Complexity of Strict Implication

Félix Bou

ABSTRACT. The aim of the present paper is to analyze the complexity of strict implication (together with falsum, conjunction and disjunction). We prove that Ladner's Theorem remains valid when we restrict the language to the strict implication fragment, and that the same holds for Hemaspaandra's Theorem. As a consequence we have that the validity problem for most standard normal modal logics is the same one than the validity problem for its strict implication fragment. We also prove that the validity problems for Visser's basic propositional logic and Visser's formal propositional logic are **PSPACE** complete. Finally, a polynomial reduction from most standard normal modal logics into its strict implication fragment is presented.

Strict implication is defined in the modal language as

$$\varphi_0 \rightarrow \varphi_1 := \quad \Box(\varphi_0 \supset \varphi_1) \quad \textit{strict implication},$$

where $\supset$ refers to material implication. Strict implication was already considered by Lewis in the birth of modal logic. Nevertheless, there is almost no complexity result in the literature for proper fragments of the modal language that are based on strict implication. One of the few exceptions is intuitionistic propositional logic [22].

On the other hand, there are a lot of complexity results for logics in the modal language. Two of the most outstanding results are Ladner's Theorem [15, Theorem 3.1] and Hemaspaandra's Theorem [21, 13]. They state, respectively, that all modal subsystems of **S4** have a **PSPACE** hard validity problem and that all modal extensions of **S4.3** have a co-**NP** complete validity problem.

In this paper we analyze the complexity of the language based on strict implication ($\rightarrow$) together with falsum ($\bot$), conjunction ($\wedge$) and disjunction ($\vee$). We notice that true ($\top$) is easily definable in this language, while neither material implication ($\supset$) nor classical negation ($\sim$) are definable. The main results of the paper say that Hemaspaandra's Theorem and Ladner's Theorem also hold when we consider the restriction to the strict implication fragment. Hence, the complexity of the validity problem for most standard normal modal logics is the same one than the complexity of the validity problem for its strict implication fragment[1].

The first section is devoted to fix the notation and to remind the complexity results for the modal language. In this section we also sketch a proof of Ladner's Theorem since we will follow the same strategy for the strict implication fragment. In Section 2 we prove Hemaspaandra's Theorem and Ladner's Theorem for the strict implication fragment, and we state some consequences of them. The aim of the third section is to show a map that is a polynomial reduction from most standard normal modal logics into its strict implication fragment. Finally, the last section is devoted to state some open problems. We notice that all the results of this paper are contained in the author's Ph.D. Dissertation [3].

1 Preliminaries

First of all, let us fix our notation. Throughout the paper we will consider two languages. One is the well-known modal language and the other is its strict implication fragment (together with falsum, conjunction and disjunction). Let Prop be an infinite set of propositions. The *modal formulas* and the *strict implication formulas* are defined, respectively, by

$$\varphi ::= p \mid \bot \mid \varphi_0 \wedge \varphi_1 \mid \varphi_0 \vee \varphi_1 \mid \varphi_0 \supset \varphi_1 \mid \Box\varphi$$

and

$$\varphi ::= p \mid \bot \mid \varphi_0 \wedge \varphi_1 \mid \varphi_0 \vee \varphi_1 \mid \varphi_0 \rightarrow \varphi_1,$$

where p ranges over elements of Prop. The set of modal formulas will be denoted by $\mathcal{L}^{mod}(\mathsf{Prop})$, and the set of strict implication formulas by $\mathcal{L}^{s}(\mathsf{Prop})$. The set of subformulas of a formula φ is denoted by $Sub(\varphi)$, and the *modal degree* $\deg(\varphi)$ of a modal formula φ is the number of nested modalities.

These formulas are to be interpreted in Krikpe models. Given a Kripke model $\mathcal{M} = \langle M, R, V\rangle$, a world $m \in M$ and a formula φ the *satisfiability relation* $\mathcal{M}, m \Vdash \varphi$ is defined as follows:

$\mathcal{M}, m \Vdash p$	iff	$m \in V(p)$
$\mathcal{M}, m \nVdash \bot$		
$\mathcal{M}, m \Vdash \varphi_0 \wedge \varphi_1$	iff	$\mathcal{M}, m \Vdash \varphi_0$ and $\mathcal{M}, m \Vdash \varphi_1$
$\mathcal{M}, m \Vdash \varphi_0 \vee \varphi_1$	iff	$\mathcal{M}, m \Vdash \varphi_0$ or $\mathcal{M}, m \Vdash \varphi_1$
$\mathcal{M}, m \Vdash \varphi_0 \supset \varphi_1$	iff	$\mathcal{M}, m \nVdash \varphi_0$ or $\mathcal{M}, m \Vdash \varphi_1$
$\mathcal{M}, m \Vdash \Box\varphi$	iff	$\forall m'(mRm' \Rightarrow \mathcal{M}, m' \Vdash \varphi)$
$\mathcal{M}, m \Vdash \varphi_0 \rightarrow \varphi_1$	iff	$\forall m'(mRm' \,\&\, \mathcal{M}, m' \Vdash \varphi_0 \Rightarrow \mathcal{M}, m' \Vdash \varphi_1)$.

[1] Besides the complexity results there are a lot of other results suggesting that the expressive power of the strict implication fragment is very close to the one of the modal language. A thorough study can be found in [3].

