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Functional Completeness for a Natural Deduction Formulation of Hybridized S5

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ABSTRACT. In this paper we give a functional completeness result for a natural deduction formulation of hybridized *S5*. Hybridized *S5* is obtained by adding to ordinary *S5* further expressive power in the form of so-called satisfaction operators and a second sort of propositional symbols called nominals.

1 Introduction

In this paper we give a functional completeness result for a natural deduction formulation of hybridized *S5*. We shall begin with making a few general remarks on the topics of hybridized *S5*, natural deduction, and functional completeness.

Hybridized *S5* is a hybrid logical version of *S5* which is obtained by adding to *S5* further expressive power in the form of so-called satisfaction operators and a second sort of propositional symbols called nominals. A nominal is assumed to be true at exactly one world, so a nominal can be considered the name of a world. Thus, in hybrid logic a name is a particular sort of propositional symbol whereas in first-order logic it is an argument to a predicate. Beside nominals, a satisfaction operator $a:$ is added for each nominal a. A satisfaction operator $a:$ makes it possible to express that a formula is true at one particular world, namely the world at which the nominal a is true. The history of hybrid logic goes back to Arthur Prior's work, more precisely, it goes back to what he called four grades of tense logical involvement. They were presented in the book [24], Chapter XI (also Chapter XI in the new edition [14]). See also [23] Chapter V.6 and Appendix B.3–4. Prior introduced nominals and satisfaction operators, and moreover, he introduced the so-called binder $\forall$ which is analogous to the standard first-order universal quantifier. We shall not consider binders in the present paper, however. See [19] for an account of Prior's work. The paper [6] makes a connection to Donald Davidson's notion of a theory of truth.

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Hybridized *S5* has a number of different applications. For example, if hybridized *S5* is extended with further modal operators, then it can be considered a natural generalisation of the so-called *description logics* used in knowledge representation, see [4]. The description logic called $\mathcal{ALC}$ is a notational variant of ordinary multi-modal logic, that is, propositional logic extended with a finite number of modal operators $\Box_1, \ldots, \Box_m$. The *concept expressions* of $\mathcal{ALC}$ simply corresponds to formulas of multi-modal logic and vice versa. Given a description logic, for example $\mathcal{ALC}$, a knowledge base is a set of metalinguistic statements expressing relationships between concepts and individuals. There are two kinds of metalinguistic statements; they are called *TBox-statements* and *ABox-statements* respectively. A TBox-statement $\phi \sqsubseteq \psi$ expresses that the concept ϕ is subsumed by the concept ψ, that is, that any individual that belongs to the extension of ϕ also belongs to the extension of ψ. An ABox-statement $\phi(a)$ expresses that the individual a belongs to the extension of the concept ϕ and an ABox-statement $R_i(a, c)$ expresses that the individual a is R_i-related to the individual c. This can all be expressed in terms of hybridized *S5* extended with further modal operators: The TBox-statement above is expressed by the formula $\Box(\phi \to \psi)$, where $\Box$ is the original *S5* modal operator, and the ABox-statements are expressed by the formulas $a : \phi$ and $a : \Diamond_i c$. Note that no binders are needed. Of course, a nominal is here considered a name of an individual. Thus, hybridized *S5* generalises description logic such that no distinction between an object language and a metalanguage is needed.

Now, natural deduction style inference rules for ordinary classical first-order logic were originally introduced by Gerhard Gentzen in [12] and later on considered by Dag Prawitz in [20, 21]. With reference to Gentzen's work, Prawitz made the following remarks on the significance of natural deduction.

> ...the essential logical content of intuitive logical operations that can be formulated in the languages considered can be understood as composed of the atomic inferences isolated by Gentzen. It is in this sense that we may understand the terminology *natural* deduction.
>
> Nevertheless, Gentzen's systems are also natural in the more superficial sense of corresponding rather well to informal practices; in other words, the structure of informal proofs are often preserved rather well when formalised within the systems of natural deduction. ([21], p. 245)

Natural deduction systems are characterised by having two different kinds of rules for each non-nullary connective; there is a kind of rules which introduces a connective and there is a kind of rules which eliminates a connective. A maximum formula in a derivation is then a formula occurrence that is both introduced by an introduction rule and eliminated by an elimination rule. A maximum formula can be considered a "detour" in the derivation and it can be removed by rewriting the derivation. Such a step is called a reduction step and it is a notable feature of natural deduction systems that they satisfy a so-called normalisation theorem which says that any derivation can be rewritten to a derivation without maximum formulas by repeated applications of reductions. Another notable feature of natural deduction systems is that they correspond closely to Gentzen systems. It can be proved that our natural deduction formulation of hybridized *S5* has both of these features. We shall, however, not consider this further in the present paper.

The notion of functional completeness we shall consider is defined as follows. Let a set of natural deduction inference rules involving a set a connectives Σ be given. Moreover, let a set of general rule-schemas involving a new connective $\sharp$ be given. The general rule-schemas can be instantiated to a set of introduction and elimination rules for $\sharp$. A functional completeness result then says that any such new connective $\sharp$ is explicitly definable in terms of the original connectives Σ in the following sense: For any formula ϕ built using the connectives $\Sigma \cup \{\sharp\}$, there exists a formula ψ built using only the original connectives Σ such that ψ can be proved to be equivalent to ϕ using the original inference rules as well as the introduction and elimination rules for the new connective $\sharp$. Thus, such a functional completeness result says that the original natural deduction system involving only the connectives Σ can define explicitly any connective with introduction and elimination rules of a certain form. Note that this notion of functional completeness is purely proof-theoretic, that is, it makes no reference to model-theoretic notions. A different notion of functional completeness which does refer to model-theoretic notions is exemplified by the result in ordinary classical propositional logic which says that any truth-functional connective is definable in terms of the connectives $\{\neg, \wedge, \top\}$. See [15] for a similar result for intuitionistic propositional logic.

With the aim of proving a functional completeness result for our natural deduction formulation of hybridized *S5*, we shall specify a set of general rule-schemas involving the new connective $\sharp$. The connective $\sharp$ takes as arguments an arbitrary, but fixed, number of nominals and an arbitrary, but again fixed, number of formulas. The rule-schemas are obviously not arbitrary; they satisfy certain requirements, one of the requirements being the so-called inversion principle (which is a prerequisite for obtaining a normalisation theorem).

