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Non-Normal Modalisation

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ABSTRACT. We study the external application of a non-normal modal logic M to a generic logic L, thus generating a modalised logic $\mathsf{M}(\mathsf{L})$. We prove the transference of completeness and decidability from M and L to $\mathsf{M}(\mathsf{L})$. Previous existing techniques that show the transference of properties in combinations of modal logics all relied on some form of normality of the modal system. Our proof is based on a technique that provides a consistency preserving mapping of L-formulas into propositional classical formulas; this technique allows us to do without normality assumptions in the modal system.

1 Introduction

In this paper we study a method of combining non-normal modal logics called *modalisation* or the external application of a modal logic M to a generic logic system L. This combination generates the modalised logic system $\mathsf{M}(\mathsf{L})$, and we analyse the transference of basic logical properties from the component logics to the combined system.

The novelty here is the technique for proving the transference of completeness and decidability, which uses a *consistency preserving mapping* into *propositional classical logic*. This mapping takes formulas of a generic logic L and generates a set of formulas in propositional classical logic, with the following property: any finite set of L-formulas is mapped into a consistent set of propositional classical formulas iff the set of L-formulas is L-consistent. This mapping allows us to show the transference of properties for non-normal modalisation.

Several combinations of modal and temporal logics have been studied in the literature, as discussed below. However, almost all of them relied on the existence of some form of normality, which may be defined either in algebraic terms or in proof-theoretical terms. In a mono-modal logic with modality $\Box$, normality implies that the following formulas are valid:

$$\Box(p \to q) \to (\Box p \to \Box q) \qquad\qquad \Box(p \wedge q) \leftrightarrow (\Box p \wedge \Box q)$$

as well as the admission of the inference rule: From $\vdash A$ infer $\vdash \Box A$.

None of these is assumed to hold in a non-normal modal logic. It turns out that the existing analysis of transference of logical properties from the

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component logic systems to combined ones in the literature relied fundamentally on the assumption of normality.

Non-normal modal systems are starting to attract the attention of research community. For example, in deontic logic with an obligation modality O [1], it has been noted that the axiom $OA \wedge OB \to O(A \wedge B)$ needs not always be respected. Furthermore, recent works in the domain of belief revision have extended the basic ontology to deal not only with belief modalities, but also with intension and feasibility modalities [9]; the latter modalities are intrinsically non-normal, and can naturally interact with other modalities. So it is highly desirable to have a theory of transference of logical properties for the combination of non-normal modal logics.

This paper starts to investigate how modal logics can be combined without assuming normality. We concentrate on a not so strong way of combining logics, known as *modalisation*, in which an *external modal logic* M is applied to a generic logic L, where the external logic is assumed to be a non-normal mono-modal logic with a 1-place connective $\Box$. We believe this is a promising step in the study of stronger combinations of non-normal modalities, as has been the case in the study of combination of normal temporal logics [5, 8].

Modalisation is a direct generalisation of the *temporalisation* process, which was previously developed for temporal logics with binary connectives U ("until") and S ("since"), initially restricted over linear time only [4], then extended to any class of flows of time [7] and then to any number of normal modal/temporal operators with any arity [8]. Surprisingly, however, the strategy for proving the transference of completeness and decidability remained the same throughout all these generalisations, as well as in the present work. The proof details, however, differed significantly and have become increasingly more complex.

Apart from the temporalisation/modalisation method, several other combinations of logics have been previously analysed in the literature. The conservativity of independently combined modal logics was presented by Thomason in [11]. Fine and Schurz [6] and Kracht and Wolter [10] have studied the transfer properties of systematically combining independently axiomatisable mono-modal systems, also called *fusion* of modal logics. The work of Fine and Schurz [6] deals with more than two independent normal modalities. A generalisation of such results for many-place multi-modal systems is presented by Wolter in [12]. The independent combination (fusion) of temporal logics was studied in [5, 7] initially for linear time and then for any class of flows of time. Wolter's approach in [10, 12] is algebraic, while all others are based on Kripke frames; even with an algebraic approach, some notion of normality was used; such a notion forbids connectives such as U ('until') and S ('since') to be used in combinations. This restriction was later weakened in [2], so as to allow for connectives temporal U and S as well as other modalities found is description logics, which are not strictly normal; however, some notion of normality, even if presented in a weaker format, still had to be imposed to obtain the transference of logical properties.

