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A Note on Relativised Products of Modal Logics

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ABSTRACT. A natural idea of reducing the strong interaction between modal operators of product logics in hope of obtaining decidable but still expressive and useful many-dimensional formalisms is to consider (not necessarily generated) subframes of product frames. Worlds are still tuples, the relations still act coordinate-wise, but not all tuples of the Cartesian product are present, and so the commutativity and Church–Rosser properties do not necessarily hold. This kind of restriction on the domains of modal operators is similar to relativisations of the quantifiers in first-order logic and algebraic logic, where it indeed results in improving the bad algorithmic behaviour. Relativised product logics are often located between the fusions and the products of their components. In this paper, we show that arbitrary and locally cubic relativised products of many standard modal logics in fact coincide with their fusions. This result can also be considered as a new many-dimensional semantical characterisation of such fusions. We give some examples of arbitrarily relativised products sitting properly between fusions and products. We also provide some observations on expanding and decreasing relativised products, and their connections with first-order modal logics.

1 Introduction

One may think of many ways of combining normal modal logics representing various aspects of an application domain. Two 'canonical' constructions, supported by a well-developed mathematical theory, are fusions [17, 6, 7] and products [8, 7].

The *fusion* $L_1 \otimes \ldots \otimes L_n$ of $n \geq 2$ propositional normal unimodal logics L_i with the boxes $\Box_i$ is the smallest multimodal logic in the language with n boxes $\Box_1, \ldots, \Box_n$ (and their duals $\Diamond_1, \ldots, \Diamond_n$) that contains all the L_i. This means that if the L_i are axiomatised by sets Ax_i of axioms, then $L_1 \otimes \ldots \otimes L_n$ is axiomatised by the union $Ax_1 \cup \ldots \cup Ax_n$. Thus, fusions are useful when the modal operators of the combined logics are not supposed to interact (see e.g. [1] which provides numerous examples of applications of fusions in description logic). It is this absence of interaction axioms that

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ensures the transfer of good algorithmic properties from the components to their fusion [17, 6, 33]. In particular, it is possible to reduce reasoning in the fusion to reasoning in the components. On the semantic level, the fusion of Kripke complete modal logics $L_1, \dots, L_n$ can be characterised by the class of all *n-frames* $\langle W, R_1, \dots, R_n \rangle$ such that each $\langle W, R_i \rangle$ is a frame for L_i [17, 6]. Note that although frames for fusions have n different accessibility relations, they cannot be regarded as 'genuinely many-dimensional' in the geometric sense.

Products of modal logics do have real many-dimensional frames. Given n Kripke frames $\mathfrak{F}_1 = \langle U_1, R_1 \rangle, \dots, \mathfrak{F}_n = \langle U_n, R_n \rangle$, their *product* $\mathfrak{F}_1 \times \dots \times \mathfrak{F}_n$ is the n-frame

$$\left\langle U_1 \times \dots \times U_n, \overline{R}_1, \dots, \overline{R}_n \right\rangle,$$

where each $\overline{R}_i$ is a binary relation on $U_1 \times \ldots \times U_n$ defined by taking

$$\langle u_1, \dots, u_n \rangle \, \overline{R}_i \, \langle u'_1, \dots, u'_n \rangle \quad \text{iff} \quad u_i R_i u'_i \text{ and } u_k = u'_k, \text{ for } k \neq i.$$

The *product* $L_1 \times \ldots \times L_n$ of Kripke complete unimodal logics $L_1, \dots, L_n$ is the logic of the class of all product frames $\mathfrak{F}_1 \times \dots \times \mathfrak{F}_n$, where each $\mathfrak{F}_i$ is a frame for L_i. (Here by 'the logic of a class' we mean the set of those modal formulas with n boxes $\Box_1, \dots, \Box_n$ that are valid in each frame of the class.) For example, $\mathbf{K}^n$ is the logic of all n-ary product frames. It is not hard to see that $\mathbf{S5}^n$ is the logic of all n-ary products of *universal* frames having the same worlds, that is, frames $\langle U, R_i \rangle$ with $R_i = U \times U$. We refer to product frames of this kind as *cubic universal product* $\mathbf{S5}^n$*-frames*. Note that the 'i-reduct' $\mathfrak{F}^{(i)} = \left\langle U_1 \times \dots \times U_n, \overline{R}_i \right\rangle$ of $\mathfrak{F}_1 \times \dots \times \mathfrak{F}_n$ is a union of n disjoint copies of $\mathfrak{F}_i$. Thus, $\mathfrak{F}^{(i)}$ and $\mathfrak{F}_i$ validate the same formulas, and so

$$L_1 \otimes \dots \otimes L_n \subseteq L_1 \times \dots \times L_n.$$

There is a strong interaction between the modal operators of product logics. Every n-ary product frame satisfies the following two properties, for each pair $i \neq j$, $i, j = 1, \dots, n$:

Commutativity:

$$\forall x \forall y \forall z \Big(\big(x\overline{R}_i y \land y\overline{R}_j z \to \exists u \; (x\overline{R}_j u \land u\overline{R}_i z) \big) \\ \land \big(x\overline{R}_j y \land y\overline{R}_i z \to \exists u \; (x\overline{R}_i u \land u\overline{R}_j z) \big) \Big).$$

Church–Rosser property:

$$\forall x \forall y \forall z \left(x\overline{R}_i y \land x\overline{R}_j z \to \exists u \; (y\overline{R}_j u \land z\overline{R}_i u) \right).$$

This means that the corresponding modal interaction formulas

$$\Box_i \Box_j p \leftrightarrow \Box_j \Box_i p \qquad \text{and} \qquad \Diamond_i \Box_j p \to \Box_j \Diamond_i p$$

belong to every n-dimensional product logic. The geometrically intuitive many-dimensional structure of product frames makes them a perfect tool for constructing formalisms suitable for, say, spatio-temporal representation and reasoning (see e.g. [34, 35]) or reasoning about the behaviour of multi-agent systems (see e.g. [4]). However, the price we have to pay for the use of products is an extremely high computational complexity—even the product of two NP-complete logics can be non-recursively enumerable (see e.g. [30, 28]). In higher dimensions practically all products of 'standard' modal logics are undecidable and non-finitely axiomatisable [16].

A natural idea of reducing the strong interaction between modal operators of product logics in hope of obtaining more 'user-friendly' but still expressive and useful many-dimensional formalisms is to consider (not necessarily generated) *subframes*[1] of product frames. Worlds are still tuples, the relations still act coordinate-wise, but not all tuples of the Cartesian product are present, and so the commutativity and Church–Rosser properties do not necessarily hold. This kind of restriction on the 'domains' of modal operators is similar to 'relativisations' of the quantifiers in first-order logic and algebraic logic, where it indeed results in improving the bad algorithmic behaviour, cf. [25, 20]. As a modification of the product construction, 'relativisation' was first suggested in [22].

This idea gives rise to the following 'product-like' combinations of logics. First, we choose a class of 'desirable' subframes of product frames. This can be any class: the class of all such subframes, the so-called 'locally cubic' frames, frames that 'expand' along one of the coordinates (see below for definitions), a class of frames satisfying some (modal or first-order) formulas, etc. Having chosen such a class $\mathcal{K}$, we then take the logic determined by those subframes of the appropriate product frames that belong to $\mathcal{K}$. Thus, each choice of $\mathcal{K}$ defines a new product-like operator on logics. More precisely, the *$\mathcal{K}$-relativised product* $(L_1 \times \ldots \times L_n)^{\mathcal{K}}$ of Kripke complete unimodal logics $L_1, \ldots, L_n$ is the logic of the class of those frames in $\mathcal{K}$ that are subframes of product frames $\mathfrak{F}_1 \times \ldots \times \mathfrak{F}_n$ such that each $\mathfrak{F}_i$ is a frame for L_i. Observe that if we choose $\mathcal{K}$ to be the class of all product frames $\mathfrak{F}_1 \times \cdots \times \mathfrak{F}_n$, where $\mathfrak{F}_i$ is a frame for L_i, then the $\mathcal{K}$-relativised product of the L_i is just their usual product.

The aim of this note may appear to be rather modest. Instead of investigating the decision and complexity problems for relativised product logics

[1] Recall that an n-frame $\langle U, S_1, \ldots, S_n \rangle$ is a *subframe* of $\langle W, R_1, \ldots, R_n \rangle$ if $U \subseteq W$ and $S_i = R_i \cap (V \times V)$, for $i = 1, \ldots, n$.

right away, we prefer to (cautiously) begin by trying to find out what kind of 'creatures' these relativised products are and how they are related to standard products and fusions. The results of investigation are somewhat surprising. As we shall see, relativised product logics are often located between the fusions and the products of their components. However, our main statements in Sections 2 and 3 show that 'arbitrary' and 'locally cubic' relativised products of many standard modal logics in fact *coincide* with their fusions (which justifies our cautious approach and gives 'automatic' solutions to the decision and complexity problems). This result can also be considered as a nice new many-dimensional semantical characterisation of such fusions. We also give some interesting (and natural) examples of 'arbitrarily' relativised products sitting properly between fusions and products. Finally, in Section 4, we provide some observations on 'expanding' and 'decreasing' relativised products, and their connections with first-order modal logics.

2 Arbitrary relativisations

We begin by considering the product operator determined by the class SF_n of *all* subframes of n-ary product frames. SF_n-relativised products of logics will be called *arbitrarily relativised products*. Since SF_n contains frames that do not satisfy commutativity and/or Church–Rosser properties, clearly we have

$$(L_1 \times \cdots \times L_n)^{\mathsf{SF}_n} \subsetneq L_1 \times \cdots \times L_n.$$

On the other hand, unlike product logics, arbitrarily relativised products do not necessarily contain the fusion of their components. For example, consider the minimal deontic logic $\mathbf{D}$, which is known to be characterised by the class of *serial* frames. Now, the formula $\Diamond_2\top$ clearly belongs to the fusion $\mathbf{K} \otimes \mathbf{D}$, but is refuted in any *finite* subframe of, say, $\langle \omega, < \rangle \times \langle \omega, < \rangle$, and so $\Diamond_2\top \notin (\mathbf{K} \times \mathbf{D})^{\mathsf{SF}_2}$. However, as we shall see below, for a large class of natural logics, arbitrarily relativised products do contain the fusions.

A Kripke complete modal logic L is called a *subframe logic* if the class of Kripke frames for L is closed under taking (not necessarily generated) subframes. (For a general theory of subframe logics consult [5, 2, 32] and references therein.) Typical examples of subframe logics are modal logics whose classes of Kripke frames are definable by universal first-order formulas, such as **K**, **Alt**, **T**, **K4**, **S4**, **S5**, **K5**, **K45**, **S4.3**, and **K4.3**. Note, however, that subframe logics like **GL**, **GL.3**, **Grz** are not first-order definable.

PROPOSITION 1. *If $L_1, \ldots, L_n$ are all subframe logics, then*

$$L_1 \otimes \ldots \otimes L_n \subseteq (L_1 \times \ldots \times L_n)^{\mathsf{SF}_n}.$$

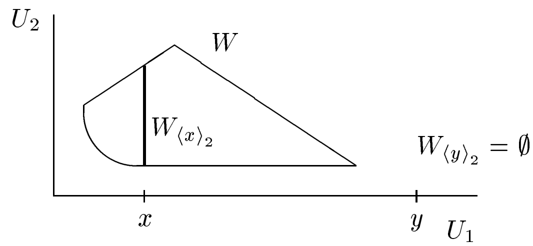


Figure 1. 'Coordinate-wise' subframes.

Proof. Suppose that an n-frame $\mathfrak{G} = \langle W, \overline{S}_1, \dots, \overline{S}_n \rangle$ is a subframe of some product frame

$$\langle U_1, R_1 \rangle \times \cdots \times \langle U_n, R_n \rangle ,$$

where $\langle U_i, R_i \rangle$ is a frame for L_i, $i = 1, \dots, n$. Fix some i, $1 \le i \le n$. For every $n-1$-tuple $\overline{u}_i = \langle u_1, \dots, u_{i-1}, u_{i+1}, \dots, u_n \rangle$ with $u_j \in U_j$, for $j \ne i$, we take the set

$$W_{\overline{u}_i} = \{\langle u_1, \dots, u_n \rangle \in W \mid u_i \in U_i,\ \langle u_1, \dots, u_{i-1}, u_{i+1}, \dots, u_n \rangle = \overline{u}_i\},$$

and let $S_{\overline{u}_i}$ be the restriction of $\overline{S}_i$ to $W_{\overline{u}_i}$, i.e., $S_{\overline{u}_i} = \overline{S}_i \cap (W_{\overline{u}_i} \times W_{\overline{u}_i})$; see Fig. 1. Then clearly we have the following:

- if $W_{\overline{u}_i}$ is not empty then $\langle W_{\overline{u}_i}, S_{\overline{u}_i} \rangle$ is isomorphic to a subframe of $\langle U_i, R_i \rangle$;
- $\langle W, \overline{S}_i \rangle$ is the disjoint union of the frames $\langle W_{\overline{u}_i}, S_{\overline{u}_i} \rangle$, for all possible $n-1$-tuples $\overline{u}_i$ with non-empty $W_{\overline{u}_i}$.

Therefore, since L_i is a subframe logic, $\langle W, \overline{S}_i \rangle$ is a frame for L_i. ■

As we shall see below, the converse of the inclusion in Proposition 1 does not always hold. However, as the following theorem shows, for many standard subframe logics, their arbitrarily relativised product coincides with their fusion. Thus, 'arbitrary relativisation' can be regarded as a 'many-dimensional' semantical characterisation of fusions of these logics.

THEOREM 2. *Let* $L_i \in \{\mathbf{K}, \mathbf{T}, \mathbf{K4}, \mathbf{S4}, \mathbf{S5}, \mathbf{S4.3}\}$, *for* $i = 1, \dots, n$. *Then*

$$(L_1 \times \cdots \times L_n)^{\mathsf{SF}_n} = L_1 \otimes \cdots \otimes L_n.$$

Proof. It is well-known (see [6, 17]) that all fusions $L_1 \otimes \cdots \otimes L_n$ mentioned in the formulation of the theorem are characterised by countable (in fact, finite) rooted n-frames $\mathfrak{G} = \langle W, S_1, \ldots, S_n \rangle$, where $\langle W, S_i \rangle$ is a frame for L_i, $i = 1, \ldots, n$. We prove now the following:

LEMMA 3. *Let $L_i \in \{\mathbf{K}, \mathbf{T}, \mathbf{K4}, \mathbf{S4}, \mathbf{S5}, \mathbf{S4.3}\}$, $i = 1, \ldots, n$, and let $\mathfrak{G} = \langle W, S_1, \ldots, S_n \rangle$ be a countable rooted n-frame such that each $\langle W, S_i \rangle$ is a frame for L_i. Then $\mathfrak{G}$ is a p-morphic image of a subframe of some product frame for $L_1 \times \cdots \times L_n$.*

Proof. First we show that every countable rooted n-frame

$$\mathfrak{G} = \langle W, S_1, \ldots, S_n \rangle$$

is a p-morphic image of a subframe of some product frame. We will construct, step-by-step, frames $\mathfrak{F}_i = \langle U_i, R_i \rangle$ $(i = 1, \ldots, n)$, a subframe $\mathfrak{H}$ of $\mathfrak{F}_1 \times \cdots \times \mathfrak{F}_n$, and a p-morphism f from $\mathfrak{H}$ onto $\mathfrak{G}$. One way of formalising this straightforward step-by-step argument is by defining a *game* $G(\mathfrak{G})$ between two players $\forall$ (male) and $\exists$ (female) over $\mathfrak{G}$. (Our game and its properties are similar to those of [14], where games are played over relation algebras. For a detailed treatment of games over many-dimensional structures consult [15].)

Define a $\mathfrak{G}$-*network* to be a tuple

$$N = \left\langle U_1^N, \ldots, U_n^N, V^N, R_1^N, \ldots, R_n^N, f^N \right\rangle$$

such that $\mathfrak{F}_i^N = \left\langle U_i^N, R_i^N \right\rangle$ are finite intransitive trees for all $i = 1, \ldots, n$, $V^N \subseteq U_1^N \times \cdots \times U_n^N$, and f^N is a homomorphism from the subframe $\mathfrak{H}^N$ of $\mathfrak{F}_1^N \times \ldots \times \mathfrak{F}_n^N$, having V^N as its set of worlds, to $\mathfrak{G}$. In other words, for all $u_1 \in U_1, \ldots, u_n \in U_n$, $i = 1, \ldots, n$, and $u_i' \in U_i$,

$$\begin{array}{l}\text{if } \langle u_1, \ldots, u_n \rangle \in V^N,\ \langle u_1, \ldots, u_{i-1}, u_i', u_{i+1}, \ldots, u_n \rangle \in V^N \text{ and } u_i R_i^N u_i' \\ \quad \text{then } \ f^N(u_1, \ldots, u_i, \ldots, u_n) S_i f^N(u_1, \ldots, u_i', \ldots, u_n).\end{array}$$

The players $\forall$ and $\exists$ build a countable 'expanding' sequence of finite $\mathfrak{G}$-networks as follows.

In round 0, $\forall$ picks the root r of $\mathfrak{G}$. $\exists$ responds with a $\mathfrak{G}$-network N_0 such that all the $U_i^{N_0}$ are singleton sets, $V^{N_0} = U_1^{N_0} \times \cdots \times U_n^{N_0}$, the relations $R_i^{N_0}$ are all empty, and f^{N_0} maps the only n-tuple in V^{N_0} to r.

Suppose now that in round j, $0 < j < \omega$, the players have already built a finite $\mathfrak{G}$-network N_{j-1}. Now player $\forall$ challenges player $\exists$ with a possible *defect* of N_{j-1} which indicates that the homomorphism $f^{N_{j-1}}$ is not a p-morphism onto $\mathfrak{G}$ yet. $\forall$ picks such a defect which consists of

- an n-tuple $\langle u_1, \ldots, u_n \rangle \in V^{N_{j-1}}$,
- a coordinate $i \in \{1, \ldots, n\}$, and
- a world w in $\mathfrak{G}$ such that $f^{N_{j-1}}(u_1, \ldots, u_n) R_i w$.

Player $\exists$ can respond in two ways. If there is some u_i' such that

$$\langle u_1, \ldots, u_i', \ldots, u_n \rangle \in V^{N_{j-1}}, \; u_i R_i^{N_{j-1}} u_i' \\ \text{and } f^{N_{j-1}}(u_1, \ldots, u_i', \ldots, u_n) = w,$$

then she responds with $N_j = N_{j-1}$. Otherwise, she responds with the following $\mathfrak{G}$-network N_j extending N_{j-1}:

- $U_i^{N_j} = U_i^{N_{j-1}} \cup \{u^+\}$, where u^+ is a fresh point,
- $R_i^{N_j} = R_i^{N_{j-1}} \cup \{\langle u_i, u^+ \rangle\}$,
- $V^{N_j} = V^{N_{j-1}} \cup \{\langle u_1, \ldots, u_{i-1}, u^+, u_{i+1}, \ldots, u_n \rangle\}$,
- $\mathfrak{F}_k^{N_j} = \mathfrak{F}_k^{N_{j-1}}$ for all $k \neq i$, and
- $f^{N_j}(u_1, \ldots, u^+, \ldots, u_n) = w$.

Observe that $\exists$ can *always* respond this way. In other words, she always has a winning strategy in the ω-long game $G(\mathfrak{G})$. It is straightforward to see that the union (in the natural sense) of the constructed $\mathfrak{G}$-networks gives the required p-morphism f from a subframe $\mathfrak{H} = \langle V, \ldots \rangle$ of a product frame $\mathfrak{F}_1 \times \cdots \times \mathfrak{F}_n$ onto $\mathfrak{G}$. This proves the lemma for $L_i = \mathbf{K}$, $i = 1, \ldots, n$.

However, in the other cases nothing guarantees that the 'coordinate' frames $\mathfrak{F}_i = \langle U_i, R_i \rangle$ are actually frames for L_i. In what follows we fix some i, $1 \leq i \leq n$, and try to transform $\mathfrak{F}_i$ into a frame for L_i keeping all other frames $\mathfrak{F}_j$ for $j \neq i$ and the set V intact. Without loss of generality we may assume that $i = 1$.

To begin with, we show that the frames $\mathfrak{F}_i$ and the subframe $\mathfrak{H} = \langle V, \ldots \rangle$ have some useful properties. First, it should be clear from the construction that

(1) for each $i = 1, \ldots, n$, the frame $\mathfrak{F}_i$ is an intransitive tree.

To formulate another property, we require an auxiliary definition. Given an odd natural number k, a sequence $\langle v^0, \ldots, v^k \rangle$ of distinct n-tuples $v^\ell = \langle v_1^\ell, \ldots, v_n^\ell \rangle$, $\ell \leq k$, from V is called a *path in V between v^0 and v^k* if the following two conditions hold:

- for each even number $\ell < k$, $v_j^\ell = v_j^{\ell+1}$ whenever $j \neq 1$, and

- for each odd number $\ell < k$, $v_1^\ell = v_1^{\ell+1}$

(see Fig. 2). We call k the *length* of this path. If in addition $v_1^0 = v_1^k$ also

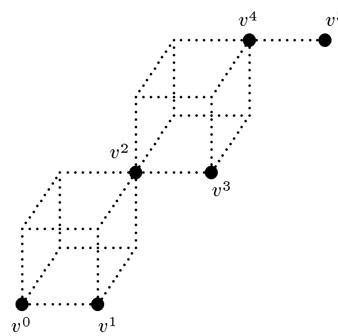


Figure 2. A 3-dimensional path of length 5.

holds then we call $\langle v^0, \dots, v^k \rangle$ a *circle in* V (since all the n-tuples are distinct in a path, this can happen only if $k \geq 3$; see Fig. 3). Observe that

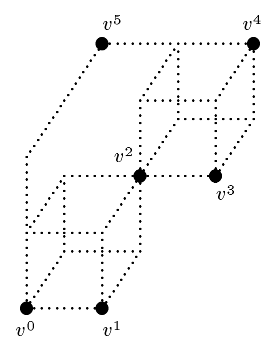


Figure 3. A 3-dimensional circle.

if $\langle v^0, \dots, v^k \rangle$ is a circle then, for every $\ell \leq k$ and every i, $1 \leq i \leq n$, there exists an $\ell' \leq k$, $\ell' \neq \ell$, such that $v_i^\ell = v_i^{\ell'}$.

The second important property is that

(2) there are no circles in V.

Indeed, suppose otherwise. Take a circle $\langle v^0, \dots, v^k \rangle$ in V and enumerate all of its n-tuples according to their 'creation time' in the game. Let v^ℓ be the last one in this list. By the rules of the game, one of the coordinates of v^ℓ must be fresh, contrary to the observation above.

Note that, as a special case of (2), we obtain that there are no *squares* in V, i.e., four distinct n-tuples of the form $\langle x, w_2, \dots, w_n \rangle$, $\langle x, w_2', \dots, w_n' \rangle$, $\langle y, w_2, \dots, w_n \rangle$, and $\langle y, w_2', \dots, w_n' \rangle$.

Now, in order to transform $\mathfrak{F}_1 = \langle U_1, R_1 \rangle$ into a frame for L_1, we will extend, step-by-step, the accessibility relation R_1 (but always leave the sets U_1, V and the frames $\mathfrak{F}_j$ for $j \neq 1$ unchanged).

First, let $L_1 = \mathbf{K4}$. Define an infinite ascending chain

$$R_1^0 \subseteq R_1^1 \subseteq \dots \subseteq R_1^m \subseteq \dots$$

of binary relations on U_1 by taking $R_1^0 = R_1$ and, for $m < \omega$,

$$R_1^{m+1} = R_1^m \cup \{\langle x_1, y_1 \rangle \in U_1 \times U_1 \mid x_1 R_1^m z_1 \text{ and } z_1 R_1^m y_1 \text{ for some } z_1 \in U_1\}.$$

For every $m < \omega$, let $\mathfrak{F}_1^m = \langle U_1, R_1^m \rangle$ and let $\mathfrak{H}^m$ be the subframe of

$$\mathfrak{F}_1^m \times \mathfrak{F}_2 \times \dots \times \mathfrak{F}_n$$

with V as its set of worlds. Finally, let

$$R_1^\infty = \bigcup_{m<\omega} R_1^m, \qquad \mathfrak{F}_1^\infty = \langle U_1, R_1^\infty \rangle,$$

and let $\mathfrak{H}^\infty$ be the corresponding subframe of $\mathfrak{F}_1^\infty \times \mathfrak{F}_2 \times \dots \times \mathfrak{F}_n$.

Clearly, $\mathfrak{F}_1^\infty$ is a frame for $\mathbf{K4}$. We are about to show that f is still a p-morphism from $\mathfrak{H}^\infty$ onto $\mathfrak{G}$. Since the 'backward' p-morphism condition always holds after extending the accessibility relation of the pre-image, it is enough to show that f is a homomorphism from $\mathfrak{H}^\infty$ onto $\mathfrak{G}$. We will prove by parallel induction on m that the following two statements hold, for all $m < \omega$ and for $x_1, y_1 \in U_1$:

(a) If $x_1 R_1^m y_1$ then there exist $x_2, \dots, x_n, y_2, \dots, y_n$ such that there is a path in V between $x = \langle x_1, x_2, \dots, x_n \rangle$ and $y = \langle y_1, y_2, \dots, y_n \rangle$.

(b) If $x_1 R_1^m y_1$ and both $w^x = \langle x_1, w_2, \dots, w_n \rangle$ and $w^y = \langle y_1, w_2, \dots, w_n \rangle$ are in V for some $w_j \in U_j$, $j = 2, \dots, n$, then $f(w^x) S_1 f(w^y)$. In other words, f is a homomorphism from $\mathfrak{H}^m$ onto $\mathfrak{G}$.

Suppose first that $m = 0$. Then, by the definition of $\mathfrak{H}$, (b) holds and there exist $w_2, \dots, w_n$ such that both $w^x = \langle x_1, w_2, \dots, w_n \rangle$ and $w^y = \langle y_1, w_2, \dots, w_n \rangle$ are in V. By (1), $x_1 \neq y_1$ and so the sequence $\langle w^x, w^y \rangle$ is a path in V as required.

Let us assume inductively that (a) and (b) hold for some $m < \omega$, and let $x_1, y_1 \in U_1$ be such that $x_1 R_1^{m+1} y_1$, but $x_1 R_1^m y_1$ does not hold. Then there is a $z_1 \in U_1$ such that $x_1 R_1^m z_1$ and $z_1 R_1^m y_1$. It is not hard to see that, by (1), x_1, y_1 and z_1 should be all distinct. By item (a) of the induction hypothesis, there are x_j, z_j, z'_j, y_j, for $j = 2, \ldots, n$, such that

- there is a path in V between the n-tuples $x = \langle x_1, x_2, \ldots, x_n \rangle$ and $z = \langle z_1, z_2, \ldots, z_n \rangle$;
- there is a path in V between the n-tuples $z' = \langle z_1, z'_2, \ldots, z'_n \rangle$ and $y = \langle y_1, y_2, \ldots, y_n \rangle$.

If $z \neq z'$ then the concatenation of these two paths gives a path between x and y. If $z = z'$ then we leave out z from the concatenated sequence, and the rest gives a path as required in (a).

For (b), suppose that $w^x = \langle x_1, w_2, \ldots, w_n \rangle$ and $w^y = \langle y_1, w_2, \ldots, w_n \rangle$ are in V for some $w_j \in U_j$, $j = 2, \ldots, n$. Let $w^z = \langle z_1, w_2, \ldots, w_n \rangle$. Consider the n-tuples x, y, z, z' given above. We claim that

(3) $\quad x = w^x, \; y = w^y$ and $z = z' = w^z$.

For suppose otherwise. Then several cases are possible. We will show that each of them means that there is a circle in V, contrary to (2). Let $\rho = \langle x, v^1, \ldots, v^k, y \rangle$ denote the path in V between x and y given by (a).

- Suppose first that $x \neq w^x$ and $y \neq w^y$. Then the concatenation of $\langle w^y, w^x \rangle$ and ρ is a circle in V.
- Suppose that $x = w^x$ and $y \neq w^y$. Then the length of ρ is ≥ 3, and so $\langle w^y, v^1, \ldots, v^k, y \rangle$ is a circle in V. The case when $x \neq w^x$ and $y = w^y$ is similar.
- Finally, suppose $x = w^x$ and $y = w^y$. If $z \neq w^z$ and $z' \neq w^z$ then the length of ρ should be ≥ 5, and $\langle v^k, v^1, \ldots, v^{k-1} \rangle$ is a circle in V. The cases when one of z and z' coincides with w^z, but the other does not, are similar.

As a consequence of (3), we obtain that w^z is in V. So by item (b) of the induction hypothesis,

$$f(w^x) S_1 f(w^z) \quad \text{and} \quad f(w^z) S_1 f(w^y).$$

Since S_1 is transitive, we have $f(w^x) S_1 f(w^y)$, which completes the proof of Lemma 3 for $L_1 = \mathbf{K4}$.

If $L_1 = \mathbf{S4}$ or $L_1 = \mathbf{T}$, we simply make all worlds of $\mathfrak{F}_1$ reflexive. Clearly, f will still be a p-morphism. In the case of $L_1 = \mathbf{S5}$, we have to 'close' $\mathfrak{F}_1$ under both transitivity and symmetry. It is not hard to see that this causes no problem, since there are no squares in V.

For $L_1 = \mathbf{S4.3}$ we need a slight modification of the above proof for $\mathbf{K4}$. We have to turn $\mathfrak{F}_1$ to a reflexive, transitive and weakly connected frame. To this end, we modify the definition of the accessibility relation R_1^{m+1} $(m < \omega)$. First, we make all the points in U_1 reflexive. Then, for all distinct $x_1, y_1 \in U_1$, we define $\langle x_1, y_1 \rangle$ to be in R_1^{m+1} iff one of the following three conditions hold:

- $x_1 R_1^m y_1$;
- there is a $z_1 \in U_1$ such that $x_1 R_1^m z_1$ and $z_1 R_1^m y_1$;
- there is a $z_1 \in U_1$ such that $z_1 R_1^m x_1$, $z_1 R_1^m y_1$, and
  - either there are no $w_2, \dots, w_n$ such that both n-tuples $w^x = \langle x_1, w_2, \dots, w_n \rangle$ and $w^y = \langle y_1, w_2, \dots, w_n \rangle$ are in V,
  - or there exist $w_2, \dots, w_n$ such that both w^x and w^y are in V, and $f(w^x) S_1 f(w^y)$ holds. (Note that although $w^x \neq w^y$, it can happen that $f(w^x) = f(w^y)$.)

