A Two-sorted Hybrid Logic Including Guarded Jumps

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ABSTRACT. In this paper we present a two-sorted hybrid logic where formulas are interpreted in spaces of sets. The logical language we consider contains names for both points and sets. Usual nominals are then realized as pairs of names, consisting of one name from each category actually. Accompanying satisfaction operators are mimicked with the aid of the global modality. And we have a family of additional hybrid operators associated with names of sets in a natural way. These connectives make jumps possible that are 'guarded' in a sense, which yields in fact more expressiveness than admitting only unrestricted jumps (by means of the global modality) and the strictly controlled ones from basic hybrid logic. Below we take the first steps towards a hybrid logic comprising the features just indicated, by proving a corresponding completeness as well as a decidability result. Moreover, we discuss an application of the new system to reasoning about knowledge.

1 Introduction

Classical modal languages play an important part in modelling and reasoning about sytems. Some applications, however, call for more expressive power. *Hybrid logic* is a useful tool for extending 'faithfully' the expressive means of usual modal logic; cf [1] for a summary regarding this. But it is not always clear *how* to hybridize a modal language. Already on the elementary stage of defining appropriate satisfaction operators there are several possibilities, depending on the source language. In the present paper, we want to illustrate this by means of a sample language we are particularly interested in because it is related to reasoning about knowledge. (As to the latter notion, cf [7] or [15].)

In [16] (and more detailedly in [6]), a bi-modal system called *topologic* was developed. The two modalities K and $\Box$ appearing there, represent *knowledge* and *effort (in the course of time)*, respectively. Formulas are interpreted in *set spaces* $(X, \mathcal{O}, V)$ consisting of a non-empty set X of *states,* a set $\mathcal{O}$ of distinguished subsets of X, which can be taken as *knowledge states* of an agent, and a valuation V determining the states where the atomic propositions are true. The operator K quantifies then across any knowledge state, whereas $\Box$ quantifies 'downward' across $\mathcal{O}$. That is, descending within $\mathcal{O}$ with respect to the set inclusion relation means acquiring knowledge, and just this is modelled by $\Box$.

It is less noticed that evolving knowledge has also a *spatial* component besides the obvious temporal one. In fact, knowledge can be viewed as *closeness* and knowledge acquisition as *getting a closer proximity* in set spaces. Thus *topology* enters the context of knowledge (justifying the name 'topologic').

Unfortunately, it turned out that several topologically interesting classes of set spaces cannot be dealt with in the framework of *topologic*. However, hybridizing the underlying language can help a lot here. Already the naïve, 'unsorted' hybrid logic of set spaces allows us to characterize neatly the classes arising from linear flows of time; cf [10] and [13].[1] And following the *sorting strategy* from [1] makes the hybrid approach much more flexible, even without having any satisfaction operators at one's disposal; cf [11].

The hybrid logic considered in this paper is a development of the one from [11]. Satisfaction operators are now integrated. Actually, we have two kinds of such operators: usual ones which can be realized with the aid of the global modality, and new 'guarded jump'–operators coming along with names of sets. For technical reasons we treat here both types of hybrid connectives simultaneously. But in general, the intended application will tell one which language to prefer. The 'guarded jump'–operators seem to be particularly suited to a certain class of intersection closed set spaces we revisit repeatedly in this paper; cf [17].

In the next section we introduce the new hybrid language of set spaces. Afterwards we present an axiomatization, add suitable proof rules and prove completeness. Finally, we argue that the new logic is decidable. All this is done for general set spaces as well as so-called *directed* ones.

2 The language

In this section we define first the syntax and semantics of the two-sorted hybrid language for set spaces indicated above. Then, we give an example proving the expressive power of this language.

We extend the basic bi-modal language for set spaces by two sets of nominals on the one hand and a family of unary modalities on the other hand. The denotation of every nominal is either a unique state or a distinguished set of states, whereas the new modalities represent the global one and the 'guarded jump'–operators belonging to set names, respectively.

[1] We let nominals denote *neighbourhood situations* there; see Definition 1 below.

Let PROP $= \{p, q, \dots\}$, $\mathrm{N}_{stat} = \{i, j, \dots\}$ and $\mathrm{N}_{sets} = \{A, B, \dots\}$ be three mutually disjoint denumerable sets of symbols called *proposition letters, names of states,* and *names of sets,* respectively. Then we define the set WFF of *well-formed formulas* over PROP $\cup \mathrm{N}_{stat} \cup \mathrm{N}_{sets}$ by the rule

$$\alpha \ ::= \ p \mid i \mid A \mid \neg\alpha \mid \alpha \wedge \beta \mid K\alpha \mid \Box\alpha \mid \mathsf{A}\alpha \mid [\epsilon_A]\alpha.$$

The modality $[\epsilon_A]$ is to be read 'in (the set denoted by) A (at the actual state)'. The missing boolean connectives $\top, \bot, \vee, \rightarrow, \leftrightarrow$ are treated as abbreviations, as needed. The duals of K, $\Box$, A and $[\epsilon_A]$ are denoted L, $\Diamond$, E and $\langle\epsilon_A\rangle$, respectively.

We give next meaning to formulas. To begin with, we define the domains where formulas will be interpreted in. We let $\mathcal{P}(X)$ designate the powerset of a given set X.

DEFINITION 1 (Set frames; set spaces with names).

1. A pair $(X, \mathcal{O})$ consisting of a non-empty set X and a set $\mathcal{O} \subseteq \mathcal{P}(X)$ of subsets of X such that $X, \emptyset \in \mathcal{O}$ is called a *set frame.*

2. Let $\mathcal{S} := (X, \mathcal{O})$ be a set frame. The set of *neighbourhood situations* of $\mathcal{S}$ is the set $\mathcal{N}_\mathcal{S} := \{x, U \mid x \in U \text{ and } U \in \mathcal{O}\}$.

3. Let $\mathcal{S}$ be a set frame as above. An *$\mathcal{S}$–valuation* is a mapping

$$V : \mathrm{PROP} \cup \mathrm{N}_{stat} \cup \mathrm{N}_{sets} \longrightarrow \mathcal{P}(X)$$

such that

   (a) $V(i)$ is either $\emptyset$ or a singleton subset of X for every $i \in \mathrm{N}_{stat}$, and

   (b) $V(A) \in \mathcal{O}$ for every $A \in \mathrm{N}_{sets}$.

4. A *set space with names* (or, in short, an *SSN*) is a triple $(X, \mathcal{O}, V)$, where $\mathcal{S} = (X, \mathcal{O})$ is a set frame and V an $\mathcal{S}$–valuation. One says then that $\mathcal{M} := (X, \mathcal{O}, V)$ is *based on* $\mathcal{S}$.

The requirement '$X \in \mathcal{O}$' from item 1 of the above definition shrinks the class of frames we are interested in not very much, but facilitates matters here and there. The condition '$\emptyset \in \mathcal{O}$' takes into account that nominals may have an empty denotation, according to item 3. This is appropriate for our purposes, but not common in standard hybrid logic.[2]

[2]Because of the presence of the global modality we could force the usual interpretation of nominals by the formulas $\mathsf{E}\,i$ and $\mathsf{E}\,A$, respectively. This would make the proceeding a bit more difficult later on, so we prefer the way just taken.

For a given SSN $\mathcal{M}$ we define now the relation of satisfaction, $\models_{\mathcal{M}}$, between neighbourhood situations of the underlying frame and formulas in WFF.

