14

Restricted Interpolation in Modal Logics

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ABSTRACT. A new variant IPR of interpolation is introduced and its interrelations with the known interpolation properties and various variants of the Beth property are investigated. It is stated that on the class of modal logics IPR follows from the usual interpolation properties CIP and IPD; on the other hand, there are modal logics that have IPR and do not have IPD. It is proved that IPR follows from the projective Beth property PB2. An algebraic equivalent of IPR, namely, the restricted amalgamation property, is found.

Interpolation theorem proved by W. Craig [4] in 1957 for the first order logic was a source of a lot of investigations devoted to interpolation problem in various logical theories [1, 3]. Interpolation is considered as a desirable and "nice" property; also it has important practical applications in computer science [3].

The most known systems of modal logic, for instance, Lewis' systems S4 and S5, Grzegorczyk's logic Grz, the logic GL of provability, the logic K4 and the least normal modal logic K possess the interpolation property. On the other hand, it appeared that this important property is rather rare. For instance, there is a continuum of normal extensions of the propositional modal logic S4 but only finitely many of these logics possess the interpolation [7]. Moreover, there are two natural variants CIP and IPD of the interpolation property which are not equivalent on the class of modal logics, more exactly, IPD is weaker than CIP.

Due to great significance of the interpolation, its weaker versions may also be useful. In the present paper we introduce a new variant IPR of interpolation that seems to be very natural. One can show that there are modal logics with IPR and without IPD. For example, such is the logic which is characterized by one 3-element linearly ordered frame. On the other hand, IPR is not trivial, for instance, one can prove that the logic S4.3 does not possess this property.

Interpolation properties are closely connected with the Beth properties which have arisen due to the well-known theorem on implicit definability

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found by E. Beth [2]. Correlations of variants of the Beth properties and interpolation are widely presented [4, 1, 15, 8, 5].

In [9] we considered two versions PB1 and PB2 of the projective Beth property and proved that in normal modal logics the property PB1 is equivalent to the Craig interpolation property CIP, whereas the weaker versions IPD and PB2 are independent.

In investigations of the projective Beth property PB2 in superintuitionistic logics [11] some interesting connections of PB2 with interpolation and amalgamation properties were found. First of all, in these logics PB2 is implied by interpolation which is equivalent to the amalgamation property AP in associated varieties of Heyting algebras. On the other hand, PB2 implies some restricted variant of amalgamation that is proved to be weaker than AP.

We extend the notion of restricted amalgamation to arbitrary varieties. Also we prove that the restricted version IPR of interpolation property is equivalent to the restricted amalgamation in varieties of modal or Heyting algebras and show that IPR follows from PB2. In addition we find an algebraic criterion for PB2 with use of the restricted amalgamation property.

1 Beth properties and interpolation

If $\mathbf{p}$ is a list of variables, let $A(\mathbf{p})$ denote a formula whose all variables are in $\mathbf{p}$, and $\mathcal{F}(\mathbf{p})$ the set of all such formulas.

Let L be a logic, $\vdash_L$ deducibility relation in L. Suppose that $\mathbf{p}$, $\mathbf{q}$, $\mathbf{q}'$ are disjoint lists of variables that do not contain x and y, $\mathbf{q}$ and $\mathbf{q}'$ are of the same length, and $A(\mathbf{p},\mathbf{q},x)$ is a formula. We say that the logic L possesses *the projective Beth property PB1*, whenever $\vdash_L A(\mathbf{p},\mathbf{q},x)\&A(\mathbf{p},\mathbf{q}',y) \to (x \leftrightarrow y)$ implies $\vdash_L A(\mathbf{p},\mathbf{q},x) \to (x \leftrightarrow B(\mathbf{p}))$ for some formula $B(\mathbf{p})$. Further, L has *the projective Beth property PB2*, if $A(\mathbf{p},\mathbf{q},x), A(\mathbf{p},\mathbf{q}',y) \vdash_L x \leftrightarrow y$ implies $A(\mathbf{p},\mathbf{q},x) \vdash_L x \leftrightarrow B(\mathbf{p})$ for some $B(\mathbf{p})$; L possesses *the Beth property B1*, whenever $\vdash_L A(\mathbf{p},x)\&A(\mathbf{p},y) \to (x \leftrightarrow y)$ implies $\vdash_L A(\mathbf{p},x) \to (x \leftrightarrow B(\mathbf{p}))$ for a suitable formula $B(\mathbf{p})$; L has *the Beth property B2* if $A(\mathbf{p},x), A(\mathbf{p},y) \vdash_L x \leftrightarrow y$ implies $A(\mathbf{p},x) \vdash_L x \leftrightarrow B(\mathbf{p})$ for a formula $B(\mathbf{p})$.

The Beth properties are closely connected with *the interpolation properties CIP* and *IPD* defined as follows (where the lists $\mathbf{p},\mathbf{q},\mathbf{r}$ are disjoint):

CIP. If $\vdash_L A(\mathbf{p},\mathbf{q}) \to B(\mathbf{p},\mathbf{r})$, then there exists a formula $C(\mathbf{p})$ such that $\vdash_L A(\mathbf{p},\mathbf{q}) \to C(\mathbf{p})$ and $\vdash_L ?\mathbf{p}) \to B(\mathbf{p},\mathbf{r})$.

IPD. If $A(\mathbf{p},\mathbf{q}) \vdash_L B(\mathbf{p},\mathbf{r})$, then there exists a formula $C(\mathbf{p})$ such that $A(\mathbf{p},\mathbf{q}) \vdash_L C(\mathbf{p})$ and $?\mathbf{p}) \vdash_L B(\mathbf{p},\mathbf{r})$.

The most known modal logics such as Lewis' systems S4 and S5, Grzegorczyk's logic Grz, the logic GL of provability, the logic K4 and the least normal modal logic K have the properties CIP and PB1. We mean that the language contains at least one propositional constant $\top$ ("truth") or $\bot$ ("false").

A *normal modal logic* is any set of modal formulas containing all the tautologies of the two-valued propositional logic and the axioms $\Box(A \to B) \to (\Box A \to \Box B)$, and closed under the inference rules R1: $A, A \to B/B$ and R2: $A/\Box A$, and the substitution rule. The set of normal modal logics containing a normal modal logic L is denoted by $NE(L)$. If L is a normal modal logic, by $\vdash_L$ we denote deducibility in L by the rules R1 and R2. The well-known deduction theorem in normal modal logics says that

$$\Gamma, A \vdash_L B \iff \Gamma \vdash_L [n]A \to B \text{ for some } n \geq 0,$$

where $[n]A \rightleftharpoons A \& \Box A \& \ldots \& \Box^n A$.