Since there are no squares in V, R_1^{m+1} is well-defined. The very same inductive proof as above shows that the frame $\mathfrak{F}_1^\infty$ obtained this way is reflexive, transitive and weakly connected, and f is still a p-morphism from $\mathfrak{H}^\infty$ onto $\mathfrak{G}$. ■

Now we can complete the proof of Theorem 2. Let $\varphi \notin L_1 \otimes \cdots \otimes L_n$. Take a countable rooted n-frame $\mathfrak{G} = \langle W, S_1, \dots, S_n \rangle$ refuting φ and such that, for every $i = 1, \dots, n$, $\langle W, S_i \rangle$ is a frame for L_i. Using Lemma 3, we can find a subframe $\mathfrak{H}$ of a product frame for $L_1 \times \cdots \times L_n$ having $\mathfrak{G}$ as its p-morphic image. It follows that $\mathfrak{H} \not\models \varphi$, and so $\varphi \notin (L_1 \times \cdots \times L_n)^{\mathsf{SF}_n}$. Thus, $(L_1 \times \cdots \times L_n)^{\mathsf{SF}_n} \subseteq L_1 \otimes \cdots \otimes L_n$. Proposition 1 gives the converse inclusion. ■

It is not clear how far Theorem 2 can be generalised. On the one hand, we conjecture that it holds for $L_i \in \{\mathbf{K4.3}, \mathbf{Grz}, \mathbf{GL}, \mathbf{GL.3}\}$ as well. For $\mathbf{K4.3}$ even Lemma 3 may hold, although a somewhat different, 'more careful' proof would be needed. However, it is not true that every countable (even finite) frame for, say, $\mathbf{Grz} \otimes \mathbf{Grz}$ is a p-morphic image of a subframe of a product of two $\mathbf{Grz}$-frames. Consider, for instance, the 2-frame $\langle \{x, y, z, w\}, R_1, R_2 \rangle$ with $x R_1 y R_2 z R_1 w R_2 x$. It is not hard to see that if this frame is a p-morphic image of a subframe of $\mathfrak{F}_1 \times \mathfrak{F}_2$ then both $\mathfrak{F}_1$ and $\mathfrak{F}_2$ must contain infinite ascending chains of distinct points, and so cannot be frames for **Grz**.

On the other hand, Theorem 2 does not hold for all subframe logics, not even for those of them that (unlike **Grz**) are characterised by universally first-order definable classes of frames. Take, for instance, the logic

$$\mathbf{K5} = \mathbf{K} \oplus \Diamond\Box p \to \Box p.$$

It is well-known (see e.g. [2]) that **K5** is Kripke complete and characterised by the class of Euclidean frames, i.e., frames $\langle W, R\rangle$ satisfying the universal (Horn) sentence

$$\forall x \forall y \forall u \, \big(R(u,x) \land R(u,y) \to R(x,y)\big).$$

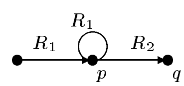
In particular, frames for **K5** have the property

$$\forall x \forall u \, \big(R(u,x) \to R(x,x)\big).$$

Now consider the formula

$$\varphi = \Diamond_1\big(p \land \Diamond_2(q \land \neg p)\big) \land \Box_1\Box_2(q \to \neg\Diamond_1 q).$$

It is clearly satisfiable in the following frame for $\mathbf{K5} \otimes \mathbf{K}$:



On the other hand, it is not hard to see that φ is not satisfiable in any subframe of a product frame for $\mathbf{K5} \times \mathbf{K}$. Therefore,

$$\mathbf{K5} \otimes \mathbf{K} \subsetneq (\mathbf{K5} \times \mathbf{K})^{\mathsf{SF}_2} \subsetneq \mathbf{K5} \times \mathbf{K}.$$

In fact, a similar statement holds for any logic $\mathbf{K} \oplus \Diamond^i\Box p \to \Box^i p$ $(i \geq 1)$ in place of **K5**. Further, the same argument shows that

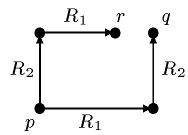
$$\mathbf{K45} \otimes \mathbf{K4} \subsetneq (\mathbf{K45} \times \mathbf{K4})^{\mathsf{SF}_2} \subsetneq \mathbf{K45} \times \mathbf{K4},$$

where $\mathbf{K45} = \mathbf{K4} \oplus \Diamond\Box p \to \Box p$.

Another kind of logics for which Theorem 2 does not hold are those having frames with a finite bound on their branching, e.g. **Alt**. Recall that $\langle W, R\rangle$ is a frame for **Alt** iff every point in W has at most one R-successor. Now consider the formula

$$\psi = p \land \Diamond_1\big(\neg p \land \Diamond_2 q\big) \land \Diamond_2\big(\neg p \land \Diamond_1 r\big) \land \Box_1\Box_2(q \to \neg r).$$

It is clearly satisfiable in the $\mathbf{Alt} \otimes \mathbf{Alt}$-frame



On the other hand, it should be clear that ψ is not satisfiable in any subframe of a frame for $\mathbf{Alt} \times \mathbf{Alt}$. Thus,

$$\mathbf{Alt} \otimes \mathbf{Alt} \subsetneq (\mathbf{Alt} \times \mathbf{Alt})^{\mathsf{SF}_2} \subsetneq \mathbf{Alt} \times \mathbf{Alt}.$$

However, in general the behaviour of arbitrarily relativised products remains unexplored. It would be of interest, for instance, to find solutions to the following problems.

Question 1. *Are arbitrarily relativised products of finitely axiomatisable logics also finitely axiomatisable (in those cases when they differ from the fusions)?*

Question 2. *Are arbitrarily relativised products of decidable logics also decidable?*

Question 3. *Find a general characterisation of those arbitrarily relativised products of logics that coincide with their fusions.*

3 Cubic and locally cubic relativisations

To motivate another kind of relativisations, let us briefly discuss a possible way of creating new, more expressive logics from products. Given the product of n unimodal logics, one may want to add new operations to $\Box_1, \dots, \Box_n$ that 'connect' the different dimensions. Perhaps the simplest and most natural operations of this sort are the *diagonal constants* d_{ij}. Given two natural numbers i and j, $1 \leq i, j \leq n$, the truth-relation for the constant d_{ij} in models over subframes of n-ary product frames is defined as follows:

$$(\mathfrak{M}, \langle u_1, \dots, u_n \rangle) \models \mathsf{d}_{ij} \quad \text{iff} \quad u_i = u_j.$$

The set of n-tuples satisfying d_{ij} is usually called the (i, j)-*diagonal element*.

Actually, the main reason for introducing such constants is to give a 'modal treatment' of equality of classical first-order logic. Let $(\mathbf{S5}^n)^=$ denote the logic (in the language with n boxes and the diagonal constants) determined by the class of cubic universal product $\mathbf{S5}^n$-frames extended with the diagonal elements (interpreting the d_{ij}). Modal algebras for this logic are called *representable cylindric algebras* and are extensively studied in the algebraic logic literature; see e.g. [13, 12, 15] and references therein. By the algebraic results of [23] and [18], $(\mathbf{S5}^n)^=$ is neither finitely axiomatisable nor decidable. Note also that $(\mathbf{S5}^n)^=$ is not a conservative extension of $\mathbf{S5}^n$ [12].