DEFINITION 2 (Satisfaction and validity). Let $\mathcal{M} = (X, \mathcal{O}, V)$ be an SSN based on the set frame $\mathcal{S} = (X, \mathcal{O})$, and let x, U a neighbourhood situation of $\mathcal{S}$. Then

$$\begin{array}{lcl}
x, U \models_{\mathcal{M}} p & :\Longleftrightarrow & x \in V(p) \\
x, U \models_{\mathcal{M}} i & :\Longleftrightarrow & x \in V(i) \\
x, U \models_{\mathcal{M}} A & :\Longleftrightarrow & V(A) = U \\
x, U \models_{\mathcal{M}} \neg\alpha & :\Longleftrightarrow & x, U \not\models_{\mathcal{M}} \alpha \\
x, U \models_{\mathcal{M}} \alpha \wedge \beta & :\Longleftrightarrow & x, U \models_{\mathcal{M}} \alpha \text{ and } x, U \models_{\mathcal{M}} \beta \\
x, U \models_{\mathcal{M}} K\alpha & :\Longleftrightarrow & y, U \models_{\mathcal{M}} \alpha \text{ for all } y \in U \\
x, U \models_{\mathcal{M}} \Box\alpha & :\Longleftrightarrow & \forall\, U' \in \mathcal{O} : (x \in U' \subseteq U \Rightarrow x, U' \models_{\mathcal{M}} \alpha) \\
x, U \models_{\mathcal{M}} \mathsf{A}\alpha & :\Longleftrightarrow & y, U' \models_{\mathcal{M}} \alpha \text{ for all } y, U' \in \mathcal{N}_{\mathcal{S}} \\
x, U \models_{\mathcal{M}} [\epsilon_A]\alpha & :\Longleftrightarrow & \text{if } x \in V(A), \text{ then } x, V(A) \models_{\mathcal{M}} \alpha,
\end{array}$$

for all $p \in$ PROP, $i \in \mathrm{N}_{stat}$, $A \in \mathrm{N}_{sets}$, and $\alpha, \beta \in$ WFF. In case $x, U \models_{\mathcal{M}} \alpha$ is true we say that α *holds in* $\mathcal{M}$ *at* the neighbourhood situation x, U.

A formula α is called *valid in* $\mathcal{M}$ ('$\mathcal{M} \models \alpha$') iff it holds in $\mathcal{M}$ at every neighbourhood situation.

The following remarks seem appropriate to elucidate this definition in some respects.

REMARK 3 (Peculiarities of the just defined language).

1. The meaning of both proposition letters and names of states is independent of neighbourhoods by definition, thus 'stable' with respect to $\Box$. This fact is reflected in two special axioms below (Axioms 6 and 11).

2. The formulas of the form $i \wedge A$, where $i \in \mathrm{N}_{stat}$ and $A \in \mathrm{N}_{sets}$, can be taken as names for elements of $\mathcal{N}_{\mathcal{S}}$. The satisfaction operator associated with such a name reads then $\mathsf{E}(i \wedge A \wedge \dots)$. I.e., pairs (i, A) can act like 'proper' nominals in set spaces with names; cf [10].

3. The last clause of Definition 2 explains what a guarded jump means for set spaces: it is allowed to jump from the neighbourhood U to the neighbourhood $V(A)$ for evaluating α there, provided that the 'guard' x, which is not affected by this jump, is also contained in the latter set. Note that $x, U \models_{\mathcal{M}} \langle\epsilon_A\rangle\alpha$ implies, in particular, that x is contained in the denotation of A.

4. The reader might wonder why a 'guarded jump'–operator belonging to a name $i \in \mathrm{N}_{stat}$ does not appear here. Such an operator is missing for the simple reason that it can be defined, in fact, by $K(i \to \dots)$.

Now, the question comes up naturally whether $[\epsilon_A]$ can be defined as well. But this is not the case for the following reason. One can 'leave' the actual neighbourhood U with the aid of the connective $[\epsilon_A]$ in a special, controlled way. However, leaving U is not at all possible by means of K or $\Box$, and not in that special way by means of A. Thus we have really got more expressiveness.

The example covered by the next definition and proposition points to an important application of the new language.

DEFINITION 4 (Directed frames and spaces). A set frame $\mathcal{S} = (X, \mathcal{O})$ is called *directed,* iff

$$\forall U_1, U_2 \in \mathcal{O}, \forall x \in X : (x \in U_1 \cap U_2 \Rightarrow \exists U \in \mathcal{O} : x \in U \subseteq U_1 \cap U_2)\,.$$

An SSN $\mathcal{M}$ is called *directed,* iff it is based on a directed frame.

PROPOSITION 5 (Expressing directedness). *Let $\mathcal{S} = (X, \mathcal{O})$ be a set frame. Then $\mathcal{S}$ is directed, iff for all SSNs $\mathcal{M}$ based on $\mathcal{S}$ we have that*

$$\mathcal{M} \models \Diamond A \wedge \Diamond B \to \Diamond K\,(\langle \epsilon_A \rangle \top \wedge \langle \epsilon_B \rangle \top) \;\; \textit{for all } A, B \in \mathrm{N}_{sets}.$$

Proof. We prove only that the condition is sufficient for directedness. (The proof of its necessity is even easier.) So, let $\mathcal{S}$ be not directed. Then there are $U_1, U_2 \in \mathcal{O}$ and $x \in X$ such that $x \in U_1 \cap U_2$, but $x \notin U$ or $U \not\subseteq U_1 \cap U_2$ for all $U \in \mathcal{O}$. Take any $\mathcal{S}$–valuation V satisfying $V(A) = U_1$ and $V(B) = U_2$, and let $\mathcal{M} := (X, \mathcal{O}, V)$. Then, $x, X \models_{\mathcal{M}} \Diamond A \wedge \Diamond B$. However, we have that $x, X \models_{\mathcal{M}} \Box L\,([\epsilon_A]\bot \vee [\epsilon_B]\bot)$ because we can find a point witnessing $U \not\subseteq V(A) \cap V(B)$ in every $U \in \mathcal{O}$ containing x. In this way we have found an SSN $\mathcal{M}$ based on $\mathcal{S}$ and an instance of the above formula schema which is not valid in $\mathcal{M}$, as desired. ■

Directed frames correspond to the topological notion of *filter base;* cf [5], Ch. I, § 6.3. Filters are crucial to the general idea of *convergence;* cf [5], Ch. I, § 7. As set spaces represent a model for the development of knowledge, directed spaces are, therefore, associated with 'converging' knowledge acquisition procedures. The *modal* logic of knowledge arising from that was studied in the paper [17]. We will come back to this system later on, at the end of the next section (and in Section 4 as well).[3]

[3] Originally, a good portion of work was devoted to the axiomatizability problem for the logic of *intersection closed spaces;* cf [6]. Later on, in [17], it was announced that this logic and the logic of directed spaces coincide.