If $\mathcal{M}, m \Vdash \varphi$ then we say that φ *is satisfied in* $\mathcal{M}$ *at* a. It is said that φ *is valid in* $\mathcal{M}$ (notation: $\mathcal{M} \Vdash \varphi$) if $\mathcal{M}, m \Vdash \varphi$ for every world $m \in M$. Given two formulas φ_0 and φ_1 we will write $\varphi_0 \equiv \varphi_1$ whenever both formulas are satisfied in the same worlds for every Kripke model. We will use the following abbreviations in the strict implication language: $\top := \bot \rightarrow \bot$, $\neg\varphi := \varphi \rightarrow \bot$ and $\Box\varphi := \top \rightarrow \varphi$. In the modal language we write $\sim\varphi := \varphi \supset \bot$, $\varphi_0 \supset\subset \varphi_1 := (\varphi_0 \supset \varphi_1) \wedge (\varphi_1 \supset \varphi_0)$ and $\Diamond\varphi := \sim\Box\sim\varphi$. Given $n \in \omega$ we denote by $\Box^n\varphi$ and $\Box^{(n)}\varphi$ the formulas

$$\overbrace{\Box \ldots \Box}^{n \text{ times}} \varphi \quad \text{and} \quad \varphi \wedge \Box\varphi \wedge \ldots \wedge \Box^n\varphi.$$

It is clear that the semantics of the non-primitive connectives is the expected one[2]. In particular,

$$\mathcal{M}, m \Vdash \sim\varphi \qquad \text{iff} \quad \mathcal{M}, m \nVdash \varphi.$$

We notice that if we add the material implication $\supset$ or the classical negation $\sim$ to $\mathcal{L}^s(\mathsf{Prop})$ then the language that we obtain has the same expressive power than $\mathcal{L}^{mod}(\mathsf{Prop})$. We also single out that the semantics on the strict implication language is precisely the standard semantics given for intuitionistic propositional logic except for the fact that we range over arbitrary Kripke models (and not only over intuitionistic models[3]).

The interest in the strict implication language comes from the fact that there is a wide range of well-known logics that can be formalized in it. As we have already pointed out the more famous example is *intuitionistic propositional logic* **IPL**, which is the set of strict implication formulas that are valid in all reflexive and transitive frames under persistent valuations [14, 5]. We recall that a valuation V is *persistent* when it satisfies that for every $p \in \mathsf{Prop}$ and every worlds m and m', if mRm' and $m \in V(p)$ then $m' \in V(p)$. Another example of logic that can be obtained in this way is *classical propositional logic* **CPL**. It is the set consisting of all strict implication formulas that are valid in all reflexive, transitive and symmetric frames under persistent valuations (or simply the set of all strict implication formulas that are valid in the reflexive frame that consists of a single point). Other examples in the literature are all superintuitionistic logics [5], some subintuitionistic logics [8, 9, 17, 25, 4], Visser's *formal propositional logic* **FPL** [24] and Visser's *basic propositional logic* **BPL** [24, 18, 19]. Let us explain which logics are the last two. We define the *Gödel translation* $\mathsf{T} : \mathcal{L}^s(\mathsf{Prop}) \longrightarrow \mathcal{L}^{mod}(\mathsf{Prop})$ using the following clauses:

[2] We notice that the semantics of $\Box$ for the modal case coincides with its semantics as a defined connective for the strict implication case. Therefore, there is no ambiguity.

[3] For this sake we do not call our language the intuitionistic propositional language. The author's opinion is that it is better to use a different name to point out this fact.

$$\text{i) } \mathsf{T}(p) = \Box p, \qquad \text{ii) } \mathsf{T}(\bot) = \bot, \qquad \text{iii) } \mathsf{T}(\varphi_0 \wedge \varphi_1) = \mathsf{T}(\varphi_0) \wedge \mathsf{T}(\varphi_1),$$
$$\text{iv) } \mathsf{T}(\varphi_0 \vee \varphi_1) = \mathsf{T}(\varphi) \vee \mathsf{T}(\varphi_1), \qquad \text{v) } \mathsf{T}(\varphi_0 \rightarrow \varphi_1) = \Box(\mathsf{T}(\varphi_0) \supset \mathsf{T}(\varphi_1)).$$

It is well-known that T is an embedding of **IPL** into both the normal modal logic **S4** and the normal modal logic **Grz**, i.e.,

$$\varphi \in \mathbf{IPL} \quad \text{iff} \quad \mathsf{T}(\varphi) \in \mathbf{S4} \quad \text{iff} \quad \mathsf{T}(\varphi) \in \mathbf{Grz}.$$

It is interesting to notice that the same map is also an embedding of **CPL** into **S5**. The logics **FPL** and **BPL** were introduced by Visser in [24] as the logics such that T is an embedding of them into **GL** and **K4**, respectively. That is,

$$\mathbf{FPL} = \{\varphi \in \mathcal{L}^s(\mathsf{Prop}) : \mathsf{T}(\varphi) \in \mathbf{GL}\},$$

and

$$\mathbf{BPL} = \{\varphi \in \mathcal{L}^s(\mathsf{Prop}) : \mathsf{T}(\varphi) \in \mathbf{K4}\}.$$

It is not hard to check that **FPL** is precisely the set of strict implication formulas that are valid in all frames that are Noetherian strict orders under persistent valuations[4], and that **BPL** is the set of strict implication formulas that are valid in all transitive frames under persistent valuations. In [24] Visser was interested in **FPL** for provability reasons, and **BPL** was simply suited for technical reasons. However, nowadays **BPL** has acquired an interest of its own since Ruitenburg has argued its philosophical interest as a constructive logic [18, 19].

We notice that there is a general method to associate a strict implication logic with a normal modal logic. The idea is to consider its strict implication fragment. Given a normal modal logic L we define its strict implication fragment L^s as the set $\{\varphi \in \mathcal{L}^s(\mathsf{Prop}) : \sigma(\varphi) \in L\}$ where σ is the translation defined by:

$$\begin{array}{lcl}
\sigma(p) & := & p \\
\sigma(\bot) & := & \bot \\
\sigma(\varphi_0 \wedge \varphi_1) & := & \sigma(\varphi_0) \wedge \sigma(\varphi_1) \\
\sigma(\varphi_0 \vee \varphi_1) & := & \sigma(\varphi_0) \vee \sigma(\varphi_1) \\
\sigma(\varphi_0 \rightarrow \varphi_1) & := & \Box(\sigma(\varphi_0) \supset \sigma(\varphi_1)).
\end{array}$$

The map σ has been considered several times in the literature. Its first appearance under this name is in [9], and since then it has been also used by other authors, e.g., Celani and Jansana [4].