This paper is structured as follows. In the second section of the paper we recapitulate the basics of hybrid logic, in the third section we introduce the natural deduction system for hybridized *S5*, and in the fourth section we prove the functional completeness theorem. In the final section we discuss related and future work.

2 Hybrid logic

In this section we recapitulate the basics of hybrid logic, and moreover, we define models for hybridized *S5*. We shall in many cases adopt the terminology of [3] and [1]. The hybrid logic we consider is obtained by adding a second sort of propositional symbols called *nominals* to ordinary modal logic. It is assumed that a countably infinite set of ordinary propositional symbols and a countably infinite set of nominals are given. The sets are assumed to be disjoint. The metavariables p, q, r, … range over ordinary propositional symbols and a, b, c, … range over nominals.

Beside nominals, an operator $a:$ called a *satisfaction operator* is added for each nominal a. The formulas of hybrid modal logic are defined by the grammar

$$S \ ::= \ p \mid a \mid S \wedge S \mid S \rightarrow S \mid \bot \mid \Box S \mid a : S$$

where p is an ordinary propositional symbol and a is a nominal. In what follows $\mathcal{H}$ denotes the language, that is, the set of formulas, defined by the grammar. Many hybrid logics involve so-called *binders*, but we shall not consider binders in the present paper. We shall make use of the following conventions. The metavariables ϕ, ψ, θ, … range over formulas. Negation, nullary conjunction, disjunction, and biimplication are defined by the conventions that $\neg\phi$ is an abbreviation for $\phi \rightarrow \bot$, $\top$ is an abbreviation for $\neg\bot$, $\phi \vee \psi$ is an abbreviation for $\neg(\neg\phi \wedge \neg\psi)$, and $\phi \leftrightarrow \psi$ is an abbreviation for $(\phi \rightarrow \psi) \wedge (\psi \rightarrow \phi)$. Similarly, $\Diamond\phi$ is an abbreviation for $\neg\Box\neg\phi$. If $\overline{a}$ is a list of pairwise distinct nominals and $\overline{c}$ is a list of nominals of the same length as $\overline{a}$, then $\psi[\overline{c}/\overline{a}]$ is the formula ψ where the nominals $\overline{c}$ have been simultaneously substituted for all occurrences of the nominals $\overline{a}$. An analogous definition is obtained if $\overline{a}$ is replaced by a list of ordinary propositional symbols and $\overline{c}$ is replaced by a list of formulas.

We now define models for hybridized *S5* (note, however, that the functional completeness result we shall prove later do not refer to any model-theoretic notions). A *model* is a tuple (W, V) where W is a non-empty set and V is a *valuation*, that is, V is a function that to each pair consisting of an element of W and an ordinary propositional symbol assigns an element of $\{0, 1\}$. The elements of W are called *worlds*. An *assignment* for a model (W, V) is a function g that to each nominal assigns an element of W. Given a model $\mathcal{M} = (W, V)$, the relation $\mathcal{M}, g, w \models \phi$ is defined by induction, where g is an assignment, w is an element of W, and ϕ is a formula.

$$
\begin{array}{rcl}
\mathcal{M}, g, w \models p & \text{iff} & V(w,p) = 1 \\
\mathcal{M}, g, w \models a & \text{iff} & w = g(a) \\
\mathcal{M}, g, w \models \phi \wedge \psi & \text{iff} & \mathcal{M}, g, w \models \phi \text{ and } \mathcal{M}, g, w \models \psi \\
\mathcal{M}, g, w \models \phi \to \psi & \text{iff} & \mathcal{M}, g, w \models \phi \text{ implies } \mathcal{M}, g, w \models \psi \\
\mathcal{M}, g, w \models \bot & \text{iff} & \text{falsum} \\
\mathcal{M}, g, w \models \Box\phi & \text{iff} & \text{for any } v \text{ in } W,\ \mathcal{M}, g, v \models \phi \\
\mathcal{M}, g, w \models a : \phi & \text{iff} & \mathcal{M}, g, g(a) \models \phi
\end{array}
$$

Of course, it is the quantification over all worlds in the clause for $\Box$ that makes this modality an *S5* modality. The quantification over all worlds also explains why the modality often is called the *universal* modality. A formula ϕ is said to be *true* at w if $\mathcal{M}, g, w \models \phi$; otherwise it is said to be *false* at w. By convention $\mathcal{M}, g \models \phi$ means $\mathcal{M}, g, w \models \phi$ for every element w of W.

3 Natural deduction for hybrid logic

Before considering natural deduction inference rules for hybridized *S5*, we shall sketch the basics of natural deduction. See [20] and [25] for further details.

A characteristic feature of natural deduction is that derivations have the form of trees. Formula occurrences at the leaves of a tree are called *assumptions* and the formula occurrence at the root of the tree is called the *end-formula*. All assumptions are annotated with numbers. An assumption is either *undischarged* or *discharged*. If an assumption is discharged, then it is discharged at a specified rule-instance which is indicated by annotating the assumption and the rule-instance with identical numbers. We shall often omit this information when no confusion can occur. A rule-instance annotated with some number discharges all undischarged assumptions that are above it and are annotated with the number in question, and moreover, are occurrences of a specified formula. Two assumptions in a derivation belong to the same *parcel* if they are annotated with the same number and are occurrences of the same formula, and moreover, either are both undischarged or have both been discharged at the same rule-instance. Thus, in this terminology rules discharge parcels. We shall make use of the standard notations

$$
\begin{array}{c}
[\phi^r] \\
\vdots\, \pi \\
\psi
\end{array}
\qquad\qquad\qquad
\begin{array}{c}
\vdots\, \tau \\
\phi \\
\vdots\, \pi \\
\psi
\end{array}
$$

which means a derivation π where ψ is the end-formula and $[\phi^r]$ is the parcel consisting of all undischarged assumptions that have the form ϕ^r,

$$\frac{c:\phi \quad c:\psi}{c:(\phi\wedge\psi)}\,(\wedge I) \qquad \frac{c:(\phi\wedge\psi)}{c:\phi}\,(\wedge E1) \quad \frac{c:(\phi\wedge\psi)}{c:\psi}\,(\wedge E2)$$

$$\begin{array}{c}[c:\phi]\\ \vdots \\ \dfrac{c:\psi}{c:(\phi\rightarrow\psi)}\,(\rightarrow I)\end{array} \qquad \frac{c:(\phi\rightarrow\psi) \quad c:\phi}{c:\psi}\,(\rightarrow E)$$

$$\frac{a:\phi}{c:a:\phi}\,(:I) \qquad \frac{c:a:\phi}{a:\phi}\,(:E)$$

$$\frac{b:\phi}{c:\Box\phi}\,(\Box I)^{*} \qquad \frac{c:\Box\phi}{f:\phi}\,(\Box E)$$

$$\begin{array}{c}[a:\neg\phi]\\ \vdots \\ \dfrac{a:\bot}{a:\phi}\,(\bot 1)^{\star}\end{array} \qquad \frac{a:\bot}{c:\bot}\,(\bot 2)$$

$$\frac{}{a:a}\,(\mathit{Ref}) \qquad \frac{a:c \quad a:\phi}{c:\phi}\,(\mathit{Nom}1)^{\star}$$