Concluding this section, we comment on the relevance of names and accompanying hybrid operators to the context of knowledge. We confine ourselves to names of sets here since the general usefulness of names of states has been sufficiently demonstrated elsewhere; cf, above all, [1]. As mentioned in the introduction already, the elements of $\mathcal{O}$ can be viewed as knowledge states of an agent, for any correspondingly given set frame $(X, \mathcal{O})$. Thus the new language supplies one with *names of knowledge states*. And the hybrid operators $[\epsilon_A]$, where $A \in \mathrm{N}_{sets}$, allow to switch over to named knowledge states as long as the point of evaluation is fixed. This additional means of expression is, therefore, perfectly in accord with the common 'external' view of knowledge in multi-agent systems, where knowledge is 'ascribed' to the agents; cf [7], Ch. 4.

3 Completeness

In this section we study a hybrid logic for the class of all set spaces with names. To start with, we propose an appropriate system of axioms. Then we formulate a couple of Gabbay–style proof rules. These and the usual ones of modal and hybrid logic lead to a logical system called GJ (indicating **g**uarded **j**umps). The completeness proof for GJ is contained in the remaining part of this section. It turns out that even *extended* completeness (in the sense of hybrid logic, cf [2], Sec. 7.3) is yielded.

The axiom schemata are arranged in four groups. First, the modal axioms for arbitrary set spaces are listed.

1. All instances of tautologies.
2. $K(\alpha \to \beta) \to (K\alpha \to K\beta)$
3. $K\alpha \to \alpha$
4. $K\alpha \to KK\alpha$
5. $L\alpha \to KL\alpha$
6. $(p \to \Box p) \wedge (\Diamond p \to p)$
7. $\Box(\alpha \to \beta) \to (\Box\alpha \to \Box\beta)$
8. $\Box\alpha \to \alpha$
9. $\Box\alpha \to \Box\Box\alpha$
10. $K\Box\alpha \to \Box K\alpha$,

where $p \in$ PROP and $\alpha, \beta \in$ WFF. Note that the schemata $2-5$ represent the usual S5 axioms of knowledge, 6 captures the aforementioned stability of the proposition letters with respect to $\Box$, $7-9$ say that $\Box$ satisfies all the S4 laws, and 10 is the *Cross Axiom* from [6];[4] see also the comments on Definition 13 below.

The second group of axioms concerns names.

11. $(i \to \Box i) \wedge (\Diamond i \to i)$
12. $i \wedge \alpha \to K(i \to \alpha)$
13. $A \to KA$
14. $\mathsf{A}(A \wedge L\alpha \to L\beta) \vee \mathsf{A}(A \wedge L\beta \to L\alpha)$
15. $K(\Diamond B \to \Diamond A) \wedge L\Diamond B \to \Box(A \to L\Diamond B)$
16. $K\Diamond A \to A$,

[4] This schema plays a subtle part in multi-agent systems where agents have *perfect recall,* i.e., do not forget anything of their own history in the course of time; see [7], notes to Ch. 8, for a more detailed discussion.

where $i \in \mathrm{N}_{stat}$, $A, B \in \mathrm{N}_{sets}$ and $\alpha, \beta \in \mathrm{WFF}$. We mention that Axioms 11 and 12 are needed to force the right interpretation of names of states in the model constructed during the proof of completeness later on, and Axioms 13 and 14 serve the same purpose regarding names of sets. The remaining two axioms ensure that really a set frame structure can be defined for that model.[5] — Actually, a further axiom can be included in this group.

$$17.\ i \wedge A \wedge \mathsf{E}(j \wedge B) \rightarrow \mathsf{E}\left(\Diamond(i \wedge A) \wedge L\Diamond(j \wedge B)\right),$$

where $i, j \in \mathrm{N}_{stat}$ and $A, B \in \mathrm{N}_{sets}$. This axiom corresponds to a certain *left-directedness* property; see Lemma 9 below. Moreover, it reflects that the set frames $(X, \mathcal{O})$ we consider are *generated* in a sense. In fact, every neighbourhood situation of the form x, X (where $x \in X$) can be viewed as a generating element, according to both the clause '$X \in \mathcal{O}$' from item 1 of Definition 1 and the semantics of K and $\Box$; see also item 5 of Definition 13 below.[6]

Each axiom of the next group deals with the global modality.

18. $\mathsf{A}(\alpha \rightarrow \beta) \rightarrow (\mathsf{A}\alpha \rightarrow \mathsf{A}\beta)$
19. $\mathsf{A}\alpha \rightarrow \alpha$
20. $\mathsf{A}\alpha \rightarrow \mathsf{A}\mathsf{A}\alpha$
21. $\alpha \rightarrow \mathsf{A}\mathsf{E}\alpha$,
22. $\mathsf{A}\alpha \rightarrow K\Box\alpha$

where $\alpha, \beta \in \mathrm{WFF}$. Here Axioms 18 – 21 mean that A is an S5 modality (like K), and Axiom 22 describes the obvious interplay between A and $K\Box$.

By the last group, the 'guarded jump'–operators are handled.

23. $[\epsilon_A](\alpha \rightarrow \beta) \rightarrow ([\epsilon_A]\alpha \rightarrow [\epsilon_A]\beta)$
24. $\Diamond(A \wedge \alpha) \rightarrow \Box\langle\epsilon_A\rangle\alpha$
25. $[\epsilon_A]A$
26. $\Diamond\langle\epsilon_A\rangle\alpha \wedge L\Diamond A \rightarrow \Diamond\alpha$
27. $\mathsf{A}\alpha \rightarrow [\epsilon_A]\alpha$,

where $A \in \mathrm{N}_{sets}$ and $\alpha, \beta \in \mathrm{WFF}$. As to the properties corresponding to Axioms 24 and 25, see item 8 of Definition 13. Axiom 26 requires an extra investigation later on. Finally, Axiom 27 expresses inclusion (like Axiom 22). – Now we define the desired logical system GJ.

DEFINITION 6 (The logic). Let GJ be the smallest set of formulas containing all of the above axiom schemata and closed under application of the following rules:

$$(\text{MODUS PONENS})\ \frac{\alpha \rightarrow \beta, \alpha}{\beta} \qquad (\Delta\text{–NECESSITATION})\ \frac{\alpha}{\Delta\alpha}$$

$$(\text{NAME}_{stat})\ \frac{j \rightarrow \beta}{\beta} \qquad (\text{NAME}_{sets})\ \frac{B \rightarrow \beta}{\beta}$$

[5] Note that, apart from Axiom 14 (which had to be reformulated since the global modality belongs now to the language) the axioms of this group were also used for proving completeness of the satisfaction operator-free fragment of our logic in the paper [11].

[6] I am grateful to Valentin Goranko who pointed an error concerning generation in the preliminary proceedings version of this paper out to me.

$$(\Delta\text{–ENRICHMENT}) \quad \frac{\mathsf{E}\,(i \wedge A \wedge \nabla(j \wedge B \wedge \alpha)) \to \beta}{\mathsf{E}(i \wedge A \wedge \nabla\alpha) \to \beta},$$

where $\alpha, \beta \in \mathrm{WFF}$, $i, j \in \mathrm{N}_{stat}$, $A, B \in \mathrm{N}_{sets}$, $\Delta \in \{K, \Box, \mathsf{A}\} \cup \{[\epsilon_A] \mid A \in \mathrm{N}_{sets}\}$, $\nabla \in \{L, \Diamond, \mathsf{E}\} \cup \{\langle\epsilon_A\rangle \mid A \in \mathrm{N}_{sets}\}$, and j, B are *new* each time (i.e., do not occur in all the other syntactic building blocks of the respective rule).

The reader can easily check that GJ is sound with respect to the class of all SSNs.

