3

Relativized Action Complement for Dynamic Logics

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ABSTRACT. Dynamic logic is becoming more and more popular as a logic of action, and is used for knowledge representation and reasoning for such distinct domains as action and change, action and time, and action and norms. We argue that to each of these three application areas the notion of 'action complement' is crucial, but that the notions of action complement found in the literature do not suffice, because of counter intuitive aspects of their semantics. As a solution to this problem we propose a range of dynamic logics with a 'relativized' action complement. These relativized complement dynamic logics have a semantics that is better suited for reasoning about action. Other results concern their relative strength and their expressiveness in terms of definability of classes of frames and models. This paper gives motivations, definitions and some initial results.

1 Introduction

Variants of dynamic logic are used more and more for representation and reasoning in the domains of action and change [13, 26], action and time [5, 30], and action and norms [24, 23, 9, 8]. In all of these domains, the notion of 'action complement' in the interpretation of reference to '*alternative action*' arises as a crucial concept. For instance, in the domain of reasoning about action and change, it is natural to consider the property that an effect is brought about exclusively by a certain action [27, 4]. Another way of saying the same thing is that alternative action cannot have that effect. This observation is central to the proposed solutions to the problem of how to express frame properties in dynamic (modal action) logics [13]. If, for the moment, we denote action complement with the symbol $-$, we can capture the property of 'exclusive effect' by a dynamic logic formula of the form: $\neg\varphi \rightarrow [-\alpha]\neg\varphi$. Intuitively it reads 'performing an action other than α from a state for which $\neg\varphi$ holds, cannot result in a state where φ holds'. So if we want to perform an action that changes φ from false to true, we can only achieve this by performing an action α (if in the given situation it is actually possible to perform such an action). The second area in which action complement, in the interpretation of alternative action, arises as a natural concept is that of temporal reasoning over action. For instance, if we consider the (liveness) property that over all possible futures eventually an action a is inevitable, we should be able to conclude that this is logically equivalent with the property that it is not possible to perform actions alternative to a forever. An important third application area is that of deontic action reasoning. Traditionally, in deontic logics that were not so much concerned with normative statements about actions α as with normative conditions about states of affairs φ, the relation between *obligation* ($O(\varphi)$) and *permission* ($P(\varphi)$) is described as $O(\varphi) \leftrightarrow \neg P(\neg\varphi)$. In the work on (dynamic) deontic action logic, initiated by Meyer [24], this property reappears as $O(\alpha) \leftrightarrow \neg P(-\alpha)$, for actions α. This identification only makes sense under a suitable interpretation of the operation $(-)$ as an action complement that refers to actions alternative to that referred to by α. Under such an interpretation the identification is read as 'obligation to perform an action α bi-implies absence of permission to perform any action alternative to α'.

In the literature, three notions of complement for dynamic (modal action) logics can be found: the above mentioned action complement used by Meyer to define his dynamic deontic logic [24], the dynamic negation defined by Van Benthem [2], and the complement with respect to the universal relation, as imported from relation algebra [14, 11, 15]. The action complement defined by Meyer does not have a straightforward interpretation in terms of standard modal action models. The complement is defined as part of an action algebra, whose relation to modal action models is not made explicit in detail. Furthermore, the complement is designed to meet some specific deontic requirements, and does not count as a solution that is applicable to the action and change or temporal domain. The dynamic negation of Van Benthem [2] does not meet the requirement concerning the intended interpretation of the complement as referring to 'alternative action'. The dynamic negation $\sim_d$ is defined as the dynamic logic test ([10]) for non-possibility to perform an action: $\sim_d \alpha \equiv_{def} ([\alpha]\bot)?$. So the dynamic negation refers to non-activity, and not to alternative action. This leaves the complement with respect to the universal relation as the most serious candidate for the modeling of the notion of alternative action. But, like we do in this paper, some authors have put some question to this view. The following is a quote from Krister Segerberg [29] that addresses the central issue:

> 'By contrast with intersection, the question concerning complement (negative action) is intricate and involves much extra-theoretical consideration: do we humans really think in terms

> of complements? Does the analysis of human languages suggest that we do? Is it not the case that the choice between two actions a and $U \setminus a$ is often a choice between a and some action b that is a proper subset of $U \setminus a$? Before these questions have been answered, this author feels a certain unease about the unrestricted acceptance of closure under complement.'

Segerberg uses the symbol U here to refer to the universal relation $S \times S$. He makes several points. One is that he is not convinced of the naturalness of a notion of action complement. We do not agree, as follows from the above discussion concerning the naturalness of the action complement for several independent application domains. The second important point made is that it is in a way counter-intuitive to define action complement with respect to the universal relation, since intuitively, choices concerning the performance of an action or its complement decide on a division of a *subset* of the universal relation space. In section 3, we try to make the uneasiness felt by Segerberg explicit by showing that the universal complement does not fit well with the three mentioned application domains. In section 4 we show how to define weaker notions of complement, that should take Segerberg's feelings of uneasiness away, and that result in logics that are more fitted to the action reasoning domain. But first we define the global setting for our logics.

2 Dynamic logics

The languages of the dynamic logics in this paper are of the form given in the following definition.

DEFINITION 1 (Modal syntax). Taking 'a' to represent arbitrary elements of a given countable set of atomic action symbols $\mathcal{A}$, and taking 'P' to represent arbitrary elements of a given countable set of proposition symbols $\mathcal{P}$, and given a finite set AC of action combinators, well formed formulas $\varphi, \psi, \ldots$ of a dynamic logic language $\mathcal{L}_{DL}(AC)$ are defined by:

$$\varphi, \psi, \ldots \quad := \quad P \mid \neg\varphi \mid \varphi \wedge \psi \mid \langle\alpha\rangle\varphi$$

where α represents arbitrary actions constructed using atomic actions a and the action combinators from the set AC.

We assume the usual syntactic extensions, including $[\alpha]\varphi \equiv_{def} \neg\langle\alpha\rangle\neg\varphi$. The above definition leaves open how complex actions α are built from atomic actions a. We consider several different sets AC of action combinators. We now define the most general action language we consider in this paper, where the symbol '$-$' functions as a 'place holder' for the range of symbols for different notions of action complement we will introduce in subsequent sections.