$*$ b does not occur in $c:\Box\phi$ or in any undischarged assumptions.
$\star$ ϕ is a propositional symbol.

Figure 1. Inference rules for $\mathbf{N}_{\mathcal{H}}$

and moreover, a derivation π where ψ is the end-formula and a derivation τ with end-formula ϕ has been substituted for each of the undischarged assumptions indicated by $[\phi^r]$. A derivation in a natural deduction system is generated from a set of inference rules from derivations consisting of a single undischarged assumption.

Another characteristic feature of natural deduction is that there are two different kinds of rules for each non-nullary connective; there is a kind of rules called *introduction rules* which introduce a connective (that is, the connective occurs in the conclusion of the rule, but not in the premises) and there is a kind of rules called *elimination rules* whicheliminate a connective (the connective occurs in the premises of the rule, but not in the conclusion). Introduction rules traditionally have names of the form $(\ldots I \ldots)$, and similarly, elimination rules traditionally have names of the form $(\ldots E \ldots)$.

We shall make use of the following conventions. The metavariables π, τ, $\ldots$ range over derivations. Formulas of the form $a : \phi$ are called *satisfaction statements*, cf. a similar notion in [2]. The metavariables Γ, Δ, $\ldots$ range over sets of satisfaction statements. A derivation π is a *derivation of* ϕ if the end-formula of π is an occurrence of ϕ and π is a *derivation from* Γ if each undischarged assumption in π is an occurrence of a formula in Γ (note that numbers annotating undischarged assumptions are ignored). If there exists a derivation of ϕ from $\emptyset$, then we simply say that ϕ is *derivable*. Moreover, $\pi[\overline{c}/\overline{a}]$ is the derivation π where each formula occurrence ψ has been replaced by $\psi[\overline{c}/\overline{a}]$.

Now, natural deduction inference rules for hybridized *S5* are given in Figure 1. The system will be denoted $\mathbf{N}_{\mathcal{H}}$. All formulas in the rules are satisfaction statements. Note that the rules $(\bot 1)$ and $(\bot 2)$ are neither introduction rules nor elimination rules (recall that $\neg\phi$ is an abbreviation for $\phi \rightarrow \bot$).

The system $\mathbf{N}_{\mathcal{H}}$ is sound and complete in the usual model-theoretic sense.

THEOREM 1 *The statements below are equivalent.*

1. *ϕ is derivable from Γ in $\mathbf{N}_{\mathcal{H}}$.*

2. *For any model $\mathcal{M}$ and any assignment g, if, for any formula $\psi \in \Gamma$, $\mathcal{M}, g \models \psi$, then $\mathcal{M}, g \models \phi$.*

Proof. Soundness and completeness of the system given in the present paper follows from the soundness and completeness result given in [7]. It is straightforward to prove that the system given here is equivalent to a mono-modal version of the system given in [7] extended with the inference rule corresponding to the condition $\forall a \forall b R(a, b)$ on the accessibility relation R. ■

Below are two examples of derivations in $\mathbf{N}_{\mathcal{H}}$.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{a : \Box(\phi \to \psi)^2}{b : (\phi \to \psi)}\,(\Box E) \qquad \dfrac{a : \Box\phi^1}{b : \phi}\,(\Box E)}{b : \psi}\,(\to E)}{a : \Box\psi}\,(\Box I)}{a : (\Box\phi \to \Box\psi)}\,(\to I)^1}{a : (\Box(\phi \to \psi) \to (\Box\phi \to \Box\psi))}\,(\to I)^2 \qquad\qquad \dfrac{\dfrac{\dfrac{\dfrac{a : \Box\neg c^1}{c : \neg c}\,(\Box E) \qquad \dfrac{}{c : c}\,(Ref)}{c : \bot}\,(\to E)}{a : \bot}\,(\bot 2)}{a : \Diamond c}\,(\to I)^1$$

The end-formula of the derivation on the left hand side is the standard modal axiom K prefixed by a satisfaction operator. The nominal b in this derivation is new. In connection with the right hand side derivation, recall that the formulas $a : \Diamond c$ and $a : (\Box\neg c \to \bot)$ are identical.

4 Functional completeness

In this section we shall prove the functional completeness result: It considers a natural deduction system for the language obtained by extending $\mathcal{H}$ with a new connective $\sharp$ that takes as arguments an arbitrary, but fixed, number of nominals and an arbitrary, but again fixed, number of formulas. The extended language will be denoted $\mathcal{H}(\sharp)$ and the corresponding natural deduction system, which is obtained by extending $\mathbf{N}_{\mathcal{H}}$ with appropriate introduction and elimination rules for the connective $\sharp$, will be denoted $\mathbf{N}_{\mathcal{H}(\sharp)}$.

Now, what does it mean to be an appropriate introduction or elimination rule? We answer this question by requiring that the introduction and elimination rules are instances of general rule-schemas for such rules. These schemas are obviously not arbitrary; they satisfy the following requirements.

1. Besides satisfaction operators, the schemas exhibit no other connectives than $\sharp$.

2. Each schema exhibits exactly one occurrence of the connective $\sharp$ such that the schema for introduction rules exhibits $\sharp$ in the conclusion and the schemas for elimination rules exhibit $\sharp$ in the premises.

3. The schemas satisfy the so-called inversion principle: A derivation of the conclusion of an elimination rule is already "contained" in the derivations of the premises if the major premiss is the conclusion of an introduction rule. (The major premiss of an elimination rule is the premiss that exhibits the connective being eliminated, in this case $\sharp$.)