In order to obtain completeness we proceed via the canonical model of GJ. To begin with, we call a maximal consistent set s of formulas

- *named* iff s contains some $i \in \mathrm{N}_{stat}$ and some $A \in \mathrm{N}_{sets}$, and
- *enriched* iff for every $\nabla \in \{L, \Diamond, \mathsf{E}\} \cup \{\langle\epsilon_A\rangle \mid A \in \mathrm{N}_{sets}\}$ we have that

  $$\mathsf{E}(i \wedge A \wedge \nabla\alpha) \in s \text{ implies } \mathsf{E}\,(i \wedge A \wedge \nabla(j \wedge B \wedge \alpha)) \in s$$

  for some $j \in \mathrm{N}_{stat}$ and $B \in \mathrm{N}_{sets}$.

Now we take a maximal consistent set Γ containing the negation γ of a given non-derivable formula, and we extend Γ to a named and enriched maximal consistent set Γ' in the language increased by enough new constants. This can be done by means of a suitable *Extended Lindenbaum Lemma* in the same way as in usual hybrid logic; cf [2], 7.25. (So, we may abstain from the details concerning this.)

Furthermore, we let $\widetilde{\mathcal{M}}$ be the submodel of the structure *generated* by Γ' of which the domain D consists of all points that are named. The accessibility relations induced on $\widetilde{\mathcal{M}}$ are denoted $\xrightarrow{\nabla}$, where $\nabla \in \{L, \Diamond, \mathsf{E}\} \cup \{\langle\epsilon_A\rangle \mid A \in \mathrm{N}_{sets}\}$. Then, we get the following *Existence Lemma;* cf [2], 7.27.

LEMMA 7. *Assume that $s \in D$ contains the formula $\nabla\alpha$ (where ∇ is as above). Then there exists some $t \in D$ satisfying $s \xrightarrow{\nabla} t$ and $\alpha \in t$.*

Proof. First, one realizes that, because of Axioms 22 and 27, the domain of the structure generated by Γ' coincides with the $\xrightarrow{\mathsf{E}}$–equivalence class of Γ'. Second, Axioms 14 and 12 imply that, given $A \in \mathrm{N}_{sets}$ and $i \in \mathrm{N}_{stat}$, then there is at most one point in D containing the formula $i \wedge A$. (The arguments for that are similar to those from the proof of [11], Lemma 2. So, we may be brief here.) Now the assertion of the lemma follows easily from the fact that Γ' is enriched. ■

We have to provide for a set frame structure next. It turns out that we cannot guarantee that $\widetilde{\mathcal{M}}$ is based on such a frame, but we will clearly exploit the properties of the accessibility relations of this model. For every $s \in D$ we let $[s] := \{t \in D \mid s \xrightarrow{L} t\}$ be the $\xrightarrow{L}$–equivalence class of s (see Axioms 3 – 5), and we define a binary relation $\preccurlyeq$ on the set $\mathcal{Q} := \{[s] \mid s \in D\}$ of all such classes by

$$[s] \preccurlyeq [t] :\Longleftrightarrow \text{ there are } s' \in [s],\, t' \in [t] \text{ satisfying } s' \xrightarrow{\Diamond} t',$$

for all $s, t \in D$. Then we have the following lemma.

LEMMA 8 (Properties of $\preccurlyeq$).

1. *The relation $\preccurlyeq$ is reflexive and transitive.*
2. *If $[s] \preccurlyeq [t]$, then every $s' \in [s]$ has at most one $\xrightarrow{\Diamond}$–successor in $[t]$.*
3. *If $[s] \preccurlyeq [t]$, then no two distinct points $s', s'' \in [s]$ have a common $\xrightarrow{\Diamond}$–successor in $[t]$.*
4. *The relation $\preccurlyeq$ is antisymmetrical.*

Proof. This is most of Lemma 3.6 from [13]. The proofs of item 1 are the same, and the proofs of item 4 are very similar here and there. Actually, Axiom 16 is applied now instead of Axiom 21 from the paper just referred to. As to the proof of items 2 and 3, see [11], Lemma 3. Note that more axioms from the first and the second group above are needed to establish the lemma. In particular, Axiom 10 and the so-called *Cross Property* derived from it,

$$s \xrightarrow{\Diamond} t \xrightarrow{L} u \Rightarrow \exists v : s \xrightarrow{L} v \xrightarrow{\Diamond} u,$$

play a crucial part.[7] We omit further details. ■

The lemma says, in particular, that $(\mathcal{Q}, \preccurlyeq)$ is a partial order. There is an additional property of $\preccurlyeq$ which is important for our purposes.

LEMMA 9. *The relation $\preccurlyeq$ is* left-directed, *i.e., for all $[s], [t] \in \mathcal{Q}$ there exists $[u] \in \mathcal{Q}$ such that $[u] \preccurlyeq [s]$ and $[u] \preccurlyeq [t]$.*

Proof. This is an easy consequence of Axiom 17, due to the fact that we are working in a named model. ■

[7](A variant of) this property is used decisively in the proof of completeness of many logics related to *topologic*.

Now suppose that $[s] \preccurlyeq [t]$. Lemma 8.2 implies that $\stackrel{\diamond}{\longrightarrow}$ induces an injective partial function $h_{[t]}^{[s]} : [s] \longrightarrow [t]$. With the aid of all such functions we construct now a model realizing γ. To this end, let X be the set of all partial functions $f : \mathcal{Q} \longrightarrow D$ of which the domain $\mathrm{dom}(f)$ is a maximal subset of $\mathcal{Q}$ satisfying the following conditions:

1. $f([s]) \in [s]$ for all $[s] \in \mathrm{dom}(f)$, and
2. for all $[s], [t] \in \mathrm{dom}(f)$ such that $[s] \preccurlyeq [t]$ it holds that $f([t]) = h_{[t]}^{[s]}([s])$.

We write $f_s := f([s])$ in case f([s]) exists. And we define

- $U_{[s]} := \{f \in X \mid f_s \text{ exists}\}$, for all $s \in D$,
- $\mathcal{O} := \{U_{[s]} \mid s \in D\} \cup \{X, \emptyset\}$, and
- $V : \mathrm{PROP} \cup \mathrm{N}_{stat} \cup \mathrm{N}_{sets} \longrightarrow \mathcal{P}(X)$ by

$$f \in V(c) \iff c \in f_s \text{ for some } s \in D \text{ such that } f_s \text{ exists,}$$

  for all $c \in \mathrm{PROP} \cup \mathrm{N}_{stat} \cup \mathrm{N}_{sets}$.

With that we obtain the following *Truth Lemma*.

LEMMA 10. *The structure $\mathcal{M} := (X, \mathcal{O}, V)$ is an SSN. Moreover, for all formulas α, functions $f \in X$, and points $s \in D$ such that $f \in U_{[s]}$, we have that*

$$(*) \quad f, U_{[s]} \models_{\mathcal{M}} \alpha \iff \alpha \in f_s.$$

Proof. First note that $X \neq \emptyset$ holds because of Zorn's Lemma. Second, V is in fact well-defined. This is due to Axioms 6, 11, 13, and Lemma 9 (among other things). Third, the property

$$U_{[t]} \subseteq U_{[s]} \text{ implies } [s] \preccurlyeq [t]$$

can be proved by means of Axiom 15 and Lemma 9; cf [11], Proposition 4. This property and Lemma 8.4 are needed for the proof of the assertion $(*)$. In fact, $(*)$ can be established now for the basic cases and all the inductive steps apart from $\alpha = [\epsilon_A]\beta$. As the arguments are similar to those from [11], proof of Lemma 5, the details are omitted here.