Interrelations of interpolation with Beth properties are presented in the following

PROPOSITION 1 [9, 12] *In the family of normal modal logics*

a) PB1 is equivalent to each of the properties CIP and B1;

b) PB1 implies the conjunction of B2 and IPD but the converse does not hold;

c) the conjunction of B2 and IPD implies PB2;

d) PB2 implies B2 but the converse is not true,

e) PB2 and IPD are independent.

We introduce a *restricted interpolation property*:

IPR. If $A(\mathbf{p},\mathbf{q}), B(\mathbf{p},\mathbf{r}) \vdash_L C(\mathbf{p})$, then there exists a formula $A'(\mathbf{p})$ such that $A(\mathbf{p},\mathbf{q}) \vdash_L A'(\mathbf{p})$ and $A'(\mathbf{p}), B(\mathbf{p},\mathbf{r}) \vdash_L C(\mathbf{p})$.

By the deduction theorem, one can easily show that IPD implies IPR in modal logics. On the other hand, in this paper we prove that PB2 implies IPR. Since PB2 does not imply IPD [12], we get that IPD does not follow from IPR.

2 Restricted interpolation and restricted amalgamation

In this section we find an algebraic equivalent of IPR. For the properties B1 and B2, and also for CIP and IPD it was done in [8].

It is well known that there exists a duality between normal modal logics and varieties of modal algebras. *A modal algebra* is an algebra $\mathbf{A} = (A, \to, \neg, \Box)$ that is a boolean algebra with respect to $\to$ and $\neg$ and, moreover, satisfies the conditions $\Box\top = \top$ and $\Box(x \to y) \leq \Box x \to \Box y$. A modal algebra $\mathbf{A}$ is called *transitive* if it satisfies an inequality $\Box x \leq \Box\Box x$; *a topoboolean algebra*, or *interior algebra* if it satisfies $\Box x \leq x$; a *diagonalizable* algebra if it satisfies $\Box(\Box x \to x) = \Box x$. It is well known that the modal logic K4 can be characterized by the variety of all transitive algebras, S4 by the variety of topoboolean algebras, GL by diagonalizable algebras.

By a *valuation* in an algebra $\mathbf{A}$, we mean, as usually, a homomorphism from the algebra of all formulas into $\mathbf{A}$.

If A is a formula, $\mathbf{A}$ a modal algebra, then $\mathbf{A} \models A$ denotes that the identity $A = \top$ is satisfied in $\mathbf{A}$. We write $\mathbf{A} \models L$ instead of $(\forall A \in L)(\mathbf{A} \models A)$. We denote $V(L) = \{\mathbf{A}|\mathbf{A} \models L\}$. Each normal modal logic L is characterized by the variety $V(L)$.

A set $\Gamma \subseteq \mathcal{F}(\mathbf{p})$ is called an *open L-theory of the language* $\mathcal{F}(\mathbf{p})$ whenever it is closed with respect to $\vdash_L$, i.e. for every formula $F(\mathbf{p})$, $\Gamma \vdash_L F(\mathbf{p})$ implies $F(\mathbf{p}) \in \Gamma$. With each open L-theory T of the language $\mathcal{F}(\mathbf{p})$, one can associate an equivalence relation on the set $\mathcal{F}(\mathbf{p})$:

$$A \sim_T B \rightleftharpoons (A \leftrightarrow B) \in T.$$

Due to the replacement lemma, $\sim_T$ is a congruence on $\mathcal{F}(\mathbf{p})$. It makes possible to define *the Lindenbaum-Tarski algebra*

$$\mathbf{A}(\mathbf{p}, T) = \mathcal{F}(\mathbf{p})/\sim_T$$

as a quotient-algebra of $\mathcal{F}(\mathbf{p})$. Let us denote $\|A\| = \{B|A \sim_T B\}$ for each formula $A = A(\mathbf{p})$. The following statement holds (see, for instance,[14]):

LEMMA 2 *Let $L \in NE(K)$. For each open L-theory T of the language $\mathcal{F}(\mathbf{p})$, the algebra $\mathbf{A}(\mathbf{p}, T)$ is in $V(L)$; the canonical mapping $\varkappa : \mathcal{F} \to \mathbf{A}(\mathbf{p}, T)$, where for every formula $A = A(\mathbf{p})$*

$$\varkappa(A) = \|A\| = A/\sim_T,$$

is a homomorphism and, moreover, $A \in T \Leftrightarrow \|A\| = \top$ in $\mathbf{A}(\mathbf{p}, T)$.

In particular, $\mathbf{A}(\mathbf{p}, L)$ is a free algebra of $V(L)$, with free generators $\|p\|$, where $p \in \mathbf{p}$.

All varieties of modal algebras possess such important properties as congruence-distributivity and congruence extension property:

CEP. If $\mathbf{A}$ is a subalgebra of $\mathbf{B}$ then every congruence Φ on $\mathbf{A}$ can be extended to a congruence Ψ on $\mathbf{B}$ such that $\Psi \cap \mathbf{A}^2 = \Phi$.

A logic L has CIP if and only if $V(L)$ is super-amalgamable, hence B1 and PB1 are also equivalent to the super-amalgamation property of the corresponding variety; IPD is equivalent to the amalgamation property AP [8]. We introduce a notion of the restricted amalgamation property RAP and prove its equivalence to IPR. Moreover, in Section 3 we find an algebraic criterion for PB2 with use of RAP.

Remind that a modal algebra is *subdirectly irreducible* (*finitely indecomposable*) if it can not be represented as a subdirect product (finite subdirect product) of its proper quotient algebras. For any class V of algebras, by $FI(V)$ and $SI(V)$ we denote the classes of finitely indecomposable and subdirectly irreducible algebras in V respectively, $FG(V)$ stands for finitely generated algebras in V.

A class V has *Amalgamation Property* if it satisfies the condition

AP. For each $\mathbf{A}, \mathbf{B}, \mathbf{C} \in V$ such that $\mathbf{A}$ is a common subalgebra of $\mathbf{B}$ and $\mathbf{C}$, there exist an algebra $\mathbf{D}$ in V and monomorphisms $\delta : \mathbf{B} \to \mathbf{D}$ and $\varepsilon : \mathbf{C} \to \mathbf{D}$ such that $\delta(x) = \varepsilon(x)$ for all $x \in \mathbf{A}$.

A class V has *Strong Amalgamation Property (StrAP)* if it satisfies AP and, moreover, $\delta(\mathbf{B}) \cap \varepsilon(\mathbf{C}) = \delta(\mathbf{A})$.