The history of the inversion principle goes back to [20]. In the formulation of the inversion principle above it is not made explicit what is meant by

$$\dfrac{\begin{array}{ccc} [\mathbf{a}_{11}:\boldsymbol{\phi}_{11}]\ldots[\mathbf{a}_{1k_1}:\boldsymbol{\phi}_{1k_1}] & & [\mathbf{a}_{n1}:\boldsymbol{\phi}_{n1}]\ldots[\mathbf{a}_{nk_n}:\boldsymbol{\phi}_{nk_n}] \\ \vdots & & \vdots \\ \mathbf{b}_1:\boldsymbol{\psi}_1 & \ldots & \mathbf{b}_n:\boldsymbol{\psi}_n \end{array}}{\mathbf{c}:\sharp(\overline{\mathbf{e}_1},\ldots,\overline{\mathbf{e}_n},\overline{\boldsymbol{\phi}_1},\ldots,\overline{\boldsymbol{\phi}_n},\overline{\boldsymbol{\psi}},\overline{\boldsymbol{\theta}})}\,(\sharp I)^*$$

$$\dfrac{\mathbf{c}:\sharp(\overline{\mathbf{e}_1},\ldots,\overline{\mathbf{e}_n},\overline{\boldsymbol{\phi}_1},\ldots,\overline{\boldsymbol{\phi}_n},\overline{\boldsymbol{\psi}},\overline{\boldsymbol{\theta}}) \quad \mathbf{a}_{i1}[\overline{\mathbf{f}_i}/\overline{\mathbf{d}_i}]:\boldsymbol{\phi}_{i1}\ldots\mathbf{a}_{ik_i}[\overline{\mathbf{f}_i}/\overline{\mathbf{d}_i}]:\boldsymbol{\phi}_{ik_i}}{\mathbf{b}_i[\overline{\mathbf{f}_i}/\overline{\mathbf{d}_i}]:\boldsymbol{\psi}_i}\,(\sharp Ei)$$

$*$ For each $i \in \{1,\ldots,n\}$, the nominals $\overline{\mathbf{d}_i}$ are pairwise distinct and do not occur in $\mathbf{c}:\sharp(\overline{\mathbf{e}_1},\ldots,\overline{\mathbf{e}_n},\overline{\boldsymbol{\phi}_1},\ldots,\overline{\boldsymbol{\phi}_n},\overline{\boldsymbol{\psi}},\overline{\boldsymbol{\theta}})$ or in any undischarged assumptions other than the specified occurrences of $\mathbf{a}_{i1}:\boldsymbol{\phi}_{i1},\ldots,\mathbf{a}_{ik_i}:\boldsymbol{\phi}_{ik_i}$.

Figure 2. The rule-schemas for $\mathbf{N}_{\mathcal{H}(\sharp)}$

requiring that some derivations "contain" a derivation of a certain formula, but it means that a derivation of the formula in question can be obtained by composition of derivations and by substitution of nominals for nominals in derivations. The rules for the standard connective $\wedge$ is an instructive example of introduction and elimination rules that satisfy the inversion principle: If the premiss of an instance of $(\wedge E1)$ is the conclusion of an introduction rule, necessarily the rule $(\wedge I)$, then the conclusion of $(\wedge E1)$ already occurs as premiss of the instance of $(\wedge I)$. Note that neither composition (of derivations) nor substitution (of nominals for nominals) is needed in this case.

Recall from the introduction of the paper that a formula occurrence that is both introduced by an introduction rule and eliminated by an elimination rule is called a maximum formula, thus, the fact that the inversion principle is satisfied suggests that such maximum formulas can be dispensed with, and this is indeed what a normalisation theorem says. We shall, however, not consider this issue further.

4.1 Rule-schemas for $\mathbf{N}_{\mathcal{H}(\sharp)}$

Below we shall give the technical details of the general rule-schemas for introduction and elimination rules of $\mathbf{N}_{\mathcal{H}(\sharp)}$. In the formulation of the rule-schemas, we shall make use of metametavariables $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, ...and $\boldsymbol{\phi}$, $\boldsymbol{\psi}$, $\boldsymbol{\theta}$, ...that range over respectively the metavariables a, b, c, ...(that in turn range over nominals) and the metavariables ϕ, ψ, θ, ...(that in turn range over formulas). Note that metametavariables are in boldface. The rule-schemas are given in Figure 2 where for each $i \in \{1,\ldots,n\}$,

$\overline{\mathbf{d}_i} = \mathbf{d}_{i1}, \ldots, \mathbf{d}_{iq_i}$, $\overline{\mathbf{e}_i} = \mathbf{e}_{i1}, \ldots, \mathbf{e}_{ir_i}$, and, $\overline{\boldsymbol{\phi}_i} = \boldsymbol{\phi}_{i1}, \ldots, \boldsymbol{\phi}_{ik_i}$, and moreover, where $\overline{\boldsymbol{\theta}} = \boldsymbol{\theta}_1, \ldots, \boldsymbol{\theta}_m$ simply are "dummy" arguments of $\sharp$. In Figure 2 we have made use of substitution of metavariables (that range over nominals) for metavariables (that also range over nominals). This is defined in analogy to the usual substitution operation where nominals are substituted for nominals. The system $\mathbf{N}_{\mathcal{H}(\sharp)}$ is obtained by extending $\mathbf{N}_{\mathcal{H}}$ with introduction and elimination rules for $\sharp$ that are instances of the schemas such that

1. the metavariables $\mathbf{c}, \overline{\mathbf{d}_1}, \ldots, \overline{\mathbf{d}_n}, \overline{\mathbf{f}_1}, \ldots, \overline{\mathbf{f}_n}, \overline{\mathbf{e}_1}, \ldots, \overline{\mathbf{e}_n}$ are pairwise distinct; and

2. $\{\mathbf{a}_{i1}, \ldots, \mathbf{a}_{ik_i}, \mathbf{b}_i\} \subseteq \{\mathbf{d}_{i1}, \ldots, \mathbf{d}_{iq_i}, \mathbf{e}_{i1}, \ldots, \mathbf{e}_{ir_i}, \mathbf{c}\}$ where $i \in \{1, \ldots, n\}$.