Finally, we treat the remaining case. We assume first that $f, U_{[s]} \models_{\mathcal{M}} [\epsilon_A]\beta$. This means that whenever $f \in V(A)$, then $f, V(A) \models_{\mathcal{M}} \beta$. We must prove that $[\epsilon_A]\beta \in f_s$ holds in this case. So, take any $t \in D$ such that $f_s \stackrel{\langle \epsilon_A \rangle}{\longrightarrow} t$. It suffices to show that $\beta \in t$. To this end we state first that we have $A \in t$, because of Axiom 25. Next, there are $u, v \in D$ such that $u \xrightarrow{\Diamond} f_s$ and $u \xrightarrow{L} v \xrightarrow{\Diamond} t$. This is true because of Lemma 9. Now Axiom 26 gives us $u \xrightarrow{\Diamond} t$. That is, f_t exists, or, in other words, $f \in V(A)$. Consequently, $f, V(A) \models_{\mathcal{M}} \beta$. Because of $V(A) = U_{[t]}$, the induction hypothesis tells us that $\beta \in f_t = t$. This is what we wanted to show.

Conversely, assume that $f, U_{[s]} \not\models_{\mathcal{M}} [\epsilon_A]\beta$. From this we conclude that

$$f \in V(A) \text{ and } f, V(A) \not\models_{\mathcal{M}} \beta.$$

Let $f_{V(A)}$ be the (uniquely determined) component of f containing A. Due to the induction hypothesis, $\beta \notin f_{V(A)}$. Thus if we can prove that $f_s \xrightarrow{\langle \epsilon_A \rangle} f_{V(A)}$, then we are done. But, according to the definition of (the elements of) X there is some $u \in D$ such that $u \xrightarrow{\Diamond} f_s$ and $u \xrightarrow{\Diamond} f_{V(A)}$. Now axiom 24 can be applied, yielding $f_s \xrightarrow{\langle \epsilon_A \rangle} f_{V(A)}$, as desired. ■

¿From the *Truth Lemma* we get the desired Completeness Theorem in the standard manner.

THEOREM 11 (Completeness). *Every formula* $\alpha \in$ WFF *that is valid in all SSNs is contained in the logic* GJ.

And we obtain even more. The fact that we are dealing with *named* objects enables us to extend the completeness proof to other classes of set spaces, by adding to the above list of axioms a suitable defining correspondent. For example, we can prove the following theorem in this way.

THEOREM 12 (Extended completeness). *Let* D *be the formula schema from Proposition 5.*[8] *Then the system* GJ + D *is sound and complete with respect to the class of all directed SSNs.*

Proof. We construct the model $\mathcal{M} = (X, \mathcal{O}, V)$ as in the proof of the previous theorem, but for the extended logic. Our aim is to show that $\mathcal{M}$ is directed. For every $f \in X$ and $U \in \mathcal{O}$ containing f we let f_U denote the unique value of f contained in U. Now assume that $f \in U_1 \cap U_2$, and let $U_1 = V(A)$ and $U_2 = V(B)$. Then, $f, X \models_{\mathcal{M}} \Diamond A \wedge \Diamond B$. Consequently, $f, X \models_{\mathcal{M}} \Diamond K(\langle \epsilon_A \rangle \top \wedge \langle \epsilon_A \rangle \top)$, according to the validity of the schema D. It follows that there is some $U \in \mathcal{O}$ such that $f, U \models_{\mathcal{M}} K(\langle \epsilon_A \rangle \top \wedge \langle \epsilon_A \rangle \top)$; note that, in particular, $f \in U$ holds. Now take any $g \in U$. Then, $g, U \models_{\mathcal{M}} \langle \epsilon_A \rangle \top \wedge \langle \epsilon_A \rangle \top$. This means that there are $u, v \in D$ such that $g_U \xrightarrow{\langle \epsilon_A \rangle} u$ and $g_U \xrightarrow{\langle \epsilon_B \rangle} v$. As $g_X \xrightarrow{L} f_X$ and $f \in V(A)$ (and $\in V(B)$, respectively), Axiom

[8] The reader will not confuse this with the schema D from standard modal logic.

26 implies $g_X \xrightarrow{\Diamond} u$ (and $g_X \xrightarrow{\Diamond} v$, respectively). Thus $g \in V(A) = U_1$ and $g \in V(B) = U_2$. This proves $U \subseteq U_1 \cap U_2$, i.e., $\mathcal{M}$ is directed. ∎

A corresponding result holds for quite a lot of different classes of set spaces.

In view of the issues of the paper [17], Theorem 12 is somewhat surprising at first glance. It was proved there that the modal theory of directed set spaces is *not* finitely axiomatizable. As we have just shown it *is* in the hybrid setting, actually;[9] see also the end of the following section.

4 Decidability

Subsequently it is shown that GJ is a decidable set of formulas. In a preparatory step, we consider a certain class of auxiliary Kripke structures which are a bit involved. These models are supposed to realize as many of the above axioms as possible.

DEFINITION 13 (GJ–models). A tupel

$$\mathfrak{M} := \left(W, \xrightarrow{L}, \xrightarrow{\Diamond}, \xrightarrow{\mathsf{E}}, \{\xrightarrow{\langle \epsilon_A \rangle}\}_{A \in \mathrm{N}_{sets}}, V \right)$$

is called a GJ–*model*, iff the following conditions are satisfied:

1. W is a non-empty set,
2. the relation $\xrightarrow{L} \subseteq W \times W$ (belonging to K) is an equivalence,
3. the relation $\xrightarrow{\Diamond} \subseteq W \times W$ (belonging to $\Box$) is reflexive and transitive,
4. for all $u, v, w \in W$ such that $u \xrightarrow{\Diamond} v \xrightarrow{L} w$ there exists $t \in W$ such that $u \xrightarrow{L} t \xrightarrow{\Diamond} w$,
5. there is some $u_0 \in W$ such that $W = \{v \mid (u_0, v) \in (\xrightarrow{L} \cup \xrightarrow{\Diamond})^*\}$ (i.e., in particular, W is *generated* by $\xrightarrow{L} \cup \xrightarrow{\Diamond}$),
6. the relation $\xrightarrow{\mathsf{E}} \subseteq W \times W$ (belonging to A) is universal,
7. $V : \mathrm{PROP} \cup \mathrm{N}_{stat} \cup \mathrm{N}_{sets} \longrightarrow \mathcal{P}(W)$ is a mapping satisfying

[9] The investigation of the part that unorthodox proof rules like the above ENRICHMENT rule play for hybrid completeness and, in particular, to what extent such rules can be taken as substitutes for infinite axiomatizations, is an actual field of research. Regarding this, cf, eg, [4].