We say that a class V has *Restricted Amalgamation Property* if it satisfies the condition:

RAP. For each $\mathbf{A}, \mathbf{B}, \mathbf{C} \in V$ such that $\mathbf{A}$ is a common subalgebra of $\mathbf{B}$ and $\mathbf{C}$ there exist an algebra $\mathbf{D}$ in V and homomorphisms $\delta : \mathbf{B} \to \mathbf{D}$, $\varepsilon : \mathbf{C} \to \mathbf{D}$ such that $\delta(x) = \varepsilon(x)$ for all $x \in \mathbf{A}$ and the restriction δ' of δ onto $\mathbf{A}$ is a monomorphism.

In the same method as in [8] we prove

THEOREM 3 *A normal modal logic L has IPR iff $V(L)$ has RAP iff $FG(V(L))$ has RAP.*

Proof. Assume that L has IPR. We prove that $V(L)$ has RAP. Let $\mathbf{A}, \mathbf{B}, \mathbf{C} \in V(L)$ and $\mathbf{A}$ be a subalgebra of both $\mathbf{B}$ and $\mathbf{C}$. We take the sets of propositional variables $\mathbf{p} = \{p_a | a \in \mathbf{A}\}$, $\mathbf{q} = \{q_b | b \in \mathbf{B} - \mathbf{A}\}$, $\mathbf{r} = \{r_c | c \in \mathbf{C} - \mathbf{A}\}$ and define the valuations $v' : \mathbf{p} \cup \mathbf{q} \to \mathbf{B}$ and $v'' : \mathbf{p} \cup \mathbf{r} \to \mathbf{C}$ by

$$v'(p_a) = v''(p_a) = a,\ v'(q_b) = b,\ v''(r_c) = c.$$

We denote

$$\begin{array}{c} T_1 = \{B | B \in \mathcal{F}(\mathbf{p}, \mathbf{q}) \text{ and } v'(B) = \top_{\mathbf{B}}\}, \\ T_2 = \{C | C \in \mathcal{F}(\mathbf{p}, \mathbf{r}) \text{ and } v''(C) = \top_{\mathbf{C}}\}, \\ T = \{D | D \in \mathcal{F}(\mathbf{p}, \mathbf{q}, \mathbf{r}) \text{ and } T_1 \cup T_2 \vdash_L D\}. \end{array} .$$

Then T_1, T_2 and T are open L-theories of the languages $\mathcal{F}(\mathbf{p}, \mathbf{q}), \mathcal{F}(\mathbf{p}, \mathbf{r})$ and $\mathcal{F}(\mathbf{p}, \mathbf{q}, \mathbf{r})$ respectively.

Let $\mathcal{F} = \mathcal{F}(\mathbf{p}, \mathbf{q}, \mathbf{r})$ and $\mathbf{D} = \mathbf{A}(\mathbf{p} \cup \mathbf{q} \cup \mathbf{r}, T) = \mathcal{F} / \sim_T$ be a Lindenbaum-Tarski algebra, where $D \sim D' \rightleftharpoons T_1, T_2 \vdash D \leftrightarrow D'$.

Due to $T_1, T_2 \subseteq T$, the mappings $\delta : \mathbf{B} \to \mathbf{D}$ and $\varepsilon : \mathbf{C} \to \mathbf{D}$ defined by $\delta(a) = \varepsilon(a) = \|p_a\|$ for $a \in \mathbf{A}$, $\delta(b) = \|q_b\|$ for $b \in \mathbf{B} - \mathbf{A}$, and $\varepsilon(c) = \|r_c\|$ for $c \in \mathbf{C} - \mathbf{A}$, are homomorphisms that coincide on $\mathbf{A}$. Assume that $\delta(a) = \delta(a')$ for some $a, a' \in \mathbf{A}$. Then we have

$$T_1, T_2 \vdash_L p_a \leftrightarrow p_{a'}.$$

Since all derivations are finite, it follows that there are some formulas $B(\mathbf{p}, \mathbf{q}) \in T_1$ and $C(\mathbf{p}, \mathbf{r}) \in T_2$ such that

$$B(\mathbf{p}, \mathbf{q}), C(\mathbf{p}, \mathbf{r}) \vdash_L p_a \leftrightarrow p_{a'}.$$

By IPR there exists $B'(\mathbf{p})$ such that

$$B(\mathbf{p}, \mathbf{q}) \vdash B'(\mathbf{p}) \text{ and } B'(\mathbf{p}), C(\mathbf{p}, \mathbf{r}) \vdash_L p_a \leftrightarrow p_{a'}.$$

We get that $v'(B(\mathbf{p}, \mathbf{q})) = v'(B'(\mathbf{p})) = \top_{\mathbf{B}}$, so $v''(B'(\mathbf{p})) = v'(B'(\mathbf{p})) = \top_{\mathbf{A}} = \top_{\mathbf{C}}$ and $B'(\mathbf{p}) \in T_2$. It follows that $(p_a \leftrightarrow p_{a'}) \in T_2$, hence $v''(p_a \leftrightarrow p_{a'}) = \top_{\mathbf{C}}$ and $a = v''(p_a) = v''(p_{a'}) = a'$. Thus $V(L)$ has RAP.

It is clear that $FG(V(L))$ has RAP whenever $V(L)$ has RAP.

Assume that $FG(V(L))$ has RAP. Suppose that $A(\mathbf{p}, \mathbf{q}), B(\mathbf{p}, \mathbf{r}) \vdash_L C(\mathbf{p})$. We consider

$$T = \{A'(\mathbf{p}) \mid A(\mathbf{p}, \mathbf{q}) \vdash_L A'(\mathbf{p})\}$$

and prove that

(1) $T, B(\mathbf{p}, \mathbf{q}) \vdash_L C(\mathbf{p})$.

Let us define

$$T_2 = \{F(\mathbf{p}, \mathbf{r}) \mid T, B(\mathbf{p}, \mathbf{r}) \vdash_L F(\mathbf{p}, \mathbf{r})\},\ T_0 = T_2 \cap \mathcal{F}(\mathbf{p}),$$

$$T_1 = \{F(\mathbf{p}, \mathbf{q}) \mid A(\mathbf{p}, \mathbf{q}), T_0 \vdash_L F(\mathbf{p}, \mathbf{q})\}.$$

We show that $T_1 \cap \mathcal{F}(\mathbf{p}) = T_0$.

Assume $F(\mathbf{p}) \in T_1$. Then $A(\mathbf{p}, \mathbf{q}), T_0 \vdash_L F(\mathbf{p})$, so by the deduction theorem we have $A(\mathbf{p}, \mathbf{q}) \vdash_L [n]A_1(\mathbf{p}) \to F(\mathbf{p})$ for some $n \geq 0$ and some $A_1(\mathbf{p}) \in T_0$. It follows that $([n]A_1(\mathbf{p}) \to F(\mathbf{p})) \in T \subseteq T_0$, $[n]A_1(\mathbf{p}) \in T_0$, and so $F(\mathbf{p}) \in T_0$. Thus we have

$$F(\mathbf{p}) \in T_2 \iff F(\mathbf{p}) \in T_0 \iff F(\mathbf{p}) \in T_1$$

for any formula $F(\mathbf{p})$.