The organisation of this paper. The rest of this paper is organised as follows. We first present the modalised system $\mathsf{M}(\mathsf{L})$ in Section 2; its language, semantics and inference system will be derived from those of M and L. We then prove the transference of completeness in Section 3; the crucial step is the definition of the consistency preserving mapping in Section 3.1, which leads directly to the transference of completeness in Section 3.2. The transference of decidability is also a consequence of the consistency preserving mapping, as shown in Section 4. We conclude with some remarks on possible application of the results here for stronger combination of logics.

2 The non-normal modalisation $\mathsf{M}(\mathsf{L})$

In this section we describe the system $\mathsf{M}(\mathsf{L})$, which is based on the temporalisation process introduced in [4]. By a *logic system* we mean a *tuple* $S = (\mathcal{L}_S, \vdash_S, \mathcal{K}_S, \models_S)$, where $\mathcal{L}_S$ is the system's language, $\vdash_S$ is an inference system, $\mathcal{K}_S$ is the system's associated class of models and $\models_S$ is the system's semantical relation between models and formulas.

The language of $\mathsf{M}(\mathsf{L})$

The language $\mathcal{L}_\mathsf{M}$ of the mono-modal system M is built from a denumerable set of atoms $\mathcal{P} = \{p_0, p_1, \ldots\}$, applying the one-place modality $\Box$ and the Boolean connectives $\neg$ (*negation*) and $\wedge$ (*conjunction*). We use A, B, C for formulas of M, φ for formulas of L and lower Greek letters for formulas of $\mathsf{M}(\mathsf{L})$, possibly with subscripts.

Very little is required of the internal logic L, except that its language is a denumerable set of finite formulas and that it has the classical Boolean connectives $\neg$ and $\wedge$ with its usual semantics. Apart from that, any other type of construct is acceptable in the language; for example, it may contain other (normal or non-normal) modalities, or it may possess quantifiers and predicates.

To avoid double parsing of modalised formulas, we partition the language of L into the sets:

- Bool_L, the set of *Boolean combinations* consists of the formulas built up from *any* other formulas with the use of the Boolean connectives $\neg$ or $\wedge$;

- $\mathrm{Mono}_{\mathsf{L}}$, the set of *monolithic formulas* is the complementary set of $\mathrm{Bool}_{\mathsf{L}}$ in the language of L.

If the internal logic L does not contain the classical connectives $\neg$ and $\wedge$, we assume that $\mathrm{Mono}_{\mathsf{L}} = \mathcal{L}_{\mathsf{L}}$ and $\mathrm{Bool}_{\mathsf{L}} = \varnothing$, so every formula in L is considered monolithic. As an example, consider the atoms $p, q \in \mathsf{L}$ and $\blacksquare$ a modal symbol in L, then $\blacksquare p$ and $\blacksquare(p \wedge q)$ are monolithic formulas whereas $\neg\blacksquare p$ and $\blacksquare p \wedge \blacksquare q$ are Boolean combinations.

The set of modalised formulas, $\mathcal{L}_{\mathsf{M}(\mathsf{L})}$, is defined as the smallest set closed under the rules:

1. If $\varphi \in \mathrm{Mono}_{\mathsf{L}}$, then $\varphi \in \mathcal{L}_{\mathsf{M}(\mathsf{L})}$;
2. If $\psi_1, \psi_2 \in \mathcal{L}_{\mathsf{M}(\mathsf{L})}$, then $\neg\psi_1 \in \mathcal{L}_{\mathsf{M}(\mathsf{L})}$ and $\psi_1 \wedge \psi_2 \in \mathcal{L}_{\mathsf{M}(\mathsf{L})}$;
3. If $\psi \in \mathcal{L}_{\mathsf{L}}$, then $\Box\psi \in \mathcal{L}_{\mathsf{M}(\mathsf{L})}$.

Note that the atoms of M are not elements of $\mathcal{L}_{\mathsf{M}(\mathsf{L})}$. We will use the connectives $\vee$, $\rightarrow$ and $\leftrightarrow$, and the constants $\top$ and $\bot$, in their usual meaning. Also, the formula $\Diamond A$ abbreviate $\neg\Box(\neg A)$. The *size* of a formula A is number of symbols it contains.

The semantics of $\mathsf{M}(\mathsf{L})$

The semantics of non-normal modal logics is here based on *minimal models* of possible worlds [3]. A minimal model for modal logic is a structure $\mathcal{M} = (W, N, V)$ such that W is a set of worlds; N is a mapping $N : W \rightarrow 2^{2^W}$, that is N associate a set of sets of worlds to each world; and $V : \mathcal{L} \rightarrow 2^W$ is a valuation that associates a set of worlds to each formula according to the following restrictions:

- $V(\neg A) = W \setminus V(A)$.
- $V(A \wedge B) = V(A) \cap V(B)$.
- $V(\Box A) = \{w \in W | V(A) \in N(w)\}$.