(a) for all $c \in \mathrm{PROP} \cup \mathrm{N}_{stat}$ and $u, v \in W$: if $u \xrightarrow{\Diamond} v$, then $u \in V(c) \iff v \in V(c)$,

(b) for all $i \in \mathrm{N}_{stat}$ and $u \in W$ there is at most one $v \in W$ such that $u \xrightarrow{L} v$ and $v \in V(i)$,

(c) for all $A \in \mathrm{N}_{sets}$, the set $V(A)$ equals either $\emptyset$ or a unique $\xrightarrow{L}$–equivalence class,

8. for every $A \in \mathrm{N}_{sets}$, the relation $\xrightarrow{\langle \epsilon_A \rangle}$ (belonging to $[\epsilon_A]$) satisfies

(a) for all $u, v \in W$: if $u \xrightarrow{\langle \epsilon_A \rangle} v$, then $v \in V(A)$, and

(b) for all $u \in W$: if there exists some $v \in V(A)$ such that $u \xrightarrow{\Diamond} v$, then $\forall w \in W : (u \xrightarrow{\Diamond} w \Rightarrow w \xrightarrow{\langle \epsilon_A \rangle} v)$.

We give a couple of comments on this definition. First, note that the fourth item above is the aforementioned Cross Property (see the proof of Lemma 8 above), which can really be taken as a certain diagram property of the relations $\xrightarrow{\Diamond}$ and $\xrightarrow{L}$. Second, item 7 suits the notion of $\mathcal{S}$–valuation from Definition 1 to GJ–models. And finally, a certain diagram property is imposed on the relations $\xrightarrow{\langle \epsilon_A \rangle}$ and $\xrightarrow{\Diamond}$, too, by (b) of the last item. As this property is occasionally referred to below, we call it the *Property (b)*.

GJ–models are closely related to SSNs. In fact, by taking neighbourhood situations as points and defining then the accessibility relations and the valuation in a way suggesting itself (in accordance with the proceeding for *topologic,* cf [6], Section 2.3), every SSN induces a GJ–model. Therefore, it is natural to try GJ–models in order to establish the finite model property for GJ.[10] Unfortunately, not every GJ–model validates necessarily all of the above axioms. But, for a start, we get at least the following partial result.

PROPOSITION 14. *Apart from, possibly, Axioms 15,16, and 26, each of the above schemata is valid in every* GJ*–model.*

Proof. The proof is rather straightforward; see [12], proof of Proposition 2, for a few details. ■

We introduce two ad hoc notations, by calling a GJ–model $\mathfrak{M}$ *(almost) faithful,* iff *all* the axioms (*except 26*) are valid in $\mathfrak{M}$. We want to show now that GJ satisfies the *Strong Finite Model Property* (cf [2], Def. 6.6) with

[10] Note that already *topologic* does *not* satisfy the fmp with respect to the class of all *set spaces;* cf [6]. Thus we cannot use much of our work from Section 3 here.

respect to the class of all faithful GJ–models. Then, the desired decidability of GJ will easily arise.

To establish this finite model property we use the method of filtration, followed by an appropriate model surgery. So, let $\alpha \in \mathrm{WFF}$ be a consistent formula for which we want to find a model of size at most $f(|\alpha|)$, where f is some computable function and $|\alpha|$ denotes the length of α. Moreover, let $\mathrm{sf}(\alpha)$ be the set of all subformulas of α. We construct a suitable filter set Σ via the following sets of formulas. We first let

$$\Sigma_0 := \begin{cases} \mathrm{sf}(\alpha) \cup \{\Box\neg A \mid A \in \mathrm{N}_{sets} \text{ occurs in } \alpha\} \\ \cup\, \{[\epsilon_A]A, [\epsilon_A]\neg A \mid A \in \mathrm{N}_{sets} \text{ occurs in } \alpha\} \\ \cup\, \{\Box\neg(A \wedge \neg\beta) \mid [\epsilon_A]\beta \in \mathrm{sf}(\alpha)\} \\ \cup\, \{\Box\neg[\epsilon_A]\neg\beta \mid A \text{ occurs in } \alpha \text{ and } \beta \in \mathrm{sf}(\alpha)\} \\ \cup\, \{[\epsilon_A]\neg\beta \mid [\epsilon_A]\beta \in \mathrm{sf}(\alpha) \text{ or } \ (A \text{ occurs in } \alpha \text{ and } \beta \in \mathrm{sf}(\alpha))\}, \end{cases}$$

and secondly $\Sigma^{\neg} := \{\neg\beta \mid \beta \in \Sigma_0\}$.[11] Then we take the set Σ' of all finite conjunctions of pairwise distinct elements of $\Sigma_0 \cup \Sigma^{\neg}$. Afterwards we close Σ' under single applications of the operator L and take, finally, the set of all subformulas of the resulting set. Let Σ be the union of all these intermediate sets of formulas. Then Σ is subformula closed, and $2^{c\cdot|\alpha|}$ is an upper bound of the cardinality of Σ (for some constant c).

Let $\mathcal{C}$ be the submodel of the canonical model of GJ generated by a maximal consistent set realizing α. Moreover, let the Kripke model

$$\mathfrak{M} := \left(W, \xrightarrow{L}, \xrightarrow{\diamond}, \xrightarrow{\mathsf{E}}, \{\xrightarrow{\langle\epsilon_A\rangle}\}_{A\in \mathrm{N}_{sets}}, V\right)$$

be obtained from $\mathcal{C}$ as follows:

- W is the filtration of the carrier set of $\mathcal{C}$ with respect to Σ,
- $\xrightarrow{L}$, $\xrightarrow{\diamond}$ and $\xrightarrow{\mathsf{E}}$ are the *smallest* filtrations, and $\xrightarrow{\langle\epsilon_A\rangle}$ is the *largest* filtration of the accessibility relations of $\mathcal{C}$ belonging to the respective modalities (where $A \in \mathrm{N}_{sets}$), and
- V is induced by the canonical valuation.

Then, exploiting the structure of the filter set Σ and the fixings for the filtrations of the relations we get two crucial propositions, of which the first one reads as follows.

PROPOSITION 15. *By modifying some of the relations $\xrightarrow{\langle\epsilon_A\rangle}$ and the valuation V suitably, the just defined model $\mathfrak{M}$ can be turned into an almost*

[11] The part the individual sets forming Σ_0 play, becomes apparent later on.

faithful GJ*–model* $\mathfrak{M}_1$ *which is semantically equivalent to* $\mathfrak{M}$ *with respect to* α.

Proof. We let $\mathfrak{M}_1$ be the model differing from $\mathfrak{M}$ in that the denotation of every element of PROP $\cup$ N$_{stat}$ $\cup$ N$_{sets}$ not occurring in α is the empty set and the corresponding relations $\xrightarrow{\langle \epsilon_A \rangle}$ are empty as well. In this way, a structure results which satisfies items 1 – 7 of Definition 13 as well as Axioms 15 and 16. This was detailedly proved in the paper [12] (proof of Proposition 3). Note that the first subset of Σ_0 is relevant to that.

We prove now that the conditions from item 8 of Definition 13 are satisfied for $\mathfrak{M}_1$. To this end, we may assume that A actually occurs in α, for otherwise the desired properties hold trivially.

(a) Let $u, v \in W$ satisfy $u \xrightarrow{\langle \epsilon_A \rangle} v$, and let $\hat{u}, \hat{v}$ be any representatives of u, v, respectively, contained in $\mathcal{C}$. Then we have that the implication $\mathcal{C}, \hat{u} \models [\epsilon_A]\beta \Rightarrow \mathcal{C}, \hat{v} \models \beta$ holds for all formulas $[\epsilon_A]\beta \in \Sigma$, due to the definition of filtration. Now, $[\epsilon_A]A$ is in Σ, and $\mathcal{C}, \hat{u} \models [\epsilon_A]A$ is in fact true because of Axiom 25. It follows that $\mathcal{C}, \hat{v} \models A$. This implies $\mathfrak{M}_1, v \models A$, i.e., $v \in V(A)$.