We have previously announced that we will prove in the next section that in most cases the complexity of the validity problem for a normal modal

[4]If we replace 'Noetherian' with 'finite' we also obtain **FPL**.

logic coincides with the complexity of the validity problem for its strict implication fragment. The following easy example shows that this is not true for all normal modal logics (as far as $\mathbf{P} \neq \mathbf{NP}$).

EXAMPLE 1. Let **Verum** be the normal modal logic given by the frame $\mathfrak{F}$ that is a single irreflexive point. Since the problem "$\varphi \in \mathbf{CPL}$?" is co-**NP** complete [7] it easily follows that "$\varphi \in \mathbf{Verum}$?" is also co-**NP** complete. On the other hand, it is not hard to prove that "$\varphi \in \mathbf{Verum}^s$?" is in **P**. It follows from the fact that the model checking for modal formulas (in particular for strict implication formulas) is in **P** together with the fact that for every $\varphi \in \mathcal{L}^s(\mathsf{Prop})$, $\varphi \in \mathbf{Verum}^s$ iff φ holds in the Kripke model over $\mathfrak{F}$ such that all propositions fail in its unique state.

In the rest of the section we recall two of the main results concerning complexity of normal modal logics. The first one is due to Hemaspaandra (*née* Spaan) [21, 13].

THEOREM 2 (Hemaspaandra). *Let* Prop *be a countable set and let* L *be a normal modal logic extending* **S4.3**. *Then, the problem "*$\varphi \in L$*?" is co-***NP** *complete.*

The other result is due to Ladner [15] and gives a lower bound of the complexity of normal modal logics that are subsystems of **S4**.

THEOREM 3 (Ladner). *Let* Prop *be a countable set and let* L *be a normal modal logic such that* $\mathbf{K} \subseteq L \subseteq \mathbf{S4}$. *Then, the problem "*$\varphi \in L$*?" is* **PSPACE** *hard.*

Using that for the validity problem of the normal modal logics **K**, **T**, **K4**, **S4**, **D**, **GL** and **Grz** it is known the existence of algorithms running in **PSPACE**, it follows that the validity problem for these normal modal logics is **PSPACE** complete.

To finish the section we review a proof of Ladner's Theorem because we will follow the same strategy in the next section. Since in the next section we want to deal with persistent valuations we cannot consider the proof given by Ladner (see [2] for a more accessible presentation of this proof). The argument that we detail is very close to the one exhibited by Halpern and Moses in [12].

Let L be a normal modal logic such that $\mathbf{K} \subseteq L \subseteq \mathbf{S4}$. Now we show a polynomial reduction from the complementary of the validity problem of the logic **QBF** of quantified Boolean formulas. It is enough because it is well known that this last problem is **PSPACE** complete [23]. Let us describe what is ***QBF***. The set of *quantified Boolean formulas*[5] consists of

[5]Sometimes these formulas have been called prenex quantified Boolean formulas. Then

expressions of the form

$$Q_0 p_0 \dots Q_{n-1} p_{n-1} \varphi(p_0, \dots, p_{n-1})$$

where each $Q_i \in \{\forall, \exists\}$, and $\varphi(p_0, \dots, p_{n-1})$ is a Boolean formula (called the *matrix*) with propositions among $p_0, \dots, p_{n-1}$. The quantifiers range over the truth values 1 (true) and 0 (false), and a quantified Boolean formula without free variables is *true* if and only if it evaluates to 1, i.e., the subformulas $\forall p\, \varphi(p)$ and $\exists p\, \varphi(p)$ are regarded to be true iff $\varphi(\top) \wedge \varphi(\bot)$ and $\varphi(\top) \vee \varphi(\bot)$ are true, respectively. For instance, $\exists p_0 \forall p_1 (p_0 \vee p_1)$ is true, while $\forall p_0 \exists p_1 \forall p_2 (p_0 \wedge p_1 \wedge p_2)$ is not true. The logic $\boldsymbol{QBF}$ is the set of quantified Boolean formulas that are true. Hence, the $\boldsymbol{QBF}$ problem is the problem of deciding, given an arbitrary quantified Boolean formula β, whether $\beta \in \boldsymbol{QBF}$.

As far as we are interested in a **PSPACE** complete problem we can restrict[6] the definition of quantified Boolean formulas to the case that $Q_0 = \exists$ and $n \geq 2$. This is obvious by the following trivial polynomial reduction:

$$\begin{gathered} Q_0 p_0 \dots Q_{n-1} p_{n-1} \varphi(p_0, \dots, p_{n-1}) \text{ is true} \quad \text{iff} \\ \exists q_0 \exists q_1 Q_0 p_0 \dots Q_{n-1} p_{n-1} \big(q_0 \wedge q_1 \wedge \varphi(p_0, \dots, p_{n-1})\big) \text{ is true,} \end{gathered}$$

where q_0 and q_1 are two new propositions. From now on we will assume that all quantified Boolean formulas satisfy the requirements $Q_0 = \exists$ and $n \geq 2$.

Now we present the promised polynomial reduction from a **PSPACE** complete problem to L. Let β be a quantified Boolean formula

$$Q_0 p_0 \dots Q_{n-1} p_{n-1} \varphi(p_0, \dots, p_{n-1})$$

with $Q_0 = \exists$ and $n \geq 2$. We consider new propositions $q_0, \dots, q_{n+1}$, and we define $g(\beta)$ as the conjunction of formulas displayed in Figure 1, where

$$\gamma_{in} := (q_i \wedge\sim q_{i+1}) \supset \big((q_0 \wedge \dots \wedge q_{i-1}) \wedge (\sim q_{i+2} \wedge \dots \wedge \sim q_{n+1}) \wedge (\sim p_i \wedge \dots \wedge \sim p_{n-1})\big),$$

$$\theta_i := (q_i \wedge \sim q_{i+1}) \supset \Diamond(q_{i+1} \wedge \sim q_{i+2}),$$

$$\delta_i := (q_i \wedge \sim q_{i+1}) \supset \big(\Diamond(q_{i+1} \wedge \sim q_{i+2} \wedge p_i) \wedge \Diamond(q_{i+1} \wedge \sim q_{i+2} \wedge \sim p_i)\big),$$

and

$$\psi_i := \big((q_i \wedge p_{i-1}) \supset \Box p_{i-1}\big) \wedge \big((q_i \wedge \sim p_{i-1}) \supset \Box \sim p_{i-1}\big).$$

---

the set of quantified Boolean formulas is defined by

$$\varphi ::= \bot \mid \top \mid p \mid \sim\varphi \mid \varphi_0 \wedge \varphi_1 \mid \forall p \varphi \mid \exists p \varphi$$

where p ranges over elements of Prop.