Note that the new connective $\sharp$ takes as arguments $r_1 + \ldots + r_n$ nominals and $k_1 + \ldots + k_n + n + m$ formulas, and also, note that there is only one schema for introduction rules but n schemas for elimination rules.

The reason why we make use of metametavariables in the definition of the rule-chemas is that we wish to be able to distinguish between (i) the case where the same metavariable occurs more than once in an inference rule, for example the rule $(\prec I)$ below where the metavariable d occurs twice, and (ii) the case where different metavariables are instantiated to the same nominal or formula, for example if the nominal c is identical to the nominal a in an instance of the rule $(: I)$ given in Figure 1. Thus, metametavariables and rule-schemas are used to speak about metavariables and rules whereas metavariables and rules are used to speak about variables and rule-instances.

The rule-schemas given in Figure 2 obviously satisfy the first two requirements listed above and it is straightforward to check that the inversion principle is satisfied. This follows from the observation that a derivation in which an occurrence of the connective $\sharp$ is both introduced by an instance of $(\sharp I)$ and eliminated by an instance of $(\sharp Ei)$ can be rewritten such that the occurrence of $\sharp$ disappears. That is, a derivation of the form

$$\dfrac{\dfrac{\begin{matrix}[a_{11}{:}\phi_{11}]\ldots[a_{1k_1}{:}\phi_{1k_1}] & & [a_{n1}{:}\phi_{n1}]\ldots[a_{nk_n}{:}\phi_{nk_n}] \\ \vdots\ \pi_1 & & \vdots\ \pi_n \\ b_1 : \psi_1 & \ldots & b_n : \psi_n\end{matrix}}{c : \sharp(\overline{e_1}, \ldots, \overline{e_n}, \overline{\phi_1}, \ldots, \overline{\phi_n}, \overline{\psi}, \overline{\theta})} \quad \begin{matrix}\vdots\ \tau_1 \\ a_{i1}[\overline{f_i}/\overline{d_i}]{:}\phi_{i1}\end{matrix} \ldots \begin{matrix}\vdots\ \tau_{k_i} \\ a_{ik_i}[\overline{f_i}/\overline{d_i}]{:}\phi_{ik_i}\end{matrix}}{b_i[\overline{f_i}/\overline{d_i}] : \psi_i}$$

can be rewritten to

$$\begin{array}{ccc} \vdots\, \tau_1 & & \vdots\, \tau_{k_i} \\ a_{i1}[\overline{f_i}/\overline{d_i}] : \phi_{i1} & \ldots & a_{ik_i}[\overline{f_i}/\overline{d_i}] : \phi_{ik_i} \\ & \vdots\, \pi_i[\overline{f_i}/\overline{d_i}] & \\ & b_i[\overline{f_i}/\overline{d_i}] : \psi_i & \end{array}$$

where the metavariables for nominals $\overline{\mathbf{d}_i}$ and $\overline{\mathbf{f}_i}$ are instantiated to the nominals $\overline{d_i}$ and $\overline{f_i}$ (see Figure 2).

We shall need the convention and the proposition below. First the convention. The *degree* of a formula is the number of occurrences of connectives in it which is different from $\perp$. Then the proposition.

PROPOSITION 2 *The rules*

$$\dfrac{\begin{array}{c}[a : \neg\phi] \\ \vdots \\ a : \perp\end{array}}{a : \phi}\,(\perp) \qquad\qquad \dfrac{a : d \quad a : \phi}{d : \phi}\,(Nom)$$

are admissible in $\mathbf{N}_{\mathcal{H}(\sharp)}$.

Proof. The proof, which makes use of the notion of the degree of a formula, is along the lines of a similar proof given in [7]. ∎

Note in the proposition above that ϕ can be any formula; not just a propositional symbol. Thus, the rule $(\perp)$ generalises the rule $(\perp 1)$ whereas (Nom) generalises $(Nom1)$. The side-conditions on the rules $(\perp 1)$ and $(Nom1)$ enables us to prove a normalisation theorem such that normal derivations satisfy a version of the subformula property. We shall, however, not consider this issue further in the present paper. In any case, the versions of the rules $(\perp 1)$ and $(Nom1)$ without side-conditions, that is, the rules $(\perp)$ and (Nom), are admissible according to the proposition above. We shall also need the replacement lemma below.

LEMMA 3 *(Replacement) If for any nominal c, $c : (\psi \leftrightarrow \theta)$ is derivable in $\mathbf{N}_{\mathcal{H}(\sharp)}$, then for any formula ϕ and any nominal d, $d : (\phi[\psi/q] \leftrightarrow \phi[\theta/q])$ is derivable in $\mathbf{N}_{\mathcal{H}(\sharp)}$.*

Proof. Induction on the degree of ϕ. ∎

4.2 Examples of rules

It is straightforward to check that the ordinary introduction and elimination rules for the connectives $\wedge$, $\rightarrow$, :, and $\Box$ given in Figure 1 are instances of the rule-schemas.

Another example of instances of the rule-schemas are the rules

$$\frac{\begin{array}{c}[d:\phi]\\ \vdots\\ d:\psi\end{array}}{c:(\phi\prec\psi)}\,(\prec I) \qquad\qquad \frac{c:(\phi\prec\psi) \quad f:\phi}{f:\psi}\,(\prec E)$$

for the binary connective $\prec$ where the rule $(\prec I)$ is equipped with the side-condition that the nominal d does not occur in $c : (\phi \prec \psi)$ or in any undischarged assumptions other than the specified occurrences of $d : \phi$ (note that we have used infix notation for the new connective instead of the usual prefix notation). The connective $\prec$ is called *strict implication* and $\phi \prec \psi$ is provably equivalent to $\Box(\phi \rightarrow \psi)$ in the sense of Lemma 5, that is, $c : ((\phi \prec \psi) \leftrightarrow \Box(\phi \rightarrow \psi))$ is derivable in $\mathbf{N}_{\mathcal{H}(\prec)}$ for any nominal c.