We write $\mathcal{M}, w \models A$ iff $w \in V(A)$. Under this view, a formula is V-associated with the set of worlds in which it holds, and the function N associates a world $w \in W$ with a set of propositions that are *necessary* at w.

Let $\mathcal{K}_{\mathsf{M}}$ be a class of models of logic M, usually defined by placing some restriction on the mapping N. Let $\mathcal{K}_{\mathsf{L}}$ be the class of models for valid formulas of L; we specify some restrictions on the semantic relation $\models_L$ for the logic L, whose class of models will be called $\mathcal{K}_L$. The basic restriction imposes that, for each $\mathcal{M} \in \mathcal{K}_{\mathsf{L}}$ and $\varphi \in \mathcal{L}_{\mathsf{L}}$ we have

$$\text{either } \mathcal{M} \models \varphi \text{ or } \mathcal{M} \models \neg\varphi.$$

To satisfy this condition in some logics, it may be necessary to adapt the notion of a class of models. For instance, if L is some modal logic with a modality $\blacksquare$, an element of $\mathcal{K}_L$ is a *pair* $\langle \mathcal{M}_{\mathsf{L}}, x \rangle$, where $\mathcal{M}_{\mathsf{L}} = (W', N', V')$ and $x \in W'$, such that either $\mathcal{M}_{\mathsf{L}}, x \models \varphi$ or $\mathcal{M}_{\mathsf{L}}, x \models \neg\varphi$; if the class of models were simply defined in terms of $\mathcal{M}_{\mathsf{L}}$, we could have a formula, say, a propositional symbol p, such that neither $\mathcal{M}_{\mathsf{L}} \models p$ nor $\mathcal{M}_{\mathsf{L}} \models \neg p$.

Finally, we can define a minimal model for the modalised logic $\mathsf{M}(\mathsf{L})$ as a structure $\mathcal{M}_{\mathsf{M}(\mathsf{L})} = (W, N, g)$, where W and N are as above, and $g : W \to \mathcal{K}_{\mathsf{L}}$ associates to each $w \in W$ a model of L. The satisfaction relation $\models$ is then defined recursively over the structure of modalised formulas:

(*i*) $\mathcal{M}_{\mathsf{M}(\mathsf{L})}, w \models \alpha$, $\alpha \in \mathrm{Mono}_{\mathsf{L}}$ iff $g(w) = \mathcal{M}_{\mathsf{L}}$ and $\mathcal{M}_{\mathsf{L}} \models \alpha$ (denoted $g(w) \models \alpha$).

(*ii*) $\mathcal{M}_{\mathsf{M}(\mathsf{L})}, w \models \neg\alpha$ iff $\mathcal{M}_{\mathsf{M}(\mathsf{L})}, w \not\models \alpha$.

(*iii*) $\mathcal{M}_{\mathsf{M}(\mathsf{L})}, w \models (\alpha \wedge \beta)$ iff $\mathcal{M}_{\mathsf{M}(\mathsf{L})}, w \models \alpha$ and $\mathcal{M}_{\mathsf{M}(\mathsf{L})}, w \models \beta$.

(*iv*) $\mathcal{M}_{\mathsf{M}(\mathsf{L})}, w \models \Box\alpha$ iff $\{w' \in W \mid \mathcal{M}_{\mathsf{M}(\mathsf{L})}, w' \models \alpha\} \in N(w)$.

A class of modalised models $\mathcal{K}_{\mathsf{M}(\mathsf{L})}$ is obtained from $\mathcal{K}_{\mathsf{L}}$ and $\mathcal{K}_{\mathsf{M}}$ by placing over modalised models $\mathcal{M}_{\mathsf{M}(\mathsf{L})} = (W, N, g)$ the same restrictions over N that are placed on the class $\mathcal{K}_{\mathsf{M}}$.

A formula is *valid* in a class $\mathcal{K}$ if it is verified at all worlds at all models over that class.

The inference system of $\mathsf{M}(\mathsf{L})$

We assume that an *inference system*, $\vdash$, for a generic logic system is a mechanism capable of recursively enumerating the set of all provable formulas of the system, here called *theorems* of the logic system.

An inference system is *sound* with respect to a class of models $\mathcal{K}$ if all its theorems are valid over $\mathcal{K}$. Conversely, it is *complete* if all valid formulas are theorems. We assume that logic L's inference system, $\vdash_{\mathsf{L}}$, is sound and complete with respect to a class $\mathcal{K}_{\mathsf{L}}$.

