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Bisimulations and Boolean Vectors

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ABSTRACT. A modal accessibility relation is just a transition relation, and so can be represented by a $\{\mathbf{0}, \mathbf{1}\}$ valued transition matrix. Starting from this observation, I first show that the machinery of matrices, over Boolean algebras more general than the two-valued one, is appropriate for investigating multi-modal semantics. Then I show that bisimulations have a rather elegant theory, when expressed in terms of transformations on Boolean vector spaces. The resulting theory is a curious hybrid, fitting between conventional modal semantics and conventional linear algebra. I don't know where the investigations begun here will ultimately wind up, but in the meantime the approach has a kind of curious charm that others may find appealing.

1 Introduction

Bisimulations are to Kripke, or transition, structures as homomorphisms are to groups. Yet while this sounds right, it cannot quite be so, since bisimulations are not maps between frames, but are relations between them. In this paper I show that a shift in the point of view, from Kripke frames to closely related Boolean vector spaces, turns bisimulations from relations to linear mappings having rather nice properties. Mono-modal frames give rise to Boolean vector spaces over the familiar two-valued Boolean algebra, while multi-modal frames bring more complex Boolean algebras into the picture. Still, the basic ideas remain the same for both the mono- and the multi-modal cases. The point of this approach is not to prove new results, but to look at well-known results in a new way, hoping that a fresh perspective will lead to fresh insights.

The paper begins with several sections discussing background. The formal work on Boolean valued vector spaces begins with Section 5.

2 Familiar background

As will be seen shortly, there is little formal difference in the algebraic treatment presented here between mono- and multi-modal logics, so I'll begin with the general case from the start. Let $\mathcal{K}$ be a non-empty set, finite or countable. We might consider the members of $\mathcal{K}$ to be *knowers*, but

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nothing depends on this. For convenience, if $\mathcal{K}$ is finite, I'll assume it is $\{1, 2, \ldots, n\}$, and if it is infinite, $\{1, 2, \ldots\}$.

A $\mathcal{K}$-*language* is a propositional modal language, built up from propositional letters (typically P, Q, $\ldots$) using the propositional connectives $\wedge$, $\vee$, $\neg$ (with $\supset$ taken as a defined connective) and the modal operators $\Diamond_k$ for each $k \in \mathcal{K}$ (with $\Box_k$ taken as a defined connective). I'll skip the obvious language details. If $\mathcal{K} = \{1\}$ I'll say the language is *mono-modal*, and otherwise, *multi-modal*.

A $\mathcal{K}$-*frame* is a tuple, $\langle \mathcal{G}, \mathcal{R}_k : k \in \mathcal{K} \rangle$, where $\mathcal{G}$ is a non-empty set and each $\mathcal{R}_k$ is a binary relation on $\mathcal{G}$. As usual, members of $\mathcal{G}$ will be referred to as possible worlds, and each $\mathcal{R}_k$ as an accessibility relation, or a transition.

If $\mathcal{F} = \langle \mathcal{G}, \mathcal{R}_k : k \in \mathcal{K} \rangle$ is a $\mathcal{K}$-frame then the structure $\langle \mathcal{F}, v \rangle$ is a $\mathcal{K}$-*model* based on this frame provided v is a mapping from members of $\mathcal{G}$ and formulas to truth values $\{\mathbf{0}, \mathbf{1}\}$ (with the usual Boolean structure) such that, for each $\Gamma \in \mathcal{G}$:

1. $v(\Gamma, \neg X) = \neg v(\Gamma, X)$
2. $v(\Gamma, X \wedge Y) = v(\Gamma, X) \wedge v(\Gamma, Y)$
3. $v(\Gamma, X \vee Y) = v(\Gamma, X) \vee v(\Gamma, Y)$
4. $v(\Gamma, \Diamond_k X) = \mathbf{1}$ if and only if $v(\Delta, X) = \mathbf{1}$ for some $\Delta \in \mathcal{G}$ such that $\Gamma \mathcal{R}_k \Delta$

Of course the behavior of v is completely determined by its behavior on propositional letters.

Next is the notion of bisimulation. I divide this into two parts, one concerning frames, the other concerning models. Customarily these are combined, but it is more convenient in the present treatment to separate the notions. In addition, I give a version that is more general than usual, allowing the bisimulation relation to be parametrized by modal operator. As it happens, it is no more work to treat this version than the usual one in the context of the paper. The usual version becomes a special case, which I designate with the terminology *standard*.

DEFINITION 1 Let $\mathcal{F}_\mathcal{G} = \langle \mathcal{G}, \mathcal{R}_k : k \in \mathcal{K} \rangle$ and $\mathcal{F}_\mathcal{H} = \langle \mathcal{H}, \mathcal{S}_k : k \in \mathcal{K} \rangle$ be two $\mathcal{K}$-frames. Also let $\mathcal{A} = \langle \mathcal{A}_k : k \in \mathcal{K} \rangle$ be a family of relations between $\mathcal{G}$ and $\mathcal{H}$. $\mathcal{A}$ is a *frame bisimulation* between $\mathcal{F}_\mathcal{G}$ and $\mathcal{F}_\mathcal{H}$ provided:

1. For all $\Gamma_1, \Gamma_2 \in \mathcal{G}$ and $\Delta_1 \in \mathcal{H}$, if $\Gamma_1 \mathcal{A}_k \Delta_1$, and $\Gamma_1 \mathcal{R}_k \Gamma_2$, then there is some $\Delta_2 \in \mathcal{H}$ such that $\Gamma_2 \mathcal{A}_k \Delta_2$ and $\Delta_1 \mathcal{S}_k \Delta_2$,
2. For all $\Delta_1, \Delta_2 \in \mathcal{H}$ and $\Gamma_1 \in \mathcal{G}$, if $\Gamma_1 \mathcal{A}_k \Delta_1$, and $\Delta_1 \mathcal{S}_k \Delta_2$, then there is some $\Gamma_2 \in \mathcal{G}$ such that $\Gamma_2 \mathcal{A}_k \Delta_2$ and $\Gamma_1 \mathcal{R}_k \Gamma_2$.

If $\mathcal{A}_j = \mathcal{A}_k$ for all $j, k \in \mathcal{K}$, I'll say $\mathcal{A}$ is a *standard* frame bisimulation. For standard frame bisimulations, I'll identify $\mathcal{A}$ with any relation in its family (all of which are the same).

If $\mathcal{K}$ consists of one element, there is no distinction between what we are calling a frame bisimulation and a standard frame bisimulation. I'll call such a case a *mono-modal* frame bisimulation. Historically it is the earliest notion of bisimulation to appear in modal logic. In the more general setting there is actually no interaction between modalities, so we have the following principle.

THEOREM 2 *Let $\mathcal{F}_\mathcal{G} = \langle \mathcal{G}, \mathcal{R}_k : k \in \mathcal{K} \rangle$ and $\mathcal{F}_\mathcal{H} = \langle \mathcal{H}, \mathcal{S}_k : k \in \mathcal{K} \rangle$ be two $\mathcal{K}$-frames, and let $\mathcal{A} = \langle \mathcal{A}_k : k \in \mathcal{K} \rangle$ be a family of relations between $\mathcal{G}$ and $\mathcal{H}$. $\mathcal{A}$ is a frame bisimulation between $\mathcal{F}_\mathcal{G}$ and $\mathcal{F}_\mathcal{H}$ if and only if, for each $k \in \mathcal{K}$, $\mathcal{A}_k$ is a mono-modal frame bisimulation between $\langle \mathcal{G}, \mathcal{R}_k \rangle$ and $\langle \mathcal{H}, \mathcal{S}_k \rangle$.*

DEFINITION 3 Let $\langle \mathcal{F}_\mathcal{G}, v_\mathcal{G} \rangle$ and $\langle \mathcal{F}_\mathcal{H}, v_\mathcal{H} \rangle$ be two $\mathcal{K}$-models, where $\mathcal{F}_\mathcal{G} = \langle \mathcal{G}, \mathcal{R}_k : k \in \mathcal{K} \rangle$ and $\mathcal{F}_\mathcal{H} = \langle \mathcal{H}, \mathcal{S}_k : k \in \mathcal{K} \rangle$. Also let $\mathcal{A}$ be a standard frame bisimulation between $\mathcal{F}_\mathcal{G}$ and $\mathcal{F}_\mathcal{H}$. $\mathcal{A}$ is a *bisimulation* if, in addition, for all $\Gamma \in \mathcal{G}$ and $\Delta \in \mathcal{H}$ with $\Gamma \mathcal{A} \Delta$ we have $v_\mathcal{G}(\Gamma, P) = v_\mathcal{H}(\Delta, P)$ for all propositional letters P.

The well-known key fact about bisimulations is the following, proved by an easy induction on formula complexity. If $\mathcal{A}$ is a bisimulation between the $\mathcal{K}$-models $\langle \mathcal{F}_\mathcal{G}, v_\mathcal{G} \rangle$ and $\langle \mathcal{F}_\mathcal{H}, v_\mathcal{H} \rangle$, and $\Gamma_1 \in \mathcal{G}$ and $\Gamma_2 \in \mathcal{H}$, then if $\Gamma_1 \mathcal{A} \Gamma_2$, $v_\mathcal{G}(\Gamma_1, X) = v_\mathcal{H}(\Gamma_2, X)$ for every $\mathcal{K}$-formula X. Much is known about bisimulations—I refer you to the standard literature for details, [1] among others.

3 Introducing Boolean algebra use

If $\mathcal{K}$ has cardinality greater than 1, we are dealing with multi-modal frames and models. These can be collapsed to mono-modal versions, provided we are willing to complicate the underlying truth-value space. I'll first introduce the notion of a Boolean valued modal model, then discuss connections with multi-modal frames.