(b) Let $u, v, \hat{u}, \hat{v}$ be as above, and let $v \in V(A)$ and $u \xrightarrow{\Diamond} v$ be satisfied. Furthermore, take any $w \in W$ such that $u \xrightarrow{\Diamond} w$. We must show that $w \xrightarrow{\langle \epsilon_A \rangle} v$. For that it suffices to prove that $\mathcal{C}, \hat{w} \models [\epsilon_A]\beta \Rightarrow \mathcal{C}, \hat{v} \models \beta$ for all formulas $[\epsilon_A]\beta \in \Sigma$, where $\hat{w}$ is any representative of w; this is true, actually, because of the fact that we are dealing with the largest filtration. So, let $\mathcal{C}, \hat{w} \models [\epsilon_A]\beta$ be valid. Then, also $\mathfrak{M}_1, w \models [\epsilon_A]\beta$ holds. Since $\xrightarrow{\Diamond}$ is a smallest filtration and $[\epsilon_A]\beta \in \Sigma$, there are representatives $\check{u}, \check{w}$ of u, w, respectively, such that both $\check{w}$ is accessible from $\check{u}$ with respect to the modality $\Box$ and $\Diamond[\epsilon_A]\beta \in \check{u}$. With the aid of Axiom 24 we conclude $\Box\neg(A \wedge \neg\beta) \in \check{u}$ from that. As this formula is contained in Σ, too, we get that $\mathfrak{M}_1, u \models \Box\neg(A \wedge \neg\beta)$. Now $\mathfrak{M}_1, v \models \beta$ follows, which implies $\mathcal{C}, \hat{v} \models \beta$, as desired.

Thus we have shown that $\mathfrak{M}_1$ is an almost faithful GJ–model. The second assertion of the proposition is obvious since we have only changed the components of $\mathfrak{M}$ by which the semantics of α is certainly not affected. ■

Another diagram property, corresponding to a weak version of Axiom 26, is established for the relations $\xrightarrow{\langle \epsilon_A \rangle}$ and $\xrightarrow{\Diamond}$ by the next lemma.

LEMMA 16. *For all* $u, v, w, x, y \in W$ *satisfying*

$$u \xrightarrow{\Diamond} v \xrightarrow{\langle \epsilon_A \rangle} w,\ u \xrightarrow{L} x \xrightarrow{\Diamond} y,\ \textit{and } y \in V(A),$$

there exists some $z \in V(A)$ *such that* $u \xrightarrow{\Diamond} z$.

Proof. Let $u, v, w, x, y \in W$ satisfy $u \xrightarrow{\Diamond} v \xrightarrow{\langle \epsilon_A \rangle} w$, $u \xrightarrow{L} x \xrightarrow{\Diamond} y$ and $y \in V(A)$. Again, we may assume that A occurs in α. In order to prove that $\exists z \in V(A) : u \xrightarrow{\Diamond} z$ holds, it suffices to show that $\Diamond A \in \hat{u}$ for some representative $\hat{u}$ of u since $\Diamond A \in \Sigma$. For that, we will use Axiom 26 with $\alpha = A$. First, we infer $\Diamond A \in \check{x}$, where $\check{x}$ is an arbitrary representative of x, from $y \in V(A)$ and $\Diamond A \in \Sigma$. Second, it is a peculiarity of our filtration that for *every* representative $\check{u}$ of u there is some representative of x which is accessible from $\check{u}$ with respect to K.[12] Thus we obtain $L\Diamond A \in \check{u}$ for all $\check{u}$. Third, we have that $\mathfrak{M}_1, w \models A$ because $\mathfrak{M}_1, v \models [\epsilon_A]A$. Thus even $\mathfrak{M}_1, v \models \langle \epsilon_A \rangle A$ is valid. Since this formula is contained in Σ, too, we infer (similarly as above) that $\Diamond \langle \epsilon_A \rangle A \in \hat{u}$ holds for some representative $\hat{u}$ of u. Now Axiom 26 can be applied, yielding $\Diamond A \in \hat{u}$, as desired. ∎

The following very special property of the model $\mathfrak{M}$ (and $\mathfrak{M}_1$, respectively) turns out to be useful for later purposes.

LEMMA 17. *Let* $A \in \mathrm{N}_{sets}$ *occur in* α, *and let* $u, v, w, x, y \in W$ *satisfy* $u \xrightarrow{\Diamond} w$, $u \xrightarrow{\Diamond} x$, $v \xrightarrow{\Diamond} w$, $v \xrightarrow{\Diamond} y$, $w \xrightarrow{\langle \epsilon_A \rangle} x$, $w \xrightarrow{\langle \epsilon_A \rangle} y$, *and* $x, y \in V(A)$. *Then,* x *and* y *cannot be distinguished by some subformula of* α.

Proof. Let $\beta \in \mathrm{sf}(\alpha)$ and assume that $\mathfrak{M}, x \models \beta$. Then, $\mathfrak{M}, w \models \langle \epsilon_A \rangle \beta$. Furthermore, let $\hat{v}, \check{v}, \hat{y}, \hat{w}$ be representatives of v, y, w, respectively, such that $\hat{v} \xrightarrow{\Diamond} \hat{w}$ and $\check{v} \xrightarrow{\Diamond} \hat{y}$. Then $\Diamond \langle \epsilon_A \rangle \beta$ is contained in $\hat{v}$ because $\langle \epsilon_A \rangle \beta$ is contained in $\hat{w}$ (see the definition of Σ). Since $y \in V(A)$, we have also that $\Diamond A \in \hat{v}$.

In the following we apply a couple of semantic arguments. Since

$$\Diamond \langle \epsilon_A \rangle \beta \wedge \Diamond A \rightarrow \Diamond (A \wedge \beta)$$

represents a validity, we conclude from Theorem 11 that this formula, and hence its right-hand side, too, is contained in $\hat{v}$. According to Axiom 24 it follows that $\Box \langle \epsilon_A \rangle \beta \in \hat{v}$. As this formula is also contained in Σ (see the definition of Σ_0), we infer $\langle \epsilon_A \rangle \beta \in \hat{y}$ from that. Now, $\langle \epsilon_A \rangle \beta \wedge A \rightarrow \beta$ is a validity, too.[13] Thus we obtain that β is contained in $\hat{y}$. That is, $\mathfrak{M}, y \models \beta$.

We may clearly reverse the roles of x and y (and of v and u, respectively). Thus the assertion of the lemma follows for reasons of symmetry. ∎

Now we are in a position to prove the second of our crucial propositions.

[12] This property was proved, eg, in [6], Sec. 2.3.

[13] Note that this is the *elimination* formula of usual hybrid logic; see [2], p 438.

PROPOSITION 18. *The model* $\mathfrak{M}_1$ *can be turned into a faithful* GJ*–model* $\mathfrak{M}'$ *which is semantically equivalent to* $\mathfrak{M}_1$ *with respect to* α.