[6] Indeed, there is no reason to adopt this restriction in the modal case, but in the strict implication case this restriction simplifies the argument.

(i) q_0

(ii) $\sim(q_1 \vee q_2 \vee \ldots \vee q_n \vee q_{n+1} \vee p_0 \vee p_1 \vee \ldots \vee p_{n-1})$

(iii) $\Diamond(q_1 \wedge \sim q_2)$

(iv) $\Box^1 \gamma_{1n} \wedge \Box^2 \gamma_{2n} \wedge \ldots \wedge \Box^n \gamma_{nn}$

(v) $\Box^1 \theta_1 \wedge \Box^2 \theta_2 \wedge \ldots \wedge \Box^{n-1} \theta_{n-1}$

(vi) $\bigwedge_{i \in \{j<n : Q_j = \forall\}} \Box^i \delta_i$

(vii) $\Box^1 \psi_1 \wedge \Box^2 (\psi_1 \wedge \psi_2) \wedge \Box^3 (\psi_1 \wedge \psi_2 \wedge \psi_3) \wedge \ldots \wedge \Box^{n-1} (\psi_1 \wedge \psi_2 \wedge \psi_3 \wedge \ldots \wedge \psi_{n-1})$

(viii) $\Box^n (q_n \supset \varphi)$

Figure 1. The modal formula $g(\beta)$

It is clear that $\deg(g(\beta)) = n$, and it is easy to check that for every quantified Boolean formula β we can compute in polynomial time the modal formula $g(\beta)$. Hence, to obtain a polynomial reduction from $\boldsymbol{QBF}^c$ to L it is enough to prove that

$$\beta \in \boldsymbol{QBF} \quad \text{iff} \quad \sim g(\beta) \notin L,$$

for every quantified Boolean formula β. This equivalence is a trivial consequence of the following lemma.

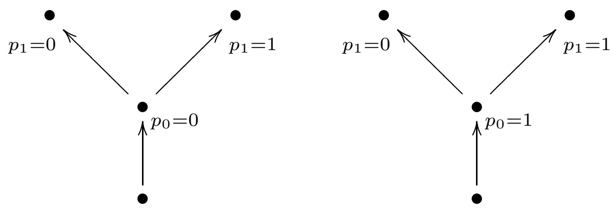
LEMMA 4. *Let β be a quantified Boolean formula*

$$Q_0 p_0 \ldots Q_{n-1} p_{n-1} \varphi(p_0, \ldots, p_{n-1})$$

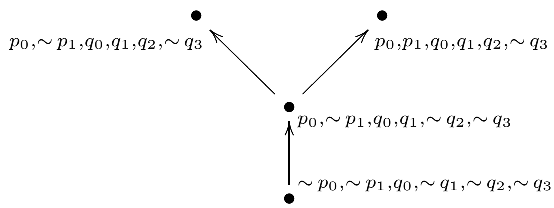
with $Q_0 = \exists$ and $n \geq 2$. The following statements are equivalent:

- *β is true.*
- *$g(\beta)$ is satisfiable in a Kripke model.*
- *$g(\beta)$ is satisfiable in a finite strict order with a persistent valuation.*
- *$g(\beta)$ is satisfiable in a finite partial order with a persistent valuation.*

Sketch of the proof. Ladner's idea is that when we are evaluating β we are essentially generating a finite number of binary 'trees' $\mathcal{T}_0, \ldots, \mathcal{T}_{k-1}$ of height n such that each one of their branches gives us a Boolean valuation



(a) The binary 'trees' generated by β



(b) A Kripke model witnessing that β is true

Figure 2. Let β be the quantified Boolean formula $\exists p_0 \forall p_1 (p_0 \vee p_1)$

in $\{p_0, \ldots, p_{n-1}\}$. These 'trees' consists of the root node, and then — working inwards along the quantifier string — each existential quantifier extends it by adding a single branch, and each universal quantifier extends it by adding two branches (In Figure 2(a) the reader can find an example). For every $j < k$, we associate a Kripke model $\mathcal{M}_j$ over the propositions $\{p_0, \ldots, p_{n-1}, q_0, \ldots, q_{n+1}\}$ with the binary 'tree' $\mathcal{T}_j$. Let us explain how to define $\mathcal{M}_j$: the universe and the accessibility relation are given by $\mathcal{T}_j$, and the behaviour of its valuation at a node of height i ($\leq n$) is the following one:

- propositions in $\{q_0, \ldots, q_i\}$ are true.
- propositions in $\{p_i, \ldots, p_n, q_{i+1}, \ldots, q_{n+1}\}$ are false.
- propositions in $\{p_0, \ldots, p_{i-1}\}$ behaves according to the Boolean valuation given by a branch containing the node.

It holds that $\mathcal{M}_j$ is a tree, and that it has a persistent valuation (In our previous example the Kripke model associated with the second 'tree' in Figure 2(a) is the one depicted in Figure 2(b)).

The connection of the previous ideas with the problem that we are interested in is that β is true iff there is $j < k$ such that the matrix φ is evaluated to 1 in all Boolean valuations recorded by the binary 'tree' $\mathcal{T}_j$ (In our previous example the witnessing 'tree' is the second one in Figure 2(a)). Now let us analyze the different implications that we have in the present lemma.