3.1 Boolean valued models

I assume the definition and basic properties of Boolean algebras are known—[7] is a very thorough reference. When working in a Boolean algebra I will use $\wedge$, $\vee$, and $\neg$ to denote the operations of meet, join, and complement. I will write $a \Rightarrow b$ for $\neg a \vee b$. I will also use $\leq$ for the standard ordering relation, $a \leq b$ iff $a \wedge b = a$ iff $a \vee b = b$, and I will use $<$ for strict ordering; $a < b$ if $a \leq b$ and $a \neq b$. The bottom element of the algebra will be denoted by $\mathbf{0}$, and the top by $\mathbf{1}$. Finally, I will use $\bigwedge$ and $\bigvee$ for the infinitary meet and join operations, when they exist.

General Assumption 1 For the rest of this paper, $\mathcal{B}$ is a complete Boolean algebra, where *complete* means that infinite, as well as finite, meets and joins exist.

In the definition below, assume we are using a mono-modal language—there is a single possibility operator.

DEFINITION 4 ($\mathcal{B}$-frames and models) A $\mathcal{B}$-frame is a pair $\mathcal{F} = \langle \mathcal{G}, \mathcal{R} \rangle$, where $\mathcal{G}$ is a non-empty set of possible worlds, as usual, and $\mathcal{R}$ is a $\mathcal{B}$-valued accessibility relation: a mapping from pairs of worlds to $\mathcal{B}$. That is, $\mathcal{R} : \mathcal{G} \times \mathcal{G} \to \mathcal{B}$.

If $\mathcal{F} = \langle \mathcal{G}, \mathcal{R} \rangle$ is a $\mathcal{B}$-frame, the structure $\mathcal{M} = \langle \mathcal{F}, v \rangle$ is a $\mathcal{B}$-model based on this frame, provided v maps members of $\mathcal{G}$ and formulas to $\mathcal{B}$ such that, for each $\Gamma \in \mathcal{G}$:

1. $v(\Gamma, \neg X) = \neg v(\Gamma, X)$
2. $v(\Gamma, X \wedge Y) = v(\Gamma, X) \wedge v(\Gamma, Y)$
3. $v(\Gamma, X \vee Y) = v(\Gamma, X) \vee v(\Gamma, Y)$
4. $v(\Gamma, \Diamond X) = \bigvee \{\mathcal{R}(\Gamma, \Delta) \wedge v(\Delta, X) \mid \Delta \in \mathcal{G}\}$

As usual, the action of v at the atomic level completely determines it for all formulas. If we assume $\Box X$ is defined to be $\neg \Diamond \neg X$, as usual, then the condition for $\Box$ becomes the following.

$$v(\Gamma, \Box X) = \bigwedge \{\mathcal{R}(\Gamma, \Delta) \Rightarrow v(\Delta, X) \mid \Delta \in \mathcal{G}\}$$

A modal model in the usual sense is simply a $\mathcal{B}$-model where $\mathcal{B}$ is the usual two-element Boolean algebra.

3.2 Connections

I will establish a connection between multi-modal models and $\mathcal{B}$-models, after which we can confine our discussion to the Boolean valued case. Suppose we have a finite or countable set $\mathcal{K}$, $\mathcal{F}_{\mathcal{K}} = \langle \mathcal{G}, \mathcal{R}_k : k \in \mathcal{K} \rangle$ is a $\mathcal{K}$-frame, and $\mathcal{M}_{\mathcal{K}} = \langle \mathcal{F}_{\mathcal{K}}, v_{\mathcal{K}} \rangle$ is a $\mathcal{K}$-model. I'll use this to create a related Boolean-valued modal model.

First, let $\mathcal{B}$ be the powerset algebra of $\mathcal{K}$. This is an atomic Boolean algebra, with the atoms being members of the form $\{n\}$, for $n \in \mathcal{K}$.

Next, instead of the $\mathcal{K}$-language appropriate for $\mathcal{M}_{\mathcal{K}}$ we want a mono-modal language. But, I enlarge this language by introducing propositional constants: for each $n \in \mathcal{K}$ let P_n be a distinct propositional constant.

I now create a $\mathcal{B}$-model as follows. Let $\mathcal{G}$ be the same set of possible worlds as in the multi-modal frame $\mathcal{F}_{\mathcal{K}}$, and let $\mathcal{R}$ be the $\mathcal{B}$-valued accessibility relation given by: $\mathcal{R}(\Gamma, \Delta) = \{k \in \mathcal{K} \mid \Gamma \mathcal{R}_k \Delta\}$. This gives us a $\mathcal{B}$-frame, $\mathcal{F} = \langle \mathcal{G}, \mathcal{R} \rangle$. I'll define a valuation v by specifying it for atomic formulas. If A is atomic and not one of the propositional constants P_n, set $v(\Gamma, A)$ to be the $\mathbf{1}$ of $\mathcal{B}$ if $v_{\mathcal{K}}(\Gamma, A)$ is true, and set $v(\Gamma, A)$ to be $\mathbf{0}$ otherwise. Finally, set $v(\Gamma, P_n) = \{n\}$. Extend v to non-atomic formulas as usual. We thus have a $\mathcal{B}$-model $\mathcal{M} = \langle \mathcal{F}, v \rangle$.

Now, define a map θ from formulas of the $\mathcal{K}$-language to formulas of the mono-modal language enlarged with the propositional constants P_n.

1. for an atomic formula A (which can not be any P_n), set $\theta(A) = A$.

2. θ is a homomorphism with respect to propositional connectives, that is $\theta(X \wedge Y) = \theta(X) \wedge \theta(Y)$, and so on.

3. $\theta(\Diamond_k X) = P_k \supset \Diamond \theta(X)$

PROPOSITION 5 *For a formula X in the $\mathcal{K}$-language, $v_{\mathcal{K}}(\Gamma, X)$ is true in the multi-modal model $\mathcal{M}_{\mathcal{K}}$ if and only if $v(\Gamma, \theta(X)) = \mathbf{1}$ in the Boolean valued model $\mathcal{M}$.*

I'll leave the proof of this to you. It is a straightforward induction on formula degree, and makes use of the observation that in Boolean valued modal modals, $v(\Gamma, X \supset Y) = \mathbf{1}$ if and only if $v(\Gamma, X) \leq v(\Gamma, Y)$.

The result above will not be needed in what follows. It serves as motivation for the consideration of Boolean valued models in place of multi-modal ones. The switch to the Boolean valued case makes an algebraic approach much simpler and more natural. In [2, 4, 3, 5] Heyting algebras were used, which are more general than Boolean algebras. Using them allowed consideration, not just of multiple modalities, as above, but also of dependencies between them. This is more than is needed here, however.

4 Introducing vector spaces

A (two-valued) Kripke frame is just a directed graph. Transition matrices are a common way of representing edges in a graph, and these relate well to modal machinery. As an example, consider the frame depicted in Figure 1. The accessibility relation is represented by the matrix

$$\begin{bmatrix} \mathbf{0} & \mathbf{1} & \mathbf{1} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{1} & \mathbf{0} \end{bmatrix}$$

where the entry in position (i, j) is $\mathbf{1}$ if there is an edge from Γ_i to Γ_j, and otherwise is $\mathbf{0}$. This is material that is familiar from many books—[6] is a recommended source.

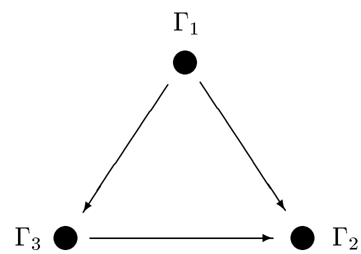


Figure 1. A classical mono-modal frame

Standard terminology is to call a set of possible worlds in a frame a *proposition*, so that in any model based on a frame the set of worlds in which a formula is true is a proposition. For our purposes, instead of working with propositions as sets, we work with propositions as *Boolean vectors*: for instance, using the frame of Figure 1 the vector $\langle \mathbf{1}, \mathbf{1}, \mathbf{0} \rangle$ corresponds to the set $\{\Gamma_1, \Gamma_2\}$; having Γ_i in the set corresponds to having $\mathbf{1}$ as the i^{th} component of the vector. Of course the introduction of vectors for this purpose depends on an arbitrary ordering of possible worlds, but any two ways of ordering will result in vector spaces that are isomorphic in obvious ways, so we can ignore this point.

The introduction of a Boolean vector space meshes well with the usual Kripke semantics. Propositional connectives correspond to straightforward Boolean operations on vectors. More interestingly, suppose matrix multiplication is defined in the usual way, but with meet ($\wedge$) replacing multiplication, and join ($\vee$) replacing addition. In the frame of Figure 1, if the vector $\langle \mathbf{1}, \mathbf{1}, \mathbf{0} \rangle$ represents the worlds at which a formula X is true (that is, X is true at Γ_1 and Γ_2), then the product of the frame transition matrix and the column vector corresponding to $\langle \mathbf{1}, \mathbf{1}, \mathbf{0} \rangle$ is a (column) vector that represents the worlds at which $\Diamond X$ is true.

$$\begin{bmatrix} \mathbf{0} & \mathbf{1} & \mathbf{1} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{1} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{1} \\ \mathbf{1} \\ \mathbf{0} \end{bmatrix} = \begin{bmatrix} \mathbf{1} \\ \mathbf{0} \\ \mathbf{1} \end{bmatrix}$$

Rather nicely, this extends directly to Boolean valued frames as well. Figure 2 shows such a frame in which the underlying space of truth values is the power set of $\{1,2,3\}$. Using this, we can still identify propositions with vectors, but now they are vectors over the powerset space. So, for instance, the vector $\langle\{1,2\},\{3\}\rangle$ could represent the status of a formula X in a particular model over this frame: at Γ_1 the truth value of X is $\{1,2\}$, and at Γ_2 it is $\{3\}$.