DEFINITION 2 (Action language).

$$\alpha, \beta, \ldots \quad := \quad a \mid \alpha \cup \beta \mid \alpha ; \beta \mid \; -\alpha \mid \alpha^{*} \mid \alpha^{\leftarrow}$$

The dynamic logic languages $\mathcal{L}_{DL}(AC)$ we consider are all conceived of as sub-languages of the most general language $\mathcal{L}_{DL}(\cup, ;, -, ^{*}, ^{\leftarrow})$ by restriction to a subset $AC \subseteq \{\cup, ;, -, ^{*}, ^{\leftarrow}\}$ of the action combinators. We now turn to the semantics.

DEFINITION 3 (Modal action models).

A modal action model $\mathcal{M} = (S, R^{\mathcal{A}}, V^{\mathcal{P}})$ is defined as follows:

- S is a nonempty set of possible states
- $R^{\mathcal{A}}$ is an action interpretation function $R^{\mathcal{A}} : \mathcal{A} \rightarrow 2^{(S \times S)}$, assigning a binary relation over $S \times S$ to each atomic action a in $\mathcal{A}$.
- $V^{\mathcal{P}}$ is a valuation function $V^{\mathcal{P}} : \mathcal{P} \rightarrow 2^{S}$ assigning to each proposition P of $\mathcal{P}$ the subset of states in S for which P is valid.

In the following definition of the semantics of formulas of a dynamic logic language $\mathcal{L}_{DL}(AC)$, we implicitly assume an extension of the interpretation function $R^{\mathcal{A}}$ for atomic actions a, to an interpretation function R for complex actions α.

DEFINITION 4 (Modal semantics, validity, logic). The modal semantics is determined by the notion of validity of a formula φ in a state s of a model $\mathcal{M} = (S, R^{\mathcal{A}}, V^{\mathcal{P}})$, denoted $\mathcal{M}, s \models \varphi$. We assume that the interpretation $R(\alpha)$ of general actions α is given.

$$\begin{array}{lll}
\mathcal{M}, s \models P & \text{iff} & s \in V^{\mathcal{P}}(P) \\
\mathcal{M}, s \models \neg\varphi & \text{iff} & \text{not } \mathcal{M}, s \models \varphi \\
\mathcal{M}, s \models \varphi \wedge \psi & \text{iff} & \mathcal{M}, s \models \varphi \text{ and } \mathcal{M}, s \models \psi \\
\mathcal{M}, s \models \langle\alpha\rangle\varphi & \text{iff} & \text{there is a } t \text{ such that } (s,t) \in R(\alpha) \text{ and } \mathcal{M}, t \models \varphi
\end{array}$$

Validity on a model $\mathcal{M}$ is defined as validity in all states of the model. If φ is valid on a model $\mathcal{M}$, we say that $\mathcal{M}$ is a model for φ. General validity of a formula φ is defined as validity on all modal action models. The subset of all generally valid formulas in $\mathcal{L}_{DL}(AC)$ is defined to be the dynamic logic over AC. This logic is symbolically referred to as $DL(AC)$.

The definition of R in terms of $R^{\mathcal{A}}$ depends on the set AC of action combinators in the modal action language. Below we give this definition of R in terms of $R^{\mathcal{A}}$ for all combinators except the action complement, since this will be the subject of subsequent sections.

DEFINITION 5. The relational interpretation R of actions α in terms of the relation $R^{\mathcal{A}}$ is given by:

$$\begin{array}{lcllcl} R(a) & = & R^{\mathcal{A}}(a) \quad \text{for } a \in \mathcal{A} & & & \\ R(\alpha \cup \beta) & = & R(\alpha) \cup R(\beta) & R(\alpha^*) & = & (R(\alpha))^* \\ R(\alpha;\beta) & = & R(\alpha) \circ R(\beta) & R(\alpha^{\leftarrow}) & = & \{(s,t) \mid (t,s) \in R(\alpha)\} \end{array}$$

The binary operation '$\circ$' denotes the sequential composition of relations, and the unary operation '*' denotes the reflexive transitive closure of a relation.

This completes the definition of the semantics of dynamic logics. The semantics enforces that $[\alpha]\varphi$ holds in a state whenever all states reachable by α obey φ, which reflects 'execution of the action α results in validity of the condition φ'. For the notion of 'logic entailment' we can choose between the local and the global variant. See [3] for details.

3 Complement with respect to the universal relation

In this section we show that in particular the extension of PDL with the complement as inherited from relation algebra [31], has properties that although useful and intuitive for relations in general, do not apply to the types of action reasoning we mentioned in the introduction. In the next section we propose weakenings that are better suited to model the action reasoning in these domains. It is straightforward to import the standard relation algebraic complement in the dynamic logics we study. We denote this complement with the symbol '$\sim$'.

DEFINITION 6 (Semantics of $\sim$-logics). $\sim$-logics are defined as modal action logics whose set AC of action combinators includes the unary action operator '$\sim$', which we call the 'universal complement'. The semantics of $\sim$-logics follows from the modal semantics of definition 4, the relational definitions for $;, \cup, ^*$ and $^{\leftarrow}$ in definition 5, and the clauses:

$$\begin{array}{lcl} R(U) & = & S \times S \\ R(\sim\alpha) & = & R(U) \setminus R(\alpha) \end{array}$$

In almost all the literature known to us concerning the complement in dynamic logics [14, 11, 25, 28, 15, 19] and related formalisms (dynamic algebras, Peirce algebras, boolean modules), this is the version of the complement that is studied. This is not surprising given that the universal complement directly corresponds to standard negation in the first-order view on modal languages. It is straightforward to define a translation T from iteration-free $\sim$-logics involving a set of propositions $\{P, Q, \ldots\}$, and a set of atomic actions $\{a, b, \ldots\}$ to a first-order logic with a set of unary predicates $\{P, Q, \ldots\}$, a set of binary predicates $\{R^a, R^b, \ldots\}$, and a set of variables $\{x, y, \ldots\}$:

$$
\begin{array}{lllllll}
T_x(P) & := & Px & & T_x(\varphi \wedge \psi) & := & T_x(\varphi) \wedge T_x(\psi) \\
T_x(\bot) & := & x \neq x & & T_x(\langle\alpha\rangle\varphi) & := & \exists y(T_{xy}(\alpha) \wedge T_y(\varphi)) \\
T_x(\neg\varphi) & := & \neg T_x(\varphi) & & T_{xy}(a) & := & R^a xy \\
T_{xy}(\alpha^{\leftarrow}) & := & T_{yx}(\alpha) & & T_{xy}(\alpha;\beta) & := & \exists z(T_{xz}(\alpha) \wedge T_{zy}(\beta)) \\
T_{xy}(\alpha \cup \beta) & := & T_{xy}(\alpha) \vee T_{xy}(\beta) & & T_{xy}(\sim\alpha) & := & \neg T_{xy}(\alpha)
\end{array}
$$