$(1 \Rightarrow 3)$: Assume that β is true. Then, there is $j < k$ such that the matrix φ is evaluated to 1 in all Boolean valuations recorded by the binary 'tree' $\mathcal{T}_j$. Now we define $\mathcal{N}_j$ as the Kripke model $\mathcal{M}_j$ except for the fact that we take the $R^{\mathcal{N}_j}$ as the transitive closure of $R^{\mathcal{M}_j}$. It is clear that $\mathcal{N}_j$ is a finite strict order with a persistent valuation; and from the fact that φ is evaluated to 1 in all Boolean valuations recorded by the binary 'tree' $\mathcal{T}_j$ it is simple to check that $\mathcal{N}_j$ satisfies $g(\beta)$ in the state that is the root of $\mathcal{M}_j$.

$(1 \Rightarrow 4)$: It is proved as the previous implication, except for the fact that now we take the reflexive-transitive closure.

$(3 \Rightarrow 2)$: Trivial.

$(4 \Rightarrow 2)$: Trivial.

$(2 \Rightarrow 1)$: We suppose that $g(\beta)$ is satisfiable in a certain Kripke model $\mathcal{M}$ at a world m. Replacing it with its unravelling [20] we can assume that $\mathcal{M}$ is a tree with root m. Using that $\deg(g(\beta)) = n$ we can assume that $\mathcal{M}$ is a tree of length n: simply remove all states with length $> n$. Using all clauses except (viii) of the definition of $g(\beta)$, it is easily verified that $\mathcal{M}$ is isomorphic to a (perhaps not generated) submodel of $\mathcal{M}_j$ for some $j < k$. By clause (viii) of the definition of $g(\beta)$, it follows that φ is evaluated to 1 in all Boolean valuations recorded by the binary 'tree' $\mathcal{T}_j$. Therefore, β is true. ■

2 New Complexity Results

In this section we prove that Hemaspaandra's Theorem and Ladner's Theorem are also true when we restrict ourselves to the strict implication fragment.

THEOREM 5. *Let* Prop *be a countable set and let L be a normal modal logic extending* **S4.3**. *Then, the problems "$\varphi \in L$?" and "$\varphi \in L^s$?" are both co-***NP** *complete.*

Proof. By Hemaspaandra's Theorem it is enough to show that "$\varphi \in L^s$?" is co-**NP** hard. It is known that the equational logic of distributive lattices is co-**NP** hard (see [10]). Given two formulas φ_0 and φ_1 using only the connectives $\wedge$ and $\vee$, it is clear that

$$\begin{array}{cl} \varphi_0 \approx \varphi_1 \text{ holds in all distributive lattices} & \text{iff} \\ \varphi_0 \approx \varphi_1 \text{ holds in all Boolean algebras} & \text{iff} \\ \varphi_0 \supset\subset \varphi_1 \in \mathbf{CPL} & \text{iff} \\ \varphi_0 \supset\subset \varphi_1 \in L & \text{iff} \\ \Box(\varphi_0 \supset\subset \varphi_1) \in L & \text{iff} \\ (\varphi_0 \to \varphi_1) \wedge (\varphi_1 \to \varphi_0) \in L^s. & \end{array}$$

Hence, we have obtained a polynomial reduction that allows us to conclude that "$\varphi \in L^s$?" is co-**NP** hard. ■

Now we state our version of Ladner's Theorem. In the proof we will use that if we restrict ourselves to quantified Boolean formulas β with matrix in conjunctive normal form, then the problem "$\beta \in \boldsymbol{QBF}$?" is also **PSPACE**-complete [1, Corollary 1.36].

THEOREM 6. *Let* Prop *be a countable set and let L be a set of $\mathcal{L}^s$-formulas such that either $\boldsymbol{K}^s \subseteq L \subseteq \boldsymbol{FPL}$ or $\boldsymbol{K}^s \subseteq L \subseteq \boldsymbol{IPL}$. Then, L is* **PSPACE** *hard.*

Proof. Let L be a set of $\mathcal{L}^s$-formulas such that either $\boldsymbol{K}^s \subseteq L \subseteq \boldsymbol{FPL}$ or $\boldsymbol{K}^s \subseteq L \subseteq \boldsymbol{IPL}$. The method of the proof is to show a polynomial reduction from a known **PSPACE** complete problem to L^c. The **PSPACE** complete problem considered is the logic $\boldsymbol{QBF}$ of quantified Boolean formulas in conjunctive normal form.

Let β be a quantified Boolean formula

$$Q_0 p_0 Q_1 p_1 \ldots Q_{n-1} p_{n-1} \varphi(p_0, \ldots, p_{n-1})$$

such that $Q_0 = \exists$, $n \geq 2$ and φ is in conjunctive normal form. Therefore, φ is of the form $(\nu_0 \supset \pi_0) \wedge \ldots \wedge (\nu_{k-1} \supset \pi_{k-1})$ where the ν's are finite (maybe empty) conjunctions of propositions and the π's are finite (maybe empty) disjunctions of propositions. Hence, the ν's and the π's are $\mathcal{L}^s$-formulas. We consider new propositions $q_0, \ldots, q_{n+1}$, and we define the following $\mathcal{L}^s$-formulas:

$$\gamma'_{in} := \left(q_i \to (q_{i+1} \vee \bigwedge_{j \in \{0,\ldots,i-1\}} q_j) \right) \wedge \bigwedge_{j \in \{i+2,\ldots,n+1\}} ((q_i \wedge q_j) \to q_{i+1}) \wedge$$

$$\wedge \bigwedge_{j \in \{i,\ldots,n-1\}} ((q_i \wedge p_j) \to q_{i+1}),$$

$$\theta'_i := (q_i \wedge (q_{i+1} \to q_{i+2})) \to q_{i+1},$$

$$\delta'_i := \left(\left(q_i \wedge \left((q_{i+1} \wedge p_i) \to q_{i+2} \right) \right) \to q_{i+1} \right) \wedge$$

(i) q_0

(ii) $q_1 \vee q_2 \vee \ldots \vee q_n \vee q_{n+1} \vee p_0 \vee p_1 \vee \ldots \vee p_{n-1}$

(iii) $q_1 \rightarrow q_2$

(iv) $\gamma'_{1n} \wedge \Box^1 \gamma'_{2n} \wedge \ldots \wedge \Box^{n-1}\gamma'_{nn}$

