Completions of Algebras and Completeness of Modal and Substructural Logics

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ABSTRACT. We will give here a brief overview of completions of algebras, and of completeness results on both modal logics and substructural logics which are obtained by using completions. Our aim is to discuss completions of algebras and completeness theorems of logics of various kinds in a uniform way, and to try to find common features among them. As will be shown in the last two sections, this is attained successfully in giving an algebraic proof of cut elimination theorems for modal and substructural logics, where a generalization of the MacNeille completions, called *quasi-completions* of algebras, plays a key role.[1]

1 Complete extensions and MacNeille completions

In the present paper, we will discuss completions of algebras for modal logics and substructural logics, and completeness theorems for them, which are obtained as logical consequences of completions. Though both canonical extensions and the MacNeille completions are considered, we will be concerned mainly about the latter. Completions of algebras are discussed in the first three sections. Many of completeness theorems of various logics, not only propositional but predicate ones, with respect to both Kripke-type semantics and algebraic one, are obtained from results on completions. The last two sections are devoted to an algebraic proof of cut elimination theorems for modal and substructural logics. We take the sequent system **LJ** of intuitionistic logic as an example. Algebraic structures for the sequent system **LJ** without cut rule are introduced in § 5, and then the completeness of this cut-free system with respect to algebraic semantics determined by these structures is shown in § 6 by using *quasi-completions* of algebras. Our final remark says that quasi-completions are generalizations

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of the MacNeille completions. Completions of residuated lattices and complete embeddings are discussed already in Ono [26], and algebraic proofs of cut elimination theorems will be studied comprehensively in the forthcoming paper Belardinelli-Jipsen-Ono [1].

Throughout the paper, we consider algebras with lattice structures. The examples are Boolean algebras with operators and residuated lattices. The formers are algebras for modal logics and the latters are for substructural logics. By a *complete* algebra, we mean an algebra whose lattice reduct is complete, i.e. an algebra in which both infinite joins and infinite meets exist always. An algebra **Q** is a *completion* of an algebra **P** if **Q** is a complete extension of **P**, i.e. **Q** is a complete algebra into which **P** is embeddable.

As is described in Givant–Venema [7], there are two different ways of getting completions of a given algebra **P**. The first is the *canonical extension* of **P**: the Boolean algebra consisting of all subsets of the set of all ultrafilters of **P** when **P** is a Boolean algebra, or the distributive lattice consisting of all upward closed subsets of the set of all prime filters of **P** when **P** is a distributive lattice. They arise from Stone representation theorem. The notion of canonical extensions is extended also to Boolean algebras with operators, and to bounded distributive lattices with operators etc. (see e.g. [13, 12, 8, 6]). The characteristics of canonical extensions are that they are atomic, but they do not always preserve infinite joins.

The second way is the Dedekind-MacNeille completion of a given algebra **P**. This completion is sometimes called simply *the completion* of **P**, since it is the minimal complete extension detemined uniquely by **P** up to isomorphism (see Sikorski [33] § 35 Theorem 35.2 for Boolean algebras, and also Ono [22] Lemma 4.5 for residuated lattices). To avoid confusions, we will call this, the *MacNeille completion* in the present paper. The MacNeille completion are discussed in e.g. [33, 18, 7] for Boolean algebras (with operators), and in e.g. [22] for residuated lattices (though a different word is used for "residuated lattices" in it). The special feature of the MacNeille completion is that it preserves all existing infinite joins and infinite meets while it is not always atomic. Also, it should be noticed that the MacNeille completion of a distributive lattice is not always distributive.

We will mainly concern with the MacNeille completion of Boolean algebras with oparators, in particular of modal algebras, and of residuated lattices in this paper. So, we start our paper by giving a brief account of the MacNeille completion of partially ordered sets and of Boolean algebras, which is defined here by using *cuts* (see § 35 Example F in [33]).

Let $\mathbf{P} = \langle P, \leq \rangle$ be a partially ordered set with the greatest element $\top$ and the least $\bot$. For each $X \subseteq P$, define $X^{\rightarrow}$ and $X^{\leftarrow}$ to be the set of upper bounds of X and of lower bounds of X, respectively. Define a map C on $\wp(\mathbf{P})$ by $CX = ((X^{\rightarrow})^{\leftarrow})$, for each $X \subseteq P$. The map C is in fact a *closure operator* on $\wp(\mathbf{P})$, i.e. it satisfies that for all X and Y in $\wp(\mathbf{P})$,

1. $X \subseteq CX$,
2. $X \subseteq Y$ implies $CX \subseteq CY$,
3. $CCX \subseteq CX$.

We say that a subset X of P is *C-closed* if $X = CX$. It is easy to see that for C-closed subsets X and Y, the intersection $X \cap Y$ of X and Y is C-closed. Now, define $X \cup_C Y$ by $C(X \cup Y)$ where $\cup$ means the union. Then, the set $C_{\mathbf{P}}$ of all C-closed subsets of P forms a lattice with respect to $\cap$ and $\cup_C$. The lattice $C_{\mathbf{P}}$ is complete since it has always the meet $\bigcap_i X_i$ and the join $C(\bigcup_i X_i)$ for any set $\{X_i\}_i$ of C-closed sets. We define a map h from $\mathbf{P}$ to $C_{\mathbf{P}}$ by $h(a) = (a]$ for each element $a \in P$, where $(a] = \{x \in P : x \leq a\}$ which is in fact C-closed. Then, h is an order isomorphism. We can show that if $\mathbf{P}$ has moreover a lattice structure, then h is a lattice isomorphism which preserves all existing infinite joins and infinite meets in P. When $\mathbf{P}$ is a Boolean algebra, then $C_{\mathbf{P}}$ becomes a complete Boolean algebra with the minimum element $\{\bot\}(= C\emptyset)$ and h is a Boolean isomorphism, if we define the complement $-X$ of a C-closed subset X by

$$-X = \{x \in P : x \leq -b \text{ for any } b \in X\},$$

where $-b$ denotes the complement of b in $\mathbf{P}$. We note that $-(-X) = CX$ holds for any subset X of P. The complete algebra $C_{\mathbf{P}}$ defined in this way is called the MacNeille completion of an algebra $\mathbf{P}$.

