A First-order Modal Logic of Strict Implication on Varying-Domain Models

Yuanzhe Yang [1]

Department of Philosophy and Religious Studies
Peking University

Abstract

In studies of first-order modal logic (FOML), in order to give a semantics on varying-domain models, we usually use free logic. However, we can also keep the classical first-order logic, but revise the semantics of the modal operator instead, as shown in works on the so-called "common sense predicate modal logic". In this paper, following the latter approach, we introduce a binary strict implication operator $\phi \Rrightarrow \psi$, which is more expressive than the common sense $\Box$-operator, and is able to characterize a lot of our reasoning about necessity (or time, knowledge, etc.) in natural language that involves existence and non-existence of objects. We offer complete axiomatizations of the FOML of $\Rrightarrow$ on the class of all varying-domain models, the class of symmetric models, as well as the class of transitive models with a special property we call the "continual existence property".

Keywords: First-order modal logic, varying-domain semantics, strict implication, completeness.

1 Introduction

In natural language, it is very common to say something like "I could have not been born", "The sun will no longer exist after ten billion years", or "He is still alive, but she does not know it" - in other words, when we are talking and reasoning about necessity, time, knowledge, etc., it is very natural to involve the existence and non-existence of objects, either explicitly or implicitly.

This motivates us to consider *first-order modal logic* (FOML for short) on *varying-domain models*. But this turns out to be a tough subject to deal with - even defining a reasonable semantics is not easy. We are facing a dilemma here: on the one hand, classical first-order logic only allows us to talk about *existent objects*; on the other hand, according to the standard semantics of modal logic, when assessing the truth value of a modal formula, we need to go from one world to its successors, where existent objects might become *non-existent* since we are working on varying-domain models - but this forces us to talk about non-existent objects. As a result, in a varying-domain semantics for FOML,

[1] 1900014924@pku.edu.cn

it seems impossible to have both classical first-order logic and the standard semantics for the modal operator at the same time.

In present studies of FOML, the most popular approach to deal with varying-domain models is to keep the standard semantics of $\Box\phi$, and employ *free logic* for the first-order part, which allows all talk about non-existent objects. [2] However, this produces conceptual absurdities: for example, at the *present* moment, it sounds strange to say either "Socrates *is* white" or "Socrates *is not* white" - one would naturally feel hesitant to give them any truth value, but according to the semantics of free logic, one of them must be true, while the other one is false. On the technical level, we would also lose classical first-order logic as a result. [3]

Is there a way to avoid using free logic, then? The answer is affirmative: we can also keep the standard semantics of first-order logic, but change the semantics of the modal operator instead. For example, consider the following alternative semantics for $\Box\phi$:

$$\mathcal{M}, w, \sigma \vDash \Box\phi \iff \text{For all } v \in W, \text{ if } wRv \text{ and } \sigma[FV(\phi)] \subseteq D_v, \text{ then } \mathcal{M}, v, \sigma \vDash \phi$$

Such a semantics was introduced by Johan van Benthem [4] , and is further studied by Jeremy Seligman [5] and also in [4] and [5] under the name "*common sense modal predicate logic*". Conceptually, such a semantics indeed makes sense in many cases: for example, when we say "Socrates would necessarily become a philosopher", we mean that in every possible world *where Socrates existed*, he would become a philosopher, and such a meaning can be precisely expressed by $\Box\mathrm{Phil}(x)$.

So far, in the literature, the common sense semantics is the only alternative semantics for the modal operator on varying-domain models. But there are also natural expressions that cannot be formalized using the common sense $\Box$. For example, consider the sentence "I could have not been born." It is true iff there is some possible world where I am not born, but such a meaning cannot be captured by the common sense modal formula $\neg\Box\mathrm{Born}(x)$, which says that there is a world *where I exist* but am not born. Thus, it is natural to consider also the following semantics for the modal operator:

$$\mathcal{M}, w, \sigma \vDash \boxdot\phi \iff \text{For all } v \in W, \text{ if } wRv, \text{ then } \sigma[FV(\phi)] \subseteq D_v \text{ and } \mathcal{M}, v, \sigma \vDash \phi$$

[2] Other approaches to deal with varying-domain models, e.g. further confining the objects we can talk about on a point or employing three-valued logic, are also discussed in books like [1] and [3]. But note that they both eventually arrive at the conclusion that we should use free logic. A more philosophical discussion is offered in [2].

[3] Of course, it is not the case that classical first-order logic is unconditionally more preferable than free logic. But nor is it easy to offer a conceptual justification for the use of free logic in FOML that is completely *non-ad hoc*.

[4] According to van Benthem, he started using this definition in his modal logic course in the 1970s. But in published works, it first appeared in [7] (and also in [6]).

[5] Unfortunately, we have not found any published work from Seligman on this subject.

With $\boxdot$, then, we can formalize the above sentence as $\neg\boxdot\text{Born}(x)$.

But even when we have both $\Box$ and $\boxdot$ in our language, the expressive power of the language still seems too weak. For example, consider the sentence "I will not live longer than you." Intuitively, it says that in every possible future where I am still alive (existing), you are also alive (existing); but neither the common sense formula $\Box(\text{Alive}(x) \to \text{Alive}(y))$ nor $\boxdot(\text{Alive}(x) \to \text{Alive}(y))$ can express such a meaning.

This motivates us to consider a more expressive modal operator. Instead of unary operators like $\Box$ and $\boxdot$, we now introduce a *binary* operator $\phi \Rrightarrow \psi$, which revives the notion of *strict implication*. Roughly, the semantics of $\phi \Rrightarrow \psi$ is that, in every possible world, if the objects mentioned by ϕ exist and ϕ is true, then the objects mentioned by ψ exist and ψ is true (we will define its semantics strictly in Section 2). While in the propositional case, strict implication can be defined by $\Box(\phi \to \psi)$, the kind of strict implication we are considering here cannot be defined by $\Box(\phi \to \psi)$ or $\boxdot(\phi \to \psi)$ - it goes beyond the expressivity of $\Box$ and $\boxdot$. Such a strict implication operator has very strong expressive power: not only can we express both $\Box$ and $\boxdot$ using $\Rrightarrow$ (we will show this in section 2), but sentences like the one we discussed in the previous paragraph can also be easily formalized. Thus, we believe the FOML of $\Rrightarrow$ is an interesting subject to study, since it offers us a powerful formal tool to reason about existence and non-existence in the modal context without resorting to free logic. [6]

The main results of this paper are a number of strongly complete axiomatizations of the FOML of $\Rrightarrow$, as summarized in the following table:

Frame properties	–	Seriality	Reflexivity
–	$\mathbf{QK}^{\Rrightarrow}$	$\mathbf{QD}^{\Rrightarrow}$	$\mathbf{QT}^{\Rrightarrow}$
Symmetry	$\mathbf{QKB}^{\Rrightarrow}$	$\mathbf{QDB}^{\Rrightarrow}$	$\mathbf{QTB}^{\Rrightarrow}$
Transitivity (+ continual existence)	$\mathbf{QK4}^{\Rrightarrow}_{\mathrm{CE}}$	$\mathbf{QD4}^{\Rrightarrow}_{\mathrm{CE}}$	$\mathbf{QS4}^{\Rrightarrow}_{\mathrm{CE}}$

The structure of the paper is as follows: In section 2, we formally introduce the language of $\Rrightarrow$ and its semantics. