$ are the most general 'approximation' of evaluation games of perfect recall that is possible without extending independence indications to $\vee$ and $\wedge$.

Semantics. The semantics is defined relative to k-ary modal structures $\mathcal{M} = (D, R_0, \ldots, R_{k-1}, \mathfrak{h})$, where $D \neq \varnothing$, $R_i \subseteq D^2$ for all $i < k$, and $\mathfrak{h} : \mathbf{prop} \longrightarrow Pow(D)$. If $\mathcal{M}$ is a k-ary modal structure and $w \in dom(\mathcal{M})$, we will call the pair $\langle \mathcal{M}, w\rangle$ a *k-ary model.* We associate with every formula φ and every k-ary model $\langle \mathcal{M}, w\rangle$ a game $G(\varphi, \mathcal{M}, w)$ between two players, $\forall$ and $\exists$. (The context will always take care that the names "$\forall$" and "$\exists$" of these players cannot be confused with universal *resp.* existential quantifier.) First we specify game rules for something a bit more complicated, namely the games $G(\varphi, \mathcal{M}, w, \pi_\varphi, \sigma_\varphi)$ proceeding from the extra information codified in the partial functions $\pi_\varphi : \omega + 1 \rightharpoonup dom(\mathcal{M})$ and $\sigma_\varphi : \omega \rightharpoonup \{0, 1\}$.

- If $\varphi \in \{p_i, \neg p_i\}$, no move is made. If $\varphi = p_i$ and $w \in \mathfrak{h}(p_i)$, or $\varphi = \neg p_i$

and $w \notin \mathfrak{h}(p_i)$, $\exists$ wins the play and $\forall$ loses it. Otherwise $\forall$ wins and $\exists$ loses.

- If $\varphi = \theta \wedge_n \psi$, $\forall$ chooses $\chi \in \{\theta, \psi\}$, and the play continues as $G(\chi, \mathcal{M}, w, \pi_\chi, \sigma_\chi)$, where $\pi_\chi := \pi_\varphi$ and $\sigma_\chi := \sigma_\varphi \cup \{(n, \flat_\chi)\}$, where $\flat_\chi = 0$ if $\chi = \theta$, and $\flat_\chi = 1$ otherwise.
- The case $\varphi = \theta \vee_n \psi$ is dual to the case for $\varphi = \theta \wedge_n \psi$.
- If $\varphi = (\Box_{i,n}/W_n)\psi$, $\forall$ chooses, if possible, $v \in dom(\mathcal{M})$ satisfying $R_i(w, v)$. The play continues as $G(\psi, \mathcal{M}, v, \pi_\psi, \sigma_\psi)$, where $\pi_\psi := \pi_\varphi \cup \{(n, v)\}$ and $\sigma_\psi := \sigma_\varphi$. If such a choice is not possible, the play ends in $\forall$'s choice `FAIL`, $\exists$ wins and $\forall$ loses. (No choice of v with $R_i(w, v)$ is possible iff w has no R_i-successor. If no other choice is possible, the player chooses by stipulation the object $\texttt{FAIL} \notin dom(\mathcal{M})$.)
- The case for $\varphi = (\Diamond_{i,n}/W_n)\psi$ is dual to the case for $\varphi = (\Box_{i,n}/W_n)\psi$.

STIPULATION 1. The game rules for $G(\varphi, \mathcal{M}, w)$ are the game rules of $G(\varphi, \mathcal{M}, w, \{(\omega, w)\}, \varnothing)$.

Evaluation of φ in $\langle \mathcal{M}, w \rangle$ begins in a situation where the point w is indexed by the ordinal ω and no conjunctive or disjunctive moves have been made: $\pi_\varphi = \{(\omega, w)\}$ and $\sigma_\varphi = \varnothing$. Thus ω will be in the domain of all functions π_χ corresponding to subformulas χ of φ. Plays of a game are also called its *histories*. An m-move play of $G(\varphi_0, \mathcal{M}, w)$ is a sequence $\langle (\pi_{\varphi_0}, \sigma_{\varphi_0}), \ldots, (\pi_{\varphi_m}, \sigma_{\varphi_m}) \rangle$, satisfying $\pi_{\varphi_0} \subseteq \ldots \subseteq \pi_{\varphi_m}$ and $\sigma_{\varphi_0} \subseteq \ldots \subseteq \sigma_{\varphi_m}$. The play can be identified with the last member of the sequence, i.e. $(\pi_{\varphi_m}, \sigma_{\varphi_m})$. *Terminal plays* are plays ending with a (negated) propositional atom φ_m, or in the choice of `FAIL` by one of the players.

A token of a subformula χ of φ is uniquely determined by fixing the left/right choices for the conjunctions/disjunctions in φ to which χ is subordinate. Hence the tokens of χ may be identified with pairs (χ, σ_χ) s.t. for some π_χ, (π_χ, σ_χ) is a play. When no confusion may arise, we refer to subformula tokens equally as subformulas. A *strategy* of a player in $G(\varphi, \mathcal{M}, w)$ is a function specifying a move for every play at which it is this player's turn to move. Hence if f is $\exists$'s strategy and (π_χ, σ_χ) is a play, $f(\pi_\chi, \sigma_\chi) \in \{0, 1\}$, if $\chi = \psi \vee_n \theta$, and $f(\pi_\chi, \sigma_\chi) \in dom(\mathcal{M}) \cup \{\texttt{FAIL}\}$, if $\chi = (\Diamond_{i,n}/W_n)\psi$.

Given a subformula (χ, σ_χ), write V_{σ_χ} for the common domain of all functions π_χ s.t. (π_χ, σ_χ) is a play of $G(\varphi, \mathcal{M}, w)$. For each subformula (χ, σ_χ) with χ of the form $(\Diamond_{i,n}/W_n)\psi$, partition the set of plays $\{(\pi_\chi, \sigma_\chi) : \pi_\chi \text{ is a function } V_{\sigma_\chi} \longrightarrow dom(\mathcal{M})\}$ into equivalence classes under the following equivalence relation $\sim_\exists^{\sigma_\chi}$: $(\pi'_\chi, \sigma_\chi) \sim_\exists^{\sigma_\chi} (\pi''_\chi, \sigma_\chi)$, if $(\forall i \in V_{\sigma_\chi})(\pi'_\chi(i) \neq \pi''_\chi(i) \implies i \in W_n)$. This partition gathers together plays at which a given $\chi := (\Diamond_{i,n}/W_n)\psi$ is evaluated and which differ at most for moves interpreting an operator $(\Box_{j,m}/W_m)$ with $m \in W_n$. Let $\sim_\exists$ be the union of all relations $\sim_\exists^{\sigma_\chi}$, where σ_χ identifies a subformula (χ, σ_χ) with χ of the form $(\Diamond_{i,n}/W_n)\psi$. Relation $\sim_\forall$ is defined similarly. A strategy f of $\exists$ is a *winning strategy* (w.s.), if there is a set S of plays such that:

- S contains the play $(\{(\omega, w)\}, \varnothing)$.
- If $h \in S$, $f(h)$ is a move according to the game rules and $f(h) \in S$.
- S is closed under all moves of $\forall$ made according to the game rules.
- f is uniform, i.e. if $h \sim_\exists h'$, then $f(h) = f(h')$.
- Every terminal play in S is a win for $\exists$.

Winning strategies of $\forall$ are defined analogously. Semantics of a formula $\varphi \in \mathbf{IFML}^+[k]$ is defined as follows:

Truth: $\mathcal{M} \models^+ \varphi[w]$ **if** there exists a w.s. for $\exists$ in $G(\varphi, \mathcal{M}, w)$.
Falsity: $\mathcal{M} \models^- \varphi[w]$ **if** there exists a w.s. for $\forall$ in $G(\varphi, \mathcal{M}, w)$.
Indeterminacy: $\mathcal{M} \models^0 \varphi[w]$ **if** $\mathcal{M} \not\models^+ \varphi[w]$ and $\mathcal{M} \not\models^- \varphi[w]$.

When no confusion may arise, we will simply write $\models$ for the relation $\models^+$.

EXAMPLE 2. Think of the unary models $\langle \mathcal{M}, m_0 \rangle$ and $\langle \mathcal{N}, n_0 \rangle$:

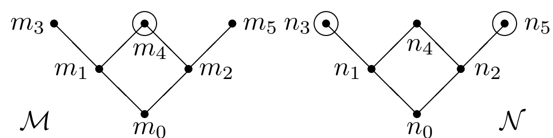

FIGURE 3

In Figure 3, the circles indicate the points making the atom p_0 true. Consider evaluating the formula $\varphi := \Box_2(\Diamond_1/\{2\})p_0$ relative to the two models.

(a) φ is true in $\langle \mathcal{M}, m_0 \rangle$. In fact, the function f defined by the condition $f(m_0, m_1) = f(m_0, m_2) = m_4$ is a w.s. for $\exists$ in $G(\varphi, \mathcal{M}, m_0)$.

(b) φ is indeterminate in $\langle \mathcal{N}, n_0 \rangle$. (**i**) Suppose for contradiction that f is a w.s. for $\exists$ in $G(\varphi, \mathcal{N}, n_0)$. Hence f is uniform. If the value $x = f(n_0, n_1) = f(n_0, n_2)$ is distinct from n_4, x constitutes an illegal move for one of the histories (n_0, n_1) and (n_0, n_2), and so f is not winning. If, again, $x = n_4$, f does not lead to a win for $\exists$ in any play of $G(\varphi, \mathcal{M}', n_0)$. (**ii**) Neither of the two possible strategies $g_1(n_0) = n_1$ and $g_1(n_0) = n_2$ of $\forall$ is winning: $\exists$ can extend both plays (n_0, n_1) and (n_0, n_2) with a choice yielding a win for her.

(c) The models $\langle \mathcal{M}, m_0 \rangle$ and $\langle \mathcal{N}, n_0 \rangle$ are bisimilar. Hence they are not distinguished by any formula of basic modal logic. ■

Known results of expressive power. If L and L' are modal logics and $\mathcal{K}$ is a class of modal structures on which the semantics of these logics is defined, L is *embeddable* in L' over $\mathcal{K}$ ($L \leq_{\mathcal{K}} L'$), if for each $\varphi \in L$ there is $\psi_\varphi \in L'$ s.t. for all $\langle \mathcal{M}, w\rangle$ with $\mathcal{M} \in \mathcal{K}$, $\mathcal{M} \models \varphi[w]$ iff $\mathcal{M} \models \psi_\varphi[w]$. *$L'$ has greater expressive power than L* over $\mathcal{K}$ ($L <_{\mathcal{K}} L'$), if $L \leq_{\mathcal{K}} L'$ but $L' \nleq_{\mathcal{K}} L$. *L and L' have the same expressive power* over $\mathcal{K}$ ($L =_{\mathcal{K}} L'$), if $L \leq_{\mathcal{K}} L'$ and $L' \leq_{\mathcal{K}} L$. (For abstract logics, the same terminology is found e.g. in [6].)

In [16] 'IF modal logic of k modality types' (or, $\mathbf{IFML}[k]$) was introduced. Its syntax can be obtained from that of $\mathbf{IFML}^+[k]$ by giving up the syntactic restrictions listed as (2) and (3) in *Section* 2, and requiring simply that modal operators may only be independent from other modal operators − whether these latter are duals of the former or not. The semantics of $\mathbf{IFML}[k]$ is completely analogous to the semantics of $\mathbf{IFML}^+[k]$. However, in the presence of diamonds indicated as independent from other diamonds, there are equivalence relations $\sim_\exists^{\sigma_\chi}$ that could not be induced in the case of $\mathbf{IFML}^+[k]$. The evaluation games of $\mathbf{IFML}[k]$ are *not* in general games of perfect recall: a player always knows the *strategy* he has followed up to any given stage in the game, but may fail to distinguish between two or more *individual moves* he himself has made in accordance to that strategy. $\mathbf{IFML}^+[k]$ is trivially embeddable in $\mathbf{IFML}[k]$. Basic modal logic of k modality types (or, $\mathbf{ML}[k]$) and basic tense logic of k temporal modality types (or, $\mathbf{TL}[k]$) are generated by closing the class of (negated) propositional atoms under $\wedge$, $\vee$ and applications of unary modal operators $\Box_i, \Diamond_i$ *resp.* $\Box_i$, $\Diamond_i$, $\Box_i^{-1}, \Diamond_i^{-1}$, with $i < k$; these modal operators have their usual semantics. Syntactically IF tense logic of perfect recall with k temporal modality types (or, $\mathbf{IFTL}^+[k]$) equals $\mathbf{IFML}^+[2k]$, and it is interpreted relative to modal structures with $2k$ accessibility relations of which the i-th and the $(i+k)$-th are each other's converses for all $i < k$, and which all are irreflexive and transitive ('temporal structures'). Let $\mathcal{C}_k$ be the class of all k-ary modal structures, and $\mathcal{T}_k$ the class of all k-ary temporal structures. Directly by (the proofs of) the corresponding theorems in [16], we have:

THEOREM 3. *Let $k \geq 1$.* **(a)** *Write $\mathbf{IFML}^+_{\mathbf{det}}[k]$ for the "determined fragment" of $\mathbf{IFML}^+[k]$, i.e. the class $\{\varphi \in \mathbf{IFML}^+[k]$: for all $\mathcal{M}, w$, either $\mathcal{M} \models^+ \varphi[w]$ or $\mathcal{M} \models^- \varphi[w]\}$. We have: $\mathbf{IFML}^+_{\mathbf{det}}[k] =_{\mathcal{C}_k} \mathbf{ML}[k]$.* **(b)** *$\mathbf{ML}[k] <_{\mathcal{C}_k} \mathbf{IFML}^+[k]$.* **(c)** *$\mathbf{TL}[k] <_{\mathcal{T}_k} \mathbf{IFTL}^+[k]$.*

Notice that Example 2 proves Theorem 3 (b). It is not difficult to show:

THEOREM 4. *For all $k \geq 1$, $\mathbf{IFML}^+[k] <_{\mathcal{C}_k} \mathbf{IFML}[k]$.*

Hence the IF modal logic of [16] is a proper extension of the IF modal logic of the present paper. However, the extra expressive power of $\mathbf{IFML}[k]$ is not needed for the main expressivity results concerning this logic: as shown by Theorem 3 (b,c), these results are obtained already within $\mathbf{IFML}^+[k]$.

3 History-Free Strategies

If φ is an $\mathbf{IFML}^+[k]$-formula and (χ, σ_χ) its subformula, write S_{σ_χ} for the set of all histories (plays) of game $G(\varphi, \mathcal{M}, w)$ that end with (χ, σ_χ). I.e., $S_{\sigma_\chi} = \{(\pi_\chi, \sigma_\chi) : (\pi_\chi, \sigma_\chi)$ is a play and π_χ is a function $V_{\sigma_\chi} \longrightarrow dom(\mathcal{M})\}$.

DEFINITION 5. (History-free strategy) A strategy f of $\exists$ is *history-free*, if for any subformula (χ, σ_χ) of φ with $\chi \in \{\theta \vee_n \psi, (\Diamond_{i,n}/W_n)\psi\}$, and any histories $(\pi'_\chi, \sigma_\chi), (\pi''_\chi, \sigma_\chi) \in S_{\sigma_\chi}$, we have:

$$\pi'_\chi(K) = \pi''_\chi(K) \Longrightarrow f(\pi'_\chi, \sigma_\chi) = f(\pi''_\chi, \sigma_\chi),$$

where $K := min\{x > n : x = \omega$ or $[O_x \prec_\varphi \vee_n \ \& \ O_x$ is of the form $\Diamond, \Box]\}$, if $\chi := \theta \vee_n \psi$; and $K := min\{x > n : x = \omega$ or $[O_x \prec_\varphi \Diamond_{i,n}/W_n \ \& \ O_x$ is of the form $\Diamond, \Box \ \& \ x \notin W_n]\}$, if $\chi := (\Diamond_{i,n}/W_n)\psi$.

Hence the value of a history-free strategy of $\exists$ for a history (π_χ, σ_χ) depends only on the last member of π_χ 'known' by $\exists$.

THEOREM 6. *Let $\varphi \in \mathbf{IFML}^+[k]$, $\mathcal{M}$ and w be arbitrary. If there is a w.s. for $\exists$ in $G(\varphi, \mathcal{M}, w)$, there is a history-free w.s. for her in this game.*

Proof. Assume $\exists$ has a w.s. f in $G(\varphi, \mathcal{M}, w)$. Given a subformula (χ, σ_χ) of φ and $y \in dom(\mathcal{M})$, consider the set $[w, y]^f_{\sigma_\chi}$ of all maps π_χ s.t. (π_χ, σ_χ) is a play of $G(\varphi, \mathcal{M}, w)$ in which $\exists$ has used f, and in which the last choice from $dom(\mathcal{M})$ is y. Whenever $[w, y]^f_{\sigma_\chi} \neq \varnothing$, choose a canonical element of this set, and denote it by $\iota([w, y]^f_{\sigma_\chi})$. Further, given a subformula (χ, σ_χ) of φ, and a play (π_ψ, σ_ψ) of $G(\varphi, \mathcal{M}, w)$ in which $\exists$ has applied f, and which satisfies $\sigma_\psi \subseteq \sigma_\chi$, think of the set $[\pi_\psi, \sigma_\psi]^f_{\sigma_\chi}$ of all maps π_χ s.t. (π_χ, σ_χ) satisfies $\pi_\psi \subseteq \pi_\chi$ and is a play of $G(\varphi, \mathcal{M}, w)$ in which $\exists$ has used f. Choose also for sets of this type a canonical representative, denoted by $\iota([\pi_\psi, \sigma_\psi]^f_{\sigma_\chi})$. Let us then define a strategy f' for $\exists$ in $G(\varphi, \mathcal{M}, w)$ as follows:

- If $\chi = \psi \vee \theta$, put $f'(\pi_\chi, \sigma_\chi) := f(\iota([w, y]^f_{\sigma_\chi}), \sigma_\chi)$, where $y := \pi_\chi(min(dom(\pi_\chi)))$.
- If $\chi = (\Diamond_{i,n}/W_n)\psi$, put $f'(\pi_\chi, \sigma_\chi) := f(\iota([\pi_\phi, \sigma_\phi]^f_{\sigma_\chi}), \sigma_\chi)$, the play (π_ϕ, σ_ϕ) being determined by the conditions (i) $\pi_\phi := \iota([w, \pi_\chi(K)]^f_{\sigma_\phi})$ and (ii) $\sigma_\phi := \{(x, \sigma_\chi(x)) : x \in dom(\sigma_\chi) \ \& \ x > K\}$, where $K := min\{x > n : x = \omega$ or $[O_x \prec_\varphi \Diamond_{i,n}/W_n \ \& \ O_x \in \{\Diamond, \Box\} \ \& \ x \notin W_n]\}$.

Observe that in the latter case (π_ϕ, σ_ϕ) can indeed be extended into a play (π_χ, σ_χ) so that $\exists$'s moves are given by the strategy f. This would not in general be possible if the syntactic restriction (3) of *Section* 2 did not hold:

then there might be operators $(\Box_{i',n'}/W_{n'}) \prec_\varphi \vee_m \prec_\varphi (\Diamond_{i,n}/W_n)$ with $n' \in W_n$ s.t. $\Box_{i',n'}$ would be interpreted in the course of the play (π_ϕ, σ_ϕ), and f would choose for $\vee_m$ a disjunct in which the operator $(\Diamond_{i,n}/W_n)$ does not appear. But when (3) is assumed, the set $[\pi_\phi, \sigma_\phi]^f_{\sigma_\chi}$ is non-empty.

CLAIM 7. f' is a w.s. for $\exists$ in $G(\varphi, \mathcal{M}, w)$.

Proof. **(1)** Let (χ, σ_χ) be any subformula of φ, and suppose $\exists$ has used f' in constructing (π_χ, σ_χ). We must show that f' gives legal extensions. For disjunctions this is trivial, so consider diamonds. The proof will require considering different plays of game $G(\varphi, \mathcal{M}, w)$; these will be schematically illustrated in Figure 4 below. (For the play (π_χ, σ_χ), cf. the sequence (w, w_1, u_2, w_3) in the figure.)

Write $\{\omega, n_1, \ldots, n_m\}$ for $dom(\pi_\chi)$, with $n_m < \ldots < n_1 < \omega$. If $\chi := (\Diamond_{i,n}/W_n)\psi$, let K be the smallest $x > n$ s.t. x is an index of a modal operator superordinate to $\Diamond_{i,n}/W_n$ with $x \notin W_n$, or if none exists, let $K := \omega$. Assume for contradiction that $\langle \pi_\chi(n_m), f'(\pi_\chi, \sigma_\chi)\rangle \notin R_i$. Write $\langle w, \pi'_\chi(n_1), \ldots, \pi'_\chi(n_{p-1}), \pi_\chi(K)\rangle$ for $\iota([w, \pi_\chi(K)]^f_{\sigma_\chi})$, $p \le m$. Hence $h_1 := (\langle w, \pi'_\chi(n_1), \ldots, \pi'_\chi(n_{p-1}), \pi_\chi(K), \pi_\chi(n_{p+1}), \ldots, \pi_\chi(n_m)\rangle, \sigma_\chi)$ is a play. (For h_1, cf. the sequence (w, v_1, u_2, w_3) in Figure 4.) Moreover, by the syntax of $\mathbf{IFML}^+[k]$ and the definition of K, all moves $\pi_\chi(n_{p+1}), \ldots, \pi_\chi(n_m)$ are by $\forall$. So h_1 is not only a play, but in particular a play that can be realized when $\exists$ applies f. But also the play h_2 for which $f'(\pi_\chi, \sigma_\chi) = f(h_2)$, $h_2 := (\langle w, \pi'_\chi(n_1), \ldots, \pi'_\chi(n_{p-1}), \pi_\chi(K), \pi''_\chi(n_{p+1}), \ldots, \pi''_\chi(n_m)\rangle, \sigma_\chi)$, is constructible when $\exists$ applies f. (For h_2, cf. the sequence (w, v_1, u_2, v_3) in Figure 4.) Since f is a w.s. and $h_1 \sim_\exists h_2$, we have that $f(h_1) = f(h_2)$. By definition, on the other hand, $f'(\pi_\chi, \sigma_\chi) = f(h_2)$, whence $\langle \pi_\chi(n_m), f(h_1)\rangle \notin R_i$. This, however, is impossible, as f is a w.s. for $\exists$.

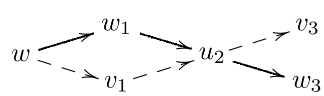

FIGURE 4

(2) f' agrees on equivalent histories: This is trivial, since the value of f' on equivalent histories is defined as the value of f on one and the same history.

(3) Clearly f' always leads to a win for $\exists$. ■

By definition f' is history-free. The statement of Theorem 6 follows. ■

Every $\mathbf{IFML}^+[k]$-formula φ has the following normal forms φ° and φ^+:

COROLLARY 8. *For all* $\varphi \in \mathbf{IFML}^+[k]$, $\mathcal{M}$, *and* w, *we have:*

$$\mathcal{M} \models \varphi^\circ[w] \iff \mathcal{M} \models \varphi[w] \iff \mathcal{M} \models \varphi^+[w],$$

φ° being obtained from φ by replacing any $(\Diamond_{i,n}/W_n)$ by $(\Diamond_{i,n}/V_n)$, and φ^+ from φ by replacing any $(\Diamond_{i,n}/W_n)$ by $(\Diamond_{i,n}/U_n)$, where V_n [resp. U_n] *is the set of indices x of boxes superordinate to $(\Diamond_{i,n}/W_n)$ in φ such that $n < x < K_n$* [resp. $x \neq K_n$], *for $K_n := min\{x > n : x = \omega$ or $[O_x \prec_\varphi \Diamond_{i,n}/W_n$ & $O_x \in \{\Diamond, \Box\}$ & $x \notin W_n]\}$.* ■

E.g., if φ is the formula $\Box_7\Diamond_6\Box_5\Diamond_4\Box_3\Box_2(\Diamond_1/\{2,5\})p_0$, it is true in precisely the same models as $\varphi^\circ := \Box_7\Diamond_6\Box_5\Diamond_4\Box_3\Box_2(\Diamond_1/\{2\})p_0$, and also true in exactly the same models as $\varphi^+ := \Box_7\Diamond_6\Box_5(\Diamond_4/\{7\})\Box_3\Box_2(\Diamond_1/\{2,5,7\})p_0$. By Theorem 6, when evaluating any of the formulas φ, φ° and φ^+, the choice for $\Diamond_1$ can be made as a function of the last known move in the relevant history, i.e. depending only on the choice for $\Box_3$. Hence allowing the choice for $\Diamond_1$ to depend not only on the move for $\Box_3$ but also on all earlier moves by the opponent (φ°), or requiring this choice to be independent from all of these earlier moves (φ^+), simply results in the truth-condition of φ.