(v) $\theta'_1 \wedge \Box^1 \theta'_2 \wedge \ldots \wedge \Box^{n-2}\theta'_{n-1}$ [7]

(vi) $\bigwedge_{i \in \{j<n : Q_j = \forall\}} \Box^{i-1}\delta'_i$ [8]

(vii) $\psi'_1 \wedge \Box^1(\psi'_1 \wedge \psi'_2) \wedge \Box^2(\psi'_1 \wedge \psi'_2 \wedge \psi'_3) \wedge \ldots \wedge \Box^{n-2}(\psi'_1 \wedge \psi'_2 \wedge \psi'_3 \wedge \ldots \wedge \psi'_{n-1})$

(viii) $\Box^{n-1}\Big(\big((q_n \wedge \nu_0) \rightarrow \pi_0\big) \wedge \big((q_n \wedge \nu_1) \rightarrow \pi_1\big) \wedge \ldots \wedge \big((q_n \wedge \nu_{k-1}) \rightarrow \pi_{k-1}\big)\Big)$

Figure 3. A list of $\mathcal{L}^s$-formulas

$$\wedge \left(\Big(q_i \wedge \big(q_{i+1} \rightarrow (p_i \vee q_{i+2})\big)\Big) \rightarrow q_{i+1} \right),$$

and

$$\psi'_i := \big(q_i \wedge p_{i-1}\big) \rightarrow \Box p_{i-1}\big) \wedge \big(q_i \rightarrow (p_{i-1} \vee \neg p_{i-1})\big).$$

It is easy to check that $\gamma'_{in} \equiv \Box\gamma_{in}$, $\theta'_i \equiv \Box\theta_i$, $\delta'_i \equiv \Box\delta_i$ and $\psi'_i \equiv \Box\psi_i$, where the formulas without $'$ are the ones considered in the proof of Ladner's Theorem (see p. 6). Now we consider the list of $\mathcal{L}^s$-formulas displayed in Figure 3. We notice that each one of them is either equivalent to the corresponding one in Figure 1 or equivalent to the classical negation $\sim$ of it. We define $f_0(\beta)$ as the conjunction of (i),(iv),(v),(vi),(vii),(viii); and we define $f_1(\beta)$ as the disjunction of (ii),(iii). Let $f(\beta)$ be the $\mathcal{L}^s$ formula $f_0(\beta) \rightarrow f_1(\beta)$. A moment of reflection shows that

$$\Diamond g(\beta) \equiv \sim f(\beta),$$

where $g(\beta)$ is the formula considered in the proof of Ladner's Theorem. Using this together with Lemma 4 it easily follows that the following statements are equivalent:

- β is true.
- $\sim f(\beta)$ is satisfiable, i.e., $f(\beta) \notin \boldsymbol{K}$.

[7] The assumption $n \geq 2$ guarantees that it is a strict implication formula.

[8] Since $Q_0 = \exists$ we know that $0 \notin \{j < n : Q_j = \forall\}$. Hence, $i - 1 \geq 0$ whenever $i \in \{j < n : Q_j = \forall\}$.

- $\sim f(\beta)$ is satisfiable in a finite strict order with a persistent valuation, i.e., $f(\beta) \notin \boldsymbol{FPL}$.
- $\sim f(\beta)$ is satisfiable in a finite partial order with a persistent valuation, i.e., $f(\beta) \notin \boldsymbol{IPL}$.

Therefore,

$$\beta \in \boldsymbol{QBF} \quad \text{iff} \quad f(\beta) \notin \boldsymbol{K} \quad \text{iff} \quad f(\beta) \notin \boldsymbol{FPL} \quad \text{iff} \quad f(\beta) \notin \boldsymbol{IPL}.$$

Thus, using the fact that either $\boldsymbol{K}^s \subseteq L \subseteq \boldsymbol{FPL}$ or $\boldsymbol{K}^s \subseteq L \subseteq \boldsymbol{IPL}$ we obtain that

$$\beta \in \boldsymbol{QBF} \quad \text{iff} \quad f(\beta) \notin L.$$

It is easy to check that the map f is a polynomial time reduction, what allows us to conclude that L is **PSPACE** hard. ■

By the last theorem it trivially follows that $\boldsymbol{K}^s$, $\boldsymbol{T}^s$, $\boldsymbol{K4}^s$, $\boldsymbol{S4}^s$, $\boldsymbol{D}^s$, $\boldsymbol{GL}^s$ and $\boldsymbol{Grz}^s$ are **PSPACE** complete. Another easy consequence of Theorem 6 is that the logics $\boldsymbol{BPL}$ and $\boldsymbol{FPL}$ are **PSPACE** complete. We know that they are in **PSPACE** by the Gödel translation. As far as the author is aware this is the first time that the complexity classes for $\boldsymbol{BPL}$ and $\boldsymbol{FPL}$ have been calculated. From this theorem it also follows that all subintuitionistic logics defined using Kripke models are **PSPACE** hard, e.g., all logics considered in [8, 9, 17, 25, 4].

3 A Polynomial Reduction from Normal Modal Logics

Up to now we have proved that most standard normal modal logics have the same complexity class as their strict implication fragment, but we have not given any polynomial reduction from the normal modal logic into its strict implication fragment. Now we present a reduction of this type that works well under certain (general) assumptions.