Let $\mathbf{P}^+$ be the MacNeille completion of a lattice $\mathbf{P}$. Then, each element X of $\mathbf{P}^+$ is a subset of P such that

1. X is downward closed and is closed under the finite join,
2. if the join $\vee X$ exists in P, then $X = \{a \in P : a \leq \vee X\}$ (and hence $\vee X \in X$).

Since each element X of $\mathbf{P}^+$ can be expressed as $X = \bigcup\{(a] : a \in X\}$, by identifying the C-closed set $(a]$ with an element $a \in P$ we can show the following:

Each element c of the MacNeille completion of a lattice $\mathbf{P}$ is represented as $c = \vee\{a \in P : a \leq c\}$.

2 Completions of residuated lattices

To make a comparison, we will next consider the MacNeille completions of residuated lattices. Residuated lattices are algebraic structures for substructural logics. For general information on substructural logics, see e.g. [28, 30, 22, 14]. To simplify the discussion, here we consider only commutative residuated lattices, i.e. algebras for substructural logics with exchange rule, or equivalently, algebras for extensions of intuitionistic linear logic (without exponentials). The contents of this section are discussed in details in Ono [22, 26].

An algebra $\mathbf{P} = \langle P, \wedge, \vee, \cdot, \rightarrow, 0, 1, \bot, \top \rangle$ is a *bounded commutative residuated lattice* if it satisfies the following:

1. $\langle P, \wedge, \vee, \bot, \top \rangle$ is a bounded lattice,
2. $\langle P, \cdot, 1 \rangle$ is a commutative monoid with the unit 1,
3. $x \cdot y \leq z$ iff $x \leq (y \rightarrow z)$.

For simplicity's sake, we omit the word "bounded" in the rest of paper. In the above definition, 0 is an arbitrary element of P by which the "negation" $-x$ of x is defined as $x \rightarrow 0$. The third condition is called the law of residuation between the monoid operation $\cdot$, sometimes called *fusion*, and the *residual* $\rightarrow$. It implies the following distributivity of the monoid operation $\cdot$ over join :

$$(1) \qquad (\vee x_i) \cdot y = y \cdot (\vee x_i) = \vee (x_i \cdot y) \;\text{ for any } x_i, y.$$

Equalities in (1) mean that if $\vee x_i$ exists, then $\vee(x_i \cdot y)$ exists also and the equalities hold. A commutative residuated lattice is *integral* if the unit 1 is equal to the greatest element $\top$ and 0 is equal to $\bot$, and it is *weakly idempotent* if $x \leq x \cdot x$ for each x. It is easy to see that a commutative residuated lattice $\mathbf{P}$ is both integral and weakly idempotent if and only if $x \cdot y = x \wedge y$ for all x, y, i.e. $\mathbf{P}$ is a Heyting algebra. Note that syntactically, integrality and weak idempotency correspond to weakening rules and contraction rules, respectively, in sequent calculi.

When we consider completions of residuated lattices, we need to manage also the monoid operation and the residual. Since residuated lattices are not always distributive, Stone representation theorem does not work. Because of this, we don't have any suitable notion of canoncal extensions of these algebras at present.

First, we will show a standard way of constructing complete commutative residuated lattices. See e.g. [22] for the details. Suppose that $\mathbf{M} = \langle M, \cdot, 1 \rangle$ is a commutative monoid. We define two operations $*$ and $\Rightarrow$ on the set of all subsets of M by

i. $X * Y = \{a \cdot b \in M : a \in X \text{ and } b \in Y\}$,
ii. $X \Rightarrow Y = \{a \in M : c \cdot a \in Y \text{ for each } c \in X\}$.

Then, we have the following proposition.

PROPOSITION 1 *For any given commutative monoid* $\mathbf{M}$ *and any subset* O *of* M, $\wp(\mathbf{M}) = \langle \wp(M), \cap, \cup, *, \Rightarrow, O, \{1\}, \emptyset, M\rangle$ *is a complete commutative residuated lattice, where* $\cap$ *and* $\cup$ *denote set intersection and union, respectively.*

A map C on a commutative residuated lattice $\mathbf{P}$ is a *closure operator* if for all $x, y \in P$

1. $x \leq Cx$,
2. $x \leq y$ implies $Cx \leq Cy$,
3. $CCx \leq Cx$,
4. $Cx \cdot Cy \leq C(x \cdot y)$.

Suppose that C is a closure operator on a commutative residuated lattice $\mathbf{P}$. An element x of P is *C-closed* when $x = Cx$ holds. Let $C(P)$ be the set of all C-closed elements of P. Though $C(P)$ is closed under both $\wedge$ and $\rightarrow$, it is not always closed under $\vee$ nor $\cdot$. So, define $\vee_C$ and $\cdot_C$ by

$$x \vee_C y = C(x \vee y) \text{ and } x \cdot_C y = C(x \cdot y).$$

Then, we have the following.

PROPOSITION 2 *Suppose that* $\mathbf{P}$ *is a commutative residuated lattice and* C *is a closure operator on* P*. Then, the algebra* $C(\mathbf{P}) = \langle C(P), \wedge, \vee_C, \cdot_C, \rightarrow, d, C1, C\bot, \top\rangle$ *forms a residuated lattice where* d *is an arbitrary* C*-closed element. If* $\mathbf{P}$ *is integral then so is* $C(\mathbf{P})$ *if we take* $C\bot$ *for* d*, if* $\mathbf{P}$ *is weakly idempotent then so is* $C(\mathbf{P})$*, and also if* $\mathbf{P}$ *is complete then so is* $C(\mathbf{P})$*.*

By Propositions 1 and 2, we have the following.

COROLLARY 3 *If* $\mathbf{M}$ *is a commutative monoid and* C *is a closure operator on* $\wp(\mathbf{M})$*, then* $\mathbf{C_M} = \langle C(\wp(M)), \cap, \cup_C, *_C, \Rightarrow, D, C(\{1\}), C(\emptyset), M\rangle$ *is a complete commutative residuated lattice where* D *is any* C*-closed subset of* M*.*

Now that we have a general way of constructing complete commutative residuated lattice from commutative monoids, we will show how to get a completion when a residuated lattice is given. For this purpose, we assume that a given commutative monoid is partially ordered. More precisely, a structure $\mathbf{M} = \langle M, \cdot, 1, \leq \rangle$ is called a *partially ordered commutative monoid* (or simply, a *po-monoid*), if $\langle M, \cdot, 1\rangle$ is a commutative monoid with the unit element 1 and $\langle M, \leq, 1\rangle$ a partially ordered set such that for any $x, y, z \in M$, $x \leq y$ implies $x \cdot z \leq y \cdot z$. We say that a po-monoid $\mathbf{M}$ is integral when the unit element 1 is equal to the greatest element, and it is weakly idempotent when $x \leq x \cdot x$ holds for every x.