Theorem 6 cannot be generalized from $\mathbf{IFML}^+[k]$ to $\mathbf{IFML}[k]$:

EXAMPLE 9. Let $\varphi := \Box_4\Box_3(\Diamond_2/\{3\})(\Diamond_1/\{2\})\top \in \mathbf{IFML}[1]$. Let us evaluate φ relative to the unary model $\langle \mathcal{M}, a\rangle$ depicted in Figure 5:

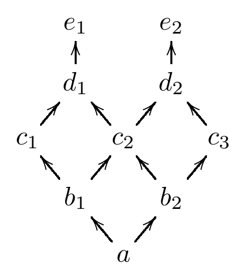

FIGURE 5

(a) To see that $\exists$ has a w.s. in $G(\varphi, \mathcal{M}, a)$, define a function f by setting: $f(a, b_1, c_1) = f(a, b_1, c_2) = d_1$, $f(a, b_2, c_2) = f(a, b_2, c_3) = d_2$, $f(a, b_1, c_1, d_1) = f(a, b_1, c_2, d_1) = e_1$, $f(a, b_2, c_2, d_2) = f(a, b_2, c_3, d_2) = e_2$. Let (a, x_1, x_2) and (a, x_1', x_2') be any plays of $G(\varphi, \mathcal{M}, a)$ s.t. $x_1 = x_1'$. So $f(a, x_1, x_2) = f(a, x_1', x_2')$. If R is the accessibility relation represented in the figure and $d = f(a, x_1, x_2)$, then $R(x_2, d)$. Further, if (a, x_1, x_2, x_3) and (a, x_1', x_2', x_3') are plays in which $\exists$ has used f and which satisfy $x_1 = x_1'$ and $x_2 = x_2'$, then $f(a, x_1, x_2, x_3) = f(a, x_1', x_2', x_3')$. And if $e = f(a, x_1, x_2, x_3)$, then $R(x_3, e)$. Hence f is a w.s. for $\exists$ in $G(\varphi, \mathcal{M}, a)$. Observe that f is *not* history-free: for both plays (a, b_1, c_2, d_1) and (a, b_2, c_2, d_2), the last move 'known' by $\exists$ is c_2, but f associates these plays with distinct choices.

(b) We claim that in fact there is *no* history-free w.s. for $\exists$ in $G(\varphi, \mathcal{M}, a)$. Suppose for contradiction that f^* is such a history-free winning strategy.

Hence in particular $f^*(a, b_1, c_2, d_1) = f^*(a, b_2, c_2, d_2)$. On the other hand, necessarily $f^*(a, b_1, c_1) = d_1$ and $f^*(a, b_2, c_3) = d_2$. But then, by the uniformity of f^*, we must further have $f^*(a, b_1, c_2) = d_1$ and $f^*(a, b_2, c_2) = d_2$. Thus (a, b_1, c_2, d_1) and (a, b_2, c_2, d_2) are both histories constructed by using f^* against a sequence of moves by $\forall$. However, the value $f^*(a, b_1, c_2, d_1) = f^*(a, b_2, c_2, d_2)$ cannot be accessed along R from both d_1 and d_2. This is a contradiction in view of f^* being a winning stratregy. We may conclude that there is no history-free w.s. for $\exists$ in game $G(\varphi, \mathcal{M}, a)$. ■

THEOREM 10. *For all $k \geq 1$, $\exists$'s winning strategies in evaluation games of* $\mathbf{IFML}[k]$ *do not in general admit of history-free normal form.* ■

Finally, let $\mathbf{IFML}^+_{12}[k]$ be the logic that results from giving up condition (3) in the definition of the syntax of $\mathbf{IFML}^+[k]$. Let us consider the impact of this condition to the existence of history-free winning strategies.

EXAMPLE 11. Let $\varphi := \Box_{0,3}\Box_{0,2} \vee_{i\in\{0,1\}} (\Diamond_{i,1}/\{2\})\top \in \mathbf{IFML}^+_{12}[2]$, and think of evaluating φ relative to the binary model $\langle \mathcal{M}, w\rangle$ of Figure 6:

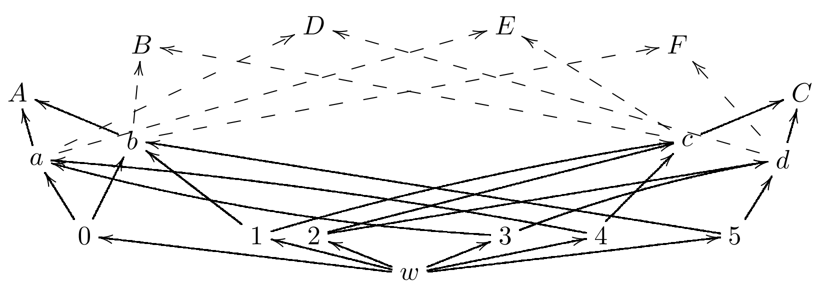

FIGURE 6

In Figure 6, the solid lines represent the accessibility relation R_0, while the dashed lines stand for the accessibility relation R_1. The domain of $\mathcal{M}$ consist of four levels: $G_0 = \{w\}$, $G_1 = \{0, 1, 2, 3, 4, 5\}$, $G_2 = \{a, b, c, d\}$ and $G_3 = \{A, B, C, D, E, F\}$. Write $Z = \{0, 2\}$.

(a) We claim that there is a w.s. for $\exists$ in $G(\varphi, \mathcal{M}, w)$. To see this, define a function f as follows: if $x \in G_1$ and $y \in G_2$ are any consecutive choices of $\forall$ in a play of $G(\varphi, \mathcal{M}, w)$, made according to the game rules, we set:

$f(w, x, y) = 0$ if $x \in Z$; and $f(w, x, y) = 1$ otherwise;
$f(w, 0, y, 0) = A$; $f(w, 2, y, 0) = C$;
$f(w, 1, y, 1) = B$; $f(w, 3, y, 1) = D$; $f(w, 4, y, 1) = E$;$f(w, 5, y, 1) = F$.

Let (w, x, y, i) and (w, x', y', i') be any plays of $G(\varphi, \mathcal{M}, w)$ s.t. $x = x'$ and $i = i'$. So $f(w, x, y, i) = f(w, x', y', i')$. Write $X := f(w, x, y, i)$. If $x \in Z$ and hence $i = 0$, we have $R_0(y, X)$. And if $x \notin Z$ and thus $i = 1$, we have

$R_1(y, X)$. We may conclude that f is a w.s. for $\exists$ in $G(\varphi, \mathcal{M}, w)$. It is readily observed that f is *not* history-free: e.g. $0 = f(w, 0, b) \neq f(w, 1, b) = 1$, while for both plays $(w, 0, b)$ and $(w, 1, b)$, the last move 'known' by $\exists$ is b.

(b) We claim there is *no* history-free w.s. for $\exists$ in $G(\varphi, \mathcal{M}, w)$. For a contradiction, suppose f^* is such a strategy. If $x \in G_2$, denote by $f^*(x) \in \{0, 1\}$ the value f^* gives, as a choice corresponding to $\vee_{i\in\{0,1\}}$, to plays ending with x. Let $x, y \in G_2$ and $z \in G_1$ be points s.t. $R_0(z, x)$, $R_0(z, y)$. If $z \in Z$, it is not possible that $f^*(x) = f^*(y) = 1$, because there is no $p \in G_3$ s.t. $R_1(x, p)$ and $R_1(y, p)$; if $z \notin Z$, it is not possible that $f^*(x) = f^*(y) = 0$, because then there is no $p \in G_3$ s.t. $R_0(x, p)$ and $R_0(y, p)$.

In fact, $f^*(x) \neq f^*(y)$. To see this, suppose for contradiction that $f^*(x) = f^*(y)$, and let x', y' be the elements in $G_2\backslash\{x, y\}$. Consider the case $z \in Z$; the case of $z \notin Z$ is similar. We must have: $(*)$ $f^*(x') = f^*(y') \neq f^*(x) = f^*(y) = 0$. For otherwise there is $v \in \{x', y'\}$ s.t. $f^*(v) = 0$ and $z' \in G_1\backslash Z$ satisfying $R_0(z', x)$, $R_0(z', v)$, which is not possible in view of what observed above. By $(*)$, $f^*(x') = f^*(y') = 1$. But there is no $p \in G_3$ s.t. $R_1(x', p)$ and $R_1(y', p)$, whence f^* is not a w.s. for $\exists$, a contradiction.