Let Prop be the set $\{p_n : n \in \omega\}$, and let Prop' be $\{p_n : n \in \omega\} \cup \{q_n : n \in \omega\} \cup \{r_\varphi : \varphi \in \mathcal{L}^{mod}(\mathsf{Prop})\}$. Then, we simultaneously define two translations $^+$ and $^-$ from $\mathcal{L}^{mod}(\mathsf{Prop})$ into $\mathcal{L}^s(\mathsf{Prop}')$:

$$\begin{array}{lcl@{\qquad\qquad}lcl}
\bot^+ & := & \bot & \bot^- & := & \top \\
\top^+ & := & \top & \top^- & := & \bot \\
p_n^+ & := & p_n & p_n^- & := & q_n \\
(\sim\varphi)^+ & := & \varphi^- & (\sim\varphi)^- & := & \varphi^+ \\
(\varphi_0 \wedge \varphi_1)^+ & := & \varphi_0^+ \wedge \varphi_1^+ & (\varphi_0 \wedge \varphi_1)^- & := & \varphi_0^- \vee \varphi_1^- \\
(\Box\varphi)^+ & := & \Box\varphi^+ & (\Box\varphi)^- & := & r_\varphi.
\end{array}$$

By a straightforward simultaneous induction it is easily verified that for every $\mathcal{L}^{mod}(\mathsf{Prop})$-formula φ with propositions among $p_0, \ldots, p_{n-1}$ and with modal degree k, it holds that

$$(1) \quad \left(\Box^{(k)}\Big(\bigwedge_{0\le i<n}(p_i \supset\subset \sim q_i) \wedge \bigwedge_{\phi\in Sub(\varphi)}(r_\phi \supset\subset \sim \Box\phi^+)\Big)\right) \supset (\varphi \supset\subset \varphi^+) \in \boldsymbol{K}$$

and

$$(2) \quad \left(\Box^{(k)}\Big(\bigwedge_{0\le i<n}(p_i \supset\subset \sim q_i) \wedge \bigwedge_{\phi\in Sub(\varphi)}(r_\phi \supset\subset \sim \Box\phi^+)\Big)\right) \supset (\sim \varphi \supset\subset \varphi^-) \in \boldsymbol{K}.$$

It is clear that the box of the modal formula

$$\bigwedge_{0\le i<n}(p_i \supset\subset \sim q_i) \wedge \bigwedge_{\phi\in Sub(\varphi)}(r_\phi \supset\subset \sim \Box\phi^+)$$

is equivalent to the formula

$$\bigwedge_{0\le i\le n}\Big(\big((p_i\wedge q_i)\to\bot\big)\wedge\Box(p_i\vee q_i)\Big)\wedge \bigwedge_{\phi\in Sub(\varphi)}\Big(\big((r_\phi\wedge\Box\phi^+)\to\bot\big)\wedge\Box(r_\phi\vee\Box\phi^+)\Big).$$

Let us call this $\mathcal{L}^s(\mathsf{Prop}')$-formula $t(\varphi)$. We define $h(\varphi)$ as the $\mathcal{L}^s(\mathsf{Prop}')$-formula

$$\Box^{(k)}t(\varphi)\to\Box\varphi^+.$$

It is easy to check that h is computable in polynomial time. Before stating the main theorem of this section we need a new definition.

DEFINITION 7. A normal modal logic L is *closed under extensions by a predecessor* if it is characterized by a class C of frames such that

> for every $\mathcal{F} \in \mathsf{C}$ and every world m in the frame $\mathcal{F}$ there is a frame $\mathcal{F}' \in \mathsf{C}$ with a world m' such that (i) m' is not an initial world in $\mathcal{F}'$ (i.e., it has a predecessor), and (ii) the subframe of $\mathcal{F}$ generated by m and the subframe of $\mathcal{F}'$ generated by m' are isomorphic.

It is obvious that if the state m is not an initial world in $\mathcal{F}$, then the previous condition trivially holds since we can take $\mathcal{F}'$ as $\mathcal{F}$, and m' as m. In particular all Kripke frame complete extensions of **T** are closed under extensions by a predecessor. An easy fact to check is that the normal modal logics **K**, **T**, **K4**, **S4**, **D**, **GL** and **Grz** are closed under extensions by a predecessor.

THEOREM 8. *Let L be a normal modal logic such that is closed under extensions by a predecessor. For every modal formula φ, it holds that*

$$\varphi \in L \quad \textit{iff} \quad h(\varphi) \in L^s.$$

Proof. $(\Rightarrow)$: Suppose that $\varphi \in L$. By (1) we obtain that

$$\Box^{(k-1)}t(\varphi) \supset \varphi^+ \in L.$$

From here it follows that $\Box^{(k)}t(\varphi) \rightarrow \Box\varphi^+ \in L$, i.e., $h(\varphi) \in L$.

$(\Leftarrow)$: Let us assume that $h(\varphi) \in L$, and let C be the class of frames given by the closure under extensions by a predecessor condition. Let $\mathcal{F} \in \mathsf{C}$, $m \in F$, and V a valuation in Prop for $\mathcal{F}$. We want to prove that $\mathcal{F}, V, m \Vdash \varphi$. Applying twice the property that C satisfies we can assume that there are states m_0 and m_1 such that $\langle m_0, m_1\rangle \in R^{\mathcal{F}}$ and $\langle m_1, m\rangle \in R^{\mathcal{F}}$. Now we extend the valuation V to a valuation V' in Prop' for $\mathcal{F}$ according to the following conditions:

- for every $n \in \omega$, $V'(q_n) := \{x \in F : x \notin V(p_n)\}$,
- for every $\phi \in \mathcal{L}^{mod}(\mathsf{Prop})$, $V'(r_\phi) := \{x \in F : \mathcal{F}, V, x \nVdash \Box\phi\}$.

By a straightforward induction it is easily verified that for every modal formula $\phi \in \mathcal{L}^{mod}(\mathsf{Prop})$ and every $x \in F$,

(3) $\quad \mathcal{F}, V, x \Vdash \phi \;$ iff $\; \mathcal{F}, V', x \Vdash \phi \;$ iff $\; \mathcal{F}, V', x \Vdash \phi^+ \;$ iff $\; \mathcal{F}, V', x \nVdash \phi^-$.

By (3) and the definition of V' it is not hard to verify that $\mathcal{F}, V' \Vdash t(\varphi)$. Using now that $\mathcal{F}, V', m_0 \Vdash h(\varphi)$ (because $h(\varphi) \in L$) and that $\mathcal{F}, V', m_1 \Vdash \Box^{(k)}t(\varphi)$ we deduce that $\mathcal{F}, V', m_1 \Vdash \Box\varphi^+$. Therefore, $\mathcal{F}, V', m \Vdash \varphi^+$. By (3) it follows that $\mathcal{F}, V, m \Vdash \varphi$. ∎

We have already seen that f is a polynomial time reduction for all normal modal logics that are closed under extensions by a predecessor. In particular it holds for **K**, **T**, **K4**, **S4**, **D**, **GL** and **Grz**.