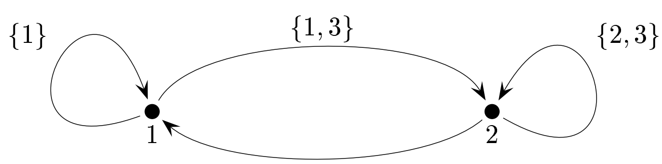


Figure 2. A Boolean-valued mono-modal frame

As we might expect, the accessibility relation can be represented by a matrix, but now with values in the powerset space. For the frame of Figure 2 the matrix is the following.

$$\begin{bmatrix} \{1\} & \{1,3\} \\ \emptyset & \{2,3\} \end{bmatrix}$$

And once again, such a matrix corresponds to the possibility operator, with matrix multiplication as application. Then, if X is represented by the vector $\langle\{1,2\},\{3\}\rangle$ as above, the following represents the status of $\Diamond X$.

$$\begin{bmatrix} \{1\} & \{1,3\} \\ \emptyset & \{2,3\} \end{bmatrix} \begin{bmatrix} \{1,2\} \\ \{3\} \end{bmatrix} = \begin{bmatrix} \{1,3\} \\ \{3\} \end{bmatrix}$$

Now it is possible to say a little more clearly what is to come in the rest of the paper. The Boolean vector methodology sketched above will be rigorously introduced. Propositions will be identified with vectors. Just as accessibility relations within frames can be represented by matrices, so too for relations *between* frames. These matrices can be thought of as mapping propositions in one frame to propositions in another. Among such matrices are those corresponding to bisimulations. What are the properties of these Boolean algebra valued matrices in general, and what is special about those that are bisimulations?

5 Boolean terminology and notation

Now the primary formal development starts. Think of the preceding sections as establishing a context—why we might be interested—but at this point I'll start fresh, with Boolean valued vector spaces themselves as the topic. Underlying everything will be a complete Boolean algebra—in the example of Figure 2 it was the powerset of $\{1, 2, 3\}$—Boolean vector spaces will be built on such a Boolean algebra.

5.1 Boolean vector spaces

Recall, $\mathcal{B}$ is a complete Boolean algebra. I do not take an abstract approach to the subject of Boolean vector spaces; a concrete representation is fine for present purposes: $\mathcal{B}^n$ is the space of n-tuples over $\mathcal{B}$. It too constitutes a Boolean algebra using pointwise operations and relations. Overloading notation, I will also denote these by $\wedge$, $\vee$, and $\neg$, and $\leq$, with $\mathbf{0}_n$ and $\mathbf{1}_n$ for bottom and top, respectively $\langle \mathbf{0}, \mathbf{0}, \ldots, \mathbf{0} \rangle$ and $\langle \mathbf{1}, \mathbf{1}, \ldots, \mathbf{1} \rangle$ (with n components). For vectors in $\mathcal{B}^n$, a dot product is defined by: $\langle b_1, \ldots, b_n \rangle \cdot \langle c_1, \ldots, c_n \rangle = (b_1 \wedge c_1) \vee \ldots \vee (b_n \wedge c_n)$. Obviously $v \cdot \mathbf{0}_n = \mathbf{0}$ and $v \cdot \mathbf{1}_n \neq \mathbf{0}$ iff $v \neq \mathbf{0}_n$. We also have $v \cdot \mathbf{1}_n = v \cdot v$, incidentally.

I will also consider $\mathcal{B}^\infty$, whose members are infinite tuples over $\mathcal{B}$, $\langle b_1, b_2, b_3, \ldots \rangle$. This corresponds to Kripke frames with a countable set of worlds. Boolean operations and orderings are still defined pointwise and are no problem. Also the dot product operation extends to $\mathcal{B}^\infty$ (recall infinite joins exist): the dot product of $\langle b_1, b_2, b_3, \ldots \rangle$ and $\langle c_1, c_2, c_3, \ldots \rangle$ is the join of the set $\{(b_1 \wedge c_1), (b_2 \wedge c_2), (b_3 \wedge c_3), \ldots\}$.

5.2 Boolean matrices

Boolean matrices are matrices in the usual sense, but with entries from $\mathcal{B}$. I'll use $\mathcal{B}_{n,m}$ to denote the collection of all $n \times m$ Boolean matrices, where either or both of n and m could be ∞. Once again overloading notation, $\wedge$, $\vee$, $\neg$ and $\leq$ are defined componentwise. It is easy to see that $\mathcal{B}_{n,m}$ itself is a Boolean algebra with respect to these operations. Generalizing notation from above, $\mathbf{1}_{n,m}$ is the Boolean $n \times m$ matrix with every entry $\mathbf{1}$, and $\mathbf{0}_{n,m}$ is the Boolean $n \times m$ matrix with every entry $\mathbf{0}$. I will use I_n for the $n \times n$ identity matrix.

For a matrix $A \in \mathcal{B}_{n,m}$, $[A]_i$ denotes row vector i of A, thought of as a member of $\mathcal{B}^m$, so that $[A]_i \in \mathcal{B}^m$. Likewise $[A]^j$ denotes column vector j of A, taken to be a member of $\mathcal{B}^n$. And $[A]_i^j$ denotes the entry in row i and column j. Then Boolean matrix multiplication is characterized in the usual way: if $B \in \mathcal{B}_{m,k}$, AB is the $n \times k$ matrix such that $[AB]_i^j = [A]_i \cdot [B]^j$. I also introduce one non-standard piece of notation.

DEFINITION 6 If β is a member of $\mathcal{B}$, by $\beta_{n,m}(a,b)$ I mean the $n \times m$ matrix all of whose entries are $\mathbf{0}$ except for the entry in row a, column b, which is β.

The following will be useful in proving some inequalities involving Boolean matrices. Item 1 is overly generous, in a sense, but using it is no more work and is often easier to apply than a stricter version would be.

LEMMA 7 *Let $A, B \in \mathcal{B}_{n,m}$.*

1. *$A \leq B$ provided $\beta_{n,m}(a,b) \leq A$ implies $\beta_{n,m}(a,b) \leq B$ for every $\beta \in \mathcal{B}$ and every $a \leq n$ and $b \leq m$.*
2. *$\beta_{m,n}(a,b)\beta_{n,k}(b,c) = \beta_{m,k}(a,c)$*

Proof. Suppose $\beta_{n,m}(a,b) \leq A$ implies $\beta_{n,m}(a,b) \leq B$ for every $\beta \in \mathcal{B}$ and every $a \leq n$ and $b \leq m$. Let $\beta = [A]_i^j$. Then obviously $\beta_{n,m}(i,j) \leq A$, so $\beta_{n,m}(i,j) \leq B$, and thus $[A]_i^j \leq [B]_i^j$. Since i and j were arbitrary, $A \leq B$. Part 2 follows directly from the definition of Boolean matrix multiplication. ■

6 Elementary matrix multiplication properties

For use later on, we need some basic properties of Boolean matrix multiplication. Some of this is analogous to matrix multiplication over a field, some is rather different. As might be expected, Boolean matrix multiplication is associative but not generally commutative. Also, multiplication distributes over $\vee$, that is, $A(B \vee C) = AB \vee AC$ and $(B \vee C)A = BA \vee CA$. If $B \leq C$ then $AB \leq AC$ and $BA \leq CA$. (All this is under the assumption that dimensions are such that the products displayed are defined, of course.) A^T denotes the transpose of A and, as usual, $(AB)^T = B^T A^T$. In addition, Boolean matrices have a number of special features not shared by matrices over a field.

THEOREM 8 *Assume $A \in \mathcal{B}_{m,n}$. Then:*

1. $A \leq AA^TA$
   $A^T \leq A^TAA^T$
2. $A\mathbf{1}_{n,k} = AA^TA\mathbf{1}_{n,k}$
   $A^T\mathbf{1}_{m,k} = A^TAA^T\mathbf{1}_{m,k}$

3. $(A^TA) \leq (A^TA)^2 \leq (A^TA)^3 \leq \ldots$
$(AA^T) \leq (AA^T)^2 \leq (AA^T)^3 \leq \ldots$

Proof. The items to be proved come in pairs. I'll show one half, the other half is obviously similar.

1. By Lemma 7, part 1, it is enough to show that if $\beta_{m,n}(a,b) \leq A$ then $\beta_{m,n}(a,b) \leq AA^TA$ for an arbitrary $\beta \in \mathcal{B}$, a and b. So, suppose $\beta_{m,n}(a,b) \leq A$. Then $\beta_{n,m}(b,a) \leq A^T$ so by Lemma 7, part 2, $\beta_{m,m}(a,a) = \beta_{m,n}(a,b)\beta_{n,m}(b,a) \leq AA^T$, and then by a similar calculation, $\beta_{m,n}(a,b) = \beta_{m,m}(a,a)\beta_{m,n}(a,b) \leq AA^TA$.

2. $A\mathbf{1}_{n,k} \leq AA^TA\mathbf{1}_{n,k}$ by part 1. Conversely, $A^TA\mathbf{1}_{n,k} \leq \mathbf{1}_{n,k}$ so $AA^TA\mathbf{1}_{n,k} \leq A\mathbf{1}_{n,k}$.

3. By part 1 we have $A \leq AA^TA$ so $A^TA \leq A^TAA^TA$. And so on.

■

Part 3 above is a strictly increasing sequence of inequalities, in the sense that for any n one can find a finite Boolean matrix A such that the sequence strictly grows for the first n steps. If one uses infinite dimension matrices, the growth can be made to continue indefinitely.