Another natural way of connecting dimensions is via so-called 'jump' modalities. Given a function $\pi : \{1, \ldots, n\} \to \{1, \ldots, n\}$ (such a map can be called a *jump*), define the truth-relation for the unary modal operator s_π in models over subframes of product frames as follows:

$$(\mathfrak{M}, \langle u_1, \ldots, u_n \rangle) \models \mathsf{s}_\pi \varphi \quad \text{iff} \quad (\mathfrak{M}, \langle u_{\pi(1)}, \ldots, u_{\pi(n)} \rangle) \models \varphi.$$

These modal operators are often called (*generalised*) *substitutions*, since (when added to $\mathbf{S5}^n$) they are the 'modal counterparts' of variable substitutions in classical first-order logic; see [20]. Note that in cubic universal product $\mathbf{S5}^n$-frames certain substitutions are expressible with the help of the boxes and the diagonal constants [12]. Various versions of the modal algebras corresponding to products of **S5** logics with substitutions and with or without diagonal constants (e.g., *polyadic* and *substitution algebras*) are studied in [9, 10, 26, 27]; see also [3, 24, 29]. Again, the algebraic results show that most of these logics are non-finitely axiomatisable and undecidable.

Arbitrary relativisations of these extensions of **S5**-products do result in new, decidable many-dimensional logics; see [25, 31]. Moreover, both the diagonal constants and the substitutions can 'detect' some properties of the set of worlds, so it makes sense to consider, for example, those frames whose sets of worlds are closed under jumps. A non-empty set W of n-tuples is called a *local n-cube* if for all maps

$$\pi : \{1, \ldots, n\} \to \{1, \ldots, n\}$$

and all $\langle u_1, \ldots, u_n \rangle \in W$, we have $\langle u_{\pi(1)}, \ldots, u_{\pi(n)} \rangle \in W$. It is easy to see that W is a local n-cube iff for every $\langle u_1, \ldots, u_n \rangle \in W$, the Cartesian power $\{u_1, \ldots, u_n\}^n$ is a subset of W, that is, W is the union of 'n-dimensional cubes.' In particular, local 2-cubes are just the reflexive and symmetric binary relations.

A set W such that $W = U^n$, for some non-empty set U, will be called an *n-cube*. Clearly, n-cubes are special cases of local n-cubes. Let

$$\begin{aligned} \mathsf{LC}_n &= \{\langle W, S_1, \ldots, S_n \rangle \in \mathsf{SF}_n \mid W \text{ is a local } n\text{-cube}\}, \\ \mathsf{C}_n &= \{\langle W, S_1, \ldots, S_n \rangle \in \mathsf{SF}_n \mid W \text{ is an } n\text{-cube}\}. \end{aligned}$$

Note that cubic universal product frames (as introduced in Section 1) belong to C_n. In general, we will refer to frames whose sets of worlds are n-cubes as *cubic*.

Locally cubic relativisations of the above extensions of **S5**-products again give new logics that are also different from the arbitrarily relativised versions. Moreover, all these 'extended relativised **S5**-products' turn out to be decidable and often finitely axiomatisable. A comprehensive treatment of relativised versions of $(\mathbf{S5}^n)^=$ and products of **S5** logics extended with substitutions can be found in [20] under the respective names of *cylindric modal logics* and *modal logics of relations*.

Note that one can also establish connections between different dimensions by introducing polyadic modal operators on product frames. This is the road taken by *arrow logics*, where a binary modal operator is considered. Relativised versions of arrow logics are among the main topics of [20]; see also references therein.

Question 4. *What can be said about extensions with diagonals and/or substitutions of arbitrarily and locally cubic relativised products of modal logics different from* **S5***?*

As mentioned in [22], decidability of these extensions of relativised $\mathbf{K}^n$ can be proved by a reduction to the $n+1$-variable packed fragment of first-order logic. According to [21], the mosaic method (which has been so successful for extensions of relativised $\mathbf{S5}^n$) can also be used to show decidability of extensions of $(L_1 \times \cdots \times L_n)^{\mathsf{LC}_n}$ and $(L_1 \times \cdots \times L_n)^{\mathsf{SF}_n}$, whenever $L_i \in \{\mathbf{K}, \mathbf{T}, \mathbf{K4}, \mathbf{S4}, \mathbf{S5}\}$.

The following two propositions show that if we do not enrich the language of n boxes, then locally cubic and cubic relativisations do not yield anything new.

PROPOSITION 4. *For all Kripke complete unimodal logics $L_1, \dots, L_n$ and all classes $\mathcal{K}$ such that $\mathsf{LC}_n \subseteq \mathcal{K} \subseteq \mathsf{SF}_n$,*

$$(L_1 \times \cdots \times L_n)^{\mathsf{LC}_n} = (L_1 \times \cdots \times L_n)^{\mathcal{K}} = (L_1 \times \cdots \times L_n)^{\mathsf{SF}_n}.$$

Proof. The inclusions

$$(L_1 \times \cdots \times L_n)^{\mathsf{LC}_n} \supseteq (L_1 \times \cdots \times L_n)^{\mathcal{K}} \supseteq (L_1 \times \cdots \times L_n)^{\mathsf{SF}_n}$$

are obvious. To prove the converse ones, take any rooted subframe $\mathfrak{H}$ of a product frame $\mathfrak{F} = \mathfrak{F}_1 \times \cdots \times \mathfrak{F}_n$, where $\mathfrak{F}_i = \langle U_i, R_i \rangle$ is a frame for L_i for each $i = 1, \dots, n$. We show that $\mathfrak{H}$ is isomorphic to a generated subframe of some $\mathfrak{G} \in \mathsf{LC}_n$, where $\mathfrak{G}$ is a subframe of some product frame $\mathfrak{F}_1^+ \times \cdots \times \mathfrak{F}_n^+$, with each $\mathfrak{F}_i^+$ being a frame for L_i. Indeed, take an isomorphic copy of $\mathfrak{F}$ such that the U_i are pairwise disjoint. By Makinson's theorem [19], for each Kripke complete unimodal logic L, either the one-element reflexive frame ($\circ$) or the one-element irreflexive frame ($\bullet$) is a frame for L. For all $i, j \in \{1, \dots, n\}$, we define binary relations R_i^j on U_j by taking

$$R_i^j = \begin{cases} R_i & \text{if } i = j, \\ \emptyset & \text{if } \bullet \text{ is a frame for } L_i, \\ \{\langle u, u \rangle \mid u \in U_j\} & \text{if } \circ \text{ is a frame for } L_i. \end{cases}$$

Now let $U = \bigcup_{1\le i\le n} U_i$. For every $i \in \{1,\dots,n\}$, set $R_i^+ = \bigcup_{1\le j\le n} R_i^j$, and take $\mathfrak{F}_i^+ = \langle U, R_i^+\rangle$. Since each $\mathfrak{F}_i^+$ is a disjoint union of L_i-frames, the product frame

$$\mathfrak{F}^+ = \mathfrak{F}_1^+ \times \cdots \times \mathfrak{F}_n^+$$

is then a frame for $L_1 \times \cdots \times L_n$. Let W denote the set of worlds of $\mathfrak{H}$. Define W^+ to be the smallest local n-cube containing W, that is,

$$W^+ = \bigcup \big\{ \{u_1,\dots,u_n\}^n \mid \langle u_1,\dots,u_n\rangle \in W \big\},$$

and let $\mathfrak{G}$ be the subframe of $\mathfrak{F}^+$ with W^+ as its set of worlds (see Fig. 4 for the case $n = 2$). Then clearly $\mathfrak{G} \in \mathsf{LC}_n$ holds, and $\mathfrak{H}$ is a subframe of $\mathfrak{G}$.