An example of rules that do not fit into the rule-schemas are the rules

$$\frac{c:\phi}{c:(\phi\vee\psi)}\,(\vee I1) \qquad \frac{c:\psi}{c:(\phi\vee\psi)}\,(\vee I2) \qquad \frac{c:(\phi\vee\psi) \quad \begin{array}{c}[c:\phi]\\ \vdots\\ \theta\end{array} \quad \begin{array}{c}[c:\psi]\\ \vdots\\ \theta\end{array}}{\theta}\,(\vee E)$$

for the connective $\vee$ which have been obtained by adapting the standard introduction and elimination rules for disjunction to hybrid logic. This is not a loss since the rules are not proof-theoretically well-behaved in the context of classical logic (in contrast to intuitionistic logic). To be precise: In the present system disjunction is taken as defined by the convention that $\phi \vee \psi$ is an abbreviation for $\neg(\neg\phi \wedge \neg\psi)$, but if disjunction instead is taken as a primitive connective, and $\mathbf{N}_{\mathcal{H}}$ is extended with the rules above, then the rule $(\perp 1)$ cannot be restricted to propositional symbols as it is in the present version of $\mathbf{N}_{\mathcal{H}}$, and this is a prerequisite for obtaining a subformula property; an important property of a natural deduction system. We shall not consider this issue further, see [20] and [7]. Note, however, that the rules above are admissible in $\mathbf{N}_{\mathcal{H}}$ as it stands.

4.3 The functional completeness theorem

We now come to the proof of functional completeness. To this end we shall prove a lemma. Before we can state and prove the lemma, we need a definition.

DEFINITION 4 Consider a connective $\sharp$ together with introduction and elimination rules which are instances of the rule-schemas given in Figure 2. For any formula

$$\alpha = \sharp(\overline{e_1}, \ldots, \overline{e_n}, \overline{\phi_1}, \ldots, \overline{\phi_n}, \overline{\psi}, \overline{\theta})$$

of $\mathcal{H}(\sharp)$, a formula χ_α of $\mathcal{H}(\sharp)$ is defined as follows. Firstly, for each $i \in \{1, \ldots, n\}$ and $l \in \{1, \ldots, q_i\}$, let ξ_{il} be the conjunction of the formulas

$$\{\phi_{ih} \mid \mathbf{a}_{ih} = \mathbf{d}_{il}\} \cup \{\neg\psi_i \mid \mathbf{b}_i = \mathbf{d}_{il}\}$$

and let ρ_{il} be the formula $\neg\Box\neg\xi_{il}$. (Note that the set $\{\neg\psi_i \mid \mathbf{b}_i = \mathbf{d}_{il}\}$ has at most one element: It contains the formula $\neg\psi_i$ if $\mathbf{b}_i = \mathbf{d}_{il}$; otherwise it is empty.) Secondly, for each $i \in \{1, \ldots, n\}$, if $\mathbf{b}_i \notin \{\mathbf{d}_{i1}, \ldots, \mathbf{d}_{iq_i}, \mathbf{c}\}$, then let $b_i = e_{ij}$ where j is such that $\mathbf{b}_i = \mathbf{e}_{ij}$ (exactly one such j is guaranteed to exist), and moreover, for each $h \in \{1, \ldots, k_i\}$, if $\mathbf{a}_{ih} \notin \{\mathbf{d}_{i1}, \ldots, \mathbf{d}_{iq_i}, \mathbf{c}\}$, then let $a_{ih} = e_{ij}$ where j is such that $\mathbf{a}_{ih} = \mathbf{e}_{ij}$ (again, exactly one such j is guaranteed to exist). Thirdly, for each $i \in \{1, \ldots, n\}$, let ρ_i be the conjunction of the formulas

$$\begin{array}{l} \{a_{ih} : \phi_{ih} \mid \mathbf{a}_{ih} \notin \{\mathbf{d}_{i1}, \ldots, \mathbf{d}_{iq_i}, \mathbf{c}\}\} \cup \{\phi_{ih} \mid \mathbf{a}_{ih} = \mathbf{c}\} \cup \\ \{b_i : \neg\psi_i \mid \mathbf{b}_i \notin \{\mathbf{d}_{i1}, \ldots, \mathbf{d}_{iq_i}, \mathbf{c}\}\} \cup \{\neg\psi_i \mid \mathbf{b}_i = \mathbf{c}\}. \end{array}$$

(Note that the set $\{\neg\psi_i \mid \mathbf{b}_i = \mathbf{c}\}$ has at most one element.) Given the definitions of ρ_i and ρ_{il}, let $\sigma_i = \neg(\rho_i \wedge \rho_{i1} \wedge \ldots \wedge \rho_{iq_i})$, and finally, let $\chi_\alpha = \sigma_1 \wedge \ldots \wedge \sigma_n$.

Note that the definition of χ_α does not depend on the identity of the nominals $\overline{e_1}, \ldots, \overline{e_n}$ and the formulas $\overline{\phi_1}, \ldots, \overline{\phi_n}, \overline{\psi}, \overline{\theta}$ in the sense that

$$\chi_\alpha = \chi_\gamma[\overline{e_1}, \ldots, \overline{e_n}/\overline{g_1}, \ldots, \overline{g_n}][\overline{\phi_1}, \ldots, \overline{\phi_n}, \overline{\psi}, \overline{\theta}/\overline{p_1}, \ldots, \overline{p_n}, \overline{q}, \overline{r}]$$

if $\gamma = \sharp(\overline{g_1}, \ldots, \overline{g_n}, \overline{p_1}, \ldots, \overline{p_n}, \overline{q}, \overline{r})$. Also, note that the number of occurrences of the connective $\sharp$ in χ_α is identical to the total number of occurrences of $\sharp$ in the formulas $\overline{\phi_1}, \ldots, \overline{\phi_n}, \overline{\psi}$.

It is instructive to take a look at some examples: If we consider the connective strict implication together with the rules $(\prec I)$ and $(\prec E)$ given in the previous subsection, and apply the definition above to the formula $\phi \prec \psi$, then $\chi_{\phi \prec \psi} = \neg(\top \wedge \neg\Box\neg(\phi \wedge \neg\psi))$. This can be compared to the rules $(\rightarrow I)$ and $(\rightarrow E)$ for the usual implication connective where $\chi_{\phi\rightarrow\psi} = \neg((\phi \wedge \neg\psi) \wedge \top)$ and the rules $(: I)$ and $(: E)$ for the satisfaction operator where $\chi_{e:\psi} = \neg((e : \neg\psi) \wedge \top)$. Now the lemma.