EXAMPLE 9 Let

$$A = \begin{bmatrix} \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{1} & \mathbf{1} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{1} & \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{1} \end{bmatrix}$$

Then

$$(A^TA) = \begin{bmatrix} \mathbf{1} & \mathbf{1} & \mathbf{0} & \mathbf{0} \\ \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{1} & \mathbf{1} & \mathbf{1} \\ \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{1} \end{bmatrix} \quad \text{and} \quad (A^TA)^2 = \begin{bmatrix} \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{0} \\ \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{1} \\ \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{1} \\ \mathbf{0} & \mathbf{1} & \mathbf{1} & \mathbf{1} \end{bmatrix} \quad \text{and}$$

$$(A^TA)^3 = \begin{bmatrix} \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{1} \\ \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{1} \\ \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{1} \\ \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{1} \end{bmatrix}$$

Thus, $(A^TA) < (A^TA)^2 < (A^TA)^3 = (A^TA)^4 = \ldots$

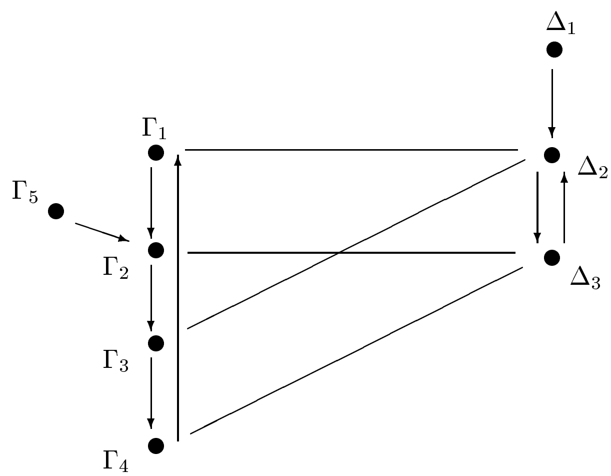


Figure 3. A bisimulation, two-valued setting

7 Bisimulation motivation

In order to motivate the work on matrices in the next several sections, a bisimulation example will be useful. In fact, one in which the underlying logic is two-valued will suffice for this purpose.

EXAMPLE 10 In Figure 3 two frames are displayed, a left-hand one with possible worlds Γ_1, Γ_2, Γ_3, Γ_4, and Γ_5, and a right-hand one with possible worlds Δ_1, Δ_2, and Δ_3. The two (mono-modal) accessibility relations are indicated by arrows.

A relation, call it $\mathcal{A}$, between the two frames is indicated by lines (not arrows) connecting worlds. It meets the conditions for being a standard frame bisimulation as given in Definition 1. It is not, of course, a bisimulation since we do not have models, but only frames, here. The additional condition for bisimulations in Definition 3 can be thought of as a restriction on the models we can base on these frames. For instance, since both Γ_1 and Γ_3 are related by $\mathcal{A}$ to Δ_2, any model based on the left-hand frame must assign the same truth values to propositional letters at both Γ_1 and Γ_3 if the condition for being a bisimulation is to apply.

Although $\mathcal{A}$ is a relation, it can be thought of as a function as well, mapping propositions in one frame to propositions in the other frame. For instance, the proposition $\{\Gamma_1, \Gamma_2, \Gamma_5\}$ maps to the proposition $\{\Delta_2, \Delta_3\}$, the set of worlds in the right frame related to the worlds of the proposition in the left frame. But since I am representing propositions, not as sets, but as vectors, I prefer to say that the vector $\langle \mathbf{1}, \mathbf{1}, \mathbf{0}, \mathbf{0}, \mathbf{1} \rangle$ maps to the vector $\langle \mathbf{0}, \mathbf{1}, \mathbf{1} \rangle$. The relation $\mathcal{A}$ can itself be represented as a matrix, A; I use a representation in which column j is the vector corresponding to the set of worlds related to Γ_j. Stated differently, $[A]_i^j = \mathbf{1}$ just in case $\Gamma_j \mathcal{A} \Delta_i$. For the present example, this gives the following matrix.

$$A = \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{1} & \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{1} & \mathbf{0} \end{bmatrix}$$

And once again, matrix multiplication provides appropriate machinery. It was noted above that the vector $\langle \mathbf{1}, \mathbf{1}, \mathbf{0}, \mathbf{0}, \mathbf{1} \rangle$ maps to the vector $\langle \mathbf{0}, \mathbf{1}, \mathbf{1} \rangle$, and in fact we have the following:

$$A \begin{bmatrix} \mathbf{1} \\ \mathbf{1} \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{1} \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ \mathbf{1} \\ \mathbf{1} \end{bmatrix}$$

Now we return to the point raised above, that if we want to base a model on one of these frames, and respect the frame bisimulation $\mathcal{A}$, we are restricted in our assignments of truth values to atomic formulas—not every proposition can be used. For instance, requiring an atomic formula P to be true only at Γ_1 will not do; since Γ_1 is related by $\mathcal{A}$ to Δ_2 which in turn is related to Γ_3, if P is taken to be true at Γ_1 we must also take it to be true at Γ_3. Rephrasing this in terms of vectors, $\langle \mathbf{1}, \mathbf{0}, \mathbf{0}, \mathbf{0}, \mathbf{0} \rangle$ is not an appropriate vector for us to be working with in the setup of Figure 3. In fact, if we map it from the left-hand frame to the right-hand one using A, then back using A^T, we get the following:

$$A^T A \begin{bmatrix} \mathbf{1} \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \end{bmatrix} = \begin{bmatrix} \mathbf{1} \\ \mathbf{0} \\ \mathbf{1} \\ \mathbf{0} \\ \mathbf{0} \end{bmatrix}$$

This is what we should expect. But here things stop. If we apply the mapping $A^T A$ to $\langle \mathbf{1}, \mathbf{0}, \mathbf{1}, \mathbf{0}, \mathbf{0} \rangle$, we simply get $\langle \mathbf{1}, \mathbf{0}, \mathbf{1}, \mathbf{0}, \mathbf{0} \rangle$ back again. Vectors that are left unchanged in this way are appropriate candidates for propositional letter assignments. I call such vectors *stable*, and the next section starts a formal investigation of them.

Example 10 was mono-modal, but multi-modal examples can be treated in the same way. Recall that Definition 1 allowed multi-modal models and a parameterized bisimulation relation. Converted to an algebraic setting, this just amounts to allowing bisimulations to be represented by matrices over Boolean algebras that are more complex than the two-valued one, just as we have represented multi-modal accessibility relations by such matrices. This is the setting you should have in mind for the following sections.

8 Linear mappings and stability

The previous section sketched background ideas informally—now it is time for the mathematical details. For this section, let $A \in \mathcal{B}_{m,n}$ (with ∞ allowed as values for m or n). A can be thought of as defining a mapping from $\mathcal{B}^n$ to $\mathcal{B}^m$ (informally, from propositions to propositions). We identify vectors with column vectors in the obvious way so that, properly speaking, A maps $\mathcal{B}_{n,1}$ to $\mathcal{B}_{m,1}$. Then, for $v \in \mathcal{B}_{n,1}$, its image under A is Av. More generally, for each k we can think of A as defining a mapping from $\mathcal{B}_{n,k}$ to $\mathcal{B}_{m,k}$, that is, $A : \mathcal{B}_{n,k} \to \mathcal{B}_{m,k}$. For a matrix $V \in \mathcal{B}_{n,k}$ its image is defined, using matrix multiplication, as AV. Of course the transpose of A maps in the reverse direction, $A^T : \mathcal{B}_{m,k} \to \mathcal{B}_{n,k}$.

In general, the transpose of a Boolean matrix won't be its inverse. In fact, having an inverse is a rare property for a Boolean matrix to possess. But as was noted in the previous section, the collection of things on which the transpose behaves like an inverse will be a collection of special interest.

DEFINITION 11 I will say a matrix $V \in \mathcal{B}_{n,k}$ is *A-stable* if $A^T AV = V$. Likewise a matrix $W \in \mathcal{B}_{m,k}$ is *A^T-stable* if $AA^T W = W$. If no confusion is likely to result, I may just use the term *stable*.

THEOREM 12 *The following are properties of stability:*

1. *If V is A-stable then AV is A^T-stable, and if W is A^T-stable then $A^T W$ is A-stable.*
2. *$\mathbf{0}_{n,k}$ is the smallest A-stable matrix in $\mathcal{B}_{n,k}$, and $\mathbf{0}_{m,k}$ is the smallest A^T-stable matrix in $\mathcal{B}_{m,k}$.*
3. *The smallest stable matrices map to each other. That is, $A\mathbf{0}_{n,k} = \mathbf{0}_{m,k}$, and $A^T \mathbf{0}_{m,k} = \mathbf{0}_{n,k}$.*
4. *$A^T A\mathbf{1}_{n,k}$ is the largest A-stable matrix in $\mathcal{B}_{n,k}$, and $AA^T \mathbf{1}_{m,k}$ is the largest A^T-stable matrix in $\mathcal{B}_{m,k}$.*
5. *The largest stable matrices map to each other. That is, $A(A^T A\mathbf{1}_{n,k}) = AA^T \mathbf{1}_{m,k}$ and $A^T(AA^T \mathbf{1}_{m,k}) = A^T A\mathbf{1}_{n,k}$.*

Proof. For each item I'll show one half; the other is similar.

1. Suppose V is A-stable, $A^T AV = V$. Then trivially $AA^T AV = AV$ so AV is A^T-stable.

2. Clearly $A^T A\mathbf{0}_{n,k} = A^T \mathbf{0}_{m,k} = \mathbf{0}_{n,k}$, so $\mathbf{0}_{n,k}$ is A-stable. It is obviously smallest.

3. Already used in previous item.

4. $A^T A(A^T A\mathbf{1}_{n,k}) = A^T(AA^T A\mathbf{1}_{n,k}) = A^T A\mathbf{1}_{n,k}$ by Theorem 8 part 2, so we have A-stability. And if V is A-stable, $V = A^T AV \leq A^T A\mathbf{1}_{n,k}$, so $A^T A\mathbf{1}_{n,k}$ is largest.

5. In one direction, $A\mathbf{1}_{n,k} \leq \mathbf{1}_{m,k}$ so $AA^T A\mathbf{1}_{n,k} \leq AA^T\mathbf{1}_{m,k}$. In the other direction, $A^T\mathbf{1}_{m,k} \leq \mathbf{1}_{n,k}$ so using Theorem 8 part 2, $AA^T\mathbf{1}_{m,k} = AA^T AA^T\mathbf{1}_{m,k} \leq AA^T A\mathbf{1}_{n,k}$.

■

I will eventually show the stable matrices themselves form a Boolean algebra, but an additional assumption will be needed. Without that, we still have the following.