We set $\mathbf{B} = \mathcal{F}(\mathbf{p},\mathbf{q})/\sim_{T_1}$, $\mathbf{C} = \mathcal{F}(\mathbf{p},\mathbf{r})/\sim_{T_2}$; let $\mathbf{A}$ be a subalgebra of $\mathbf{B}$ generated by $\mathbf{p}/T_1$. Then $\mathbf{A}$ is embeddable into $\mathbf{C}$ by a monomorphism $\varphi(p/\sim_{T_1}) = p/\sim_{T_2}$ for $p \in \mathbf{p}$. It is clear from Lemma 2 that $\mathbf{A},\mathbf{B},\mathbf{C} \in V(L)$. By RAP there exist $\mathbf{D} \in V(L)$ and homomorphisms $\delta : \mathbf{B} \to \mathbf{D}$, $\varepsilon : \mathbf{C} \to \mathbf{D}$ such that for every $p \in \mathbf{p}$ we have $\delta(p/\sim_{T_1}) = \varepsilon\varphi(p/\sim_{T_1}) = \varepsilon(p/\sim_{T_2})$ and the restriction δ' of δ onto $\mathbf{A}$ is a monomorphism.

Let us consider the valuation

$$\begin{aligned} v(p) &= \delta(p/\sim_{T_1}) \text{ for } p \in \mathbf{p}, \\ v(q) &= \delta(q/\sim_{T_1}) \text{ for } q \in \mathbf{q}, \ . \\ v(r) &= \varepsilon(r/\sim_{T_2}) \text{ for } r \in \mathbf{r}. \end{aligned}$$

We note that $A(\mathbf{p},\mathbf{q}) \in T_1$, so $v(A(\mathbf{p},\mathbf{q})) = \delta(A(\mathbf{p},\mathbf{q})/\sim_{T_1}) = \top_{\mathbf{D}}$. In addition, $B(\mathbf{p},\mathbf{r}) \in T_2$, so $v(B(\mathbf{p},\mathbf{r})) = \varepsilon(B(\mathbf{p},\mathbf{r})/\sim_{T_2}) = \top_{\mathbf{D}}$. Due to $A(\mathbf{p},\mathbf{q}), B(\mathbf{p},\mathbf{r}) \vdash_L C(\mathbf{p})$, we obtain $v(C(\mathbf{p})) = \delta(C(\mathbf{p})/\sim_{T_1}) = \top_{\mathbf{D}}$. Since the restriction of δ onto $\mathbf{A}$ is a monomorphism, we get $C(\mathbf{p})/\sim_{T_1} = \top_{\mathbf{B}}$ and $C(\mathbf{p}) \in T_1$, so $C(\mathbf{p}) \in T_2$. Thus $T, B(\mathbf{p},\mathbf{q}) \vdash_L C(\mathbf{p})$, and we proved (1).

Since all derivations are finite, there exist a finite subset Γ of T such that $\Gamma, B(\mathbf{p},\mathbf{q}) \vdash_L C(\mathbf{p})$. Taking the conjunction of all formulas in Γ as $A'(\mathbf{p})$, we obtain $A(\mathbf{p},\mathbf{q}) \vdash_L A'(\mathbf{p})$ and $A'(\mathbf{p}), B(\mathbf{p},\mathbf{q}) \vdash_L C(\mathbf{p})$. ■

We bring out various equivalents of RAP in

PROPOSITION 4 *For every variety V of modal algebras the following are equivalent:*

(1) V has RAP;

(2) for any finitely indecomposable $\mathbf{A},\mathbf{B},\mathbf{C}$ in V such that $\mathbf{A}$ is a subalgebra of both $\mathbf{B}$ and $\mathbf{C}$ there exist $\mathbf{D}$ in V and homomorphisms $g : \mathbf{B} \to \mathbf{D}$, $h : \mathbf{C} \to \mathbf{D}$ such that $g(z) = h(z)$ for all z in $\mathbf{A}$ and the restriction of g onto $\mathbf{A}$ is a monomorphism;

(3) for any subdirectly irreducible $\mathbf{A},\mathbf{B},\mathbf{C}$ in V such that $\mathbf{A}$ is a subalgebra of both $\mathbf{B}$ and $\mathbf{C}$ there exist $\mathbf{D}$ in V and homomorphisms $g : \mathbf{B} \to \mathbf{D}$, $h : \mathbf{C} \to \mathbf{D}$ such that $g(z) = h(z)$ for all z in $\mathbf{A}$ and the restriction of g onto $\mathbf{A}$ is a monomorphism;

(4) $SI(V)$ has RAP;

(5) for any finitely generated and finitely indecomposable $\mathbf{A},\mathbf{B},\mathbf{C}$ in V such that $\mathbf{A}$ is a subalgebra of both $\mathbf{B}$ and $\mathbf{C}$ and for any $b, c \in \mathbf{A}$, $b \neq c$, there exist a subdirectly irreducible algebra $\mathbf{D}$ in V and homomorphisms $g : \mathbf{B} \to \mathbf{D}$, $h : \mathbf{C} \to \mathbf{D}$ such that $g(z) = h(z)$ for all z in $\mathbf{A}$ and $g(b) \neq g(c)$.

Proof. The proposition can be proved by analogy with the proof of the similar statement concerning the amalgamation property AP ([8], Theorem 2). It is clear that (1) $\Rightarrow$ (2) $\Rightarrow$ (3). We prove

(3) $\Rightarrow$ (4). Let $\mathbf{A}, \mathbf{B}, \mathbf{C} \in SI(V)$ such that $\mathbf{A}$ is a subalgebra of both $\mathbf{B}$ and $\mathbf{C}$. Then there exist an algebra $\mathbf{D}$ in V and homomorphisms $g : \mathbf{B} \to \mathbf{D}$, $h : \mathbf{C} \to \mathbf{D}$ such that $g(z) = h(z)$ for all z in $\mathbf{A}$ and the restriction g' of g onto $\mathbf{A}$ is a monomorphism. Since $g(\mathbf{A})$ is subdirectly irreducible, there exists a pair (a, a') of elements in $g(\mathbf{A})$ such that $a \neq a'$ and every non-zero congruence on $g(\mathbf{A})$ contains (a, a'). Take any congruence Φ on $\mathbf{D}$ which is maximal among the congruences on $\mathbf{D}$ not containing (a, a'). Then the algebra $\mathbf{D}' = \mathbf{D}/\Phi$ is subdirectly irreducible and the homomorphisms $g_1 : \mathbf{B} \to \mathbf{D}'$ and $h_1 : \mathbf{C} \to \mathbf{D}$, where $g_1(b) = g(b)/\Phi$ and $h_1(c) = h(c)/\Phi$, coincide on $\mathbf{A}$. Moreover, the restriction of g_1 onto $\mathbf{A}$ is a monomorphism since $(a, a') \notin \Phi$.