LEMMA 5 *Let a formula* $\alpha = \sharp(\overline{e_1}, \ldots, \overline{e_n}, \overline{\phi_1}, \ldots, \overline{\phi_n}, \overline{\psi}, \overline{\theta})$ *of* $\mathcal{H}(\sharp)$ *be given. For any nominal* c*,* $c : (\alpha \leftrightarrow \chi_\alpha)$ *is derivable in* $\mathbf{N}_{\mathcal{H}(\sharp)}$.

Proof. For each $i \in \{1, \ldots, n\}$, let $\overline{f_i} = f_{i1}, \ldots, f_{iq_i}$ be a list of pairwise distinct new nominals and let

$$b'_i = \begin{cases} b_i & \text{if } \mathbf{b}_i \notin \{\mathbf{d}_{i1}, \ldots, \mathbf{d}_{iq_i}, \mathbf{c}\}, \\ f_{il} & \text{if } \mathbf{b}_i = \mathbf{d}_{il}, \\ c & \text{if } \mathbf{b}_i = \mathbf{c}, \end{cases}$$

and moreover, for each $h \in \{1, \ldots, k_i\}$, let

$$a'_{ih} = \begin{cases} a_{ih} & \text{if } \mathbf{a}_{ih} \notin \{\mathbf{d}_{i1}, \ldots, \mathbf{d}_{iq_i}, \mathbf{c}\}, \\ f_{il} & \text{if } \mathbf{a}_{ih} = \mathbf{d}_{il}, \\ c & \text{if } \mathbf{a}_{ih} = \mathbf{c}. \end{cases}$$

(Recall that b_i and a_{ih} were defined together with the formula χ_α.) We first prove the implication from right to left: It follows from the rule ($\sharp I$) that $c : \alpha$ is derivable from $\{c : \chi_\alpha\}$ if for each $i \in \{1, \ldots, n\}$, the formula $b'_i : \psi_i$ is derivable from $\{a'_{i1} : \phi_{i1}, \ldots, a'_{ik_i} : \phi_{ik_i}, c : \sigma_i\}$. But this is the case if $c : \perp$ is derivable from $\Gamma_i \cup \{c : \sigma_i\}$ where

$$\Gamma_i = \{a'_{i1} : \phi_{i1}, \ldots, a'_{ik_i} : \phi_{ik_i}, b'_i : \neg\psi_i\},$$

that is, if the formulas $c : \rho_{i1}, \ldots, c : \rho_{iq_i}$, and $c : \rho_i$ all are derivable from Γ_i. Clearly, $c : \rho_i$ is derivable from Γ_i. Consider now any $l \in \{1, \ldots, q_i\}$. The formula $c : \rho_{il}$ is derivable from Γ_i if $c : \perp$ is derivable from $\Gamma_i \cup \{c : \Box\neg\xi_{il}\}$. Note that $f_{il} : \neg\xi_{il}$ is derivable from $\{c : \Box\neg\xi_{il}\}$ by the rule ($\Box E$). On the other hand, it is straightforward to check that $f_{il} : \xi_{il}$ is derivable from Γ_i. We therefore conclude that $c : \perp$ is derivable from $\Gamma_i \cup \{c : \Box\neg\xi_{il}\}$.

We now prove the implication from left to right: The formula $c : \chi_\alpha$ is derivable from the formula $\{c : \alpha\}$ if for each $i \in \{1, \ldots, n\}$, $c : \perp$ is derivable from $\{c : \alpha, c : \rho_i, c : \rho_{i1}, \ldots, c : \rho_{iq_i}\}$. Note that for any $h \in \{1, \ldots, k_i\}$, if $\mathbf{a}_{ih} \notin \{\mathbf{d}_{i1}, \ldots, \mathbf{d}_{iq_i}, \mathbf{c}\}$, then $a'_{ih} = a_{ih}$ and $a_{ih} : \phi_{ih}$ is derivable from $\{c : \rho_i\}$. Moreover, if $\mathbf{a}_{ih} = \mathbf{c}$, then $a'_{ih} = c$ and $c : \phi_{ih}$ is derivable from $\{c : \rho_i\}$. Furthermore, for any $l \in \{1, \ldots, q_i\}$, if $\mathbf{a}_{ih} = \mathbf{d}_{il}$, then $a'_{ih} = f_{il}$ and $f_{il} : \phi_{ih}$ is derivable from $\{f_{il} : \xi_{il}\}$. Thus, by the rule ($\sharp Ei$), $b'_i : \psi_i$ is derivable from

$$\Delta_i = \{c : \alpha, c : \rho_i, f_{i1} : \xi_{i1}, \ldots, f_{iq_i} : \xi_{iq_i}\}.$$

We now split up in three cases. The first case is where $\mathbf{b}_i = \mathbf{d}_{il_1}$ for some $l_1 \in \{1, \ldots, q_i\}$. Then $b'_i = f_{il_1}$ and $f_{il_1} : \neg\psi_i$ is derivable from $\{f_{il_1} : \xi_{il_1}\}$. Thus, $f_{il_1} : \perp$ is derivable from Δ_i wherefore the formula $f_{il_1} : \neg\xi_{il_1}$ is derivable from $\Delta_i - \{f_{il_1} : \xi_{il_1}\}$ and so is $c : \Box\neg\xi_{il_1}$ by the rule ($\Box I$). Therefore $c : \perp$ is derivable from $(\Delta_i - \{f_{il_1} : \xi_{il_1}\}) \cup \{c : \rho_{il_1}\}$. We are done if $q_i = 1$. Otherwise, let $l_2 \in \{1, \ldots, q_i\}$ be such that $l_2 \neq l_1$. Then $f_{il_2} : \neg\xi_{il_2}$ is derivable from $(\Delta_i - \{f_{il_1} : \xi_{il_1}, f_{il_2} : \xi_{il_2}\}) \cup \{c : \rho_{il_1}\}$ and so is $c : \Box\neg\xi_{il_2}$ by the rule $(\Box I)$. Therefore $c : \bot$ is derivable from $(\Delta_i - \{f_{il_1} : \xi_{il_1}, f_{il_2} : \xi_{il_2}\}) \cup \{c : \rho_{il_1}, c : \rho_{il_2}\}$. We are done if $q_i = 2$. Otherwise, etc. This step is carried out q_i times in total. The second case is where $\mathbf{b}_i \notin \{\mathbf{d}_{i1}, \ldots, \mathbf{d}_{iq_i}, \mathbf{c}\}$. Then $b'_i = b_i$ and $b_i : \neg\psi_i$ is derivable from $\{c : \rho_i\}$. Thus, $c : \bot$ is derivable from Δ_i. We are done if $q_i = 0$. Otherwise, etc. The rest of this case is analogous to the first case. The third case is where $\mathbf{b}_i = \mathbf{c}$. Then $b'_i = c$ and $c : \neg\psi_i$ is derivable from $\{c : \rho_i\}$. Again, $c : \bot$ is derivable from Δ_i. The rest of the third case is analogous to the second case. ■

We are now ready for functional completeness.