Proof. We subject $\mathfrak{M}_1$ to two changes. First, all the points that are both contained in $V(A)$ and indistinguishable by α in the sense of Lemma 17, are identified; moreover, all the arrows that already exist are kept (and no new ones are introduced). The reader can easily convince himself or herself that an almost faithful GJ–model $\mathfrak{M}_2$ results from that which is semantically equivalent to $\mathfrak{M}_1$ with respect to α.

Second, we leave out some of the $\xrightarrow{\langle\epsilon_A\rangle}$–connections. We explain which ones. Let u_0 be a generating element of $\mathfrak{M}_2$. For every u contained in the $\xrightarrow{L}$–equivalence class $[u_0]$ of u_0 and satisfying $u \xrightarrow{\Diamond} v$ for some $v \in V(A)$, we do the following: For all w such that $u \xrightarrow{\Diamond} w$ we take the $\xrightarrow{\langle\epsilon_A\rangle}$–connection from w to v. Afterwards we cross out all the other $\xrightarrow{\langle\epsilon_A\rangle}$–connections. Let $\mathfrak{M}'$ be the resulting model.

We have to convince the reader that $\mathfrak{M}'$ is correctly defined. That is, an arrow $\xrightarrow{\langle\epsilon_A\rangle}$ from w to v really exists (w and v as above). But this is clear from the Property (b) established in the proof of Proposition 15.

The decisive property we have just achieved for the relation $\xrightarrow{\langle\epsilon_A\rangle}$ is *partial functionality,* which is a consequence of Lemma 17. In fact, Lemma 17 implies that two points having both a $\xrightarrow{\Diamond}$–successor in $V(A)$ and a common $\xrightarrow{\Diamond}$–successor, do have a *unique* $\xrightarrow{\Diamond}$–successor in $V(A)$ (with regard to $\mathfrak{M}'$). Therefore, exactly one point in $V(A)$ is assigned via $\xrightarrow{\langle\epsilon_A\rangle}$ to every w having a $\xrightarrow{\Diamond}$–predecessor u in $[u_0]$ which points into $V(A)$ by means of $\xrightarrow{\Diamond}$.

But we must be careful, for necessary $\xrightarrow{\langle\epsilon_A\rangle}$–connections could have been deleted above. We give now reasons for that this is not the case. So, let v be any point and assume that $v \xrightarrow{\langle\epsilon_A\rangle} w$ holds in the model $\mathfrak{M}_2$, for some point w. Take some $u \in [u_0]$ such that $u \xrightarrow{\Diamond} v$. Then there exists some $z \in V(A)$ such that $u \xrightarrow{\Diamond} z$, because of Lemma 16. Due to the Property (b), there is also an arrow $\xrightarrow{\langle\epsilon_A\rangle}$ from v to z, i.e., to a previously chosen element of $V(A)$. This means that not every arrow $\xrightarrow{\langle\epsilon_A\rangle}$ leaving v was deleted.

By an inductive argument we can show now that any subformula of α holds in the model $\mathfrak{M}'$ at some point u, iff it holds in $\mathfrak{M}_2$ at u. As to the proof of this assertion, we confine ourselves to the case $[\epsilon_A]\beta$ of the induction. However, regarding this we remark only that the following property is required for that:

$$(*) \quad \text{if } \exists v : u \xrightarrow{\langle \epsilon_A \rangle} v, \text{ then } (\mathfrak{M}, u \models \langle \epsilon_A \rangle \beta \iff \mathfrak{M}, u \models [\epsilon_A]\beta).$$

In order to establish $(*)$, one has to apply a semantic argument once again.[14] Actually, $\langle \epsilon_A \rangle \beta \rightarrow [\epsilon_A]\beta$ is a valid formula, thus a theorem of GJ. Now the last set constituting Σ_0 (see above) plays its part, yielding the critical left-to-right direction of $(*)$.

On the basis of the properties of $\mathfrak{M}'$ we have just proved, Axiom 26 is easily seen to be valid there. And $\mathfrak{M}'$ is clearly an almost faithful GJ–model. This finishes the proof of the proposition. ■

The previous results imply that α is satisfiable in a finite model of the axioms, of which the size is in $O\left(2^{2^{c \cdot |\alpha|}}\right)$ (where $c \in \mathbb{N}$ is some constant). This gives us the first main issue of this section.

THEOREM 19 (Decidability). *The logic* GJ *is decidable.*

Now we turn to the system GJ + D. By changing the filter set Σ_0 slightly and performing the same filtration-and-model-surgery procedure as above, we obtain the following theorem.

THEOREM 20 (Extended decidability). *The logic* GJ + D *is decidable.*

Proof. We have to guarantee that the schema D remains valid by passing to the accordingly defined filtrated structure. Again, only the nominals occurring in the consistent formula α for which we want to construct a finite model have to be taken into account in the process. So, let

$$\Sigma_0^{\mathrm{D}} := \Sigma_0 \cup \{K\left(\langle \epsilon_A \rangle \top \wedge \langle \epsilon_B \rangle \top\right) \mid A, B \in \mathrm{N}_{sets} \text{ occur in } \alpha\},$$

and let Σ^{D} be derived from Σ_0^{D} as Σ was derived from Σ_0 above. Then, the proof of the validity of D in the final model (i.e., the counterpart of $\mathfrak{M}'$) can easily be done in the same manner as, eg, the proof of Property (b) in Proposition 15. We may, therefore, omit further details. ■

As an immediate consequence of Theorem 20 we get that the modal logic of directed spaces is decidable, too.

COROLLARY 21. *The modal logic of directed spaces is decidable.*

This result constitutes a striking application of hybrid logic to reasoning about knowledge. In fact, it solves the decidability problem raised in [17], Sec. 6.

[14] Note that $(*)$ resembles the *Fun–Lemma* used in the completeness proof for linear time temporal logic; cf [9], 9.9.

Concluding this section we comment on the complexity of the satisfiability problem of GJ. Unfortunately, we can say only little about this. The above decidability proof yields obviously a very large upper bound. On the other hand, Theorem 4.5 from [3] gives us good reason for suspecting that EXPTIME is a lower bound.[15]

5 Summary, comparison, and outlook for further options

In the present paper we developed the fundamental matters of a two-sorted hybrid logic, GJ, for set spaces with names. A family of new 'guarded jump'–operators assigned to names of sets was considered, in particular. We obtained a Completeness as well as a Decidability Theorem for GJ, and we could extend these results to the logic of directed spaces (which are significant to reasoning 'topologically' about knowledge).

It should be remarked that hybrid logics for spaces of sets received little attention in the literature up to now. At least, a (usual) hybrid extension of the *classical* topological semantics of modal logic (going back to [14]) was briefly considered in [8] (with regard to expressiveness).

Which other classes of set spaces should be studied from the point of view of hybrid logic? Concluding this paper, we touch on the class of frames $(X, \mathcal{O})$ where $\mathcal{O}$ is *closed under complementation*. This class is possibly of some interest to various applications. Addressing the complement of the actual neighbourhood at a situation x, U means, in particluar, jumping completely outside U. Thus a 'complementation operator' appears, which goes beyond all the previous abilities. Now, it is not too hard to write down some formulas describing certain properties of such an operator and its interaction with K and $\Box$. In order to obtain nice meta-results, however, it seems more promising to connect complementation operators with names. In a forthcoming paper we will develop a system like that.

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[15] Concerning this, note once again that $[\epsilon_A]$ is a partially functional modality, for every $A \in \mathrm{N}_{sets}$.

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