With induction we can prove that for any $\varphi \in \mathcal{L}_{DL}(^{\leftarrow}, ;, \cup, \sim)$ it holds that $\mathcal{M}, s \models \varphi$ iff $\mathcal{M} \models_{FOL} T_x(\varphi)[s]$ (s is assigned to the free variable x). Iteration free $\sim$-logics are thus subsumed by first-order logic. This means that they inherit the compactness property from FOL: any non-satisfiable infinite set has a non-satisfiable finite subset.

We now give a set of action connectives that can be introduced as syntactic abbreviations in terms of the complement ($\sim$) and the choice ($\cup$).

DEFINITION 7. The 'universal relation U', the 'impossible action $fail$', the 'concurrent action $\alpha \cap \beta$', the 'subsumption relation $\alpha \subseteq \beta$', and the 'equivalence action $\alpha \doteqdot \beta$' are defined through the following syntactic extensions on $\cup$ and $\sim$ (We assume that unary operators bind stronger than binary ones, and freely add parentheses whenever ambiguity in the reading of relational formulas may arise.):

$$
\begin{array}{llllll}
\alpha \cap \beta & \equiv_{def} & \sim(\sim\alpha \cup \sim\beta) & fail & \equiv_{def} & \alpha \cap \sim\alpha \\
\alpha \subseteq \beta & \equiv_{def} & \sim\alpha \cup \beta & U & \equiv_{def} & \alpha \cup \sim\alpha \\
\alpha \doteqdot \beta & \equiv_{def} & (\alpha \subseteq \beta) \cap (\beta \subseteq \alpha) & & &
\end{array}
$$

Thus, an action operation like $\cap$ does not have to be introduced explicitly in a modal action logic whenever complement and choice are already included[1].

Introduction of the universal complement in modal logic enhances expressiveness considerably ([14]). However, we argue from semantic intuitions

[1] If we have the dynamic logic test in the language, the action $fail$ can also be defined as: $fail \equiv_{def} \bot?$.

that the universal complement is inappropriate for a modal logic of action. Of course this does not mean that it cannot be a useful construct in other applications of modal logic (e.g. epistemic reasoning).

Our first point is that the universal relation U does not correspond to the notion of 'any action', an interpretation it should have if the complement were to reflect the notion of 'alternative action'. The intuition for the notion of 'any action' is that it subsumes all possible actions, but not *more*. But the relation U reaches any state, including the ones that are not reachable by any action. In our view, this makes the U too strong to represent the notion of 'any action'. Nevertheless, some authors on action reasoning in PDL refer to the relation U as the 'any action' [26].

We now go on to show what counter intuitive consequences the use of the complement with respect to the universal relation would have in the three application domains we mentioned.

For temporal reasoning over action, it is intuitive to assume that time evolves by the performance of actions. It follows that only if states of affairs can be achieved through action, they belong to the possible futures that have to be considered. Then it is clear that the universal relation U that reaches all possible states cannot function as an action, since then many states that are not reachable should have to be considered as possible future states of affairs.

The universal complement is also not suited to reason about action (non-) effects. A suitable notion of action complement ('$-$') should enable the expression of frame properties. In particular we claimed that it should be possible to use formulas of the form $\neg\varphi \to [-\alpha]\neg\varphi$ to express that the condition φ can only be changed by α. But his cannot be achieved by using the universal complement. Again, it is too strong. The universal complement in the expression $\neg\varphi \to [\sim\alpha]\neg\varphi$ does not only say that actions other than (not involving) α cannot have φ as a result, it says in addition that all states in the state-space that embody the result φ can actually be reached by α ($[|\alpha|]\varphi \equiv_{def} [\sim\alpha]\neg\varphi$ is also called 'the window operator' [1, 17]). This follows the semantics of the universal complement: any state can be reached from any other state by either α or $\sim\alpha$, and reachability by $\sim\alpha$ contradicts $[\sim\alpha]\neg\varphi$. This means that the universal complement is not suited to encode Reiter's approach to specifying frame properties [13, 27] into modal action logic.

In the deontic context, the universal complement is not useful for similar reasons. We do not give the arguments here, but only mention that they were reason for Meyer to define his version of the action complement [24].

Note that we do not claim that a modal *formula* of the form $[\sim\alpha]\varphi$ does not have an intuitive reading in terms of the action α. We only claim that the operator $\sim$ does not have an intuitive reading as an action operation. An intuitive reading of $[\sim\alpha]\varphi$ in terms of action is 'φ holds in all states that are not the result of performing the action α'. But this does not give a reading of $\sim$ as an action combinator, because $\sim\alpha$ is *not* an action: it may include 'transitions' that do not correspond to any action at all.