Suppose that C is a closure operator on $\wp(\mathbf{M})$ for a po-monoid $\mathbf{M}$, in which every C-closed subset X is *downward closed*, i.e. $x \in X$ and $y \leq x$ imply $y \in X$ for any $x, y \in M$. In this case, both the integrality and the weak idempotency are preserved under the construction of $\mathbf{C_M}$ from $\mathbf{M}$. More precisely, when $\mathbf{M}$ is integral, $C(\{1\}) = M$ holds, and hence the residuated lattice $\mathbf{C_M}$ with $D = C(\emptyset)$ becomes also integral.

For a given commutative residuated lattice $\mathbf{P}$, consider its po-monoid reduct $\mathbf{P}^*$. Then this determines a complete commutative residuated lattice $\mathbf{C}_{\mathbf{P}^*}$. The next theorem tells us when a complete embedding from $\mathbf{P}$ to $\mathbf{C}_{\mathbf{P}^*}$ exists. We show the theorem in a more general setting.

For a given po-monoid $\mathbf{M}$ and a closure operator C on $\wp(\mathbf{M})$, define a subset U of M by $U = \{b \in M : (b] \text{ is } C\text{-closed}\}$, where $(b] = \{x \in M : x \leq b\}$. We assume now that the following *closure condition* holds for C: For any C-closed subset X,

1. X is downward closed,
2. if $a_i \in U$ for each $i \in I$ and moreover if $\vee_i a_i$ exists and belongs to U, then $\vee_i a_i \in X$ whenever $a_i \in X$ for each $i \in I$.

THEOREM 4 *Suppose that* $\mathbf{M}$ *is a po-monoid and* C *is a closure operator on* $\wp(\mathbf{M})$ *for which closure conditions hold. Then the map* $h : U \to C(\wp(M))$ *defined by* $h(b) = (b]$ *is a complete embedding, i.e. an order isomorphism which preserves all existing products and residuals in* U*, and also all existing (infinite) joins and meets in* U*.*

For the proof, see [26]. Let us consider the MacNeille completion of a given commutative residuated lattice $\mathbf{P}$. That is, take its po-monoid reduct $\mathbf{P}^*$ and define a map C^0 on $\wp(\mathbf{P}^*)$ by $C^0 X = ((X^{\rightarrow})^{\leftarrow})$ for each $X \subseteq P$. Then, we can show that C^0 becomes a closure operator and that the set U is equal to P. In this case, the second item of the closure condition becomes equal to the condition that if $a_i \in X$ for each $i \in I$ and $\vee_i a_i$ exists then $\vee_i a_i \in X$. Thus, we can show that C^0 satisfies the closure condition. Therefore, we have the following by Theorem 4.

THEOREM 5 *Each (integral, weakly idempotent) commutative residuated lattice can be embedded into its MacNeille completion by a residuated lattice isomorphism which preserves all existing infinite joins and meets.*

We note that the converse of Corollary 3 holds. That is, any complete commutative residuated lattice $\mathbf{P}$ is isomorphic to $\mathbf{C_M}$ for a commutative monoid $\mathbf{M}$ with a closure operator C on $\wp(\mathbf{M})$. In fact, it suffices to take ${\mathbf{C}^0}_{\mathbf{P}^*}$ for $\mathbf{C_M}$ since the MacNeille completion of a complete commutative residuated lattice $\mathbf{P}$ is isomorphic to $\mathbf{P}$.

The above result on the MacNeille completion of commutative residuated lattices can be extended to that of commutative residuated lattices with *exponentials*, where exponentials are particular modal operations (see [22, 25] for the details). The result is extended also to commutative residuated lattices with weaker modalities (see Bucalo [4]).

THEOREM 6 *Each commutative residuated lattice with exponentials has the MacNeille completion.*

We remark that results mentioned in this section hold for Heyting algebras, and in particular for Boolean algebras, since Heyting algebras are just commutative, integral and weakly idempotent residuated algebras. Note that $x \cdot y = x \wedge y$ holds for x, y in any Heyting algebra and therefore $X * Y = X \cap Y$ holds for all downward closed sets X and Y.

3 Complete additivity, conjugates and residuated pairs

By an *operator* on a Boolean algebra we mean an operation which preserves finite joins. An operator is *completely additive* if it preserves all existing joins. Boolean algebras with completely additive operators play key roles in the study of Boolean algebras with operators. For each Boolean algebra with operators $\mathbf{P}$ there exists a complete and atomic Boolean algebra with completely additive operators $\mathbf{Q}$ which is a canonical extension of $\mathbf{P}$. Any complete and atomic Boolean algebra with completely additive operators can be represented as a complex algebra determined by its atom structure (see Jónsson-Tarski [13], Jónsson [12]).

Also, any Boolean algebra with complete additive operators $\mathbf{P}$ has the MacNeille completion $\mathbf{P}^+$, since each complete additive operator h of $\mathbf{P}$ can be extended to a complete additive operator h^+ of $\mathbf{P}^+$ (see Monk [18], Givant-Venema [7]). Here, for a given n-ary operation h of $\mathbf{P}$, the operation h^+ of $\mathbf{P}$ is defined by

$$h^+(\mathbf{c}) = \vee\{h(\mathbf{a}) : \mathbf{a} \leq \mathbf{c} \text{ with } \mathbf{a} \in P^n\}$$

and is called the *complete extension* of h. Preservation theorems under canonical extensions and the MacNeille completions are discussed in [13, 12, 18, 7]. See also [9].