We will assume that the modal logic M inference system, $\vdash_{\mathsf{M}}$, is given in an axiomatic form, consisting of a set of *axioms* and a set of inference rules. In fact, all we have to assume of $\vdash_{\mathsf{M}}$ is the validity of propositional classical tautologies, the admissibility of Modus Ponens and the following inference rule: if $\vdash_{\mathsf{M}} A \leftrightarrow B$ then $\vdash_{\mathsf{M}} \Box A \leftrightarrow \Box B$.

We include those rules for they are valid in any minimal model, and they define a system that is sound and complete with respect to the class of all minimal models. On the other hand, by forcing other inference rule and axioms on logic M, some restriction is imposed on the structure of the class of models; see [3].

The combined inference system of $\mathsf{M}(\mathsf{L})$ is denoted by $\vdash_{\mathsf{M}(\mathsf{L})}$ and consists of the following elements:

- The axioms and inference rules of $\vdash_{\mathsf{M}}$;
- The inference rule *Preserve*: For every formula φ in L, if $\vdash_{\mathsf{L}} \varphi$ then $\vdash_{M(L)} \varphi$.

As usual, a formula φ is *consistent* if $\nvdash \neg\varphi$.

In [4] it was shown that if L is sound over a class $\mathcal{K}_{\mathsf{L}}$ and M is sound over a class $\mathcal{K}_{\mathsf{M}}$, then the inference system of $\vdash_{\mathsf{M}(\mathsf{L})}$ is sound over the combined class $\mathcal{K}_{\mathsf{M}(\mathsf{L})}$; no extra restrictions were made on the nature of M. That is, soundness transfers over modalisation. In the following, we investigate the transference of completeness and decidability.

3 Completeness of $\mathsf{M}(\mathsf{L})$

In this section we show that the non-normal modalisation of sound and complete logics preserves completeness. The proof strategy is the same that has been used in our previous works of temporalisation [4, 7, 8] and is illustrated in Figure 1. We start with a consistent $\mathsf{M}(\mathsf{L})$-formula φ, translate it into a consistent modal formula A in M; then completeness of M over $\mathcal{K}_{\mathsf{M}}$ gives us a model for A; after some model manipulation using the completeness of L, we obtain a model for φ in $\mathcal{K}_{\mathsf{M}(\mathsf{L})}$, thus deriving the transference of completeness from L and M to $\mathsf{M}(\mathsf{L})$.

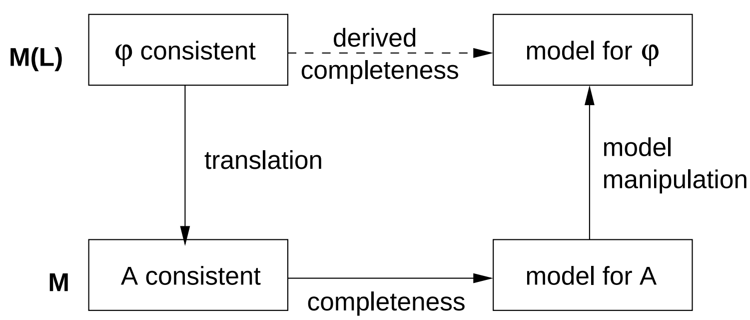

Figure 1. Completeness proof strategy

The difficulty in our proof lies in finding an adequate translation σ such that φ is $\mathsf{M}(\mathsf{L})$-consistent iff $A = \sigma(\varphi)$ is M-consistent and such that any model for $\sigma(\varphi)$ can be easily transformed into a model for φ. The simple idea of mapping each monolithic formula of $\mathsf{M}(\mathsf{L})$ to a propositional variable does not work. To see that, suppose the logic L is a modal logic with modality $\blacksquare$ and suppose that the axiom $\mathbf{T}$ holds for L:

$$\vdash_{\mathsf{L}} \blacksquare p \rightarrow p$$

Now suppose that $\varphi = \blacksquare p \rightarrow p$ is a $\mathsf{M}(\mathsf{L})$ formula; by the inference rule Preserve, φ is a theorem. If we simply map all monolithic subformulas to atoms, we end up with a formula in M of the form $\sigma(\varphi) = q_1 \rightarrow q_2$, where q_1, q_2 are "new" atoms, and M-models of $\sigma(\varphi)$ may make the atom q_1 true but q_2 false at some world, so not all M-models of $\sigma(\varphi)$ can be directly transformed into a $\mathsf{M}(\mathsf{L})$-model of φ. A translation mapping has to guarantee that the mapped formulas behaves exactly the same way as the original formulas in terms of satisfiability and validity.