4 Open Problems

Halpern proved in [11] that Ladner's Theorem for normal modal logics holds even in the case that there is a single proposition (see also [6]). On the other hand, in this paper we have seen that the strict implication language satisfies a version of Ladner's Theorem. So, a natural question is whether in the case that there is a single proposition the strict implication language also has a certain version of Ladner's Theorem. We cannot give the same version than in Theorem 6 because it is known that intuitionistic propositional logic with one variable is decidable in linear time [16]. But perhaps it is possible to give the same version when there are only two propositions. Indeed, as far as the author knows it is still open the characterization of the complexity of **IPL** with two variables (see [5, p. 564]). The only result in this direction already proved is in [3] and claims that $\mathbf{K}^s$ is **PSPACE** hard even when Prop is empty.

Another interesting open question is what happens when the strict implication is alone (i.e., without falsum, conjunction and disjunction). One of the few results known for the pure strict implication fragment is that in the case of **IPL** this fragment is also **PSPACE** complete [22].

Acknowledgments

I would like to thank Nick Bezhanishvili for some previous discussions. I also owe my gratitude to Valentin Shehtman for some remarks during the conference. The research presented in this paper was partially supported by Catalan grant 2001SGR-00017 and by Spanish grant BFM2001-3329.

BIBLIOGRAPHY

[1] J. L. Balcázar, J. Díaz, and J. Gabarró. *Structural complexity. I.* Texts in Theortical Computer Science. An EATCS Series. Springer-Verlag, Berlin, second edition, 1995.

[2] P. Blackburn, M. de Rijke, and Y. Venema. *Modal logic.* Number 53 in Cambridge Tracts in Theoretical Computer Science. Cambridge University Press, Cambridge, 2001.

[3] F. Bou. *Strict-Weak Languages. An analysis of strict implication.* Ph. D. Dissertation, University of Barcelona, 2004.

[4] S. Celani and R. Jansana. A closer look at some subintuitionistic logics. *Notre Dame Journal of Formal Logic*, 42(4):225–255, 2001.

[5] A. Chagrov and M. Zakharyaschev. *Modal Logic*, volume 35 of *Oxford Logic Guides.* Oxford University Press, 1997.

[6] A. V. Chagrov and M. N. Rybakov. How many variables does one needs to prove PSPACE-hardness of modal logics. volume 4 of *Advances in Modal Logic*, pages 71–82. 2003.

[7] S. A. Cook. The complexity of theorem-proving procedures. In *ACM, Proc. 3rd ann. ACM Sympos. Theory Computing, Shaker Heights, Ohio 1971, 151-158.* 1971.

[8] G. Corsi. Weak logics with strict implication. *Zeitschrift für Mathematische Logik und Grundlagen der Mathematik*, 33:389–406, 1987.

[9] K. Došen. Modal translations in K and D. In M. de Rijke, editor, *Diamond and Defaults*, pages 103–127. Kluwer Academic Publishers, 1993.

[10] R. Freese. Algorithms in finite, finitely presented and free lattices. Preprint, July 1999.

[11] J. Y. Halpern. The effect of bounding the number of primitive propositions and the depth of nesting on the complexity of modal logic. *Artificial Intelligence*, 75(2):361–372, 1995.

[12] J. Y. Halpern and Y. Moses. A guide to completeness and complexity for modal logics of knowledge and belief. *Artificial Intelligence*, 54:319–379, 1992.

[13] E. Hemaspaandra. The price of universality. *Notre Dame Journal of Formal Logic*, 37:174–203, 1996.

[14] S. Kripke. Semantical analysis of intuitionistic logic I. In J. M. Crossley and M. A. E. Dummet, editors, *Formal Systems and Recursive Functions*, pages 92–130. North-Holland, Amsterdam, 1965.
[15] R. E. Ladner. The computational complexity of provability in systems of modal propositional logic. *SIAM J. Comput.*, 6(3):467–480, 1977.
[16] I. Nishimura. On formulas of one variable in intuitionistic propositional calculus. *The Journal of Symbolic Logic*, 25:327–331 (1962), 1960.
[17] G. Restall. Subintuitionistic logics. *Notre Dame Journal of Formal Logic*, 35(1):116–129, 1994.
[18] W. Ruitenburg. Constructive logic and the paradoxes. *Modern Logic*, 1:207–301, 1991.
[19] W. Ruitenburg. Basic logic and Fregean set theory. In H. Barendregt, M. Bezem, and J. W. Klop, editors, *Dirk van Dalen Festschrift, Questiones Infinitae*, volume 5, pages 121–142. Department of Philosophy, Utrecht University, 1993.
[20] H. Sahlqvist. Completeness and Correspondence in First and Second Order Semantics for Modal Logic. In S. Kanger, editor, *Proceedings of the third Scandinavian logic symposium*, pages 110–143. North-Holland, Amsterdam, 1975.
[21] E. Spaan. *Complexity of modal logics.* Ph. D. Dissertation, Institute for Logic, Language and Computation, 1993.
[22] R. Statman. Intuitionistic propositional logic is polynomial-space complete. *Theoretical Computer Science*, 9(1):67–72, 1979.
[23] L. J. Stockmeyer and A. R. Meyer. Word problems requiring exponential time: preliminary report. In *Fifth Annual ACM Symposium on Theory of Computing (Austin, Tex., 1973)*, pages 1–9. Assoc. Comput. Mach., New York, 1973.
[24] A. Visser. A propositional logic with explicit fixed points. *Studia Logica*, 40:155–175, 1981.
[25] H. Wansing. Displaying as temporalizing. Sequent systems for subintuitionistic logic. In S. Akama, editor, *Logic, Language and Computation*, pages 159–178. Kluwer, 1997.

Félix Bou
JAIST, Nomi, Ishikawa, 923-1292, Japan
bou@jaist.ac.jp