So either $f^*(a) = f^*(c) = 0$ and $f^*(b) = f^*(d) = 1$, or $f^*(a) = f^*(c) = 1$ and $f^*(b) = f^*(d) = 0$. In both cases we can find $x, y \in G_2$ and $z \in G_1\backslash Z$ such that $R_0(x, z)$, $R_0(y, z)$ and $f^*(x) = f^*(y) = 0$. Again, there is no $p \in G_3$ such that $R_0(x, p)$ and $R_0(y, p)$, a contradiction. We may conclude that there is no history-free w.s. for $\exists$ in $G(\varphi, \mathcal{M}, w)$. $\blacksquare$

THEOREM 12. *For all $k \geq 2$, $\exists$'s winning strategies in evaluation games of $\mathbf{IFML}^+_{12}[k]$ do not in general admit of history-free normal form.* $\blacksquare$

4 Satisfiability Problem

Supposing $\varphi \in \mathbf{IFML}^+[k]$ is satisfied in $\langle \mathcal{M}, w\rangle$, we (1) construct a tree $\mathcal{T}$ out of $\langle \mathcal{M}, w\rangle$; (2) generate a labeled subtree $\mathcal{T}^f$ of $\mathcal{T}$; (3) produce a model $\langle \mathcal{M}^*, w\rangle$ of φ out of $\mathcal{T}^f$; (4) generate from $\langle \mathcal{M}^*, w\rangle$ a finite model $\langle \mathcal{N}, v\rangle$ of φ, whose size has a recursive upper bound w.r.t. $\sharp[\varphi]$. For the rest of the section, let φ be any satisfiable $\mathbf{IFML}^+[k]$-formula, and $\langle \mathcal{M}, w\rangle$ its model.

DEFINITION 13. The *modal depth* of an $\mathbf{IFML}^+[k]$-formula is defined as follows: $md(p_i) = 0 = md(\neg p_i)$; $md(\varphi \vee_n \psi) = max\{md(\varphi), md(\psi)\} = md(\varphi \wedge_n \psi)$; and $md((\Diamond_{i,n}/W_n)\varphi) = md(\varphi) + 1 = md((\Box_{i,n}/W_n)\varphi)$.

Towards a finite model: preparations. For each $n \leq md(\varphi)$, let T^n_w be the set of sequences $(w, i_1, s_1, \ldots, i_m, s_m)$ s.t. $R_{i_1}(w, s_1), \ldots, R_{i_m}(s_{m-1}, s_m)$ for some $m \leq n$ and some $i_1, \ldots, i_m \in \{0, \ldots, k-1\}$. Further, define relations R'_i $(i < k)$ on the set $\bigcup_{n\leq md(\varphi)} T^n_w$ as follows: $(w, i_1, s_1, \ldots, i_m, s_m)\ R'_i\ (w, i'_1, s'_1, \ldots, i'_{m'}, s'_{m'})$ if $m' = m+1$ and $(i_j, s_j) = (i'_j, s'_j)$ $[0 \leq j \leq m]$ and $i'_{m'} = i$. Finally, fix an assignment $\mathfrak{h}'$ by putting, for all $p_i \in \mathbf{prop}$: $(w, i_1, s_1, \ldots, i_m, s_m) \in \mathfrak{h}'(p_i)$ if $s_m \in \mathfrak{h}(p_i)$. We have obtained a tree structure $\langle \mathcal{T}, w\rangle = \langle \bigcup_{n \leq md(\varphi)} T_w^n, R_0', \ldots, R_{k-1}', \mathfrak{h}', w\rangle$. In general, $\langle \mathcal{T}, w\rangle$ is *not* a model of φ.

DEFINITION 14. A φ-*label* is a set $\{\sigma_{\varphi_1}, \ldots, \sigma_{\varphi_n}\}$ of partial functions $\sigma_{\varphi_i} : \omega \rightharpoonup \{0,1\}$ such that φ_i is a subformula of φ, and $dom(\sigma_{\varphi_i})$ consists of indices of disjunctions and conjunctions that are superordinate to φ_i in φ. When no confusion may arise, we refer to φ-labels as labels.

We will write $\dot{x} := (w, s_0, \ldots, s_m)$, if $x = (w, i_0, s_0, \ldots, i_m, s_m)$. Let f be a w.s. of $\exists$ in game $G(\varphi, \mathcal{M}, w)$, and define a labeling on $\mathcal{T}$ as follows:

- First let all points in $\mathcal{T}$ be labeled with the empty label $\emptyset$. Then label the root w of $\mathcal{T}$ with $\{\emptyset\}$, and go on as follows.
- If a point x in $\mathcal{T}$ has the label $\Sigma = \{\sigma_{\varphi_1}, \ldots, \sigma_{\varphi_N}\}$ with $\varphi_m = (\Box_{i,n}/W_n)\psi$ $(1 \leq m \leq N)$, and if Σ' is the label of an R_i-successor of x, replace Σ' by $\Sigma' \cup \{\sigma_\psi\}$, where $\sigma_\psi := \sigma_{\varphi_m}$.
- If a point x in $\mathcal{T}$ has the label $\Sigma = \{\sigma_{\varphi_1}, \ldots, \sigma_{\varphi_N}\}$ with $\varphi_m = (\psi \wedge_j \theta)$ $(1 \leq m \leq N)$, replace the label Σ itself by $(\Sigma \backslash \{\sigma_{\varphi_m}\}) \cup \{\sigma_\psi, \sigma_\theta\}$, where $\sigma_\psi := \sigma_{\varphi_m} \cup \{(j,0)\}$ and $\sigma_\theta := \sigma_{\varphi_m} \cup \{(j,1)\}$.
- If a point x in $\mathcal{T}$ has the label $\Sigma = \{\sigma_{\varphi_1}, \ldots, \sigma_{\varphi_N}\}$ with $\varphi_m = (\Diamond_{i,n}/W_n)\psi$ $(1 \leq m \leq N)$, take the R_i-successor $y := f(\dot{x}, \sigma_{\varphi_m})$ of x, and if Σ' is the label of y, replace Σ' by $\Sigma' \cup \{\sigma_\psi\}$, where $\sigma_\psi := \sigma_{\varphi_m}$.
- If a point x in $\mathcal{T}$ has the label $\Sigma = \{\sigma_{\varphi_1}, \ldots, \sigma_{\varphi_N}\}$ with $\varphi_m = (\psi \vee_j \theta)$ $(1 \leq m \leq N)$, take the disjunct $\chi \in \{\psi, \theta\}$ given by $f(\dot{x}, \sigma_{\varphi_m}) \in \{0,1\}$, and replace the label Σ itself by $(\Sigma \backslash \{\sigma_{\varphi_m}\}) \cup \{\sigma_\chi\}$, where $\sigma_\chi := \sigma_{\varphi_m} \cup \{(j, f(\dot{x}, \sigma_{\varphi_m}))\}$.

We define $\mathcal{T}^f$ as the subtree of $\mathcal{T}$ containing precisely those of its elements that have a non-empty label under the above labeling.

Next we turn the labeled tree $\mathcal{T}^f$ into a modal structure $\mathcal{M}^*$ that will provide a model for φ. Specify $\mathcal{M}^* := \langle D^*, R_0^*, \ldots, R_{k-1}^*, \mathfrak{h}^*\rangle$ by setting:

$$
\begin{aligned}
& D^* := \{w\} \ \cup \ \{(n,i,s) : \text{for some } (w, i_1, s_1, \ldots, i_m, s_m) \in T^f, \\
& \qquad\quad n = m \ \ \& \ \ i = i_m \ \ \& \ \ s = s_m\}. \\
& \forall j < k\text{: } (n,i,s)\, R_j^*\, (n', i', s') \ \ \textbf{if} \ \ n' = n{+}1 \ \ \& \ \ j = i' \ \ \& \ \ R_{i'}(s, s').^1 \\
& \forall p_j \in \mathbf{prop}\text{: } (n,i,s) \in \mathfrak{h}^*(p_j) \ \ \textbf{if} \ \ s \in \mathfrak{h}(p_j).
\end{aligned}
$$

CLAIM 15. $\mathcal{M}^* \models \varphi[w]$. ■

[1] Here $R_{i'}$ is the relation of type i' of the original model $\langle \mathcal{M}, w\rangle$.

Finite model $\langle \mathcal{N}, v\rangle$ of φ. We move on to build up a *finite* model $\langle \mathcal{N}, v\rangle$ of φ out of the model $\langle \mathcal{M}^*, w\rangle$.