3.1 The consistency preserving map

Let $\Phi = \{\varphi_1, \ldots, \varphi_n\}$ be a set of monolithic L-formulas. We will map them into propositional classical formulas $A_1, \ldots, A_n$ as defined below built from propositions $q_1, \ldots, q_n$; since M extends classical logic, this will be the basic step for mapping a $\mathsf{M}(\mathsf{L})$ formula ψ built from the elements of Φ into a M formula $\sigma(\psi)$. The translation σ from $\mathsf{M}(\mathsf{L})$-formulas to M-formulas is then recursively defined as:

- $\sigma(\varphi_i) = A_i$;
- $\sigma(\neg\psi) = \neg\sigma(\psi)$;
- $\sigma(\psi_1 \wedge \psi_2) = \sigma(\psi_1) \wedge \sigma(\psi_2)$;
- $\sigma(\Box\psi) = \Box\sigma(\psi)$.

It remains to define the propositional classical formulas $A_1, \ldots, A_n$. This mapping of L formulas into classical ones is in fact independent of the modalisation process; we will show it guarantees that consistency is preserved through this mapping. Before we do that, we need some definitions.

If $\Phi = \{\varphi_1, \ldots, \varphi_n\}$, we write $\sigma(\Phi)$ for the set $\{\sigma(\varphi) | \varphi \in \Phi\}$. Let $\mathrm{Lit}(\Phi) = \Phi \cup \{\neg\varphi | \varphi \in \Phi\}$ be the set of *literals* of Φ. Let $\mathrm{Cons}(\Phi) = \{\bigwedge \varphi_i | \varphi_i \in \mathrm{Lit}(\Phi) \text{ and } \nvdash_{\mathsf{L}} \neg \bigwedge \varphi_i\}$.

Fix an enumeration of Φ, $\epsilon = \varphi_1, \ldots, \varphi_n$, and suppose we have variables $q_1, \ldots, q_n$ in M. We then recursively define $A_1, \ldots, A_n$ in the following way. Each $A_i = \sigma(\varphi_i)$ will be a boolean combination of $q_1, \ldots, q_i$.

For the base case, we define A_1 as follows:

- $A_1 = q_1 \wedge \neg q_1$, if $\varphi_1 \notin \mathrm{Cons}(\Phi_1)$, i.e. $\vdash_{\mathsf{L}} \neg\varphi_1$;
- $A_1 = q_1 \vee \neg q_1$, if $\neg\varphi_1 \notin \mathrm{Cons}(\Phi_1)$, i.e. $\vdash_{\mathsf{L}} \varphi_1$;

- $A_1 = q_1$, otherwise.

Suppose $A_1, \dots, A_m$ is defined for $m < n$ and define $\Phi_m = \{\varphi_1, \dots, \varphi_m\}$. As we did for the base case, the definition of A_{m+1} will have to analyse three cases of combinations based on Φ_m, namely those that imply φ_{m+1}, those that imply $\neg\varphi_{m+1}$, and those of which φ_{m+1} is independent.

We start by defining the (transformation of) combinations that imply φ_{m+1}:

$$B^+(\Phi_m) = \bigvee \{\sigma(\psi) | \psi \in \mathrm{Cons}(\Phi_m) \text{ and } \vdash_{\mathsf{L}} \psi \to \varphi_{m+1}\}.$$

$B^+(\Phi_m)$ is a disjunction of the σ-mapped elements of Φ_m that imply φ_{m+1}. $B^+(\Phi_m)$ may have several disjuncts, a single one or even none (in which case $B^+(\Phi_m) = \bot$). Similarly, define:

$$B^-(\Phi_m) = \bigvee \left\{ \begin{array}{c} \sigma(\psi) | \psi \in \mathrm{Cons}(\Phi_m) \text{ and} \\ \vdash_{\mathsf{L}} \psi \to \neg\varphi_{m+1} \end{array} \right\}.$$

$B^-(\Phi_m)$ is a disjunction of the σ-mapped elements of Φ_m that imply $\neg\varphi_{m+1}$. Finally, for the elements of $\mathrm{Cons}(\Phi_m)$ that imply neither φ_{m+1} nor $\neg\varphi_{m+1}$, define:

$$\begin{array}{c} B^0(\Phi_m) = \bigvee\{\sigma(\psi) \wedge q_{m+1} | \psi \in \mathrm{Cons}(\Phi_m) \\ \text{and} \\ \psi \wedge \varphi_{m+1}, \psi \wedge \neg\varphi_{m+1} \in \mathrm{Cons}(\Phi_{m+1})\}. \end{array}$$