4 Relativized complement dynamic logics

The universal complement introduces an aspect that is not in the spirit of modal logic: globalness. This aspect of the complement is inherited from relation algebra, where the semantic view is also global. The complement we consider in this section is faithful to the idea of locality in modal semantics. This results in a better fit with the interpretation of the modal language as a logic of action. As a side-effect we also expect better complexities for the logics with a relativized action complement. The action language comprising the new, relativized complement is only concerned with the relations over the part of the state space that is reachable. It assumes that in general, from any given state at least part of the state space cannot be reached through any action, and that the part of the state space that *is* reachable may vary from state to state. Then, the general intuition for the alternative complement, denoted $\wr^{I}$, is that it is taken with respect to all possible relations over this reachable state space. But what may be considered reachable, depends on what action combinators are in the action language. This explains the term 'relativized'. The complement operation is relativized with respect to the part of the state space in a modal action model that (1) is the minimal relation space containing all atomic actions and that (2) is closed under application of the action combinators of the dynamic logic language (the complement operation itself excluded). Thus, if we allow iteration in the action language, the complement space is reflexive, and if we allow converse, the complement space is symmetric. Therefore we cannot give, as for the universal complement, a general definition of the new complement for all dynamic logics; each dynamic logic comes with its own interpretation for the complement. All in all we define 6 versions of the relativized complement: $\wr^{K}, \wr^{B}, \wr^{S4}, \wr^{K4}, \wr^{B4}, \wr^{S5}$, one for each dynamic logic involving a specific set of action combinators. The annotations give information about the nature of the relation space with respect to which the complement is taken, where we adopt standard terminology from modal logic to refer to transitivity, reflexivity etc. Intuitively, this relation space with respect to which the relativized complement is taken reflects the space of possible alternative complex actions. That this space is relative to the action combinators in the action language makes sense, since what can be done 'alternatively' depends on the way in which the action syntax allows us to combine actions into more complex ones.

As a consequence of this scattering of interpretations, some action operators that can be syntactically defined in terms of the complement also get alternative interpretations (any^I, $\subseteq^I$). Other combinators undergo no effective change in interpretation ($\cup, \cap, ;, ^*, ^\leftarrow, \mathit{fail}$).

To our knowledge the notion of relativized complement has not been studied before in modal logic. Only in the work of De Giacomo [12] a complement appears that is equivalent with $\wr^K$.

4.1 Semantics of $\wr^I$-logics

DEFINITION 8 (Semantics of $\wr^I$-logics). Let $\wr^K, \wr^B, \wr^{S4}, \wr^{K4}, \wr^{B4}, \wr^{S5}$, be action complement connectives for the logics $DL(\cup, \wr^K)$, $DL(^\leftarrow, \cup, \wr^B)$, $DL(;, \cup, \wr^{K4})$, $DL(^*, ;, \cup, \wr^{S4})$, $DL(^\leftarrow, ;, \cup, \wr^{B4})$, $DL(^*, ^\leftarrow, ;, \cup, \wr^{S5})$ respectively. We refer to this type of logics as $\wr^I$-logics. The semantics of each $\wr^I$-logic is determined by the modal semantics of definition 4, together with a relational interpretation function R for actions, that follows by selection of the relevant clauses of definition 5 in combination with the relevant clause for the relativized action from the following list (the unary operation '$^+$' denotes the transitive closure of a relation):

$$
\begin{array}{lrcl}
\text{for } DL(\cup, \wr^K): & R(any^K) & = & \bigcup_{a \in \mathcal{A}} R(a) \\
\text{for } DL(^\leftarrow, \cup, \wr^B): & R(any^B) & = & \bigcup_{a \in \mathcal{A}} (R(a) \cup R(a^\leftarrow)) \\
\text{for } DL(;, \cup, \wr^{K4}): & R(any^{K4}) & = & (\bigcup_{a \in \mathcal{A}} R(a))^+ \\
\text{for } DL(^*, ;, \cup, \wr^{S4}): & R(any^{S4}) & = & (\bigcup_{a \in \mathcal{A}} R(a))^* \\
\text{for } DL(^\leftarrow, ;, \cup, \wr^{B4}): & R(any^{B4}) & = & (\bigcup_{a \in \mathcal{A}} (R(a) \cup R(a^\leftarrow)))^+ \\
\text{for } DL(^*, ^\leftarrow, ;, \cup, \wr^{S5}): & R(any^{S5}) & = & (\bigcup_{a \in \mathcal{A}} (R(a) \cup R(a^\leftarrow)))^*
\end{array}
$$

$$R(\wr^I \alpha) = R(any^I) \setminus R(\alpha)$$

The logics of definition 8 are clearly not the only possible $\wr^I$-logics. But many $\wr^I$-logics are rather eccentric. Examples are $DL(\wr^K)$ (the complement is only applicable to atomic actions), which has the same complement as $DL(\cup, \wr^K)$, and $DL(^*, \cup, \wr^{S4})$, which has the same complement as $DL(^*, ;, \cup, \wr^{S4})$. We do not consider these logics, since they seem to have less relevancy for the application to reasoning about action.

We introduce the relativized *any* and the relativized *subsumption action* as syntactic extensions by defining them in terms of the relativized complement ($\wr^I$) and the choice ($\cup$). Note that we already defined the relativized any as part of the definition of the relativized action complement above. But we identify dynamic logics by reference to their base-combinators, that is, sets of action combinators that are not definable in terms of others. We can define the 'any' in terms of the complement, but not the other way round.

DEFINITION 9. The relativized any, and the relativized subsumption action are defined through the following syntactic extensions on $\cup$ and $\wr^I$:

$$\begin{array}{lll} \alpha \subseteq^I \beta & \equiv_{def} & \wr^I \alpha \cup \beta \\ any^I & \equiv_{def} & \alpha \cup \wr^I \alpha \text{ for an arbitrarily chosen } \alpha \end{array}$$

The relativized versions of the *any* and the *subsumption action* differ from their non-relativized counterparts. The relativized *any* does not reach the complete state space, but only the part that is reachable through (complex) action, as determined by the action language. The next propositions states that the *intersection, equivalence action* and *fail* have the same interpretation as their relativized counterparts.