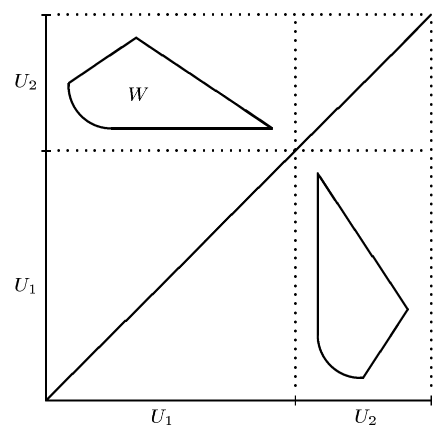


Figure 4. The smallest local 2-cube containing W.

It is not hard to see that $\mathfrak{H}$ is in fact a *generated* subframe of $\mathfrak{G}$, because $W^+ \cap (U_1 \times \cdots \times U_n) = W$. ∎

PROPOSITION 5. *For all subframe logics $L_1,\dots,L_n$,*

$$(L_1 \times \cdots \times L_n)^{\mathsf{C}_n} = L_1 \times \cdots \times L_n.$$

Proof. The inclusion $L_1 \times \cdots \times L_n \subseteq (L_1 \times \cdots \times L_n)^{\mathsf{C}_n}$ is easy, since the L_i are subframe logics and every cubic subframe of a product frame is in fact a product of some subframes of the components.

To prove the converse, we show that every frame $\mathfrak{F} = \mathfrak{F}_1 \times \cdots \times \mathfrak{F}_n$, with each $\mathfrak{F}_i$ being a frame for L_i, is a p-morphic image of a cubic product frame, that is, a frame $\mathfrak{G} = \mathfrak{G}_1 \times \cdots \times \mathfrak{G}_n$ such that every $\mathfrak{G}_i$ has the same set of worlds and each $\mathfrak{G}_i$ is a frame for L_i. Indeed, take a cardinal $\kappa \geq \max_{1 \leq i \leq n} |\mathfrak{F}_i|$ and let $\mathfrak{G}_i$ be the disjoint union of κ-many copies of $\mathfrak{F}_i$. Then $|\mathfrak{G}_i| = \kappa$ and $\mathfrak{F}_i$ is a p-morphic image of $\mathfrak{G}_i$, whenever $1 \leq i \leq n$. Thus, it is easy to see (cf. e.g. [8]) that the 'product' of these p-morphisms gives a p-morphism from $\mathfrak{G}$ onto $\mathfrak{F}$. Since all the $\mathfrak{G}_i$ are of the same cardinality, we may assume that they are frames over the same set of worlds. ■

4 Expanding and decreasing relativisations

First-order modal and intuitionistic logics as well as modal description logics motivate our third group of relativisations. Fix a subset N of $\{1, \ldots, n\}$. An n-frame $\mathfrak{G} = \langle W, S_1, \ldots, S_n \rangle$ is called an *N-expanding* (or *N-decreasing*) *relativised product frame* if there are frames $\mathfrak{F}_1 = \langle U_1, R_1 \rangle, \ldots, \mathfrak{F}_n = \langle U_n, R_n \rangle$ such that

- $\mathfrak{G}$ is a subframe of $\mathfrak{F}_1 \times \cdots \times \mathfrak{F}_n$;
- for all $\langle w_1, \ldots, w_n \rangle \in W$, $j \in N$, and $u \in U_j$,

$$\text{if } w_j R_j u \text{ (or } u R_j w_j) \text{ then } \langle w_1, \ldots, w_{j-1}, u, w_{j+1}, \ldots, w_n \rangle \in W.$$

If $N = \{1\}$ then we call $\mathfrak{G}$ an (n-ary) *expanding* (*decreasing*) *relativised product frame*. Examples of decreasing relativised product frames are the two-dimensional frames for the Halpern–Shoham interval temporal logic (they are also $\{2\}$-expanding); see [11, 20]. In what follows we consider only expanding relativisations. The reader should have no problem in reformulating all notions and results for the case of decreasing ones.

Define EX_n to be the class of all n-ary expanding relativised product frames. In case $n = 2$, we omit the subscript and write EX.

It is easy to see that every expanding relativised product frame satisfies 'left' commutativity

$$\forall x \forall y \forall z \left(x R_i y \wedge y R_1 z \rightarrow \exists u \left(x R_1 u \wedge u R_i z \right) \right),$$

and Church–Rosser properties between coordinates 1 and i, for all $i = 2, \ldots, n$. Therefore, the formulas $\Box_1 \Box_i p \rightarrow \Box_i \Box_1 p$ and $\Diamond_1 \Box_i p \rightarrow \Box_i \Diamond_1 p$ are valid in expanding relativised product frames for all $i = 2, \ldots, n$.

Let us consider first the axiomatisation problem for two-dimensional expanding relativisations. Given logics L_1 and L_2, define

$$[L_1, L_2]^{\mathsf{EX}} = (L_1 \otimes L_2) \ \oplus \ \Box_1 \Box_2 p \rightarrow \Box_2 \Box_1 p \ \oplus \ \Diamond_1 \Box_2 p \rightarrow \Box_2 \Diamond_1 p.$$

A formula

$$\forall x \forall y \forall \bar{z} \left(\psi(x, y, \bar{z}) \rightarrow R(x, y) \right)$$

of the first-order language with a binary predicate R is called a *universal Horn sentence* if $\psi(x,y,\bar{z})$ is built up from atoms using only $\wedge$ and $\vee$. We call a unimodal formula φ a *Horn formula*, if there is a universal Horn sentence χ_φ such that for all frames $\mathfrak{F}$,

$$\mathfrak{F} \models \varphi \qquad \text{iff} \qquad \mathfrak{F} \models \chi_\varphi.$$

A unimodal formula is called *variable free* if it contains no propositional variables, i.e., all of its atomic subformulas are constants $\bot$ or $\top$. We call a unimodal logic *Horn axiomatisable* if it is axiomatisable by only Horn and variable-free formulas. Examples of Horn axiomatisable logics are **K**, **D**, **K4**, **K5**, **S4**, **KD45**, **T**, **B**, **S5**.

THEOREM 6. *Let L_1 and L_2 be Kripke complete logics such that $L_1 \in \{\mathbf{K},\mathbf{T},\mathbf{K4},\mathbf{S4},\mathbf{S5}\}$ and L_2 is Horn axiomatisable. Then*

$$(L_1 \times L_2)^{\mathsf{EX}} = [L_1, L_2]^{\mathsf{EX}}.$$

Proof. It is easy to see that if $\Box_1\Box_2 p \to \Box_2\Box_1 p$ and $\Diamond_1\Box_2 p \to \Box_2\Diamond_1 p$ are valid in a frame $\mathfrak{F} = \langle W, R_1, R_2\rangle$ with symmetric R_1, then $\Box_1\Box_2 p \leftrightarrow \Box_2\Box_1 p$ is also valid in $\mathfrak{F}$. By a result of [8], we then have

$$(\mathbf{S5} \times L_2)^{\mathsf{EX}} = \mathbf{S5} \times L_2 = [\mathbf{S5}, L_2] = [\mathbf{S5}, L_2]^{\mathsf{EX}}$$

(here $[\mathbf{S5}, L_2] = (\mathbf{S5} \otimes L_2) \;\oplus\; \Box_1\Box_2 p \leftrightarrow \Box_2\Box_1 p \;\oplus\; \Diamond_1\Box_2 p \to \Box_2\Diamond_1 p$). In the other cases one can show, by a step-by-step construction similar to the one in the proof of Lemma 3, that every countable rooted 2-frame validating $\Box_1\Box_2 p \to \Box_2\Box_1 p$ and $\Diamond_1\Box_2 p \to \Box_2\Diamond_1 p$ is a p-morphic image of an expanding relativised product frame. Then, as L_2 is Horn definable, we can add the missing pairs to R_1 and R_2, if needed (see [8] for a similar proof). By adding new pairs to R_1 we are not forced to extend the set of worlds, since $L_1 \in \{\mathbf{T},\mathbf{K4},\mathbf{S4}\}$. ■