The presence of q_{m+1} in $B^0(\Phi_m)$ provides the freedom to choose the truth value of A_{m+1}. So, finally, we define A_{m+1} as:

$$A_{m+1} = \neg B^-(\Phi_m) \wedge (B^+(\Phi_m) \vee B^0(\Phi_m))$$

Note that some of the B-elements may be absent from the formula in case it is an empty disjunction. In particular, if $B^0(\Phi_m)$ is empty we will not have q_{m+1}, meaning that the value of φ_{m+1} is totally determined by $\varphi_1, \dots, \varphi_m$.

With this construction, if a non-normal modal model satisfies a disjunct of $B^+(\Phi_m)$, it will not satisfy $B^-(\Phi_m)$, and hence it satisfies A_{m+1}. Conversely, by satisfying $B^-(\Phi_m)$, A_{m+1} is falsified. If neither $B^+(\Phi_m)$ nor $B^-(\Phi_m)$ is satisfied, the new atom q_{m+1} determines the value of A_{m+1}.

We now revisit the previous example of mapping the axiom $\mathbf{T}$, $\blacksquare p \to p$, from L to classical propositional logic. We have $\Phi = \{\blacksquare p, p\}$, and $\mathrm{Cons}(\Phi) = \{\blacksquare p \wedge p, \neg\blacksquare p \wedge p, \neg\blacksquare p \wedge \neg p\}$. Fix an enumeration $\epsilon = \blacksquare p, p$; in this case $\mathrm{Cons}(\Phi_1) = \{\blacksquare p, \neg\blacksquare p\}$ and by the definition above $A_1 = \sigma(\blacksquare p) = q_1$. To find A_2, note that $B^+(\Phi_1) = q_1$ (for $\vdash_{\mathsf{L}} \blacksquare p \to p$), $B^-(\Phi_1) = \bot$ and

$B^0(\Phi_1) = \neg q_1 \wedge q_2$. Hence $A_2 = q_1 \vee (\neg q_1 \wedge q_2)$, which is classically equivalent to $\neg q_1 \to q_2$; note that the consistent combinations of A_1 and A_2 are exactly the consistent combinations of $\blacksquare p$ and p. In this case, $\sigma(\blacksquare p \to p) = q_1 \to (\neg q_1 \to q_2)$, which is a classical tautology, as desired for the mapping of an L-theorem.

Had we chosen the other enumeration of monolithic formulas ($\epsilon' = p, \blacksquare p$) we would have obtained different values: $A_1 = \sigma(p) = q_1$, $A_2 = \sigma(\blacksquare p) = q_1 \wedge q_2$; however, this leads to $\sigma(\blacksquare p \to p) = (q_1 \wedge q_2) \to q_1$, which is also a tautology, as desired.

The following lemma shows that, indeed, this construction always works and will be useful for the completeness proof.

LEMMA 1 *Let ψ be a conjunction of literals of a set of L-formulas $\Phi = \{\varphi_1, \ldots, \varphi_n\}$. Then ψ is L-consistent iff $\sigma(\psi)$ is classically consistent.*

Proof. By induction on $m < n$. For $m = 1$, the result follows immediately from the definition of A_1. Now suppose the result is valid for Φ_m; we will prove it for Φ_{m+1}. All we have to care for are formulas ψ in which φ_{m+1} occurs, that is $\psi = \mu \wedge \varphi_{m+1}$ or $\psi = \mu \wedge \neg\varphi_{m+1}$.

Suppose $\sigma(\psi)$ is classically consistent. By the definition of σ, $\sigma(\mu)$ is classically consistent so the induction hypothesis gives us that μ is L-consistent. Hence $\sigma(\mu)$ is an element of one of the disjunctions $B^+(\Phi_m)$, $B^-(\Phi_m)$ or $B^0(\Phi_m)$.