PROPOSITION 10.
For any two actions α and β in the action language of a $\wr^I$-logic the following holds for the relational interpretations $R(\alpha)$ and $R(\beta)$ on a modal action model $\mathcal{M} = (S, R^{\mathcal{A}}, V^{\mathcal{P}})$:

$$\begin{array}{lllllll} R(\alpha \cap \beta) & = & R(\alpha \cap^I \beta) & \textit{with} & \alpha \cap^I \beta & \equiv_{def} & \wr^I(\wr^I \alpha \cup \wr^I \beta) \\ R(\alpha \doteqdot \beta) & = & R(\alpha \doteqdot^I \beta) & \textit{with} & \alpha \doteqdot^I \beta & \equiv_{def} & (\alpha \subseteq^I \beta) \cap (\beta \subseteq^I \alpha) \\ R(fail) & = & R(fail^I) & \textit{with} & fail^I & \equiv_{def} & \alpha \cap \wr^I \alpha \textit{ for an} \\ & & & & & & \textit{arbitrarily chosen } \alpha \end{array}$$

Proof. We prove the first equivalence. From the action semantics for the universal complement and the relativized complements it follows that $R(\wr^I \alpha) = R({\sim}(\alpha \cup {\sim} any^I))$. Therefore, in the definition for $\alpha \cap^I \beta$ we can substitute the action ${\sim}(\alpha \cup {\sim} any^I)$ for all occurrences of the action $\wr^I \alpha$. This returns ${\sim}({\sim}(\alpha \cup {\sim} any^I) \cup {\sim}(\beta \cup {\sim} any^I) \cup {\sim} any^I)$. By applying standard boolean properties we arrive at the equivalent ${\sim}({\sim}((\alpha \cup {\sim} any^I) \cap any^I) \cup {\sim}((\beta \cup {\sim} any^I) \cap any^I))$. By again applying standard boolean properties we arrive at (1) ${\sim}({\sim}((\alpha \cap any^I) \cup ({\sim} any^I \cap any^I)) \cup {\sim}((\beta \cap any^I) \cup ({\sim} any^I \cap any^I)))$. Now we focus on the actions (2) $\alpha \cap any^I$ and (3) ${\sim} any^I \cap any^I$. Action 3 is equivalent with the impossible action $fail$, as follows from the meaning of $\sim$. Action 2 is equivalent with α, since any^I is the action that subsumes any other action: for any α it holds that $R(\alpha) \subseteq R(any^I)$. Substitution of these equivalent actions into action 1 results in ${\sim}({\sim}\alpha \cup {\sim}\beta)$, which is equivalent with $\alpha \cap \beta$. ∎

For any specific $\wr^I$-logic, we talk of its $\wr^I$-reduced sub-logic when we mean the logic that results by removal of the complement, and we talk of its $\sim$-logic counterpart, when we mean the corresponding logic where the $\wr^I$-operator is replaced by the $\sim$-operator. In the same way we talk about $\sim$-logic counterpart formulas, validities, validity schemes, etc.

We will prove that $\wr^I$-logics are subsumed by their $\sim$-logic counterparts (under syntactic replacement of the complements). In this sense they are 'weaker'. But (with one exception) it is not the case that $\wr^I$-logics can be syntactically defined by their $\sim$-logic counterparts. We will prove this for the two weakest $\wr^I$-logics.

PROPOSITION 11.
For the logics $DL(\cup, \wr^K)$ and $DL(\cup, ^{\leftarrow}, \wr^B)$, the relativized complement operator $\langle \wr^I \alpha \rangle \varphi$ is not syntactically definable in their $\sim$-logic counterparts.

Proof. From the translation to first-order logic we gave in section 3 follows compactness for iteration-free $\sim$-logics. The relativized complements $\wr^K$ and $\wr^B$ are not compact. For the logic $DL(\cup, \wr^K)$, it is straightforward to verify that the set $\{\langle \wr^K a \rangle \neg P, [b]P, [c]P, \ldots\}$ (where $\{a, b, c, \ldots\} = \mathcal{A}$), is not satisfiable while all of its finite sub-sets are. For the logic $DL(\cup, ^{\leftarrow}, \wr^B)$, the same holds for $\{\langle \wr^B (a \cup a^{\leftarrow}) \rangle \neg P, [b]P, [b^{\leftarrow}]P, [c]P, [c^{\leftarrow}]P, \ldots\}$. ■

4.2 Validities

In this section we study the relations between validities of $\wr^I$-logics and validities of their $\sim$-logic counterparts.

THEOREM 12. *Under syntactic replacement of $\wr^I$ with $\sim$, each validity for a given $\wr^I$-logic turns into a validity for its $\sim$-logic counterpart.*

Proof. Through negative demonstration. Assume that the formula $\varphi^{\wr^I}$ is a $\wr^I$-validity (so it is not a validity scheme). Denote its $\sim$-logic counterpart formula by $\varphi^{\sim}$. Now assume that $\varphi^{\sim}$ is not a validity. Under this assumption, we show how to construct a model on which $\varphi^{\wr^I}$ is not valid, thereby proving the theorem. If $\varphi^{\sim}$ is not a validity, it follows that there is a model $\mathcal{M}$ and a state s such that $\mathcal{M}, s \not\models \varphi^{\sim}$. Define $aa(\varphi^{\sim})$ to be the set of atomic actions occurring in the formula $\varphi^{\sim}$. Now construct the model $\mathcal{M}'$, by adding to $\mathcal{M}$ the interpretation of an atomic action r such that $r \notin aa(\varphi^{\sim})$ and $R^{\mathcal{A}}(r) = S \times S \setminus any^I$. It follows that $\mathcal{M}', s \not\models \varphi^{\sim}$, because the addition of the new relation does not alter the truth-condition for $\varphi^{\sim}$: (1) all actions in $aa(\varphi^{\sim})$ are equally interpreted in $\mathcal{M}$ and $\mathcal{M}'$, and (2) all action connectives are equally interpreted, in particular, addition of $R^{\mathcal{A}}(r)$ does not in any way change the interpretation of actions of the form $\sim \alpha$, because the universal relation is not affected, (3) all valuations of propositions remain as they are. But now, due to the addition of the extra relation, for any a such that $a \in aa(\varphi^{\sim})$, the interpretation of the relation $\wr^I a$ on $\mathcal{M}'$, is exactly the same as the interpretation of $\sim a$ on $\mathcal{M}$. It is easy to see that this does not only hold for the interpretation of negated atomic actions, but that the interpretation of any complex action $\wr^I \alpha$ on $\mathcal{M}'$, is exactly the same as the interpretation of $\sim \alpha$ on $\mathcal{M}$. But then, from the information that $\mathcal{M}'$ and $\mathcal{M}$ agree on all valuations of atomic proposition, it follows that the truth conditions for a formula $\varphi^{\wr^I}$ on $\mathcal{M}'$ is necessarily equal to the truth condition for $\varphi^{\sim}$ on $\mathcal{M}$. So it holds that $\mathcal{M}', s \not\models \varphi^{\wr^I}$. So there is a model, namely $\mathcal{M}'$, for which the formula $\varphi^{\wr^I}$ is not valid. This contradicts the assumption we started off with. ■

A second theorem gives inclusion relations between separate $\wr^I$-logics.