THEOREM 13 *Let $V, W \in \mathcal{B}_{n,k}$, and assume both are A-stable.*

1. $A(V \vee W) = AV \vee AW$

2. $V \vee W$ is A-stable

Similar results obtain for A^T as well.

Proof. The arguments are as follows

1. This is immediate since Boolean matrix multiplication always distributes over $\vee$.

2. $A^T A(V \vee W) = A^T(AV \vee AW) = A^T AV \vee A^T AW = V \vee W$.

■

9 The final general assumption

So far we have been working with arbitrary (complete) Boolean algebras. But algebras arising from multi-modal logics are rather special. One way of saying it is that they are powerset algebras, as the discussion in Section 3 illustrated. Another way of saying it is that they are *atomic*.

DEFINITION 14 For the Boolean algebra $\mathcal{B}$:

1. $a \in \mathcal{B}$ is an *atom* if $a \neq \mathbf{0}$ and there is no $b \in \mathcal{B}$ such that $\mathbf{0} < b < a$.
2. $\mathcal{B}$ is *atomic* if for each $x \in \mathcal{B}$ other than $\mathbf{0}$ there is an atom a such that $a \leq x$.

Atoms have many useful features, several of which we will need. If β is an atom and $\beta \leq x \vee y$ then $\beta \leq x$ or $\beta \leq y$. If α and β are different atoms, $\alpha \wedge \beta = \mathbf{0}$, while of course $\alpha \wedge \alpha = \alpha$. If $\mathcal{B}$ is finite, it is automatically complete and atomic. Whether finite or not, if $\mathcal{B}$ is atomic and complete, each member is the join of the collection of atoms below it. This implies that a complete, atomic Boolean algebra is isomorphic to the collection of all subsets of a set, namely the set of atoms.

General Assumption 2 From now on, $\mathcal{B}$ is both complete and atomic.

LEMMA 15 *Assume $A \in \mathcal{B}_{m,n}$. Then for any $V \in \mathcal{B}_{n,k}$ and $W \in \mathcal{B}_{k,n}$*

1. $V \wedge A^T W \leq A^T AV$
2. $W \wedge AV \leq AA^T W$

Proof. I'll only show item 1. Fix a and b—I'll show $[V \wedge A^T W]_a^b \leq [A^T AV]_a^b$. And since each member of $\mathcal{B}$ is the join of the family of atoms below it, it is enough to show that for any atom β if $\beta \leq [V \wedge A^T W]_a^b$ then $\beta \leq [A^T AV]_a^b$. Assume $\beta \leq [V]_a^b$ and $\beta \leq [A^T W]_a^b$. From the second of these, $\beta \leq [A^T]_a \cdot [W]^b$, that is, $\beta \leq \bigvee_c \{[A^T]_a^c \wedge [W]_c^b\}$. Since β is an atom, for some c, $\beta \leq [A^T]_a^c \wedge [W]_c^b$, so in particular, $\beta \leq [A^T]_a^c$. It follows that $\beta_{n,m}(a,c) \leq A^T$. Then $\beta_{m,n}(c,a) \leq A$, so by Lemma 7, part 2, $\beta_{n,n}(a,a) \leq A^T A$. Then again, since $\beta_{n,k}(a,b) \leq V$, $\beta_{n,k}(a,b) \leq A^T AV$, so $\beta \leq [A^T AV]_a^b$. ■

Now Theorem 13 for joins can have a companion for meets.

THEOREM 16 *Let $A \in \mathcal{B}_{m,n}$, $V, W \in \mathcal{B}_{n,k}$, and assume both V and W are A-stable.*

1. $A(V \wedge W) = AV \wedge AW$

2. $V \wedge W$ *is* A*-stable*

Similar results obtain for A^T *as well.*

Proof. The arguments are as follows

1. First, $V \wedge W \leq V$, so $A(V \wedge W) \leq AV$. Similarly $A(V \wedge W) \leq AW$. So $A(V \wedge W) \leq AV \wedge AW$.

   Next, V is A-stable so AV is A^T-stable, and consequently $AV \leq AA^T\mathbf{1}_{m,k}$ by Theorem 12 part 4. Similarly $AW \leq AA^T\mathbf{1}_{m,k}$. Now

$$\begin{aligned} AV \wedge AW &= AV \wedge AW \wedge AA^T\mathbf{1}_{m,k} && (1)\\ &\leq AA^T(AV \wedge AW) && (2)\\ &\leq A(A^T AV \wedge A^T AW) && (3)\\ &= A(V \wedge W) && (4)\end{aligned}$$

   Here (2) is by Lemma 15 part 2, taking W of that Lemma to be $AV \wedge AW$ and V of that Lemma to be $A^T\mathbf{1}_{m,k}$. For (3) we apply the first half of the argument, using A^T.

2. Similar to part 2 of Theorem 13.

■

10 Negation and stability

Throughout this section $A \in \mathcal{B}_{m,n}$, so that for each k, $A : \mathcal{B}_{n,k} \to \mathcal{B}_{m,k}$ and $A^T : \mathcal{B}_{m,k} \to \mathcal{B}_{n,k}$. Also I'll systematically use V as a member of $\mathcal{B}_{n,k}$. The goal of this section is to show that A-stable matrices have a 'natural' notion of negation. First, a few minor preliminary items. Recall the notation of Definition 6.

LEMMA 17 *If* $\beta_{m,k}(a,b) \leq A\beta_{n,k}(c,b)$ *then* $\beta_{n,k}(c,b) \leq A^T\beta_{m,k}(a,b)$.

Proof.
All entries of $\beta_{m,k}(a,b)$ are $\mathbf{0}$ except for β in row a, column b. Consequently if $\beta_{m,k}(a,b) \leq A\beta_{n,k}(c,b)$ it must be that $\beta \leq [A]^c_a$. This in turn implies $\beta \leq [A^T]^a_c$, or $\beta_{n,m}(c,a) \leq A^T$. Consequently $\beta_{n,k}(c,b) = \beta_{n,m}(c,a)\beta_{m,k}(a,b) \leq A^T\beta_{m,k}(a,b)$. ■

LEMMA 18 *Let* V *be* A*-stable. Then*

1. $AV \wedge A(\neg V) = \mathbf{0}_{m,k}$

2. $A^T A(\neg V) \leq \neg V$

3. $A^T A(\neg V) = \neg V \wedge A^T A \mathbf{1}_{n,k}$

Proof. The arguments are as follows.

1. This makes use of an infinite distributive law that holds in complete Boolean algebras, see [7, Vol. 1, Lemma 1.33]. I'll suppose $AV \wedge A(\neg V) \neq \mathbf{0}_{m,k}$ and derive a contradiction. By the supposition, for some a and b the entry in row a and column b is non-zero. Then, using the distributive law:

$$\begin{aligned}
\mathbf{0} &< [AV \wedge A(\neg V)]_a^b \\
&= [AV]_a^b \wedge [A(\neg V)]_a^b \\
&= [A]_a \cdot [V]^b \wedge [A]_a \cdot [\neg V]^b \\
&= \bigvee_c \{[A]_a^c \wedge [V]_c^b\} \wedge \bigvee_d \{[A]_a^d \wedge [\neg V]_d^b\} \\
&= \bigvee_{c,d} \{[A]_a^c \wedge [V]_c^b \wedge [A]_a^d \wedge [\neg V]_d^b\}
\end{aligned}$$

Of course not every member of this join can be $\mathbf{0}$. Let us say that for a, b, c, d,

$$[A]_a^c \wedge [V]_c^b \wedge [A]_a^d \wedge [\neg V]_d^b = \beta > \mathbf{0}$$

From this we have

$$\beta_{m,n}(a, c) \leq A \tag{5}$$
$$\beta_{n,k}(c, b) \leq V \tag{6}$$
$$\beta_{m,n}(a, d) \leq A \tag{7}$$
$$\beta_{n,k}(d, b) \leq \neg V \tag{8}$$

Now, from (5) and (7), and Lemma 7 part 2 we have

$$\beta_{m,k}(a, b) = \beta_{m,n}(a, c)\beta_{n,k}(c, b) \leq A\beta_{n,k}(c, b) \tag{9}$$
$$\beta_{m,k}(a, b) = \beta_{m,n}(a, d)\beta_{n,k}(d, b) \leq A\beta_{n,k}(d, b) \tag{10}$$

From (10) and Lemma 17 we have

$$\beta_{n,k}(d, b) \leq A^T \beta_{m,k}(a, b) \tag{11}$$

And then

$$\begin{aligned}
\beta_{n,k}(d,b) &\leq A^T\beta_{m,k}(a,b) && \text{by (11)} && (12)\\
&\leq A^T A\beta_{n,k}(c,b) && \text{by (9)} && (13)\\
&\leq A^T AV && \text{by (6)} && (14)\\
&= V && \text{by stability} && (15)
\end{aligned}$$

But this contradicts (8).

2. By part 1, $AV \wedge A(\neg V) = \mathbf{0}_{m,k}$ so $A(\neg V) \leq \neg(AV)$ and hence $A^T A(\neg V) \leq A^T(\neg AV)$. Also AV is A^T-stable so by part 1 again (for A^T), $A^T(AV) \wedge A^T(\neg AV) = \mathbf{0}_{n,k}$ so $A^T(\neg AV) \leq \neg A^T(AV)$. Combining things, $A^T A(\neg V) \leq \neg A^T(AV) = \neg V$.

3. $V \vee \neg V = \mathbf{1}_{n,k}$ so $A^T AV \vee A^T A(\neg V) = A^T A\mathbf{1}_{n,k}$. But V is stable, so $V \vee A^T A(\neg V) = A^T A\mathbf{1}_{n,k}$, and hence $\neg V \wedge A^T A\mathbf{1}_{n,k} \leq A^T A(\neg V)$. In the other direction, by part 2 $A^T A(\neg V) \leq \neg V$, and of course $A^T A(\neg V) \leq A^T A\mathbf{1}_{n,k}$ and so $A^T A(\neg V) \leq \neg V \wedge A^T A\mathbf{1}_{n,k}$.