In this section we will give some remarks on the complete additive operations. First recall that the monoid operation $\cdot$ in the previous section satisfies the equality (1) in the previous section, which means the complete additivity of $\cdot$. The complete extension of this monoid operation $\cdot$ becomes the monoid operation in the MacNeille completion. But, to show the complete additivity of this complete extension, the standard argument used in Boolean algebras with complete additive operators does not work, since the proof of complete additivity in Boolean algebras needs the *complete distributivity* of (infinite) joins over meet (see the proof of Lemmas 12 and 13 in [7]), which does not hold always in residuated lattices. The complete additivity of the monoid operation comes in fact from the law of residuation.

We can explain this in a general setting, based on a result from *residuation theory* (see Blyth-Janowitz [3]). Suppose that $\mathbf{M}$ and $\mathbf{N}$ are partially ordered sets and that $f : M \to N$ and $g : N \to M$ are maps. If

$$f(x) \leq y \Leftrightarrow x \leq g(y) \quad \text{for all } x \in M \text{ and } y \in N,$$

then f and g are said to form a *reduated pair*, and the left member f is said to be *residuated*. Then, we can show the following.

PROPOSITION 7 *Suppose that* $f : M \to N$ *and* $g : N \to M$ *form a residuated pair. Then,*

1. $f(\vee_i x_i) = \vee_i f(x_i)$ *for* $x_i \in M$,
2. $g(\wedge_j y_j) = \wedge_j g(y_j)$ *for* $y_j \in N$.

Here, the equlities mean that if $\vee_i x_i$ $(\wedge_j y_j)$ *exists, then* $\vee_i f(x_i)$ $(\wedge_j g(y_j)$, *respectively* $)$ *exists and these equlities hold.*

Recall that the law of residuation says that $x \cdot y \leq z$ iff $x \leq (y \to z)$ for all x, y, z. For a fixed y, define maps f and g by $f(x) = x \cdot y$ and $g(z) = y \to z$. Then f and g form a residuated pair, and therefore f preserves all existing joins, i.e. it is completely additive, and g preserves all existing meets.

Let us compare the notion of residuated pairs with that of *conjugates* introduced in Jónsson-Tarski [13]. Two unary operations h and k on a Boolean algebra are *conjugates* if

$$x \wedge h(y) = 0 \Leftrightarrow k(x) \wedge y = 0 \text{ for all } x, y.$$

In Boolean algebras, this is equivalent to say that h and k^d form a residuated pair, where k^d is the *dual* of k, which is defined as $k^d(x) = -k(-x)$ for all x. Thus, by Proposition 7 we can derive the following well-known result (see e.g. Lemma 19 in [7]).

COROLLARY 8 *If h and k are conjugates then they are completely additive.*

Lemma 20 in [7] says that if h and k are conjugates then so are their complete extensions h^+ and k^+. On the other hand, while the monoid operation $*$ of the MacNeille completion of a given commutative residuated lattice $\mathbf{P}$ is the complete extension of the monoid operation $\cdot$ in $\mathbf{P}$, the residual of $*$ is not always equal to the complete extension of the residual of $\cdot$.

4 Completeness theorems

As consequences of results on completions of algebras in the previous sections, we can derive various types of completeness theorems for modal and substructural logics. A clue to the completeness is that desired properties of the original algebra, e.g. the validity of some axioms, can be transferred from it to its completions. In the following, $\supset, \wedge, \vee$ and $\neg$ denote logical connectives implication, conjunction, disjunction and negation, respectively.

Needless to say, preservation theorems under canonical extensions of modal algebras offer Kripke completeness for many of modal propositional logics. (See e.g. [5] for general information.) The argument goes as follows: Let $\mathcal{V}$ be the class of all modal algebras for a given modal logic $\mathbf{L}$. If φ is not provable in $\mathbf{L}$ then it is not valid in an algebra $\mathbf{P}$ in $\mathcal{V}$ (for instance in the Lindenbaum algebra of $\mathbf{L}$) by the algebraic completeness of $\mathbf{L}$. Here the validity of formulas is defined by using *valuations* on $\mathbf{P}$, in which each logical connective is interpreted as its corresponding algebraic operation. Let $\tilde{\mathbf{P}}$ be the canonical extension of $\mathbf{P}$. Since $\mathbf{P}$ is embedded into $\tilde{\mathbf{P}}$, φ is neither valid in $\tilde{\mathbf{P}}$. Since the validity of formulas in $\tilde{\mathbf{P}}$ is the same as the validity of those in the *canonical Kripke structure* of $\mathbf{P}$, φ is not valid in the canonical Kripke structure of $\mathbf{P}$. Moreover, when $\tilde{\mathbf{P}}$ belongs to $\mathcal{V}$, the canonical Kripke structure of $\mathbf{P}$ becomes a Kripke structure for $\mathbf{L}$. Thus, we have the Kripke completeness of $\mathbf{L}$.

The similar argument does not work for Kripke completeness of modal predicate logics, since quantifiers are usually interpreted as infinite joins and meets in algebras but canonical extensions do not preserve infinite joins. In fact, we have a lot of examples of modal predicate logics which are Kripke incomplete (see [21, 31])

However, in a restricted case, canonical extensions in a modified form still work for modal and intermediate predicate logics. The basic idea used here is usually stated as *Rasiowa-Sikorski lemma*. Originally the lemma is proved for Boolean algebras (see e.g. § 24 of [33]). It is extended in [10, 29] to Heyting algebras satisfying the following distributivity: $\wedge_i(a_i \vee b) = (\wedge_i a_i) \vee b$ for any b and any a_i such that $\wedge_i a_i$ exists. Rasiowa-Sikorski lemma says that if Q is a fixed *countable* set of infinite joins and meets in a Boolean algebra $\mathbf{P}$, there exists a complete Boolean algebra $\mathbf{P}'$ and an embedding f from $\mathbf{P}$ into $\mathbf{P}'$ which preserves all infinite joins and meets in Q. To make a given first-order formula φ invalid in $\mathbf{P}'$, when φ is invalid in $\mathbf{P}$, preserving countably many infinite joins and meets is enough since φ contains finitely many quantifiers. In Tanaka-Ono [34], we succeeded to extend this embedding result to modal algebras with a completely additive modal operator $\Diamond$.