(4) $\Rightarrow$ (5). Let $\mathbf{A}, \mathbf{B}, \mathbf{C} \in V$, $\mathbf{A}$ be a subalgebra of both $\mathbf{B}$ and $\mathbf{C}$, and $b, c \in \mathbf{A}$, $b \neq c$. We take any congruence Φ on $\mathbf{B}$ which is maximal among the congruences on $\mathbf{B}$ not containing (b, c). Then $\Phi_0 = \Phi \cap \mathbf{A}^2$ is a congruence on $\mathbf{A}$. By CEP it may be extended to a congruence Ψ on $\mathbf{C}$ such that $\Psi \cap \mathbf{A}^2 = \Phi_0$. By Zorn's lemma we can take Ψ to be maximal among the congruences with this property. We note that $(b, c) \notin \Psi$ and any proper extension of Ψ must contain the pair (b, c). It means that the algebras $\mathbf{A}' = \mathbf{A}/\Phi_0$, $\mathbf{B}' = \mathbf{B}/\Phi$ and $\mathbf{C}' = \mathbf{C}/\Psi$ are subdirectly irreducible. Moreover, $\mathbf{A}'$ is embeddable into $\mathbf{B}'$ by a monomorphism $\varphi(a/\Phi_0) = a/\Phi$ and into $\mathbf{C}'$ by a monomorphism $\psi(a/\Phi_0) = a/\Psi$. Since $SI(V)$ has RAP, there exist $\mathbf{D} \in SI(V)$ and homomorphisms $g : \mathbf{B}' \to \mathbf{D}$, $h : \mathbf{C}' \to \mathbf{D}$ such that $g\varphi(z) = h\psi(z)$ for all z in $\mathbf{A}'$ and the restriction of $g\varphi$ onto $\mathbf{A}'$ is a monomorphism.

We define $g'(y) = g(y/\Phi)$ for $y \in \mathbf{B}$ and $h'(z) = h(z/\Psi)$ for $z \in \mathbf{C}$. Then g' and h' are the required homomorphisms.

(5) $\Rightarrow$ (1). Due to Theorem 3 it is sufficient to prove that $FG(V)$ has RAP. Let $\mathbf{A}, \mathbf{B}, \mathbf{C} \in FG(V)$ and $\mathbf{A}$ be a subalgebra of both $\mathbf{B}$ and $\mathbf{C}$. For any $b, c \in \mathbf{A}$, $b \neq c$, we take a congruence $\Phi(b, c)$ on $\mathbf{B}$ that is maximal among the congruences on $\mathbf{B}$ not containing the pair (b, c). By analogy with the proof of (4)$\Rightarrow$(5) we construct a congruence $\Psi(b, c)$ on $\mathbf{C}$ that is maximal among the congruences Ψ on $\mathbf{C}$ such that $\Psi \cap \mathbf{A}^2 = \Phi(b, c) \cap \mathbf{A}^2 = \Phi_0$. Then the algebra $\mathbf{A}' = \mathbf{A}/\Phi_0$ is embeddable into both $\mathbf{B}' = \mathbf{B}/\Phi(b, c)$ and $\mathbf{C}' = \mathbf{C}/\Psi(b, c)$, and all the three algebras are in $FGFI(V)$. Also we have $b/\Phi_0 \neq c/\Phi_0$. By (5) there exist an algebra $\mathbf{D}_{bc}$ and homomorphisms $g_{bc} : \mathbf{B}' \to \mathbf{D}_{bc}$ and $h_{bc} : \mathbf{C}' \to \mathbf{D}_{bc}$ such that

$g_{bc}(x/\Phi(b, c)) = h_{bc}(x/\Psi(b, c))$ for all $x \in \mathbf{A}$ and

$g_{bc}(b/\Phi(b, c)) \neq g_{bc}(c/\Phi(b, c))$.

We take $\mathbf{D} = \prod_{b,c \in \mathbf{A}; b \neq c} \mathbf{D}_{bc}$ and define $g : \mathbf{B} \to \mathbf{D}$ and $h : \mathbf{C} \to \mathbf{D}$ as follows:

$$g(y)(b, c) = g_{bc}(y/\Phi(b, c)) \text{ for } y \in \mathbf{B},$$

$$h(z)(b,c) = h_{bc}(z/\Psi(b,c)) \text{ for } z \in \mathbf{C}.$$

Then $g(x) = h(x)$ for all $x \in \mathbf{A}$. Moreover, $g(x) \neq g(x')$ for all $x, x' \in \mathbf{A}$ such that $x \neq x'$, since $g_{xx'}(x/\Phi(x,x')) \neq g_{xx'}(x'/\Phi(x,x'))$, and so $g(x)(x,x') \neq g(x'(x,x')$. Thus g and h are the required homomorphisms. ■

For transitive modal algebras we bring out some additional equivalents of RAP in Section 4.

3 A criterion for PB2 in modal logics

In [10] we found an algebraic equivalent of PB2 that is so-called strong epimorphisms surjectivity SES. It was proved in [11] that SES implies RAP in varieties of Heyting algebras. The analogous statement holds for varieties of transitive modal algebras [12]. In this section we extend this result to all modal algebras and find a useful criterion for PB2 with use of RAP.

We say that a class V has *Strong Epimorphisms Surjectivity* if it satisfies

(SES) For each $\mathbf{A}, \mathbf{B}$ in V, for every monomorphism $\alpha : \mathbf{A} \to \mathbf{B}$ and for every $x \in \mathbf{B} - \alpha(\mathbf{A})$ there exist $\mathbf{C} \in V$ and homomorphisms $\beta : \mathbf{B} \to \mathbf{C}$, $\gamma : \mathbf{B} \to \mathbf{C}$ such that $\beta\alpha = \gamma\alpha$ and $\beta(x) \neq \gamma(x)$.

We introduced this property in [9] and proved

THEOREM 5 [10] *Let L be a normal modal logic. Then L possesses the projective Beth property PB2 iff $V(L)$ has the property SES.*

It was proved in [5] that this equivalence holds for a large class of algebraizable logics. Also we need

THEOREM 6 ([10], Theorem 3.8) *For each normal modal logic L the following are equivalent:*
(1) L has the property PB2;
(2) for any finitely generated and finitely indecomposable $\mathbf{A}, \mathbf{B}, \mathbf{C}$ in $V(L)$, for every monomorphisms $\beta : \mathbf{A} \to \mathbf{B}$, $\gamma : \mathbf{A} \to \mathbf{C}$ and for any $b \in \mathbf{B}, c \in \mathbf{C}$, if $\neg(\exists z \in \mathbf{A})(b = \beta(z) \& \gamma(z) = c)$ then there exist a subdirectly irreducible algebra $\mathbf{D}$ in $V(L)$ and homomorphisms $\delta : \mathbf{B} \times \mathbf{C} \to \mathbf{D}$, $\varepsilon : \mathbf{B} \times \mathbf{C} \to \mathbf{D}$ such that $\delta(b,c) \neq \varepsilon(b,c)$ and $\delta(\beta(z), \gamma(z)) = \varepsilon(\beta(z), \gamma(z))$ for all z in $\mathbf{A}$.