OBSERVATION 16. **(a)** For a fixed φ, the number m_φ of different φ-labels is finite: if r is the number of subformulas of φ, and n is the maximal number of nested conjunctions and disjunctions in φ, then $m_\varphi \leq 2^{r\cdot 2^n}$. **(b)** The labeling of $\mathcal{T}^f$ induces a labeling of $\mathcal{M}^*$. The label of an element $(n, i, s) \in D^*$ is obtained as the union of the labels of elements of the set $T_{(n,i,s)} := \{(w, i_1, s_1, \ldots, i_m, s_m) \in T^f : m = n \;\;\&\;\; i_m = i \;\;\&\;\; s_m = s\}$. ■

To construct $\langle \mathcal{N}, v\rangle$, we partition $dom(\mathcal{M}^*)$ into $md(\varphi)$ 'levels', the level of an element $(N, i, s) \in D^*$ being given by its 'height', N. On each level we 'identify' as many elements as it goes without affecting the fact that the resulting structure is a model of φ. The identification yields a finite structure. Write $M := md(\varphi)$. We define structures $\mathcal{M}^N := (D^N, R_0^N, \ldots, R_{k-1}^N, \mathfrak{h}^N)$ $[N := 1, \ldots, M]$. First we define an equivalence relation $\equiv_M$ on D^*:

$$\begin{aligned}&(n, i, s) \;\equiv_M\; (n', i', s') \;\; \textbf{if} \;\; (n, i, s) = (n', i', s') \;\vee \\ &n = n' = M \quad \& \quad i = i' \quad \& \quad \forall p_j \in \textbf{prop}[s \in \mathfrak{h}(p_j) \Longleftrightarrow s' \in \mathfrak{h}(p_j)].\end{aligned}$$

The modal structure $\mathcal{M}^M$ is then specified by putting:

$$\begin{aligned}&D^M := D^*/\equiv_M. \\ &[\xi]_{\equiv_M} \, R_j^M \, [\zeta]_{\equiv_M} \quad \textbf{if} \quad \text{there is } \zeta' \text{ s.t. } \zeta \equiv_M \zeta' \quad \& \quad R_j^*(\xi, \zeta'). \\ &\text{For all } p_i \in \textbf{prop}: \;\; [\xi]_{\equiv_M} \in \mathfrak{h}^M(p_i) \;\; \textbf{if} \;\; \xi \in \mathfrak{h}^*(p_i).\end{aligned}$$

It can be checked that the definitions of $\mathfrak{h}^M$ and the R_j^M are independent of the choices of representatives of the equivalence classes involved.

OBSERVATION 17. The size of the set $S_M := \{\xi \in D^M : \xi = [(M, i, s)]_{\equiv_M}$ for some $i, s\}$ is finite. In fact, it has at most $k \cdot 2^{p_\varphi}$ elements, where p_φ is the number of propositional atoms actually appearing in φ. ■

CLAIM 18. There is a history-free w.s. for $\exists$ in $G(\varphi, \mathcal{M}^M, [w]_{\equiv_M})$. ■

Fix $N > 1$. Let $D^{M+1} := D^*$. Suppose that if $N \leq K \leq M$, we already have defined: **(a)** an equivalence relation $\equiv_K \subseteq (D^{K+1})^2$; **(b)** a modal structure $\mathcal{M}^K := (D^K, R_0^K, \ldots, R_{k-1}^K, \mathfrak{h}^K)$ with $D^K := D^{K+1}/\equiv_K$; and **(c)** a history-free w.s. f^K for $\exists$ in $G^K := G(\varphi, \mathcal{M}^K, [[w]_{\equiv_M} \ldots]_{\equiv_K})$. We go on to define an equivalence relation $\equiv_{N-1}$ on D^N by putting $\xi \equiv_{N-1} \xi'$ if and only if either $\xi = \xi'$, or else there are n, i, s, n', i', s' such that

$$n = n' = N{-}1 \;\&\; \xi = [[(n, i, s)]_{\equiv_M} \ldots]_{\equiv_N} \;\&\; \xi' = [[(n', i', s')]_{\equiv_M} \ldots]_{\equiv_N},$$

and the following conditions (i) to (v) hold:

(i) $i = i'$.

(ii) For all $p_j \in \mathbf{prop} : s \in \mathfrak{h}^N(p_j) \Longleftrightarrow s' \in \mathfrak{h}^N(p_j)$.

(iii) The points (n, i, s) and (n', i', s') have the same label in $\mathcal{M}^*$.

(iv) For all $i < k$ and all ζ: $R_i^N(\xi, \zeta) \Longleftrightarrow R_i^N(\xi', \zeta)$.

(v) For all subformulas (χ, σ_χ) of φ, and all plays (π'_χ, σ_χ) and $(\pi''_\chi, \sigma_\chi)$ of G^N in which $\exists$ has applied f^N, if the last choices from the domain 'known' to $\exists$ are, respectively, $\pi'_\chi(K) = \xi$ and $\pi''_\chi(K) = \xi'$, and for all $l < K$, $\pi'_\chi(l) = \pi''_\chi(l)$, then $f^N(\pi'_\chi, \sigma_\chi) = f^N(\pi''_\chi, \sigma_\chi)$.

For condition (v), notice that given any subformula (χ, σ_χ) of φ, and any plays (π'_χ, σ_χ) and $(\pi''_\chi, \sigma_\chi)$ of G^N, we necessarily have: $dom(\pi'_\chi) = dom(\pi''_\chi)$. Further, the index K of the last modal operator 'known' to $\exists$ is obtained as follows: $K := min\{x > n : x = \omega$ or $[O_x \prec_\varphi \vee_n$ & O_x is of the form $\Diamond, \Box]\}$, if $\chi := \psi \vee_n \theta$; and $K := min\{x > n : x = \omega$ or $[O_x \prec_\varphi \Diamond_{i,n}/W_n$ & O_x is of the form $\Diamond, \Box$ & $x \notin W_n]\}$, if $\chi := (\Diamond_{i,n}/W_n)\psi$. As f^N is history-free, its values on the plays (π'_χ, σ_χ) and $(\pi''_\chi, \sigma_\chi)$ do not depend on the values $\pi'_\chi(l)$ *resp.* $\pi''_\chi(l)$ for $l > K$. Define $\mathcal{M}^{N-1}$ by setting:

$D^{N-1} := D^N / \equiv_{N-1}$.
$[\xi]_{\equiv_{N-1}} R_j^{N-1} [\zeta]_{\equiv_{N-1}}$ **if** there is ζ' s.t. $\zeta \equiv_{N-1} \zeta'$ and $R_j^N(\xi, \zeta')$.
For all $p_i \in \mathbf{prop}$: $[\xi]_{\equiv_{N-1}} \in \mathfrak{h}^{N-1}(p_i)$ **if** $\xi \in \mathfrak{h}^N(p_i)$.

It can be checked that the definition of the modal structure $\mathcal{M}^{N-1}$ is independent of the choices for representatives of the equivalence classes involved.

CLAIM 19. Let $1 \leq K \leq M$. **(a)** The size of the set $S_K = \{\xi \in D^K : \xi = [[(K, i, s)]_{\equiv_M} \ldots]_{\equiv_K}$ for some $i, s\}$ has a recursive upper bound w.r.t. the length of φ. **(b)** There is a history-free w.s. for $\exists$ in G^K.

Proof. The inductive proof of **(b)** is straightforward. For **(a)**, let us write $N := k \cdot 2^{p_\varphi} \cdot m_\varphi$, where p_φ is the number of propositional atoms appearing in φ, and m_φ is the number of distinct φ-labels. Let $c(K)$ denote the size of the set S_K $(K := 1, \ldots, M)$. By Observation 17, $c(M) \leq N$. For $K := 1, \ldots, M-1$, we have by the construction of $\mathcal{M}^K$: $c(K) \leq N \cdot 2^{c(K+1)} \cdot \Pi_{i=K+1}^M c(i)$. Hence statement (a) follows. ∎

To finish the construction, we put: $\mathcal{N} := \mathcal{M}^1$ and $v := [[w]_{\equiv_M} \ldots]_{\equiv_1}$.