THEOREM 6 (Functional completeness) *For any formula ϕ in $\mathcal{H}(\sharp)$, there exists a formula ψ in $\mathcal{H}$ such that for any nominal c, $c : (\phi \leftrightarrow \psi)$ is derivable in $\mathbf{N}_{\mathcal{H}(\sharp)}$.*

Proof. Induction on the number of occurrences of the connective $\sharp$ in ϕ where we use Lemma 5 and Lemma 3. Note that replacing an occurrence of $\alpha = \sharp(\overline{e_1}, \ldots, \overline{e_n}, \overline{\phi_1}, \ldots, \overline{\phi_n}, \overline{\psi}, \overline{\theta})$ in a formula by an occurrence of the formula χ_α decreases the number of occurrences of the connective $\sharp$ by at least one, cf. the definition of χ_α. ■

5 Related and further work

There is a vast literature on the proof-theory of the modal logic *S5*. It has turned out to be difficult to formulate natural deduction and Gentzen systems for *S5* without introducing metalinguistic machinery. The history of this problem goes back to the papers [17, 18] where a counter-example to cut-elimination is given for an otherwise very natural and straightforward Gentzen formulation of *S5*. See [5] for an example of a cut-free Gentzen system for *S5* where metalinguistic machinery is avoided (at the expense of having to make use of a "non-local" side-condition on an inference rule, that is, a side-condition that does not just refer to the premiss of the rule, but to the whole derivation of the premiss). See the book [16] for an example of a cut-free Gentzen formulation of *S5* that makes use of metalinguistic machinery, namely indexed formulas. This system can be considered a reformulation in Gentzen style of Fitting's prefixed tableau system for *S5* given in [9]. Fitting's metalinguistic approach is at a general level in line with Gabbay's labelled deduction systems, see [10].

It should be mentioned that various other Gentzen and natural deduction style formulations for *S5* have been suggested, notable here are formulations in terms of Belnap's display logic and Dǒsen's higher-order sequents. However, these formulations differ considerably from Gentzen's original sequent and natural deduction systems for classical logic, and moreover, they are relatively complicated from a technical point of view. (Although it has to be acknowledged that display sequents as well as higher-order sequents were introduced as natural generalisations of Gentzen's notion of a sequent. They are intended to allow a uniform sequent-style formulation of many different logics.) An overview can be found in [28].

So what is it that enables us to formulate natural deduction rules for hybridized *S5* without having to turn to metalinguistic machinery when it is difficult in the context of ordinary *S5*? One feature of hybridization appears to be crucial in the formulation of the introduction and elimination rules for the *S5* modality: We can express that a formula ϕ is true at a world a, indeed, this is what is expressed by the formula $a : \phi$. But this is exactly the interpretation of the indexes used in the metalingustic approach, so by hybridizing *S5*, we have in the the terminology of [2] internalized the metalanguage in the object language.

Our functional completeness results are in line with a result given in [22] for natural deduction formulations of intuitionistic and classical propositional logic. See also [30] where a functional completeness result is given which is similar to the result of [22] but within a somewhat different framework. Furthermore, see [27]. In [29] functional completeness is proved for a number of classical modal and tense logics which are presentable in terms of Belnap's display logic.

It is straightforward to check that hybridized *S5* has the same expressive power over models as monadic first-order logic with equality, one variable, and a countably infinite set of constants but no function symbols. That is, a model and an assignment in the usual sense for this first-order logic correspond to a model, an assignment, and a world as appropriate for hybridized *S5* and there exists truth preserving translations in both directions between the logics. So our functional completeness result for hybridized *S5* can be seen as a functional completeness result for the mentioned first-order logic, just couched in modal-logical terms. It is, by the way, notable that the expressive power at the level of frames obtained by adding nominals as well as the universal modality to ordinary mono-modal logic is the same as the expressive power obtained by adding the difference operator, this is shown in [11] (the difference operator is the modal operator which expresses that a formula is true in all worlds which are different from the actual world). See [8] for a further investigation of the difference operator as an additional operator and see [13] for an investigation of the universal modality as an additional operator. Also, see [26] for relevant work in connection with counting modalities (a counting modality expresses that a formula is true in at least n distinct worlds, where n is some fixed number).

5.1 Further work

We are currently investigating the extent to which the approach for obtaining the functional completeness result of the present paper can be extended to other hybrid logics.

One line of work is the following. We have very recently proved a functional completeness result for hybridized *S5* where the rule-schemas for the connective $\sharp$ strongly generalises those considered in the present paper. We have moreover proved a similar functional completeness result for a natural deduction formulation of the hybrid logic of inequality (the hybrid logic of inequality is identical to hybridized *S5* except that the universal modality has been replaced by the difference operator). This work will be published elsewhere. The hybrid logic of inequality is more expressive than hybridized *S5* since the universal modality is definable in terms of the difference operator. Given the strongly generalised rule-schemas, different conditions are imposed on the rules for $\sharp$ depending on which logic is under consideration. Of course, the more expressive power the logic has, the weaker the conditions on the rules are. One possible continuation of this line of work is to consider modal operators with stronger expressive power than that of the difference operator (in combination with weaker conditions on the rules for $\sharp$). One such option is a version of the counting modalities mentioned above.

Another possible line of work concerns the natural deduction formulation of multi-modal hybrid logic given in [7]. The system considered in that paper is a hybridized version of multi-modal K which can be extended with additional inference rules corresponding to conditions on the accessibility relations expressed by so-called geometric theories. One technical obstacle is that the introduction rule for a modal operator given in [7] does not only exhibit the modal operator in the conclusion, but also in the discharged assumptions. Thus, it is implicit, rather than explicit, in the terminology of [22], adapted to our natural deduction formulations of hybrid logic.

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