The following characterization of finitely indecomposable modal algebras was found in [8], Lemma 2.3 (where $[n]x = x \& \Box x \& \ldots \& \Box^n x$).

LEMMA 7 *A modal algebra $\mathbf{A}$ is finitely indecomposable if and only if for any elements $a \neq \top$ and $b \neq \top$ in $\mathbf{A}$ there exists some n such that $[n]a \vee [n]b \neq \top$.*

We prove the following

LEMMA 8 *Let* $\mathbf{B}, \mathbf{C}$, $\mathbf{D}$ *be modal algebras,* $\mathbf{D}$ *subdirectly irreducible and* $h : \mathbf{B} \times \mathbf{C} \to \mathbf{D}$ *a homomorphism. Then just one of the following conditions is fulfilled:*

1) $h(\top, \bot) = \top$, $h(x, y) = h(x, \top)$ *for all* $x \in \mathbf{B}, y \in \mathbf{C}$, *and* $\alpha(x) = h(x, \top)$ *is a homomorphism from* $\mathbf{B}$ *into* $\mathbf{D}$,

2) $h(\bot, \top) = \top$, $h(x, y) = h(\top, y)$ *for all* $x \in \mathbf{B}, y \in \mathbf{C}$, *and* $\beta(y) = h(\top, y)$ *is a homomorphism from* $\mathbf{C}$ *into* $\mathbf{D}$.

Proof. Suppose $h(\top, \bot) \neq \top$ and $h(\bot, \top) \neq \top$. Since $\mathbf{D}$ is finitely indecomposable, by Lemma 7 there exists n such that

$$[n]h(\top, \bot) \vee [n]h(\bot, \top) < \top,$$

where $[n]x = x \& \Box x \& \ldots \& \Box^n x$. On the other hand,
$[n]h(\top, \bot) \vee [n]h(\bot, \top) = h([n](\top, \bot) \vee [n](\bot, \top)) = h((\top, \bot) \vee (\bot, \top)) = h(\top, \top) = \top$,
because of $[n](\top, \bot) = ([n]\top, [n]\bot) = (\top, \bot)$, and so we get a contradiction.

Assume $h(\top, \bot) = \top$. Then for every $y \in \mathbf{C}$ we have $h(\top, y) \geq h(\top, \bot) = \top$, and for every $x \in \mathbf{B}$:

$$h(x, y) = h(x, \top) \& h(\top, y) = h(x, \top).$$

It is easy to check that in this case $\alpha(x) = h(x, \top)$ is a homomorphism.

Another case $h(\bot, \top) = \top$ is considered by analogy. ■

The lemma 8 is essentially used in the proof of the following

THEOREM 9 *For any variety* V *of modal algebras, SES is equivalent to conjunction of two conditions:*

(a) FI(V) has SES;

(b) for any finitely generated and finitely indecomposable $\mathbf{A}, \mathbf{B}, \mathbf{C}$ *in* V *such that* $\mathbf{A}$ *is a subalgebra of both* $\mathbf{B}$ *and* $\mathbf{C}$ *and for any* $b, c \in \mathbf{A}$, $b \neq c$, *there exist a subdirectly irreducible algebra* $\mathbf{D}$ *in* V *and homomorphisms* $g : \mathbf{B} \to \mathbf{D}$, $h : \mathbf{C} \to \mathbf{D}$ *such that* $g(z) = h(z)$ *for all* z *in* $\mathbf{A}$ *and* $g(b) \neq g(c)$.

Proof. In Theorem 6(2) take β and γ to be identical embeddings so that $\mathbf{A}$ is a subalgebra of both $\mathbf{B}$ and $\mathbf{C}$. Then this condition is equivalent to:

$(2')$ for any finitely generated and finitely indecomposable $\mathbf{A}, \mathbf{B}, \mathbf{C}$ in V such that $\mathbf{A}$ is a subalgebra of both $\mathbf{B}$ and $\mathbf{C}$ and for any $b \in \mathbf{B}, c \in \mathbf{C}$, if $\neg(\exists z \in \mathbf{A})(b = z \& z = c)$, then there exist a subdirectly irreducible algebra

$\mathbf{D}$ in V and homomorphisms $\delta : \mathbf{B} \times \mathbf{C} \to \mathbf{D}$, $\varepsilon : \mathbf{B} \times \mathbf{C} \to \mathbf{D}$ such that $\delta(b,c) \neq \varepsilon(b,c)$ and $\delta(z,z) = \varepsilon(z,z)$ for all z in $\mathbf{A}$.

First note that conjunction of (a) and (b) implies (2′). Assume the premise of (2′) is satisfied. If $b \in \mathbf{B} - \mathbf{A}$ or $c \in \mathbf{C} - \mathbf{A}$, then one easily constructs homomorphisms δ and ε by (a). One should take two homomorphisms from $\mathbf{B}$ or, respectively, from $\mathbf{C}$ into a suitable algebra in $SI(V)$ and then take their compositions with projections from $\mathbf{B} \times \mathbf{C}$ onto $\mathbf{B}$ or $\mathbf{C}$.

Now let $b, c \in \mathbf{A}$. Then we have $b \neq c$, so by (b) there exist a subdirectly irreducible algebra $\mathbf{D}$ in V and homomorphisms $g : \mathbf{B} \to \mathbf{D}$, $h : \mathbf{C} \to \mathbf{D}$ such that $g(b) \neq g(c)$, and $g(z) = h(z)$ for all z in $\mathbf{A}$. We set

$$\delta(x,y) = g(x) \text{ and } \varepsilon(x,y) = h(x)$$

for all $x \in \mathbf{B}$, $y \in \mathbf{C}$. Then we have

$$\delta(b,c) = g(b) \neq g(c) = h(c) = \varepsilon(b,c)$$

and for all $z \in \mathbf{A}$:

$$\delta(z,z) = g(z) = h(z) = \varepsilon(z,z),$$

and thus (2′) is satisfied.