For a normal modal propositional logic $\mathbf{M}$ over $\mathbf{K}$, let $\mathbf{Q}$-$\mathbf{M}$ be the minimum predicate extension of $\mathbf{M}$. Let BF be the Barcan formula $\forall x \Box \psi \supset \Box \forall x \psi$, which expresses the complete additivity of $\Diamond$ in its dual form. Let $\mathbf{Q}$-$\mathbf{M}$+BF be the modal predicate logic obtained from $\mathbf{Q}$-$\mathbf{M}$ by adding the axiom scheme BF. We say that $\mathbf{M}$ is *universal* if the class of Kripke structures for $\mathbf{M}$ is representable by a set of universal first-order sentences. Then we have the following [34].

THEOREM 9 *If a normal modal propositional logic* $\mathbf{M}$ *over* $\mathbf{K}$ *is Kripke complete and universal then* $\mathbf{Q}$-$\mathbf{M}$+BF *is Kripke complete.*

By using Rasiowa-Sikorski Lemma for Heyting algebras with the distributivity mentioned above, we can also give an alternative proof of the Kripke completeness result by Shimura [32] on minimal predicate extensions of some intermediate logics with the axiom of constant domain (see also [34]). Here, the axiom of constant domain means formulas of the form $\forall x(\varphi(x) \vee \psi) \supset (\forall x \varphi(x) \vee \psi)$, where ψ does not contain any free occurrence of x. Shimura showed the following.

THEOREM 10 *If an intermediate propositional logic* $\mathbf{J}$ *is either tabular or a subframe logic, the minimal predicate extension with the axiom of constant domain* $\mathbf{Q}$-$\mathbf{J}$+CD *is Kripke complete.*

In contrast with propositional logics, predicate logics are not always complete with respect to algebraic semantics (see [21, 31, 25]). But, using preservation theorems in [7], we can derive also a similar result on the algebraic completeness of modal predicate logics of the form $\mathbf{Q}$-$\mathbf{M}$+BF where

$\mathbf{M}$ is axiomatized by a set of Sahlqvist formulas obtained from Sahlqvist equations discussed in [7] (see [34]).

Next we will show how completeness results for substructural logics can be derived from completions of residuated lattices. First, we discuss Kripke-type semantics for substructural logics introduced in Ono-Komori [27].

A structure $\mathbf{M} = \langle M, \cdot, 0, 1, \vee, \leq \rangle$ is a *semilattice-ordered, commutative integral monoid* (or, simply a *so-monoid*) if it satisfies the following:

1. $\langle M, \cdot, 1, \leq \rangle$ is a po-monoid with the smallest element 0,
2. the join $x \vee y$ exists for every $x, y \in M$,
3. the monoid operation $\cdot$ is finitely additive, i.e. $x \cdot (y \vee z) = (x \cdot y) \vee (x \cdot z)$ holds.

Note here that so-monoids in [27] are defined by using reverse orders. A nonempty subset X of a so-monoid $\mathbf{M}$ is an *ideal*, if (1) X is downward closed, and (2) if both a and b are in X, then $a \vee b \in X$. We define a operation C on $\wp(M)$ by the condition that $C(X)$ is the smallest ideal containing X. Then, we can show that C is a closure operator on the complete commutative residuated lattice $\wp(\mathbf{M})$.

Let $\mathbf{FL_{ew}}$ be the substructural logic obtained from intuitionistic propositional logic by deleting contraction rule. In algebraic terms, $\mathbf{FL_{ew}}$ is the logic complete with respect to the class of all *integral* commutative residuated lattices. *Kripke frames* for $\mathbf{FL_{ew}}$ introduced in [27] are nothing but integral so-monoids. A *valuation* on a Kripke frame $\mathbf{M}$ is any map v from the set of all propositional variables to $C(\wp(M))$. Then, v can be naturally extended to a map from the set of all formulas to $C(\wp(M))$, since $\mathbf{C_M}$ forms an integral commutative residuated lattice. Note that in [27], the notation $a \models \varphi$ is used instead of writing $a \in v(\varphi)$, where $\models$ corresponds to a given valuation v, and also that the words *strong valuations* and *total strong frames* are used instead of the words valuations and Kripke frames, respectively. A formula φ is *valid* in a Kripke frame $\mathbf{M}$ ($\mathbf{M} \models \varphi$, in symbol) if $a \models \varphi$ for every $a \in M$.

From the definition, it is obvious that for any Kripke frame $\mathbf{M}$ and any formula φ, $\mathbf{M} \models \varphi$ if and only if φ is valid in the integral commutative residuated lattice $\mathbf{C_M}$. Now, we take the so-monoid reduct $\mathbf{P}^\dagger$ of a given integral commutative residuated lattice $\mathbf{P}$ for $\mathbf{M}$, and apply Theorem 4 in § 2. Note that U is equal to P in the present case and each C-closed set satisfies the closure condition (for finite joins).

COROLLARY 11 *Every integral commutative residuated lattice* $\mathbf{P}$ *can be embedded into a complete, commutative integral residuated lattice* $\mathbf{C}_{\mathbf{P}^\dagger}$.

This is the essence of the proof of the completeness of $\mathbf{FL_{ew}}$ with respect to Kripke frames given in [27]. Now, we have the following.

COROLLARY 12 *The substructural propositional logic* $\mathbf{FL_{ew}}$ *is Kripke-complete.*

Another type of completeness results on substructural predicate logics follows from Theorems 5 and 6 in § 2, on the MacNeille completions of commutative residuated lattices (with exponentials). In fact, we can show the following (see Ono [22] for the details).

COROLLARY 13 *Intuitionistic linear predicate logic* $\mathbf{FL_e}$ *(with exponentials) is complete with respect to the class of complete, commutative residuated lattices (with exponentials). Similarly, the substructural predicate logic* $\mathbf{FL_{ew}}(\mathbf{FL_{ec}})$ *is complete with respect to the class of complete, integral (weakly idempotent) commutative residuated lattices, where* $\mathbf{FL_{ew}}(\mathbf{FL_{ec}})$ *is obtained from* $\mathbf{FL_e}$ *by adding weakening rule (contraction rule, respectively). This holds also for the case with exponentials.*

5 Algebras for cut-free systems

In this section, we will show how techniques of completions of algebras can be applied to algebraic proof of cut elimination theorems for modal and substructural logics. Our idea is based on results by Maehara [16] and Okada [19] (see also Jipsen-Tsinakis [11]), and the topics in this section will be discussed comprehensively in our forthcoming paper [1] with Belardinelli and Jipsen.