THEOREM 13.

Under replacement of complements in formulas, the following inclusion relation between $\wr^I$-logics holds. In the picture, logics are represented by the type of complement they endorse, and arrows denote inclusion of validities.

$$\begin{tikzcd}
 & \wr^{K4} \arrow[r] \arrow[ddr] & \wr^{S4} \arrow[dr] & \\
\wr^{K} \arrow[ur] \arrow[dr] & & & \wr^{S5} \\
 & \wr^{B} \arrow[r] & \wr^{B4} \arrow[ur] &
\end{tikzcd}$$

Proof. We can use the same proof strategy as for theorem 12. If we want to prove that for a logic encompassing the complement $\wr^I$, the validities are included in a logic encompassing the complement $\wr^j$, we make use of the addition of a relation $R^{\mathcal{A}}(r)$ for an atomic action r, defined as $R^{\mathcal{A}}(r) = any^j \setminus any^I$. On all other aspects the proofs are completely analogous to the proof of theorem 12. This implies that inclusion of logics is determined by inclusion of the respective complement spaces. So the above partial order simply follows definition 8 by inspecting the inclusion relations between the spaces with respect to which the respective complements are defined. ■

It is clear right away that the above inclusions are strict, since stronger logics encompass supersets of non-syntactically definable action combinators. But it is illustrative to see how the logics can also be distinguished by validity schemes concerning action combinators that are available in (almost) all relativized complement logics, such as the $\cup$, the $\wr^I$ and the ;. First we note that many validity schemes are shared by all $\wr^I$-logics and $\sim$-logics. One example is the property K-R:

PROPOSITION 14.

For all $\wr^I$-logics we have the following validity scheme.

$$\textit{K-R} \qquad \models \langle\alpha\rangle\varphi \wedge [\wr^I\beta]\neg\varphi \rightarrow \langle\alpha \cap \beta\rangle\varphi$$

Proof. Consider an arbitrary model $\mathcal{M}$ and an arbitrary state s. In case that $\mathcal{M}, s \not\models \langle\alpha\rangle\varphi$ or $\mathcal{M}, s \not\models [\wr^I\beta]\neg\varphi$ it holds trivially that $\mathcal{M}, s \models \langle\alpha\rangle\varphi \wedge [\wr^I\beta]\neg\varphi \rightarrow \langle\alpha \cap \beta\rangle\varphi$. So, we assume that (1) $\mathcal{M}, s \models \langle\alpha\rangle\varphi$ and (2) $\mathcal{M}, s \models [\wr^I\beta]\neg\varphi$. From (1) it follows that $\alpha \neq \mathit{fail}$ and that there is a state t such that $(s,t) \in R(\alpha)$ and (1') $\mathcal{M}, t \models \varphi$. Now either $(s,t) \in R(\beta)$ or $(s,t) \in R(\wr^I\beta)$ ($\wr^I\beta \cup \beta$ covers all states reachable through complex action). But from $(s,t) \in R(\wr^I\beta)$ and property 2 we would have to conclude that $\mathcal{M}, t \models \neg\varphi$, which contradicts 1'. Then, from $(s,t) \in R(\beta)$ and $(s,t) \in R(\alpha)$ it follows that $(s,t) \in R(\alpha\cap\beta)$, which together with 1' gives $\mathcal{M}, s \models \langle\alpha\cap\beta\rangle\varphi$. ■

Other properties are valid for some $\wr^I$-logics and invalid for others. We mention the following relevant validity schemes for $\wr^I$-logics:

Trans-any	$\models \langle any^i\rangle\langle any^i\rangle\varphi \rightarrow \langle any^i\rangle\varphi$
Symm-any	$\models \varphi \rightarrow [any^i]\langle any^i\rangle\varphi$
Refl-any	$\models \varphi \rightarrow \langle any^i\rangle\varphi$
NegSeq-R	$\models \langle\alpha\rangle[\wr^i\beta]\varphi \rightarrow [\wr^i(\alpha;\beta)]\varphi$

Now the following proposition lists a table saying which properties hold for which logic.

PROPOSITION 15 (validity schemes of $\wr^I$-logics).

MAL variant(s)	*Trans-any*	*Symm-any*	*Refl-any*	*NegSeq-R*
$(\cup, \wr^K)$	*no*	*no*	*no*	*no*
$(^\leftarrow, \cup, \wr^B)$	*no*	*yes*	*no*	*no*
$(;, \cup, \wr^{K4})$	*yes*	*no*	*no*	*no*
$(^*, ;, \cup, \wr^{S4})$	*yes*	*no*	*yes*	*no*
$(^\leftarrow, ;, \cup, \wr^{B4})$	*yes*	*yes*	*no*	*yes*
$(^*, ^\leftarrow, ;, \cup, \wr^{S5})$	*yes*	*yes*	*yes*	*yes*

Proof. We only verify the soundness of the scheme NegSeq-R for the strongest logic $DL(^*, ^\leftarrow, ;, \cup, \wr^{S5})$. We prove NegSeq-R from negative demonstration: assume there is a model $\mathcal{M}$ with a state s such that (1) $\mathcal{M}, s \models$

$\langle\alpha\rangle[\wr^{S5}\beta]\varphi$ and (2) $\mathcal{M}, s \models \neg[\wr^{S5}(\alpha;\beta)]\varphi$. From (1) it follows that $\alpha \neq fail$ and that there is a state t such that $(s,t) \in R(\alpha)$ and (1') $\mathcal{M}, t \models [\wr^{S5}\beta]\varphi$. Property 2 is equivalent with (2') $\mathcal{M}, s \models \langle\wr^{S5}(\alpha;\beta)\rangle\neg\varphi$. Now there are two cases: $\beta = fail$ or $\beta \neq fail$.