THEOREM 20. *Let* $k \geq 1$. **(a)** $\mathbf{IFML}^+[k]$ *has the strong finite model property.* **(b)** *The satisfiability problem of* $\mathbf{IFML}^+[k]$ *is decidable.*

Proof. Statement **(b)** is immediate from **(a)**. For **(a)**, suppose $\mathcal{M} \models \varphi[w]$. Let $\langle \mathcal{N}, v \rangle$ be the model constructed out of $\langle \mathcal{M}, w \rangle$ as explained above. By Claim 19 (b), $\mathcal{N} \models \varphi[v]$. And by Claim 19 (a), the size of $\mathcal{N}$ has a recursive upper bound w.r.t. the length of φ. ∎

5 Validity Problem

$\mathbf{IFML}^+[k]$-formulas are in negation normal form: the symbol $\neg$ appears only as prefixed to propositional atoms. Closing $\mathbf{IFML}^+[k]$ under $\neg$, and interpreting $\neg$ as initiating a 'role switch' between the players of evaluation games (from verifier to falsifier and *vice versa*), we have: $\exists$ [$\forall$] has a w.s. in $G(\neg\varphi)$ iff $\forall$ [*resp.* $\exists$] has a w.s. in $G(\varphi)$. The negation hence interpreted game-theoretically may be termed *dual negation*. As there are indeterminate $\mathbf{IFML}^+[k]$-formulas (formulas s.t. neither $\exists$ nor $\forall$ has a w.s. in the associated evaluation game), $\neg$ cannot have the force of *contradictory negation* ($\sim$), whose semantics is simply: $\sim\varphi$ is true in $\langle\mathcal{M}, w\rangle$ iff φ is *not* true in $\langle\mathcal{M}, w\rangle$. So if φ is indeterminate in $\langle\mathcal{M}, w\rangle$, $\sim\varphi$ is true therein.

FACT 21. The satisfiability and validity problems of $\mathbf{IFML}^+[k]$ are *not* each others' duals.

Proof. If they were duals, we would have: φ is valid iff $\neg\varphi$ is not satisfiable. Let $\psi \in \mathbf{IFML}^+[k]$ be any formula which is indeterminate in a model $\langle\mathcal{M}, w\rangle$. Then the formula $\neg(\psi \vee \neg\psi)$ is not satisfiable, but still the formula $(\psi \vee \neg\psi)$ is not valid, since it is indeterminate in $\langle\mathcal{M}, w\rangle$. ■

We prove that the validity problem of $\mathbf{IFML}^+[k]$ is decidable by reducing it to the validity problem of $\mathbf{ML}[k]$. By Fact 21, this result neither implies nor is implied by Theorem 20. We define a syntactic transformation $\varphi \mapsto \varphi^*$ among $\mathbf{IFML}^+[k]$-formulas. Recall the definitions of syntactic tree and path in a syntactic tree from *Section* 2. If $\varphi \in \mathbf{IFML}^+[k]$ and $r \geq 0$, consider all paths of the form $(*)$ $\langle(\Box_{i',n'}/W_{n'}), \beta_1, \dots, \beta_r, (\Diamond_{i'',n''}/W_{n''})\rangle$ in its syntactic tree, where $\beta_i \in \{\vee, \wedge\}$ and the diamond is independent from the box (i.e. $n' \in W_{n''}$). For each such path, replace the full subtree with the diamond at its root by the symbol $\perp$. The result of these replacements is denoted by φ^*.

CLAIM 22. For any $\varphi \in \mathbf{IFML}[k]^+$, φ is valid $\Leftrightarrow$ φ^* is valid.

Proof. Let $\varphi \in \mathbf{IFML}[k]^+$ be arbitrary. The validity of φ^* trivially implies the validity of φ. For the converse, suppose φ is valid. We assume that the syntactic tree of φ contains at least one path of the form $(*)$: otherwise there is nothing to prove. Let $\langle\mathcal{M}, w\rangle = \langle D, R_0, \dots, R_{k-1}, \mathfrak{h}, w\rangle$ be an arbitrary model. We must show that $\mathcal{M} \models \varphi^*[w]$. Now let $N := 2^{2^{\sharp[\varphi]}}$. We construct a structure $\mathcal{M}^* = (D^*, R^*_0, \dots, R^*_{k-1}, \mathfrak{h}^*)$ as follows:

$$D^* := \{(\eta, d) : d \in D \text{ and } \eta : n \to N \text{ for some } n < \omega\}$$
$$(\eta, d)\ R^*_i\ (\eta', d') \quad \textbf{if} \quad (d\ R_i\ d' \ \&\ \eta' = \eta^\frown\langle k\rangle \text{ for some } k < N)$$
$$(\eta, d) \in \mathfrak{h}^*(p) \quad \textbf{if} \quad d \in \mathfrak{h}(p).$$

Because φ is valid, we have in particular: $\mathcal{M}^* \models \varphi[(\emptyset, w)]$. Let f be a w.s. of $\exists$ in the corresponding evaluation game. We prove:

SUBCLAIM 23. For any $d_1, \ldots, d_n \in D$ there are numbers $m_1, \ldots, m_n < N$ s.t. we have: *if* a play of $G(\varphi, \mathcal{M}^*, (\emptyset, w))$ is played so that (i) $\exists$ has followed the strategy f, and (ii) $\forall$ has chosen for each box an element $(\eta^\frown\langle m_i\rangle, d_i)$, where i is the least $j \leq n$ with $m_j \notin dom(\eta)$, and the previous element chosen from D^* is (η, d) for some $d \in D$, *then* assuming that the play has gone according to the rules, there is no path of the form $(*)$ in the syntactic tree of φ s.t. the play has proceeded to the diamond $(\Diamond_{i'',n''}/W_{n''})$.

Proof of the Subclaim. The proof is by induction on the number n of boxes reached in a play of game $G(\varphi, \mathcal{M}^*, (\emptyset, w))$. The base case of 0 boxes holds trivially. Suppose then that the claim holds for the case of n boxes. Let h be a play of game $G(\varphi, \mathcal{M}^*, (\emptyset, w))$ involving choices interpreting $n+1$ boxes. Suppose that it is $\forall$'s turn to move at h to interpret the $(n+1)$-th box. Assume that $\exists$ has followed f when playing h. Given that the first n moves from D^* by $\forall$ are $(\eta^\frown\langle m_i\rangle, d_i)$ $[i := 1, \ldots, n]$ and that they are made according to the game rules, by the induction hypothesis we may assume that the numbers m_i are chosen so that the play h has not proceeded along any path of the form $(*)$ to a diamond $(\Diamond_{i'',n''}/W_{n''})$.

Let $d_{n+1} \in D$ be any element s.t. $d \ R_i \ d_{n+1}$, where $d \in D$ is given by the previous choice (η, d) from D^* in h. Consider the different elements $(\eta^\frown\langle p\rangle, d_{n+1})$ with $p < N$ that $\forall$ may choose, following the game rules, to interpret the $(n+1)$-th box $(\Box_{i',n'}/W_{n'})$. Suppose from $(\Box_{i',n'}/W_{n'})$ there begins a path of the form $(*)$ in the syntactic tree of φ. The sequence $\beta_1, \ldots, \beta_r$ contains some finite number $m \leq r$ of conjunctions; so there are at most $2^{2^m} < 2^{2^{\sharp[\varphi]}}$ different ways in which the values of the strategy f can be chosen for the disjunctions (if any) among $\beta_1, \ldots, \beta_r$. Hence there are numbers $p_1, p_2 < N$ with $p_1 \neq p_2$, and plays h_1, h_2, s.t. up to the box $(\Box_{i',n'}/W_{n'})$, h_1 and h_2 both coincide with h; in h_1 the box is interpreted as $(\eta^\frown\langle p_1\rangle, d_{n+1})$ and in h_2 as $(\eta^\frown\langle p_2\rangle, d_{n+1})$; and proceeding from both points $(\eta^\frown\langle p_1\rangle, d_{n+1})$ and $(\eta^\frown\langle p_2\rangle, d_{n+1})$, the strategy f gives the same choices for the disjunctions (if any) in $\beta_1, \ldots, \beta_r$. Choose $m_{n+1} := min\{p_1, p_2\}$. For a contradiction, suppose there are choices for the conjunctions in $\beta_1, \ldots, \beta_r$, and replies to them by f, so that these choices $\overline{b} \in \{0,1\}^r$ extend both h_1 and h_2 in such a way that the diamond $(\Diamond_{i'',n''}/W_{n''})$, which is independent from the box $(\Box_{i',n'}/W_{n'})$, is reached. As f is uniform, $f(h_1{}^\frown\overline{b}) = f(h_2{}^\frown\overline{b})$. But then f is not a w.s. for $\exists$, since from distinct points in $\mathcal{M}^*$ only distinct points are accessible along R_i^*. ■

Given the truth of Subclaim 23, the strategy f clearly induces a w.s. for $\exists$ in game $G(\varphi, \mathcal{M}, w)$, and so $\mathcal{M} \models \varphi^*[w]$. Hence Claim 22 follows. ■