■

Now we head to the main material. If V is A-stable, it does not follow that $\neg V$ will also be. But there is a suitable candidate for a negation that is stable. The form is suggested by the following—recall, $A^T A\mathbf{1}_{n,k}$ is the largest A-stable $n \times k$ matrix.

THEOREM 19 *If V is A-stable,* $V = V \wedge A^T A\mathbf{1}_{n,k}$

Proof. Trivially $V \wedge A^T A\mathbf{1}_{n,k} \leq V$. The other direction follows from Theorem 12 part 4. ■

Now, the following definition should seem reasonable. The Theorem that follows it justifies this sense of reasonableness.

DEFINITION 20 $\overline{V} = \neg V \wedge A^T A\mathbf{1}_{n,k}$. I'll call $\overline{V}$ the *stable negation* of V.

THEOREM 21 *Let V be A-stable. Then*

1. $\overline{V} = A^T A(\neg V)$

2. $\overline{V}$ *is A-stable*

3. $\overline{\overline{V}} = V$

4. $V \wedge \overline{V} = \mathbf{0}_{n,k}$

5. $V \vee \overline{V} = A^T A \mathbf{1}_{n,k}$

6. $A\overline{V} = \overline{AV}$

7. $A(\neg V) = A(\overline{V})$

8. $\overline{V}$ *is the largest matrix* W *such that* W *is* A*-stable and* $W \leq \neg V$

Proof. The various parts are as follows.

1. This is a restatement of Lemma 18, part 3.

2. In one direction:

$$A^T A(\neg V \wedge A^T A \mathbf{1}_{n,k}) \leq A^T A \neg V \wedge A^T A A^T A \mathbf{1}_{n,k} \quad (16)$$
$$\leq \neg V \wedge A^T A A^T A \mathbf{1}_{n,k} \quad (17)$$
$$= \neg V \wedge A^T A \mathbf{1}_{n,k} \quad (18)$$

   In this, (16) is a standard item (see the beginning of the proof of Theorem 16 part 1); then (17) is by Lemma 18 part 2; and (18) is by Theorem 8 part 2.

   In the other direction, $A^T A(\neg V) \leq A^T A A^T A(\neg V)$ by Theorem 8 part 3, then use part 1.

3. $\overline{\overline{V}} = \neg \overline{V} \wedge A^T A \mathbf{1}_{n,k} = \neg(\neg V \wedge A^T A \mathbf{1}_{n,k}) \wedge A^T A \mathbf{1}_{n,k} = (V \vee \neg A^T A \mathbf{1}_{n,k}) \wedge A^T A \mathbf{1}_{n,k} = V \wedge A^T A \mathbf{1}_{n,k} = V$. The last step is by Theorem 19.

4. Straightforward.

5. $V \vee \overline{V} = V \vee (\neg V \wedge A^T A \mathbf{1}_{n,k}) = (V \vee \neg V) \wedge (V \vee A^T A \mathbf{1}_{n,k}) = \mathbf{1}_{n,k} \wedge A^T A \mathbf{1}_{n,k} = A^T A \mathbf{1}_{n,k}$. Note: along the way we used Theorem 12 part 4.

6. $V \wedge \overline{V} = \mathbf{0}_{n,k}$ and both V and $\overline{V}$ are A-stable, so $AV \wedge A\overline{V} = A(V \wedge \overline{V}) = A\mathbf{0}_{n,k} = \mathbf{0}_{m,k}$ and hence $A\overline{V} \leq \neg(AV)$. Also $A\overline{V} \leq AA^T \mathbf{1}_{m,k}$ (Theorem 12 part 4), so $A\overline{V} \leq \neg(AV) \wedge AA^T \mathbf{1}_{m,k} = \overline{AV}$.

   In the other direction, $V \vee \overline{V} = A^T A \mathbf{1}_{n,k}$ so using Theorem 12 part 5, $AV \vee A\overline{V} = A(V \vee \overline{V}) = AA^T A \mathbf{1}_{n,k} = AA^T \mathbf{1}_{m,k}$, so $\neg(AV) \wedge AA^T \mathbf{1}_{m,k} \leq A\overline{V}$, or $\overline{AV} \leq A\overline{V}$.

7. From the definition of stable negation, $\overline{V} \leq \neg V$, so $A(\overline{V}) \leq A(\neg V)$. By Lemma 18 part 2, $A^T A(\neg V) \leq \neg V$, so $AA^T A(\neg V) \leq A(\neg V)$. And by Theorem 8 part 1, $A(\neg V) \leq AA^T A(\neg V)$, so $AA^T A(\neg V) = A(\neg V)$. Then $A(\neg V)$ is A^T-stable, so $A(\neg V) \leq AA^T \mathbf{1}_{n,k}$ by Theorem 12 part 4. By Lemma 18 part 1, $AV \wedge A(\neg V) = \mathbf{0}_{n,k}$ so $A(\neg V) \leq \neg(AV)$. Thus $A(\neg V) \leq \neg(AV) \wedge AA^T \mathbf{1}_{n,k}$, so using the definition of stable negation with respect to A^T instead of A, $A(\neg V) \leq \overline{AV} = A\overline{V}$ by part 6 of this Theorem.

■

11 Summary so far

Once again, let $A \in \mathcal{B}_{m,n}$. Various results about A-stability have been shown, and it is time to collect them together.

1. The A-stable members of $\mathcal{B}_{n,k}$ constitute a Boolean algebra with meet and join being the $\wedge$ and $\vee$ of $\mathcal{B}_{n,k}$, with complementation being stable negation, and with bottom and top being $\mathbf{0}_{n,k}$ and $A^T A \mathbf{1}_{n,k}$.
2. Likewise the A^T-stable members of $\mathcal{B}_{m,k}$ constitute a Boolean algebra.
3. A is an isomorphism from the Boolean algebra of A-stable members of $\mathcal{B}_{n,k}$ onto the A^T-stable members of $\mathcal{B}_{m,k}$, with A^T as its inverse.

12 Boolean bisimulations

Consider again Example 10. There are two mono-modal frames displayed, each with its accessibility relation. Representing these by transition matrices we get

$$R = \begin{bmatrix} \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{0} \\ \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix} \qquad S = \begin{bmatrix} \mathbf{0} & \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{1} \\ \mathbf{0} & \mathbf{1} & \mathbf{0} \end{bmatrix}$$

for the left and right frames respectively. Earlier I also specified a matrix, A, corresponding to the frame bisimulation relation $\mathcal{A}$. Question: what conditions on A, R, and S tell us that $\mathcal{A}$ is, in fact, a (standard) frame bisimulation? Answer: the conditions are $AR \leq SA$ and $SA^T \leq RA^T$. It is easily checked that these inequalities hold in the special case of Example 10. In Section 13 I'll prove these are the inequalities that characterize bisimulations not only in the mono-modal case but also in the multi-modal one. Until then, I'll investigate matrices satisfying the inequalities for their own sakes.

DEFINITION 22 Let $A \in \mathcal{B}_{m,n}$, $R \in \mathcal{B}_{n,n}$, and $S \in \mathcal{B}_{m,m}$. I'll call A a *Boolean bisimulation* from R to S if:

1. $AR \leq SA$
2. $A^T S \leq RA^T$

There are some elementary properties of Boolean bisimulations, whose proofs are immediate and so are omitted.

THEOREM 23 *Boolean bisimulation is an equivalence relation, in the following sense.*

1. *The identity matrix I_n in $\mathcal{B}_{n,n}$ is a Boolean bisimulation from R to R.*
2. *If A is a Boolean bisimulation from R to S, then A^T is a Boolean bisimulation from S to R.*
3. *If A is a Boolean bisimulation from R to S and A' is a Boolean bisimulation from S to U, then $A'A$ is a Boolean bisimulation from R to U.*

Boolean bisimulations always exist, in an uninteresting way, because $\mathbf{0}_{m,n}$ is a Boolean bisimulation. Also there is always a largest Boolean bisimulation, the disjunction of all Boolean bisimulations. This is an immediate consequence of the fact that multiplication of Boolean matrices distributes over disjunction.

13 Connections

In the previous section the notion of Boolean bisimulation was defined, and there is also the usual notion of bisimulation, extended somewhat in Definition 1. It is time to connect these. I'll begin with frame bisimulations in a mono-modal setting (which are trivially standard), then move on to the more general situation.

THEOREM 24 *Let $\mathcal{F}_\mathcal{G} = \langle \mathcal{G}, \mathcal{R} \rangle$ and $\mathcal{F}_\mathcal{H} = \langle \mathcal{H}, \mathcal{S} \rangle$ be two mono-modal frames, and let $\mathcal{A}$ be a relation between $\mathcal{G}$ and $\mathcal{H}$. Let $\mathcal{B}$ be the two-member Boolean algebra, with elements $\{\mathbf{0}, \mathbf{1}\}$. Assume fixed enumerations $\{\Gamma_1, \Gamma_2, \ldots\}$ of $\mathcal{G}$ and $\{\Delta_1, \Delta_2, \ldots\}$ of $\mathcal{H}$. Let R be the transition matrix for $\mathcal{F}_\mathcal{G}$, that is $[R]_i^j = \mathbf{1}$ iff $\Gamma_i \mathcal{R} \Gamma_j$, and similarly let S be the transition matrix for $\mathcal{F}_\mathcal{H}$. Finally let A be the matrix corresponding to the relation $\mathcal{A}$, so that $[A]_i^j = \mathbf{1}$ iff $\Gamma_j \mathcal{A} \Delta_i$.*

$\mathcal{A}$ is a frame bisimulation between $\mathcal{F}_\mathcal{G}$ and $\mathcal{F}_\mathcal{H}$ iff A is a Boolean bisimulation from R to S.