If $\beta = fail$ property 2' reduces to $\mathcal{M}, s \models \langle any^{S5}\rangle\neg\varphi$. This is in direct contradiction with semantic property 1', that for $\beta = fail$ reduces to $\mathcal{M}, t \models [any^{S5}]\varphi$. Since the any^{S5} reaches over all connected states, including the ones reachable by converse action, the contradiction strikes notwithstanding the circumstance that this second validity holds for another state. This is also why the property NegSeq-R does not hold for $DL(^*, ; , \cup, \wr^{S4})$: the action any^{S4} does not have this 'global' power.

If $\beta \neq fail$, there is a state u such that $(s,u) \in R(\wr^{S5}(\alpha;\beta))$ and (3) $\mathcal{M}, u \models \neg\varphi$. Now either $(t,u) \in R(\beta)$ or $(t,u) \in R(\wr^{S5}\beta)$ (again, $\wr^{S5}\beta \cup \beta$ reaches all connected states). But from $(t,u) \in R(\wr^{S5}\beta)$ and property 1' we would have to conclude that $\mathcal{M}, u \models \varphi$, which contradicts 3. And from $(t,u) \in R(\beta)$ together with $(s,t) \in R(\alpha)$ we get $(s,u) \in R(\alpha;\beta)$, which directly contradicts $(s,u) \in R(\wr^{S5}(\alpha;\beta))$. ■

It might come as a surprise that the scheme $\varphi \rightarrow \langle any^i\rangle\varphi$, representing reflexivity, only holds in the logic with $\wr^{S4}$ and $\wr^{S5}$, and not for the logic with $\wr^{B4}$. The complement space for the logic with $\wr^{B4}$ is both transitive and symmetric. Transitivity and symmetry together imply reflexivity. But in the present setting, transitivity and symmetry only holds for the reachable relation space, which results in a relativized reflexivity with respect to the complete state space: reflexivity only holds for worlds from which some action is possible.

The table not only lists properties that distinguish $\wr^I$-logics mutually, it also forms a comparison of $\wr^I$-logics with their $\sim$-logic counterparts. For instance, it shows that $DL(\cup, \wr^K)$ is considerably weaker than its sibling $DL(\cup, \sim)$, because it does not support transitivity, or reflexivity or symmetry for the any^K. But the logics lower in the table are closer to their $\sim$-logic counterparts. And for the strongest logic we can prove the following theorem.

THEOREM 16. *Under syntactic interchange of occurrences of $\wr^{S5}$ and $\sim$, the logics $DL(\wr^{S5}, \cup, ;, ^*, ^{\leftarrow})$ and $DL(\sim, \cup, ;, ^*, ^{\leftarrow})$ are equivalent, that is, they encompass the same set of validities.*

Proof. Theorem 12 states that under syntactic interchange of occurrences of $\wr^{S5}$ and $\sim$, it holds that $DL(\wr^{S5}, \cup, ;, ^*, ^{\leftarrow}) \subseteq DL(\sim, \cup, ;, ^*, ^{\leftarrow})$. Here we have to prove that also $DL(\wr^{S5}, \cup, ;, ^*, ^{\leftarrow}) \supseteq DL(\sim, \cup, ;, ^*, ^{\leftarrow})$. Again we rely on a proof through negative demonstration. Assume that $\varphi^{\sim}$ is a $\sim$-validity. Denote its $\wr^{S5}$-logic counterpart formula by $\varphi^{\wr^{S5}}$. Now assume that $\varphi^{\wr^{S5}}$ is not a validity. Under this assumption, we show how to construct a model on which $\varphi^{\sim}$ is not valid, thereby proving the theorem. If $\varphi^{\wr^{S5}}$ is not a validity, it follows that there is a model $\mathcal{M}$ and a state s such that $\mathcal{M}, s \not\models \varphi^{\wr^{S5}}$. Define $aa(\varphi^{\wr^{S5}})$ to be the set of atomic actions occurring in the formula $\varphi^{\wr^{S5}}$. Now construct the model $\mathcal{M}'$, by contraction to the generated submodel of state s (simply remove all non-reachable states, where reachability also accounts for the converse direction of relations). It follows that $\mathcal{M}', s \not\models \Phi^{\wr^{S5}}$, since the truth function does not get another value by this contraction. But now, for any a such that $a \in aa(\varphi^{\wr^{S5}})$, the interpretation of the relation $\sim a$ on $\mathcal{M}'$, corresponds one to one to the interpretation of $\wr^{S5}a$ on $\mathcal{M}$. It is easy to see that actually the interpretation of any complex relation and well-formed formula using $\sim$ on $\mathcal{M}'$ corresponds one to one to the interpretation of a complex relation and well-formed formula using $\wr^{S5}$ on $\mathcal{M}$. But then it holds that $\mathcal{M}', s \not\models \varphi^{\sim}$, where $\varphi^{\sim}$ corresponds to $\varphi^{\wr^{S5}}$ with $\sim$ in place for $\wr^{S5}$. It follows that there is a model $\mathcal{M}'$ for which the scheme $\varphi^{\sim}$ is not valid. This contradicts the assumption we started off with. ■

4.3 Expressiveness

We developed $\wr^{I}$-logics specifically to reason about concurrency, equivalence and complement of action. And we want the logics to be strong enough to capture all the relevant aspects of the reasoning with these concepts. It is well-known that intersection frames are not definable in standard modal logics (see e.g. [16]). This points to an inadequacy of standard modal logics for reasoning about concurrency, because it shows that not all consequences of the decision to model concurrency through intersection are 'felt' at the level of validities. $\wr^{I}$-logics do not suffer this inadequacy since they are strong enough to define intersection, action equivalence and action complement both at the level of frames *and* at the level of models. In particular, action equivalence is defined at the level of models by the formula $[\wr^{I}(\alpha \doteqdot \beta)]\bot$ saying that any two actions α and β whose relational interpretations do not coincide, are not possible.

4.4 Reasoning about action using the relativized complement

To stress the intuitive adequacy of the complement to reason about action, we show how it can be used in the three application domains we mentioned in the introduction. First we show that it can be used to encode Reiter's approach to the representation of frame formulas. We argued that the universal complement is not suitable for the encoding of Reiter's solution to the frame problem in modal action logics, because the expression $\neg\varphi \rightarrow [\sim\alpha]\neg\varphi$ does not only say that actions other than (not involving) α cannot have φ as a result, it says in addition that all states in the state-space that embody the result φ can actually be reached by α. The relativized complement version of this expression, i.e. $\neg\varphi \rightarrow [\wr^I \alpha]\neg\varphi$ does not have this unintended additional consequence: only for the states that are reachable through complex action it holds that if φ holds in them, they can be reached by α. Note that we can introduce a relativized version of the window operator ([1, 17]) to abbreviate $[\wr^I \alpha]\neg\varphi$.