Our method works well for a wide variety of sequent systems of nonclassical logics, including Gentzen's systems **LK** and **LJ** for classical logic and intuitionistic one, not only for propositional logics but for predicate logics. We will give here a proof of cut elimination theorem for **LJ** as a typical example. By a slight modification of it, we can get a similar algebraic proof of cut elimination theorems for modal logics.

To lighten a burden to readers, we will try to minimize explanations of sequent systems and their syntactic properties. When necessary, consult a survey article [24], written from a syntactic point of view, for more information on cut elimination theorems for nonclassical logics.

Each sequent of **LJ** is of the form $\Gamma \Rightarrow \alpha$, where Γ is a (possibly empty) sequence of formulas separated by "commas", and α a formula. The negation $\neg\alpha$ is defined to be the abbreviation of $\alpha \supset 0$, where 0 denotes the constant symbol expressing the falsehood. The system **LJ** consists of initial sequents and rules of inference. Each initial sequent is one of the following:

1) $\alpha \Rightarrow \alpha$, 2) $0, \Gamma \Rightarrow \delta$. There are two kinds of rules of inference. The first is rules for logical connectives. For instance, rules for $\supset$ and $\wedge$ consist of the following:

$$\frac{\Gamma \Rightarrow \alpha \quad \beta, \Sigma \Rightarrow \delta}{\alpha \supset \beta, \Gamma, \Sigma \Rightarrow \delta} \ (\supset\Rightarrow) \qquad \frac{\Gamma, \alpha \Rightarrow \beta}{\Gamma \Rightarrow \alpha \supset \beta} \ (\Rightarrow\supset)$$

$$\frac{\alpha, \Gamma \Rightarrow \delta}{\alpha \wedge \beta, \Gamma \Rightarrow \delta} \ (\wedge 1 \Rightarrow) \qquad \frac{\beta, \Gamma \Rightarrow \delta}{\alpha \wedge \beta, \Gamma \Rightarrow \delta} \ (\wedge 2 \Rightarrow)$$

$$\frac{\Gamma \Rightarrow \alpha \quad \Gamma \Rightarrow \beta}{\Gamma \Rightarrow \alpha \wedge \beta} \ (\Rightarrow \wedge)$$

The second is structural rules, consisting of weakening rule, contraction rule, exchange rule, and also cut rule. Here, we show weakening rule and cut rule.

$$\frac{\Gamma \Rightarrow \delta}{\alpha, \Gamma \Rightarrow \delta} \ (w \Rightarrow) \qquad \frac{\Gamma \Rightarrow \alpha \quad \alpha, \Sigma \Rightarrow \delta}{\Gamma, \Sigma \Rightarrow \delta} \ (cut)$$

Cut elimination theorem for **LJ** says that if a sequent $\Gamma \Rightarrow \alpha$ is provable in **LJ** then it is provable in **LJ** without using cuts. Many important syntactic properties, including the decidability for the case of propositional logics, follows from cut elimination theorem (see e.g. [24]).

It is well-known that intuitionistic propositonal logic is complete with respect to the class of Heyting algebras. Corresponding to the sequent system **LJ** *without cut*, we introduce algebraic structures, called *semi-Heyting structures*. For a given nonempty set Q, let Q^* be the set of all finite (possibly empty) subsets of Q. The empty set is denoted here by ε. For elements x and y of Q^*, xy denotes the set-theoretic union of x and y. When y is in particular a singleton set $\{c\}$ with $c \in Q$, xy is also written as xc. Obviously, Q^* forms a commutative idempotent monoid with the unit ε. In the following, letters x, y, z, u, v are used for expressing elements in Q^*, and letters a, b, c, d are used for expressing elements in Q.

A structure $\mathbf{Q} = \langle Q, \preceq, \wedge, \vee, \rightarrow, 0 \rangle$ is a *semi-Heyting structure* if it satisfies the following conditions: For all $x, y \in Q^*$ and all $a, b, c \in Q$,

- $a \preceq a$,
- $0x \preceq c$,
- $x \preceq c$ implies $yx \preceq c$,
- $x \preceq a$ and $by \preceq c$ imply $(a \rightarrow b)xy \preceq c$,

- $ax \preceq b$ implies $x \preceq a \to b$,
- $ax \preceq c$ and $bx \preceq c$ imply $(a \vee b)x \preceq c$,
- $x \preceq a$ implies $x \preceq a \vee b$,
- $x \preceq b$ implies $x \preceq a \vee b$,
- $ax \preceq c$ implies $(a \wedge b)x \preceq c$,
- $bx \preceq c$ implies $(a \wedge b)x \preceq c$
- $x \preceq a$ and $x \preceq b$ imply $x \preceq a \wedge b$.

Each of these conditions corresponds to either one of initial sequents or one of rules of inference except cut rule, and vice versa. Exchange rule and contraction rule are incorporated into the definition of Q^*, while weakening rule is expressed as the third condition. Thus, we have the following by considering the *free* semi-Heyting structure for the cut-free **LJ**. Here, t_α denotes the algebraic term corresponding to a given formula α. An *inequality* is an expression of the form $\langle s_1, \ldots, s_m \rangle \preceq t$ for terms $s_1, \ldots, s_m, t$. An inequality $\langle s_1, \ldots, s_m \rangle \preceq t$ is said to hold in a semi-Heyting structure $\mathbf{Q}$, if for every valuation h on $\mathbf{Q}$ $\{h(s_1), \ldots, h(s_m)\} \preceq h(t)$ holds in $\mathbf{Q}$.