Proof. First assume $\mathcal{A}$ is a frame bisimulation between $\mathcal{F}_{\mathcal{G}}$ and $\mathcal{F}_{\mathcal{H}}$. To show A is a Boolean bisimulation, I'll show $AR \le SA$; the other inequality is similar. Suppose $[AR]_i^j = \mathbf{1}$, I'll show $[SA]_i^j = \mathbf{1}$. Since $[AR]_i^j = \mathbf{1}$ we have $[A]_i \cdot [R]^j = \mathbf{1}$, and so for some k, $[A]_i^k = [R]_k^j = \mathbf{1}$. Since $[A]_i^k = \mathbf{1}$, $\Gamma_k \mathcal{A} \Delta_i$, and since $[R]_k^j = \mathbf{1}$, $\Gamma_k \mathcal{R} \Gamma_j$. Then by the definition of frame bisimulation, there must be a world Δ_n with $\Delta_i \mathcal{S} \Delta_n$ and $\Gamma_j \mathcal{A} \Delta_n$. But then $[S]_i^n = \mathbf{1}$ and $[A]_n^j = \mathbf{1}$. It follows that $[SA]_i^j = \mathbf{1}$.

Next, assume A is a Boolean bisimulation from R to S; I'll show $\mathcal{A}$ is a frame bisimulation—actually I'll show one of the two bisimulation conditions, the other is similar The argument is essentially that of the previous paragraph, reversed. So, suppose $\Gamma_j, \Gamma_k \in \mathcal{G}$, $\Delta_i \in \mathcal{H}$, $\Gamma_k \mathcal{A} \Delta_i$, and $\Gamma_k \mathcal{R} \Gamma_j$. From the first of these relation instances, $[A]_i^k = \mathbf{1}$ and from the second, $[R]_k^j = \mathbf{1}$. But then $[AR]_i^j = \mathbf{1}$ and, since A is a Boolean bisimulation, $AR \le SA$, so $[SA]_i^j = \mathbf{1}$. It follows that for some n, $[S]_i^n = [A]_n^j = \mathbf{1}$, and hence for the world $\Delta_n \in \mathcal{H}$, $\Gamma_j \mathcal{A} \Delta_n$ and $\Delta_i \mathcal{S} \Delta_n$, which is what was to be shown. ■

Before extending this Theorem to the multi-modal setting, a small but useful detour is needed. We need a notion of scalar multiplication for Boolean matrices.

DEFINITION 25 If $b \in \mathcal{B}$ and M is a matrix, by bM I mean the matrix such that $[bM]_i^j = b \wedge [M]_i^j$. That is, in bM each component of M has been replaced by its meet with b.

Using scalar multiplication, there is a kind of normal form for matrices over $\mathcal{B}$. Recall, $\mathcal{B}$ is atomic and complete; take $\{a_1, a_2, \ldots\}$ to be the set of atoms. If $A \in \mathcal{B}_{m,n}$, there are matrices A_{a_1}, A_{a_2}, …, whose entries are all in $\{\mathbf{0}, \mathbf{1}\}$, such that $A = a_1 A_{a_1} \vee a_2 A_{a_2} \vee \ldots$. Rather than a formal proof of this, an example should suffice. Consider the following matrix from Section 4, where the Boolean algebra is all subsets of $\{1,2,3\}$ and the atoms are $\{1\}$, $\{2\}$, and $\{3\}$. The matrix is:

$$\begin{bmatrix} \{1\} & \{1,3\} \\ \emptyset & \{2,3\} \end{bmatrix}$$

This can be written as follows.

$$\{1\}\begin{bmatrix} \mathbf{1} & \mathbf{1} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \vee \{2\}\begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{1} \end{bmatrix} \vee \{3\}\begin{bmatrix} \mathbf{0} & \mathbf{1} \\ \mathbf{0} & \mathbf{1} \end{bmatrix}$$

With normal forms available, we can move to the main item—the multi-modal case.

THEOREM 26 *Let $\mathcal{K}$ be finite or countable. Let $\mathcal{F}_{\mathcal{G}} = \langle \mathcal{G}, \mathcal{R}_i : i \in \mathcal{K}\rangle$ and $\mathcal{F}_{\mathcal{H}} = \langle \mathcal{H}, \mathcal{S}_i : i \in \mathcal{K}\rangle$ be two $\mathcal{K}$-frames, and let $\mathcal{A} = \langle \mathcal{A}_i : i \in \mathcal{K}\rangle$ be a family of relations between $\mathcal{G}$ and $\mathcal{H}$. Assume $\mathcal{G} = \{\Gamma_1, \Gamma_2, \ldots\}$ and $\mathcal{H} = \{\Delta_1, \Delta_2, \ldots\}$. Let $\mathcal{B}$ be the powerset Boolean algebra whose elements are the subsets of $\mathcal{K}$. Let R be the $\mathcal{B}$-valued matrix with $[R]_i^j = \{k \in \mathcal{K} \mid \Gamma_i \mathcal{R}_k \Gamma_j\}$, and similarly let S be the matrix corresponding to $\mathcal{S}$. Finally let A be the $\mathcal{B}$-valued matrix such that $[A]_i^j = \{k \in \mathcal{K} \mid \Gamma_j \mathcal{A}_k \Delta_i\}$.*

$\mathcal{A}$ is a frame bisimulation between $\mathcal{F}_{\mathcal{G}}$ and $\mathcal{F}_{\mathcal{H}}$ iff A is a Boolean bisimulation from R to S.

Proof. By Theorem 2, $\mathcal{A}$ is a frame bisimulation between $\mathcal{F}_{\mathcal{G}}$ and $\mathcal{F}_{\mathcal{H}}$ if and only if $\mathcal{A}_k$ is a mono-modal frame bisimulation between $\langle \mathcal{G}, \mathcal{R}_k\rangle$ and $\langle \mathcal{H}, \mathcal{S}_k\rangle$, for each $k \in \mathcal{K}$. Let R_k be the transition matrix corresponding to $\mathcal{R}_k$, with entries from $\{\mathbf{0}, \mathbf{1}\}$, let S_k similarly correspond to $\mathcal{S}_k$, and A_k correspond to $\mathcal{A}_k$. Then, by Theorem 24, $\mathcal{A}$ is a frame bisimulation if and only if $A_k R_k \leq S_k A_k$ and $A_k^T S_k \leq R_k A_k^T$, for each $k \in \mathcal{K}$. It remains to show these families of inequalities are equivalent to $AR \leq SA$ and $A^T S \leq RA^T$.

It is easy to see that the normal forms for R, S, and A are:

$$R = \bigvee_{k \in \mathcal{K}} \{k\} R_k \quad \text{and} \quad S = \bigvee_{k \in \mathcal{K}} \{k\} S_k \quad \text{and} \quad A = \bigvee_{k \in \mathcal{K}} \{k\} A_k$$

Now, $AR \leq SA$ if and only if

$$\bigvee_{j \in \mathcal{K}} \{j\} A_j \bigvee_{k \in \mathcal{K}} \{k\} R_k \leq \bigvee_{j \in \mathcal{K}} \{j\} S_j \bigvee_{k \in \mathcal{K}} \{k\} A_k$$

Making use of distributivity, this can be shown equivalent to

$$\bigvee_{j,k \in \mathcal{K}} (\{j\} \wedge \{k\}) A_j R_k \leq \bigvee_{j,k \in \mathcal{K}} (\{j\} \wedge \{k\}) S_j A_k$$

This in turn is equivalent to

$$\bigvee_{k \in \mathcal{K}} \{k\} A_k R_k \leq \bigvee_{k \in \mathcal{K}} \{k\} S_k A_k$$

Finally, this is equivalent to

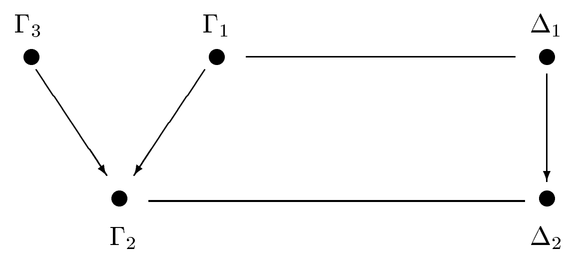
$$A_k R_k \leq S_k A_k \text{ for each } k \in \mathcal{K}$$

In a similar way, $A^T S \leq RA^T$ is equivalent to $A_k^T S_k \leq R_k A_k^T$ for every $k \in \mathcal{K}$. These equivalences complete the proof. ∎

14 The modal operator

Suppose we have two Boolean valued frames and a bisimulation between them. Then various results of earlier sections ensure that stable vectors are well-behaved with respect to $\wedge$, $\vee$, and stable negation. But what about $\Diamond$? Its application can turn stable vectors into non-stable ones, as the following shows.

EXAMPLE 27 The diagram below shows two mono-modal Kripke frames, one on the left and one on the right, with a relation between them which is, in fact, a standard frame bisimulation.



Suppose, in the left frame, we assign the atomic formula P to be true just at Γ_2 and in the right frame just at Δ_2. This meets the conditions of Definition 3 for bisimulation—the worlds at which P is true are related by the frame bisimulation. But, in the left frame $\Diamond P$ will be true at both Γ_1 and Γ_3, and in the right at Δ_1, but only Γ_1 and Δ_1 are related by the bisimulation. This is awkward, but not a serious problem—Γ_3 does not take part in the bisimulation so we can, in effect, ignore it. The question is, how to do that in a way consistent with the theory developed so far.

To make better connection with the present paper, I'll convert this example to an algebraic one. Let R be the transition matrix corresponding to the accessibility relation of the left frame above, and let S correspond to the right frame accessibility relation. Also, let A represent the bisimulation. Then we have the following.

$$R = \begin{bmatrix} \mathbf{0} & \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{1} & \mathbf{0} \end{bmatrix} \quad \text{and} \quad S = \begin{bmatrix} \mathbf{0} & \mathbf{1} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \quad \text{and} \quad A = \begin{bmatrix} \mathbf{1} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{1} & \mathbf{0} \end{bmatrix}$$

A is a Boolean bisimulation from R to S; in fact $AR = SA$, while $A^T S < RA^T$. Above we took P to be true in the left frame just at Γ_2. Algebraically,

P is assigned the vector (proposition) $V = \langle \mathbf{0}, \mathbf{1}, \mathbf{0} \rangle$. It is easy to check that this is A-stable. But,

$$RV = \begin{bmatrix} \mathbf{0} & \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{1} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{0} \\ \mathbf{1} \\ \mathbf{0} \end{bmatrix} = \begin{bmatrix} \mathbf{1} \\ \mathbf{0} \\ \mathbf{1} \end{bmatrix}$$

and $\langle \mathbf{1}, \mathbf{0}, \mathbf{1} \rangle$ is *not* A-stable.