THEOREM 24. *For all $k \geq 1$, $\mathbf{IFML}^{+}[k]$-validity is decidable.*

Proof. Define a syntactic transformation $\varphi \mapsto \varphi^{-}$ from $\mathbf{IFML}[k]^{+}$-formulas to $\mathbf{ML}[k]$-formulas by letting φ^{-} be the result of replacing any operator of the form $(O_{i,n}/W_n)$ in the formula φ by $O_{i,n}$. By Theorem 6 it is immediate that for any $\varphi \in \mathbf{IFML}[k]^{+}$ and for any model $\langle \mathcal{M}, w \rangle$, $\mathcal{M} \models \varphi^{*}[w]$ iff $\mathcal{M} \models (\varphi^{*})^{-}[w]$, where * stands for the above-defined transformation $\varphi \mapsto \varphi^{*}$ among $\mathbf{IFML}^{+}[k]$-formulas. The transformations $\varphi \mapsto \varphi^{*} \mapsto (\varphi^{*})^{-}$ are effective; so to decide whether a given $\varphi \in \mathbf{IFML}^{+}[k]$ is valid, it suffices to apply an algorithm deciding $\mathbf{ML}[k]$-validity to the formula $(\varphi^{*})^{-}$. ■

COROLLARY 25. *$\mathbf{IFML}^{+}[k]$-validity is decidable in PSPACE.*

Proof. The statement follows from the proof of Theorem 24 by the fact that the complexity of $\mathbf{ML}[k]$-validity is in PSPACE (cf. [8]). ■

6 Concluding Remarks

Related work. *Independence-Friendly (IF) first-order logic* was introduced by Hintikka and Sandu in [10]. Its basic properties were studied by Sandu in [13]. The idea behind this logic is to allow arbitrary relations of logical (in)dependence between quantifiers. One way of thinking how to accomplish this is by separating the syntactic relation of subordinateness and the semantic relation of logical dependence, so that a quantifier can be logically independent from some quantifiers in whose syntactic scope it nevertheless lies. IF first-order sentences coincide in expressive power with Σ_1^1-sentences, whence IF first-order logic is a proper extension of first-order logic. This logic has been discussed in connection with foundations of mathematics, e.g. in the monograph [9] of Hintikka, and in the papers [11] and [12] by Hyttinen and Sandu. Sandu and Väänänen showed in [15] how to extend the idea of informational independence from the case of quantifiers to propositional connectives. Bradfield was in [2] the first to define a version of IF modal logic; his logic was designed to be used when studying transition systems with concurrency. With Fröschle, he studied further properties of this logic in [5]. In [3] and [4], Bradfield has extended IF first-order logic into an IF fixpoint logic, and shown how to obtain an IF modal μ-calculus.

Motivation. Given the existence of IF first-order logic and the fact that its expressive power exceeds that of usual first-order logic, a systematic question arises: how to formulate an IF modal logic. A natural approach is to modify basic modal logic ($\mathbf{ML}[k]$) so as to allow modal operators to be logically independent from syntactically superordinate dual operators. The relevant notion of logical independence is implemented by the requirement of uniformity of players' winning strategies. In the corresponding evaluation games, a player may be ignorant of earlier moves of the opponent, but always remembers his or her own previous moves. Allowing this kind of independence in a modal logical setting typically allows expressing the existence of a *common* endpoint of several transitions: for instance the requirement that various computations end in the *same* state becomes expressible. More general motivation for studying IF modal logics is that they provide one possible, and arguably natural, way of producing relatively expressive modal languages. Such languages are interesting not only for modal logic itself, but in particular for finite model theory, in which modal logics are of interest in connection with different definability issues (cf., e.g., [7]).

Changing $\mathbf{ML}[k]$ by allowing more logical independencies results in powerful logics. As noted in the present paper, $\mathbf{IFML}^+[k]$ has greater expressive power than $\mathbf{ML}[k]$. On the other hand, it is possible to show that this logic is translatable into first-order logic. (In [17] a translation is provided to a fragment of $\mathbf{IFML}^+[k]$; the proof of the general result can be found in [14].) By contrast, in [17] it is shown that generalizing $\mathbf{IFML}^+[k]$ by allowing diamonds (boxes) to be independent from conjunctions (disjunctions) results in a modal logic which is no longer translatable into first-order logic, but has in fact some genuine *second-order* expressive power (i.e. can express properties of modal structures which can be expressed of the corresponding first-order structures only by genuine second-order formulas).

Open problems. In the present paper the 'perfect recall' fragment $\mathbf{IFML}^+[k]$ of the IF modal logic (**IFML**) of [16] was studied. Which of the two logics better deserves to be termed 'IF modal logic'? The answer depends on whether or not it makes sense to speak of a diamond (box) being *logically independent* from another diamond (*resp.* box). It is not obvious how to separate cases where the game-theoretical framework merely serves to model logical phenomena from cases where this tool adds to the picture something that does not really belong to the domain of logic.

The upper bound for the complexity of $\mathbf{IFML}^+[k]$-satisfiability given by the proof of Theorem 20 is far from feasible. (The obtained recursive bound of the size of the finite model of φ has the form of tower function w.r.t. the length of φ.) The complexity is at least PSPACE, because $\mathbf{ML}[k]$ is embeddable in $\mathbf{IFML}^+[k]$, and $\mathbf{ML}[k]$ has PSPACE-complete satisfiability problem w.r.t. $\mathcal{C}_k$ (cf. [8]). The exact complexity of $\mathbf{IFML}^+[k]$-satisfiability is an open question. It is also an open problem whether the satisfiability and validity problems of the logics $\mathbf{IFML}^+_{12}[k]$ and $\mathbf{IFML}[k]$ are decidable. However, if their satisfiability problems are decidable at all, by Theorems 12 and 10 this cannot be proven in the same way as Theorem 20, i.e. relying on the existence of history-free winning strategies. We pointed out in *Section* 2 that IF modal logic of perfect recall would be most naturally formulated as a logic whose semantics obeys the principle "a player never forgets a move he has once known." This would involve allowing disjunctions (conjunctions) independent from boxes (diamonds). Studying this general version of IF modal logic of perfect recall is left for future research.

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BIBLIOGRAPHY

[1] P. Blackburn & al., 2002. *Modal Logic*, Cambridge University Press, Cambridge.
[2] J. Bradfield, 2000. "Independence: Logics and Concurrency," LNCS 1862.
[3] J. Bradfield, 2004a. "Parity of Imperfection" in *Proc. CSL 2003*, LNCS 2803, 72-85.
[4] J. Bradfield, 2004b. "On Independence-Friendly Fixpoint Logics," *Philosophia Scientiae* 8(2), 125-44.
[5] J. Bradfield & S. Fröschle, 2002. "Independence-Friendly Modal Logic and True Concurrency" in *Nordic Journal of Computing*, vol. 9, No. 2, 102-117.
[6] H.-D. Ebbinghaus & J. Flum, 1999. *Finite Model Theory*, Springer-Verlag.
[7] E. Grädel & al., 2004. *Finite Model Theory and Its Applications*, Springer-Verlag.
[8] J. Halpern & Y. Moses, 1992. "A guide to completeness and complexity for modal logics of knowledge and belief", *Artificial Intelligence* 54(3), 319-79.
[9] J. Hintikka, 1996. *The Principles of Mathematics Revisited*, Cambridge University Press, New York.
[10] J. Hintikka & G. Sandu, 1989. "Informational Independence as a Semantical Phenomenon," in Fenstad et al. (eds.), *Logic, Methodology and Philosophy of Science*, vol. 8, Amsterdam, North-Holland, 571-589.
[11] T. Hyttinen & G. Sandu, 2000. "Henkin Quantifiers and the Definability of Truth," *Journal of Philosophical Logic*, vol. 29, 507-527.
[12] T. Hyttinen & G. Sandu, 2001. "IF-logic and foundations of mathematics," *Synthese*, vol. 126, 37-47.
[13] G. Sandu, 1993. "On the Logic of Informational Independence and its Applications," *Journal of Philosophical Logic*, 22, 29-60.
[14] G. Sandu & T. Tulenheimo, 2005. *Logics of Imperfect Information*, manuscript.
[15] G. Sandu & J. Väänänen, 1992. "Partially ordered connectives," *Zeitschrift für Matematische Logik und Grundlagen der Mathematik*, 38, 361-372.
[16] T. Tulenheimo, 2003. "On IF Modal Logic and its Expressive Power" in Balbiani & al. (eds.): *Advances in Modal Logic*, vol. 4, King's College Publications, London, 475-98.
[17] T. Tulenheimo, 2004. *Independence-Friendly Modal Logic*, Philosophical Studies from the University of Helsinki vol. 4 (Doctoral dissertation).

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