To show that the defined complements can be suitably applied to the temporal domain, we define mixed action / time properties using the relativized action complement. To accomplish this, we add modal fixed-point operators to the logic $DL(\wr^K, \cup)$. The resulting μ-calculus (see [6] for more on μ-calculi) enables the expression of mixed dynamic / temporal modalities. We identify 'on all paths the condition φ is preserved until action α is performed' (notation $A(\varphi\, U\, \alpha)$), with 'after any finite number of performances of $\wr^K \alpha$ the condition φ holds'. Through similar identifications we define 'on all paths we keep on performing αs until the condition φ is met' (notation $A(\alpha\, U\, \varphi)$), and 'on all paths we keep on performing αs until the action β is performed' (notation $A(\alpha\, U\, \beta)$). Note that we define non-reflexive versions of these operators.

DEFINITION 17 (Mixed action / time until operators).

$$\begin{array}{lll} A(\varphi\, U\, \alpha) & \equiv_{def} & [\wr^K \alpha]\, \mu Z.\, \varphi \wedge [\wr^K \alpha] Z \\ A(\alpha\, U\, \varphi) & \equiv_{def} & [\wr^K \alpha]\bot \wedge [\alpha]\, \mu Z.\, \neg\varphi \rightarrow ([\wr^K \alpha]\bot \wedge [\alpha] Z) \\ A(\alpha\, U\, \beta) & \equiv_{def} & [\wr^K \alpha]\bot \wedge [\alpha]\, \mu Z.\, [\wr^K (\alpha \cup \beta)]\bot \wedge [\alpha] Z \end{array}$$

Finally, we show how to define deontic modal action logics using the relativized complement (see [7] for details). We will only consider, what we call, 'goal norms'. The class of goal norms takes an action norm to be a norm about the result / goal / effect of an action, while normative repercussions for sequential sub-actions are absent. Thus, goal norms may only be violated in the result state of an action; violations cannot occur in the process of reaching this state. We denote norms of this types with the symbols $O_\odot(\alpha), P_\odot(\alpha)$ and $F_\odot(\alpha)$, where O, P and F stand for Obligation, Permission and Prohibition ('Forbiddeness') respectively, and the subscript '$\odot$' depicts that norms concern action goals. For goal norms it holds, for instance, that each of the formulas $P_\odot(a; b) \wedge \neg P_\odot(a)$, $F_\odot(a) \wedge \neg F_\odot(a; b)$, and $O_\odot(a; b) \wedge \neg O_\odot(a)$ is consistent. Thus, it can be permitted to perform $a; b$, while at the same time a is forbidden. The point is that it is the result of the action that matters: action $a; b$ is permitted, because it brings about a result that is allowed. This view on norms as compliance to action goals was first adopted by Meyer [24]. Meyer identifies the violation of an action goal norm with the validity of a violation proposition V in the resulting state of the action. With the introduction of this special proposition, we may define the following reductions of the modal operators for permission, prohibition and obligation, where (a) free choice permission for α is identified as the absence of a possibility to do α in a way that results in a violation, (b) free choice prohibition for α is identified as the possibility to do α in a way that results in a violation, and (c) free choice obligation for α is identified as the conjunction of free choice permission for α and the occurrence of a violation after any action performance not involving α:

DEFINITION 18 (A reduction to $\wr^I$-logics).

$$\begin{array}{ll} (a) & P_{\odot}(\alpha) \equiv_{def} \neg\langle\alpha\rangle V \\ (b) & F_{\odot}(\alpha) \equiv_{def} \langle\alpha\rangle V \\ (c) & O_{\odot}(\alpha) \equiv_{def} P_{\odot}(\alpha) \wedge [\wr^I \alpha] V \end{array}$$

The reduction of definition 18 is easily adapted in order to satisfy other requirements. For instance, we can adapt the reduction in such a way that we exchange free choice norms for imposed choice norms (see e.g. [18, 7] for the distinction). But the interesting aspect of the above reduction is that it gives a consistent logic of free choice permission. In the deontic logic literature it is generally assumed that the introduction of free choice permission leads to inconsistent or degenerate logics [20].

5 Conclusions and future work

In this paper we argued that the more traditional action complement with respect to the universal relation is not suited for many action representation tasks that use dynamic logics. As example domains of such representation and reasoning tasks, we mentioned the domains of (1) action and time, (2) action and norms (deontic logic), and (3) action and change. For all of these domains, the notion of action complement was argued to be important. As an alternative to the traditional complement we defined a series of dynamic logics exhibiting a relativized action complement. Logics with this complement feature a number of attractive properties: (1) semantic intuitiveness, (2) syntactic definability of the standard intersection connective, and (3) definability of intersection, complement and action equivalence at the level of models.

We also expect better complexity properties for the relativized complement logics (as compared to the complexities of their universal complement logic counterparts). But so far we have not studied complexity issues in any depth. We base our expectation on the general insight that relativization may result in better complexity properties. This is demonstrated by the work of Marx [21] on relativized relation algebras. Relativization can be seen as an adaptation of the semantics of formulas that allows for more models, which should make the satisfaction problem easier. Having more models corresponds with having less validities. The validities of proposition 15 are examples of properties that hold for $\sim$-logics but not necessary for their $\wr^I$-logic counterparts.

To prove complexity results we expect that we can use the local character of the relativized complement. This local character enables us to define generalized tree properties such as defined by Marx and Venema [22]. A particular interesting question is whether the logic $DL(;,\cup,\wr^{K4})$ is decidable. The presence of sequence is likely to cause undecidability. Transitivity introduces undecidability in first order generalizations of modal logic such as the two variable and guarded fragments. But for some traditional incarnations of modal logic that were developed for the temporal and process domains, transitivity is a common feature that does not cause undecidability.

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