Conversely, assume V has SES. It is clear that SES in V implies SES in $FI(V)$, so (1) is satisfied. Prove that (2′) implies (b). We take finitely generated and finitely indecomposable $\mathbf{A}, \mathbf{B}, \mathbf{C}$ in V such that $\mathbf{A}$ is a subalgebra of both $\mathbf{B}$ and $\mathbf{C}$. Let $b \neq c$ for some $b, c \in \mathbf{A}$. By (2′) there exist a subdirectly irreducible algebra $\mathbf{D}$ in V and homomorphisms $\delta : \mathbf{B} \times \mathbf{C} \to \mathbf{D}$, $\varepsilon : \mathbf{B} \times \mathbf{C} \to \mathbf{D}$ such that $\delta(b,c) \neq \varepsilon(b,c)$ and $\delta(z,z) = \varepsilon(z,z)$ for all z in $\mathbf{A}$. Then by Lemma 8 we have two possibilities:

Case 1. $\delta(\top, \bot) = \top$, $\delta(x,y) = \delta(x,\top)$ for all $x \in \mathbf{B}, y \in \mathbf{C}$, and $g(x) = \delta(x,\top)$ is a homomorphism of $\mathbf{B}$ into $\mathbf{D}$.

Case 2. $\delta(\bot, \top) = \top$, $\delta(x,y) = \delta(\top, y)$ for all $x \in \mathbf{B}, y \in \mathbf{C}$, and $h(y) = \delta(\top, y)$ is a homomorphism from $\mathbf{C}$ into $\mathbf{D}$.

The analogous alternative holds for ε.

In Case 1 we have $\delta(b,c) = \delta(b,b) = \varepsilon(b,b) \neq \varepsilon(b,c)$, so there exists x such that $\varepsilon(x,b) \neq \varepsilon(x,c)$. By Lemma 8 it follows that $h(y) = \varepsilon(x,y)$ is a homomorphism of $\mathbf{C}$ into $\mathbf{D}$, and $h(z) = \varepsilon(z,z) = \delta(z,z) = g(z)$ for all $z \in \mathbf{A}$. Moreover, $g(b) = \delta(b,c) \neq \varepsilon(b,c) = h(c) = g(c)$, i.e., $g(b) \neq g(c)$.

In Case 2 we have $\delta(b,c) = \delta(c,c) = \varepsilon(c,c) \neq \varepsilon(b,c)$. So $g(x) = \varepsilon(x,\top)$ is a homomorphism of $\mathbf{B}$ into $\mathbf{D}$ and, moreover, $h(c) = \delta(b,c) \neq \varepsilon(b,c) = g(b)$. In addition, $h(z) = \delta(z,z) = \varepsilon(z,z) = g(z)$ for all $z \in \mathbf{A}$, so $g(c) = h(c) \neq g(b)$. It completes the proof. ∎

From Proposition 4 and Theorem 9 we immediately get

COROLLARY 10 *SES implies RAP in any variety of modal algebras.*

By Proposition 4 and Theorems 5 and 9 we obtain the following criterion

THEOREM 11 *Let L be a normal modal logic. Then L has PB2 if and only if $SI(V)$ has RAP and $FI(V)$ has SES.*

4 Two definitions of restricted amalgamation

In [11] we defined RAP for Heyting algebras in another way and proved that this property follows from PB2. An analogous notion was introduced in [12] for transitive modal algebras which characterize the modal system K4 = K+($\Box A \to \Box\Box A$). We state that for varieties of transitive algebras, the definition of restricted amalgamation introduced in Section 2 is equivalent to the definition in [12].

Recall that a modal algebra is *transitive* if it satisfied the condition $\Box x \leq \Box\Box x$. Let $[*]x = x \& \Box x$. We say that an element a of a transitive algebra $\mathbf{A}$ is an *opremum* of $\mathbf{A}$ if $a \neq \top$, $[*]a = a$ and $(\forall x \in \mathbf{A})(x \neq \top \Rightarrow [*]x \leq a)$. Remind that a transitive algebra is subdirectly irreducible iff it has an opremum; it is easy to see that an opremum is unique.

The following two lemmas were proved in [12]:

LEMMA 12 *Let $\mathbf{A}$ be a transitive algebra, $a, b \in \mathbf{A}$ and $a = [*]a$, $b = [*]b$. If $a \not\leq b$ then there exist a subdirectly irreducible $\mathbf{C}$ with an opremum ω and a homomorphism $f : \mathbf{A} \to \mathbf{C}$ such that $f(a) = \top$ and $f(b) = \omega$.*

LEMMA 13 *Let $\mathbf{A}$ be a subdirectly irreducible transitive algebra with an opremum ω and h a homomorphism from $\mathbf{A}$ into $\mathbf{B}$ such that $h(\omega) < \top$. Then h is a monomorphism.*

Say that a monomorphism $\alpha : \mathbf{A} \to \mathbf{B}$ of two subdirectly irreducible transitive algebras is *o-preserving* if it maps the opremum of $\mathbf{A}$ into opremum of $\mathbf{B}$. The *Restricted Amalgamation Property* of a class V was defined in [12] as follows:

(RAP^*) for any o-preserving monomorphisms $\beta : \mathbf{A} \to \mathbf{B}$ and $\gamma : \mathbf{A} \to \mathbf{C}$ of subdirectly irreducible algebras in V there exist an algebra $\mathbf{D}$ in V and monomorphisms $\delta : \mathbf{B} \to \mathbf{D}$, $\varepsilon : \mathbf{C} \to \mathbf{D}$ such that $\delta\beta = \varepsilon\gamma$.

It is evident that AP implies implies RAP^*, and for any variety V of transitive algebras, RAP^* is equivalent to the property:

for each subdirectly irreducible $\mathbf{A}, \mathbf{B}, \mathbf{C} \in V$ such that $\mathbf{A}$ is a common subalgebra of $\mathbf{B}$ and $\mathbf{C}$ and all three algebras have the same opremum, there exist an algebra $\mathbf{D}$ in V and monomorphisms $\delta : \mathbf{B} \to \mathbf{D}$, $\varepsilon : \mathbf{C} \to \mathbf{D}$ such that $\delta(x) = \varepsilon(x)$ for all $x \in \mathbf{A}$.

Also we proved in [12] that the following lemma holds:

LEMMA 14 *For all varieties of transitive algebras, RAP^* is equivalent to the property:*

for any o-preserving monomorphisms $\beta : \mathbf{A} \to \mathbf{B}$ and $\gamma : \mathbf{A} \to \mathbf{C}$ of subdirectly irreducible algebras in V there exist a subdirectly irreducible algebra $\mathbf{D}$ in V and o-preserving monomorphisms $\delta : \mathbf{B} \to \mathbf{D}$, $\varepsilon : \mathbf{C} \to \mathbf{D}$ such that $\delta\beta = \varepsilon\gamma$.