LEMMA 14 *For any sequent $\alpha_1, \ldots, \alpha_m \Rightarrow \beta$, $\alpha_1, \ldots, \alpha_m \Rightarrow \beta$ is provable in* **LJ** *without using cut rule if and only if $\langle t_{\alpha_1}, \ldots, t_{\alpha_m} \rangle \preceq t_\beta$ holds in every semi-Heyting structure.*

Suppose that $\mathbf{P} = \langle P, \wedge, \vee, \to, 0 \rangle$ is a Heyting algebra with the lattice order $\leq$. We define the relation $\preceq$ by the condition that $\{a_1, \ldots, a_m\} \preceq c$ if and only if $(a_1 \wedge \ldots \wedge a_m) \leq c$ holds in $\mathbf{P}$. Then, $\mathbf{P}' = \langle P, \preceq, \wedge, \vee, \to, 0 \rangle$ becomes a semi-Heyting structure. Thus, each Heyting algebra can be regarded naturally as a semi-Heyting structure by identifying $\mathbf{P}$ with $\mathbf{P}'$.

Conversely, suppose that the relation $\preceq$ of a given semi-Heyting structure $\mathbf{Q}$ satisfies moreover the following *transitivity condition*:

$x \preceq a$ and $ay \preceq c$ imply $xy \preceq c$,

which is an algebraic expression of cut rule. Then, we have the following.

1. The relation $\preceq_0$ which is the restriction of $\preceq$ between two elements of Q is transitive and hence is a quasi order,
2. $\{a_1, \ldots, a_m\} \preceq c$ if and only if $(a_1 \wedge \ldots \wedge a_m) \preceq c$,
3. $ax \preceq b$ if and only if $x \preceq a \to b$.

To show 3, it suffices to show the if-part. From $a \preceq a$ and $b \preceq b$, $(a \to b) \wedge a \preceq b$ follows. So, if $x \preceq a \to b$ then we can show that $ax = xa \preceq (a \to b) \wedge a$. Then, by the transitivity condition we have $ax \preceq b$. Now, if the relation $\preceq$ of a semi-Heyting structure $\mathbf{Q}$ satisfies the transitivity condition and moreover if $\preceq_0$ is a partial order, $\mathbf{Q}_0 = \langle Q, \wedge, \vee, \to, 0\rangle$ becomes a Heyting algebra with the lattice order $\preceq_0$.

Since any Heyting algebra can be regarded as a semi-Heyting structure in this way, we can see that for all terms $s_1, \ldots, s_m$ and t, if an inequality $\langle s_1, \ldots, s_m\rangle \preceq t$ holds for every semi-Heyting structure then $(s_1 \wedge \ldots \wedge s_m) \leq t$ holds for every Heyting algebra. Our Theorem 16, proved in the following, says that the converse holds also. This is the algebraic content of cut elimination theorem for **LJ**.

6 Quasi-completions and cut elimination theorems

Now suppose that an inequality $\langle s_1, \ldots, s_m\rangle \preceq t$ fails in a semi-Heyting structure $\mathbf{Q}$ under a valuation f, i.e. $\{f(s_1), \ldots, f(s_m)\} \preceq f(t)$ does not hold in $\mathbf{Q}$. Since Q^* forms a commutative monoid, we have that $\mathbf{D}_{\mathbf{Q}^*}$ is a complete commutative residuated lattice for any closure operator D on $\wp(Q^*)$ by using Corollary 3 in § 2.

Now we define a special closure operator C on $\wp(Q^*)$ as follows. For $x \in Q^*$ and $a \in Q$, define

$$[x; a] = \{w \in Q^* : xw \preceq a\}.$$

Define an operation C on $\wp(Q^*)$ as follows: For each subset X of Q^*,

$$C(X) = \bigcap\{[x; a] : X \subseteq [x; a] \text{ for } x \in Q^* \text{ and } a \in Q\ \}.$$

We can show that C is actually a closure operator in our sense, which satisfies $C(\{\varepsilon\}) = Q^*$. Moreover, we can show that $X \cap Y = X * Y$ whenever both X and Y are C-closed. In fact, it is obvious that $X \cap Y \subseteq X * Y$. Conversely, suppose that $xy \in X * Y$ where $x \in X$ and $y \in Y$. Suppose that $X = \bigcap_i \{[z_i; a_i] : i \in I\}$. Then, for each $i \in I$, $xz_i \preceq a_i$ and hence $xyz_i \preceq a_i$. This implies that $xy \in X$. Similarly, $xy \in Y$. Hence, $xy \in X \cap Y$. Thus, it turns out that $\mathbf{C}_{\mathbf{Q}^*}$ is a complete Heyting algebra. The Heyting algebra $\mathbf{C}_{\mathbf{Q}^*}$ is uniquely determined by a given semi-Heyting structure $\mathbf{Q}$. So we call it the *quasi-completion* of $\mathbf{Q}$. Corollary 19 below will explain the reason why we call it in this way.

Let us define a map $k : Q \to C(\wp(Q^*))$ by $k(a) = [a]$, where $[a] = [\varepsilon; a]$, i.e. $[a] = \{w \in Q^* : w \preceq a\}$. The following theorem can be shown essentially in the same way as the proof of Lemma 7.3 in [11] (see also Maehara [16] and Okada [19]).

THEOREM 15 *Suppose that $a, b \in Q$ and that U and V are arbitrary C-closed subsets of Q^* such that $a \in U \subseteq k(a)$ and $b \in V \subseteq k(b)$. Then for each $\star \in \{\wedge, \vee, \rightarrow\}$, $a \star b \in U \star_C V \subseteq k(a \star b)$, where $\star_C$ denotes $\cap, \cup_C$ and $\Rightarrow$, respectively. Thus, in particular $a \star b \in k(a) \star_C k(b) \subseteq k(a \star b)$.*

Now we will show the following theorem.