If $\sigma(\mu)$ is in $B^+(\Phi_m)$, then $\vdash_{\mathsf{L}} \mu \to \varphi_{m+1}$. We want to show that $\psi = \mu \wedge \varphi_{m+1}$, in which case ψ is clearly consistent. For contradiction, suppose that $\psi = \mu \wedge \neg\varphi_{m+1}$. Suppose v is a classical valuation that satisfies $\sigma(\mu)$. By the definition of $A_{m+1} = \sigma(\varphi_{m+1})$, $\sigma(\mu)$ is a disjunct in $B^+(\Phi_m)$, thus satisfied by v. We show that v does not satisfy $B^-(\Phi_m)$. Indeed, if there is a conjunct in $B^-(\Phi_m)$ satisfied by v, then there is a conjunction of literals μ' in Φ_m such that $\vdash_{\mathsf{L}} \mu' \to \neg\varphi_{m+1}$. Since μ is consistent, we also have that $\vdash_{\mathsf{L}} \mu \to \neg\mu'$, so the induction hypothesis gives us that v cannot satisfy μ', and hence v cannot satisfy $B^-(\Phi_m)$. It follows that v satisfies $\sigma(\varphi_{m+1})$ and thus falsifies $\sigma(\psi)$. On the other hand, any valuation that falsifies $\sigma(\mu)$ also falsifies $\sigma(\psi)$. So any valuation falsifies $\sigma(\psi)$, contradicting its consistency. We have thus proved that $\psi = \mu \wedge \varphi_{m+1}$, and ψ is consistent.

If $\sigma(\mu)$ is in $B^-(\Phi_m)$, by an analogous argument, if $\sigma(\mu)$ is satisfied, $\sigma(\psi_{m+1})$ is falsified, so the only possibility is that $\psi = \mu \wedge \neg\varphi_{m+1}$. But then $\vdash_{\mathsf{L}} \mu \to \neg\varphi_{m+1}$ and ψ is consistent.

Finally, if $\sigma(\mu)$ is in $B^0(\Phi_m)$, then by definition both $\mu \wedge \varphi_{m+1}$ and $\mu \wedge \neg\varphi_{m+1}$ are consistent.

Suppose now that ψ is L-consistent. Then μ is also L-consistent and by induction hypothesis, $\sigma(\mu)$ is classically consistent. Also, $\sigma(\mu)$ is an element of one of the disjunctions $B^+(\Phi_m)$, $B^-(\Phi_m)$ or $B^0(\Phi_m)$ and we have three possibilities to analyse in a manner totally analogous as in the previous direction. Details omitted. ■

We now generalise Lemma 1 to full $\mathsf{M}(\mathsf{L})$.

LEMMA 2 *Let ψ be a $\mathsf{M}(\mathsf{L})$-formula and let Φ be the set of its monolithic subformulas, over which the mapping σ is defined. Then ψ is $\mathsf{M}(\mathsf{L})$-consistent iff $\sigma(\psi)$ is M-consistent.*

Proof. ($\Rightarrow$) Suppose, for contradiction, that $\sigma(\psi)$ is inconsistent, that is, $\vdash_{\mathsf{M}} \neg\sigma(\psi)$. If φ is a subformula of ψ and a formula of L, then by Lemma 1 $\sigma(\psi)$ is a classical tautology iff φ is a theorem of L and of $\mathsf{M}(\mathsf{L})$ by the inference rule *Preserve*. Furthermore, all other M inference step can be copied in $\mathsf{M}(\mathsf{L})$. Therefore, the deduction of $\vdash_{\mathsf{M}} \neg\sigma(\psi)$ can be simulated in $\mathsf{M}(\mathsf{L})$, so as to prove $\vdash_{\mathsf{M}(\mathsf{L})} \neg\psi$, contradicting the consistency of ψ.

($\Leftarrow$) Suppose now that ψ is $\mathsf{M}(\mathsf{L})$-inconsistent. Consider a deduction of $\vdash_{\mathsf{M}(\mathsf{L})} \neg\psi$; suppose it contains an instance of the rule preserve, in which for a formula φ built from the elements of Φ, $\vdash_{\mathsf{L}} \varphi$ is inferred. Then, by Lemma 1, $\sigma(\varphi)$ is a classical tautology, that is also a theorem of M. All other inference steps of $\mathsf{M}(\mathsf{L})$ can be copied as inference steps in M, so that the deduction of $\vdash_{\mathsf{M}(\mathsf{L})} \neg\psi$ can be transformed into a deduction of $\vdash_{\mathsf{M}} \neg\sigma(\psi)$. ■

This result leads us to a proof of transference of completeness.