We use this lemma and Proposition 4 in order to prove

THEOREM 15 *For any variety of Heyting algebras or transitive modal algebras, RAP and RAP^* are equivalent.*

Proof. Let V have RAP. Assume that $\mathbf{A}, \mathbf{B}$ and $\mathbf{C}$ are subdirectly irreducible algebras in V such that $\mathbf{A}$ is a common subalgebra of $\mathbf{B}$ and $\mathbf{C}$ and all three algebras have the same opremum ω. Then there exist $\mathbf{D} \in V$ and two homomorphisms $\delta : \mathbf{B} \to \mathbf{D}$, $\varepsilon : \mathbf{C} \to \mathbf{D}$ such that $\delta(x) = \varepsilon(x)$ for all $x \in \mathbf{A}$ and the restriction of δ to $\mathbf{A}$ is a monomorphism, so $\delta(\omega) = \varepsilon(\omega) < \top$. Then δ and ε are monomorphisms by Lemma 13, and thus V satisfies RAP^*.

Conversely, assume V has RAP^*. We prove that $SI(V)$ has RAP, then V will possess RAP by Proposition 4(3). Let us take $\mathbf{A}, \mathbf{B}$ and $\mathbf{C}$ in $SI(V)$ such that $\mathbf{A}$ is a common subalgebra of $\mathbf{B}$ and $\mathbf{C}$ and let ω be an opremum of $\mathbf{A}$. By Lemma 12 there exists a homomorphism h from $\mathbf{B}$ onto a subdirectly irreducible algebra $\mathbf{B}'$ such that $h(\omega)$ is an opremum in $\mathbf{B}'$. Similarly, there exists a homomorphism g from $\mathbf{C}$ onto some $\mathbf{C}'$ in $SI(V)$ such that $g(\omega)$ is an opremum in $\mathbf{C}'$. By Lemma 13, the restrictions h' of h to $\mathbf{A}$ and g' of g to $\mathbf{A}$ are monomorphisms of $\mathbf{A}$ into $\mathbf{B}'$ and $\mathbf{C}'$ respectively.

By Lemma 14 there exist $\mathbf{D} \in SI(V)$ and o-preserving monomorphisms $\delta : \mathbf{B}' \to \mathbf{D}$, $\varepsilon : \mathbf{C}' \to \mathbf{D}$ such that $\delta h' = \varepsilon g'$. Then δh and εg are the homomorphisms from $\mathbf{B}$ and $\mathbf{C}$ into $\mathbf{D}$ as required. ∎

5 Concluding remarks

Summarizing the results of this paper together with those of [8] and [10], we see that for all normal modal logics and varieties of modal algebras the following hold:

(1) IPD & B2 $\iff$ StrAP,

(2) IPD & B2 $\Rightarrow$ PB2 $\iff$ SES,

(3) PB2 $\not\Rightarrow$ IPD $\iff$ AP, IPD $\not\Rightarrow$ PB2, PB2 $\Rightarrow$ B2,

(4) PB2 $\Rightarrow$ IPR $\iff$ RAP,

(5) IPD $\Rightarrow$ IPR, IPR & B2 $\not\Rightarrow$ IPD,

(6) IPR $\not\Rightarrow$ B2, B2 $\not\Rightarrow$ IPR.

In fact, most of these implications are valid for arbitrary varieties which are congruence-distributive, have congruence extension property and satisfy an additional condition: any subalgebra of a subdirectly irreducible algebra is finitely indecomposable. The proofs will be given in a separate paper. In particular, all of (1)-(6) hold for logics without contraction characterized by varieties of residuated lattices which are investigated in [6].

It is known that all varieties of Heyting algebras, of relatively pseudo-complemented lattices and of transitive modal algebras have the Beth property B2. For those varieties and corresponding logics we obtain:

IPD $\iff$ AP $\iff$ StrAP,

IPD $\Rightarrow$ PB2 $\iff$ SES,

PB2 $\not\Rightarrow$ IPD,

PB2 $\Rightarrow$ IPR $\iff$ RAP.

We note that the implication from PB2 to IPR in modal logics follows from purely algebraic Corollary 3.6. It would be interesting to find a direct proof in logical terms.

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BIBLIOGRAPHY

[1] *J.Barwise, S.Feferman, eds*. Model-Theoretic Logics. New York: Springer-Verlag, 1985.

[2] *E.W.Beth*. On Padoa's method in the theory of definitions. Indagationes Math. 15, no. 4 (1953), 330-339.

[3] *J.Bicarregui, T.Dimitrakos, D.Gabbay, T.Maibaum*. Interpolation in Practical Formal Development. Logic Journal of the IGPL, 9, no. 2 (2001), 247-259.

[4] *W.Craig*. Three uses of Herbrand-Gentzen theorem in relating model theory and proof theory. J. Symbolic Logic, 22 (1957), 269-285.

[5] *E.Hoogland*. Definability and Interpolation. Model-theoretic investigations. ILLC Dissertation Series DS-2001-05, Amsterdam, 2001.

[6] *T.Kowalski, H.Ono*. Splittings in the variety of residuated lattices. Algebra Universalis, 44 (2000), 283-298.

[7] *L.L.Maksimova*. Interpolation theorems in modal logics and amalgamable varieties of topoboolean algebras. Algebra i Logika, 18, no. 5 (1979), 556–586.

[8] *L.L.Maksimova*. Modal Logics and Varieties of Modal Algebras: the Beth Property, Interpolation and Amalgamation. Algebra and Logic, 31, no.2 (1992), 145-166.

[9] *L.Maksimova*. Explicit and Implicit Definability in Modal and Related Logics. Bulletin of Section of Logic, 27, no. 1/2 (1998), 36–39.

[10] *L.L.Maksimova*. Projective Beth properties in modal and superintuitionistic logics. Algebra and Logic, 38 (1999), 316-333.

[11] *L.L.Maksimova*. Intuitionistic Logic and Implicit Definability. Annals of Pure and Applied Logic, 105 (2000), 83–102.
[12] *L. Maksimova*. Projective Beth's Properties in Infinite Slice Extensions of K4. In: F.Wolter, H.Wansing, M. de Rijke, M.Zakharyaschev, eds. Advances in Modal Logics, Volume 3, World Scientific, Singapore, 2002, 349-363.
[13] *A.I.Maltsev*. Algebraic systems. Moscow: Nauka, 1970.
[14] *H.Rasiowa, R.Sikorski*. The Mathematics of Metamathematics. Warszawa: PWN, 1963.
[15] *I.Sain*. Beth's and Craig's Properties via Epimorphisms and Amalgamation in Algebraic Logic. In: *C.H.Bergman, R.D.Maddux, D.I.Pigozzi (ed)*, Algebraic Logic and Universal Algebra in Computer Science. Lecture Notes in Computer Science, 425 (1990), Springer Verlag, 209–226.

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