THEOREM 16 *For all terms $s_1, \ldots, s_m$ and t, if $(s_1 \wedge \ldots \wedge s_m) \leq t$ holds in every Heyting algebra, then $\langle s_1, \ldots, s_m \rangle \preceq t$ holds in every semi-Heyting structure.*

Suppose that $\langle s_1, \ldots, s_m \rangle \preceq t$ doesn't hold in a semi-Heyting structure $\mathbf{Q}$. Then, there exists a valuation f on $\mathbf{Q}$ such that $\{f(s_1), \ldots, f(s_m)\} \preceq f(t)$ is not true in $\mathbf{Q}$. We define a valuation g on the quasi-completion $\mathbf{C}_{\mathbf{Q}^*}$ of $\mathbf{Q}$ as follows. For each variable q appearing in one of $s_1, \ldots, s_m$ and t, $g(q) = k(f(q))$. (For the constant 0 we suppose that $f(0) = 0$ for f, and define that $g(0) = C\emptyset$ ($\subseteq k(0)$).) Then, using Theorem 15, we can prove with the unit ε the following by induction on the length of the term r. with the unit ε

LEMMA 17 *For each subterm r of one of $s_1, \ldots, s_m$ and t, $\{f(r)\} \in g(r) \subseteq k(f(r))$.*

Suppose that $(s_1 \wedge \ldots \wedge s_m) \leq t$ holds in $\mathbf{C}_{\mathbf{Q}^*}$. Then $(g(s_1) \cap \ldots \cap g(s_m)) \subseteq g(t)$, in particular. Using Lemma 17, $\{f(s_i)\} \in g(s_i)$ for each i and hence $\{f(s_1), \ldots, f(s_m)\} \in g(s_i)$. Thus,

$$\{f(s_1), \ldots, f(s_m)\} \in (g(s_1) \cap \ldots \cap g(s_m)) \subseteq g(t) \subseteq k(f(t)),$$

by Lemma 17. But this implies $\{f(s_1), \ldots, f(s_m)\} \preceq f(t)$, which is a contradiction. Therefore, the inequality $(s_1 \wedge \ldots \wedge s_m) \leq t$ doesn't hold in the Heyting algebra $\mathbf{C}_{\mathbf{Q}^*}$. This completes the proof.

COROLLARY 18 *For all terms $s_1, \ldots, s_m$ and t, $(s_1 \wedge \ldots \wedge s_m) \leq t$ holds in every Heyting algebra if and only if $\langle s_1, \ldots, s_m \rangle \preceq t$ follows from the system of basic "inequalities" by using rules for "inequalities" described in the definition of semi-Heyting structures. Thus, the sequent system* **LJ** *without cut rule is complete with respect to the class of all Heyting algebras.*

As shown essentially in [16], in order to discuss sequent systems like **LK** where each sequent is of the form $\Gamma \Rightarrow \Delta$ with sequences of formulas Γ and Δ, we need to make an obvious modification of the definition of $\preceq$. That is, $\preceq$ must be defined as a binary relation on Q^*. Also, instead of taking $[x; a]$ we need to define $[x; y]$ for $x, y \in Q^*$ by

$$[x; y] = \{\langle u, w\rangle \in (Q^*)^2 : xu \preceq yw\},$$

and then define a closure operator C on $\wp((Q^*)^2)$ by using them. For **LK**, we define $\sim X$ for each subset X of $(Q^*)^2$ by

$$\sim X = \bigcap\{[x; y] : \langle x, y\rangle \in X\},$$

which plays a role of the complement of X. In fact, we can show that $\sim\sim X = C(X)$ for any X, and therefore the law of double negation holds when each C-closed X. For modal logics, we introduce $\Diamond X$ for a subset X of $(Q^*)^2$ by

$$\Diamond X = \bigcap\{[\Box x; \Diamond y] : X \subseteq [x; y]\},$$

where $\Box x$ ($\Diamond x$) is the element $\{\Box a_1, \ldots, \Box a_m\}$ ($\{\Diamond a_1, \ldots, \Diamond a_m\}$, respectively) of Q^* when x is $\{a_1, \ldots, a_m\}$. By using this, we can get an algebraic proof of cut elimination theorem for sequent systems of some of basic modal logics, including **K**, **KT** and **S4**.

By modifying our algebraic proof of cut elimination theorems and using ideas developed by Lafont and Okada-Terui [15, 20], we can show moreover the finite model property of a logic which has a cut-free sequent system, if every unprovable formula of it has a finite *proof search tree* in the sequent system. The details will be discussed in [1]. Usually, the finite model property is shown in order to derive the decidability. So, the above statement may sound strange, as we use the existence of decision procedures to show the finite model property. But, this has happened in the study of substructural logics, where decidability results on most of basic substructural logics were shown much earlier than those of the finite model property (see [23, 17, 15, 20]). Another promising way of obtaining the finite model property of substructural logics is given in the paper Blok-van Alten [2], where the *finite embeddability property* of the class of algebras for a given substructural logic is used.

In the last, we will remark that the quasi-completion of any Heyting algebra $\mathbf{P}$ is nothing but the MacNeille completion. As mentioned above, each Heyting algebra $\mathbf{P} = \langle P, \wedge, \vee, \rightarrow, 0, 1\rangle$ with the lattice order $\leq$ determines naturally a a semi-Heyting structure $\mathbf{P}'$, by identifying each element $z = \{a_1, \ldots, a_m\}$ in P^* with an element $\tilde{z} = a_1 \wedge \ldots \wedge a_m$ in P, and taking $\leq$ for $\preceq$.

Under this identification, each set of the form $[x; a]$ will be regarded as a subset $\{b \in P : b \wedge \tilde{x} \leq a\}$, or equivalently $\{b \in P : b \leq \tilde{x} \rightarrow a\}$, which we write also as $[\tilde{x} \rightarrow a]$ by abuse of symbols. Of course, each set of the form $[c]$ for $c \in P$ is downward closed since $\leq$ is transitive. Thus, we can define an operation C on $\wp(P)$, instead of $\wp(P^*)$ by

$C(X) = \bigcap\{[c] : X \subseteq [c]$ for an element $c \in P\}$ for each subset X of P.

This is equivalent to the condition that $C(X) = ((X)^{\rightarrow})^{\leftarrow}$. Thus, the quasi-completion of a Heyting algebra $\mathbf{P}$ is identified with the Heyting algebra $\mathbf{C_P}$, which is the MacNeille completion of $\mathbf{P}$. Moreover, in the present case, by using $a \star b \in k(a) \star_C k(b)$ in Theorem 15, we have $k(a \star b) \subseteq k(a) \star_C k(b)$ and hence $k(a \star b) = k(a) \star_C k(b)$. Since k is injective, we have the following when identifying a Heyting algebra $\mathbf{P}$ with the corresponding semi-Heyting structure $\mathbf{P}'$.

COROLLARY 19 *The quasi-completion of a Heyting algebra is equal to the MacNeille completion, and moreover the map k defined by $k(a) = [a]$ for each $a \in P$ is an embedding of $\mathbf{P}$ into $\mathbf{C_P}$.*

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