3.2 Transference of completeness

THEOREM 3 *If L and M are complete logics, so is $\mathsf{M}(\mathsf{L})$.*

Proof. Let ψ be a consistent $\mathsf{M}(\mathsf{L})$-formula and let Φ be the set of all monolithic subformulas of ψ. By Lemma 2, $\sigma(\psi)$ is M-consistent, so by the completeness of M there is a model $\mathcal{M}_{\mathsf{M}} = (W, N, V)$ with $\mathcal{M}_{\mathsf{M}} \models \sigma(\psi)$. We build a $\mathsf{M}(\mathsf{L})$ model $\mathcal{M}_{\mathsf{M}(\mathsf{L})} = (W, N, g)$ in the following way: for all $w \in W$ and $\varphi \in \Phi$, $g(w) \models_{\mathsf{L}} \varphi$ iff $w \in V(\sigma(\varphi))$. To see that this is indeed a model, consider for each $w \in W$ the elements $\varphi_i \in \Phi$ such that $\sigma(\varphi)$ are satisfied by V in w; by Lemma 1 all such φ_i are simultaneously L-consistent and by completeness of L there is a model $\mathcal{M}_{\mathsf{L}} = g(w)$ that simultaneously satisfies all of them.

A simple structural induction now shows that $\mathcal{M}_{\mathsf{M}(\mathsf{L})} \models \psi$ iff $\mathcal{M}_{\mathsf{M}} \models \sigma(\psi)$. Therefore $\mathcal{M}_{\mathsf{M}(\mathsf{L})} = (W, N, g)$ is a model for ψ and $\mathsf{M}(\mathsf{L})$ is complete over the same class of models that M is. ■

4 Decidability

We now show the transference of decidability. A logic is *decidable* if there is an algorithm that, given any formula in that logic, decides in a finite number of steps if that formula is a theorem of the logic.

We first note that Lemma 2 gives us immediately that a formula ψ is a $\mathsf{M}(\mathsf{L})$-theorem iff $\sigma(\psi)$ is a M-theorem.

THEOREM 4 *If the logics* L *and* M *are complete and decidable, so is* $\mathsf{M}(\mathsf{L})$.

Proof. Give a $\mathsf{M}(\mathsf{L})$-formula ψ, we can algorithmically construct the set Φ of all its monolithic formulas. Since Φ is finite and L is decidable, we can construct the set Cons(Φ) and the M-formula $\sigma(\psi)$. We then apply the decision procedure for logic M over $\sigma(\psi)$ and by Lemma 2 we know that ψ is a $\mathsf{M}(\mathsf{L})$-theorem iff $\sigma(\psi)$ is a M-theorem. ∎

4.1 Complexity

As for complexity, the decision procedure above gives us the following. Let $c_{\mathsf{L}}(n)$ and $c_{\mathsf{M}}(n)$ be the complexities of the decision procedure in L and M, respectively.

If n is the size of the $\mathsf{M}(\mathsf{L})$-formula ψ, the size of the set Φ is $O(n)$ and the number of elements potentially needed in the construction of Cons(Φ) is $O(2^n)$. Then the construction of $\sigma(\psi)$ can be done in time $O(2^n \times c_{\mathsf{L}}(n))$. Note, however, that by the construction of Cons(Φ), each element of Cons(Φ) will generate a disjuct in $\sigma(\psi)$, so the size of $\sigma(\psi)$ is $O(2^n)$. As a consequence, the decision procedure of M is applied to $\sigma(\psi)$ in time $O(c_{\mathsf{M}}(2^n))$.

As a result, we have the following.

LEMMA 5 *The time complexity of the decision procedure in Theorem 4 is:*

$$O(2^n \times c_{\mathsf{L}}(n) + c_{\mathsf{M}}(2^n)).$$

With regards to the space complexity, because the size of Cons(Φ) is $O(2^n)$, so even if the complexity of the decision procedure of logics M and L are in PSPACE, the space of the decision procedure for $\mathsf{M}(\mathsf{L})$ above will be in EXPSPACE.

LEMMA 6 *If the logics* L *and* M *are in PSPACE, then* $\mathsf{M}(\mathsf{L})$ *is in EXPSPACE.*

Of course, this does not rule out the possibility of existing a different decision procedure for $\mathsf{M}(\mathsf{L})$ that places it in PSPACE. In other words, the discussion above does not provide a lower bound for the complexity of $\mathsf{M}(\mathsf{L})$.

5 Conclusion

We have presented an original method that maps formulas of a logic L into boolean combinations of propositional classical formulas, so as to preserve validity and consistency. This mapping was then used to prove the transference of completeness and decidability of the modalisation $\mathsf{M}(\mathsf{L})$ that applies non-normal modal logic M to a generic logic L.

For the future, we hope to be able to explore such mapping for other logics, and for translating a logic into another. Also, as done previously, we want to explore the use of successive modalisations as a means to study the *independent combination* or *fusion* of two non-normal modal logics. Another possible path of development is the generalisation of the non-normal modalisation process to modal logics with many connectives and with connectives with any finite arity, following the steps of [12, 8].

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