In Example 27, Γ_3 is not really relevant to the bisimulation, so what we want to do is eliminate it from consideration. Our solution parallels the treatment of negation in Section 10—instead of RV, we work with the largest stable matrix below it. And this has the same familiar form it did when negation was involved.

Section assumption For the rest of this section, $A \in \mathcal{B}_{m,n}$ so that for each k, $A : \mathcal{B}_{n,k} \to \mathcal{B}_{m,k}$ and $A^T : \mathcal{B}_{m,k} \to \mathcal{B}_{n,k}$. Further, I'll assume A is a Boolean bisimulation from $R \in \mathcal{B}_{n,n}$ to $S \in \mathcal{B}_{m,m}$.

DEFINITION 28 $R \circ V = RV \wedge A^T A \mathbf{1}_{n,k}$.

THEOREM 29 *Let V be A-stable. Then*

1. $R \circ V = A^T A(RV)$
2. $A(R \circ V) = A(RV)$
3. $A(R \circ V) = S \circ (AV)$
4. $R \circ V$ *is A-stable*
5. $R \circ V$ *is the largest matrix W such that W is A-stable and $W \leq RV$*

Proof. Since A is a Boolean bisimulation and V is A-stable, $A^T A(RV) \leq A^T SAV \leq RA^T AV = V$. Of course $A^T ARV \leq A^T A \mathbf{1}_{n,k}$, and so $A^T ARV \leq RV \wedge A^T A \mathbf{1}_{n,k}$. We also have $RV \wedge A^T A \mathbf{1}_{n,k} \leq A^T ARV$ by Lemma 15. Combining things, we have item 1.

Using item 1, $A(R \circ V) = AA^T ARV \geq A(RV)$ by Theorem 8. Also, $A(R \circ V) = A(RV \wedge A^T A \mathbf{1}_{n,k}) \leq A(RV)$, and we have item 2.

Using items 1 and 2 (including their counterparts for S) and the fact that A is a Boolean bisimulation, $A(R \circ V) = AA^T ARV \leq AA^T SAV = S \circ (AV)$. Also using the fact that V is stable, $S \circ (AV) = AA^T SAV \leq ARA^T AV = ARV = A(R \circ V)$. These give us item 3.

$A^T A(R \circ V) = A^T A(RV \wedge A^T A \mathbf{1}_{n,k}) \leq A^T A(RV) = R \circ V$. Also, $R \circ V = A^T A(RV) \leq A^T A A^T A R V = A^T A(R \circ V)$ by Theorem 8 again. So we have item 4.

It is trivial that $R \circ V \leq RV$. And, if W is A-stable and $W \leq RV$, then $W = A^T A W \leq A^T A R V = R \circ V$, so we have item 5. ■

Note that part 2 of this theorem says that, with respect to bisimulation, $R \circ V$ and RV behave alike. This is analogous to a similar result concerning stable negation: $A\overline{V} = A(\neg V)$, which was established earlier.

15 Bisimulations, models, and formulas

The definition of Boolean bisimulation corresponds to *frame* bisimulation. To turn a frame bisimulation into a bisimulation, as in Definition 3, models must be based on the frames involved—that is, assignments of truth values to atomic formulas at worlds must be given. But not just any assignment will do, since the bisimulation relation must be respected. In the usual two-valued, mono-modal setting, worlds related by a bisimulation must make the same atomic formulas true, and an analogous condition is needed in the Boolean valued case too.

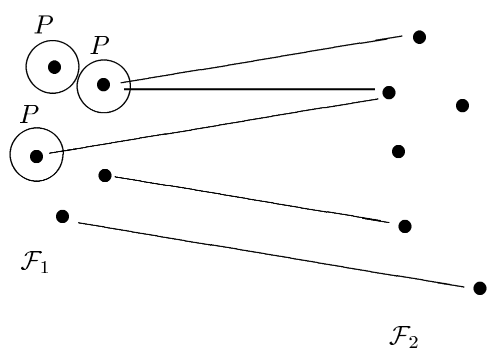


Figure 4. Bisimulations and models

Consider the example displayed in Figure 4, where two mono-modal frames, $\mathcal{F}_1$ and $\mathcal{F}_2$, and a frame bisimulation are indicated (accessibility relations have not been shown). Suppose we want to make the atomic formula P true at the worlds indicated in frame $\mathcal{F}_1$. Then we are required to have P true at the worlds related to them in $\mathcal{F}_2$ (there are two such worlds in this example), and in turn worlds related to these in $\mathcal{F}_1$ must have P

true, that is, they must be part of the original set of worlds where P was to be true. Similar considerations apply to more general Boolean valued cases as well. All this corresponds to an easily expressible condition on vectors. We must require that values assigned to atomic formulas be what I will call *weakly stable.*

DEFINITION 30 V is *weakly stable* (with respect to A) if $A^TAV \leq V$.

Weak stability is too weak to have attractive algebraic features on its own. But note again the example in Figure 4. The set of worlds at which P is true in $\mathcal{F}_1$ is weakly stable, but it is not stable. One of the three worlds is, in an obvious sense, irrelevant to the bisimulation. But if we shift our attention to the set of worlds in $\mathcal{F}_2$ that are related to these three, we have a two-element set that is, in fact, not just weakly stable, but stable. Indeed, this always happens. Suppose V is weakly stable. Then $A^TAV \leq V$, and so $AA^TAV \leq AV$. But also $AV \leq AA^TAV$ by Theorem 8, and so $AA^TAV = AV$, that is, AV is stable. Since we are interested in behavior under bisimulation, and since weak stability turns into stability after one shift from a model to another, we simplify things by requiring stability from the start.

From now on, given two Boolean valued frames and a bisimulation between them, in basing models on these frames, *we require that the values assigned to atomic formulas must be stable vectors.*

Finally, it is time to return to the original reason for introducing bisimulations into modal logic—they preserve formula truth. In an algebraic setting this becomes especially simple, as the formulation in this section will show. For the rest of this section I assume the following.

1. R and S are (transition) matrices, with $R \in \mathcal{B}_{n,n}$ and $S \in \mathcal{B}_{m,m}$.

2. $A \in \mathcal{B}_{m,n}$ is a Boolean bisimulation from R to S.

L is a mono-modal language, but as we have seen, since $\mathcal{B}$ can be a more complex Boolean algebra than $\{\mathbf{0}, \mathbf{1}\}$, we have the effect of a multi-modal language and semantics.

Let v be a mapping from propositional letters of L to A-*stable* vectors in $\mathcal{B}^n$, and let w likewise be a mapping from propositional letters of L to A^T-*stable* vectors in $\mathcal{B}^m$. ($\mathcal{B}^n$ is identified with $\mathcal{B}_{n,1}$ and $\mathcal{B}^m$ with $\mathcal{B}_{m,1}$, as usual.) Both v and w are extended to mappings from arbitrary formulas in L as follows.

1. $v(X \wedge Y) = v(X) \wedge v(Y)$
   $w(X \wedge Y) = w(X) \wedge w(Y)$

2. $v(X \vee Y) = v(X) \vee v(Y)$
   $w(X \vee Y) = w(X) \vee w(Y)$

3. $v(\neg X) = \overline{v(X)}$ (using stable negation in $\mathcal{B}^n$)
   $w(\neg X) = \overline{w(X)}$ (using stable negation in $\mathcal{B}^m$)

4. $v(\Diamond X) = R \circ v(X)$
   $w(\Diamond X) = S \circ w(X)$

Using results from Theorems 13, 16, 21, and 29, for any formula X, $v(X)$ is A-stable and $w(X)$ is A^T-stable. And further, we have a central result on bisimulations, in the following form.

THEOREM 31 *If $A(v(P)) = w(P)$ and $A^T(w(P)) = v(P)$ for propositional letters P of L, then for any formula X, $A(v(X)) = w(X)$ and $A^T(w(X)) = v(X)$.*

16 Conclusion

An algebraic approach to modal semantics, and bisimulation in particular, has been presented. It can be carried further, but details must remain for another paper. I'll sketch a few items, to illustrate the possibilities.

There is a special class of bisimulations known as *P-morphisms* that, historically, were investigated before the more general notion of bisimulation was introduced. These can be characterized easily in the algebraic setting: a matrix A is a *P-morphism onto* if $I_n = A^T A$ and $I_m \leq AA^T$. The expected results follow easily from this characterization.

It is possible to 'multiply' frames, by forming Cartesian products. The algebraic counterpart of this is the tensor product, a standard operation in other contexts. It relates quite well to Boolean bisimulations. In a similar way, the disjoint union operation on frames corresponds to the notion of direct sum, familiar from linear algebra. That is to say, we are seeing operations that are long-known and well-understood, but in a less familiar context.

Bisimulations also arise in other contexts, of course. One such place is automata theory. For instance, to show that the usual algorithm for converting a non-deterministic automaton to a deterministic version is correct, one essentially shows the resulting automaton bisimulates the original one. For some time Dexter Kozen has been developing an approach to automata theory that makes central use of matrices, in ways that are strikingly similar to what was presented here. An extensive set of notes presenting this work can be found on his web site, www.cs.cornell.edu/kozen/, under the heading *CS786 S02 Introduction to Kleene Algebra*.

It is not unreasonable to hope that by looking at modal machinery from an algebraic point of view we will achieve additional insight and understanding, based on the work of generations of mathematicians who have gone before.

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