Question 5. *What can be said about axiomatisations of higher-dimensional expanding relativised products?*

As to decidability, expanding relativisations can be reduced to products. Let φ be a multimodal formula of the language with n boxes and let e be a propositional variable which does not occur in φ. Define by induction on the construction of φ a multimodal formula φ^e as follows:

$$\begin{aligned}
p^e &= p \quad (p \text{ a propositional variable}),\\
(\psi \wedge \chi)^e &= \psi^e \wedge \chi^e,\\
(\neg\psi)^e &= \neg\psi^e,\\
(\Box_1\psi)^e &= \Box_1\psi^e,\\
(\Box_i\psi)^e &= \Box_i(e \to \psi^e) \quad (i = 2,\ldots,n).
\end{aligned}$$

A straightforward induction on the construction of φ proves the following:

THEOREM 7. *For all Kripke complete unimodal logics $L_1,\dots,L_n$ and all multimodal formulas φ, the following conditions are equivalent:*

- $\varphi \in (L_1 \times \cdots \times L_n)^{\mathsf{EX}_n}$;
- $\Big(e \wedge \Box_1^{\leq md(\varphi)}\mathsf{M}_{(2,n)}^{\leq md(\varphi)}(e \to \Box_1 e)\Big) \to \varphi^e \ \in\ L_1 \times \cdots \times L_n,$

where $md(\varphi)$ is the modal depth of φ, $\mathsf{M}_{(2,n)}^{\leq 0}\psi = \psi$ and

$$\mathsf{M}_{(2,n)}^{\leq k+1}\psi = \mathsf{M}_{(2,n)}^{\leq k}\psi \wedge \bigwedge_{i=2}^{n} \Box_i \mathsf{M}_{(2,n)}^{\leq k}.$$

In particular, for $n = 2$,

$$\varphi \in (L_1 \times L_2)^{\mathsf{EX}} \quad \textit{iff} \quad \Big(e \wedge \Box_1^{\leq md(\varphi)}\Box_2^{\leq md(\varphi)}(e \to \Box_1 e)\Big) \to \varphi^e \in L_1 \times L_2.$$

As a consequence of this theorem we obtain that expanding relativised products are decidable in all those cases when the corresponding products are decidable. Unfortunately, for $n \geq 3$ the only known cases of this kind are products of tabular and **Alt** logics [8, 16].

Question 6. *Does the decidability of an expanding relativised product logic imply that the corresponding product logic is decidable as well?*

Let us conclude this section by observing the (lack of) connections between expanding relativised products and finite variable fragments of first-order modal logics with expanding domains. To begin with, we reduce product logics of the form

$$L \times \overbrace{\mathbf{S5} \times \cdots \times \mathbf{S5}}^{n}$$

to n-variable fragments of first-order modal logics $\mathbf{Q}L$ with *constant domains*. Define a translation $\dagger$ from the multimodal language with boxes $\Box_1,\dots,\Box_{n+1}$ to unimodal first-order formulas with variables $x_1,\dots,x_n$ by taking

$$\begin{array}{rcl}
p_i^\dagger & = & P_i(x_1,\dots,x_n)\\
(\varphi \wedge \psi)^\dagger & = & \varphi^\dagger \wedge \psi^\dagger,\\
(\neg\varphi)^\dagger & = & \neg\varphi^\dagger,\\
(\Box_1\psi)^\dagger & = & \Box\psi^\dagger,\\
(\Box_j\psi)^\dagger & = & \forall x_{j-1}\psi^\dagger, \qquad \text{for } j = 2,\dots,n+1.
\end{array}$$

It is not hard to see that the product logic $L \times \mathbf{S5} \times \cdots \times \mathbf{S5}$ is determined by product frames of the form $\mathfrak{F} \times \mathfrak{G} \times \cdots \times \mathfrak{G}$, where $\mathfrak{F}$ is a frame for L and $\mathfrak{G} = \langle V, V \times V \rangle$ for some non-empty set V. Now, there is a one-to-one correspondence between (propositional) modal models $\mathfrak{M}$ based on such product frames and first-order Kripke models of the form $\mathfrak{N} = \langle \mathfrak{F}, V, I \rangle$ where, for all $w \in W$,

$$I(w) = \left\langle V, P_0^{I(w)}, \dots \right\rangle$$

is a first-order structure such that, for all $a_1, \dots, a_n \in V$,

$$\langle a_1, \dots, a_n \rangle \in P_i^{I(w)} \quad \text{iff} \quad (\mathfrak{M}, \langle w, a_1, \dots, a_n \rangle) \models p_i.$$

It should be clear that in fact for *all* multimodal formulas φ of the language with $n+1$ boxes, we have

$$I(w) \models \varphi^{\dagger}[a_1, \dots, a_n] \quad \text{iff} \quad (\mathfrak{M}, \langle w, a_1, \dots, a_n \rangle) \models \varphi,$$

for all $a_1, \dots, a_n \in V$ and $w \in W$. As a consequence we obtain the following:

PROPOSITION 8. *Let L be a Kripke complete unimodal logic. Then for every formula φ,*

$$\varphi \in L \times \overbrace{\mathbf{S5} \times \cdots \times \mathbf{S5}}^{n} \qquad \textit{iff} \qquad \varphi^{\dagger} \in \mathbf{Q}L.$$

On the one hand, it is readily checked that if $n = 1$ then the translation † reduces $(L \times \mathbf{S5})^{\mathsf{EX}}$ to the 1-variable fragment of the first-order modal logic $\mathbf{Q}L_e$ having models with *expanding domains*.

On the other hand, as far as we see, for $n \geq 3$ there is no such reduction of expanding relativised products of the form

$$(4) \qquad (L \times \overbrace{\mathbf{S5} \times \cdots \times \mathbf{S5}}^{n})^{\mathsf{EX}_{n+1}}$$

to $\mathbf{Q}L_e$, since quantifiers $\forall x_i$ and $\forall x_j$ of the latter always commute, while there is no interaction between the boxes $\Box_i$ and $\Box_j$ of the former whenever $i \neq j$ and $i, j > 1$. An alternative approach could be to consider the (2-dimensional) product of L and the multimodal[2] logic $\mathbf{S5}^n$ and then take instead of (4) the 2-dimensional expanding relativised product

$$(L \times \mathbf{S5}^n)^{\mathsf{EX}}.$$

Note that in general it is not known whether the product operator is associative. In particular, for $n \geq 3$ it is not known whether 2-dimensional product

[2]Products of multimodal logics can be defined similarly to the unimodal case.

logic $L \times \mathbf{S5}^n$ and the $n+1$-dimensional product logic $L \times \mathbf{S5} \times \cdots \times \mathbf{S5}$ are the same. Moreover, since we do not know how frames for $\mathbf{S5}^n$ look like when $n \geq 3$ (it is not even decidable whether a finite n-frame is frame for $\mathbf{S5}^n$ [16]), it is not clear how to turn a model for $(L \times \mathbf{S5}^n)^{\mathsf{EX}}$ into a model for $\mathbf{Q}L_e$.

For $n = 2$ we do have a characterisation of $\mathbf{S5} \times \mathbf{S5}$-frames. As is shown in [8], each countable rooted frame for $\mathbf{S5} \times \mathbf{S5}$ is in fact a p-morphic image of a product of two $\mathbf{S5}$-frames. Therefore, it is not hard to see that we have the required reduction: for every formula φ,

$$\varphi \in (L \times (\mathbf{S5} \times \mathbf{S5}))^{\mathsf{EX}} \qquad \text{iff} \qquad \varphi^{\dagger} \in \mathbf{Q}L_e.$$

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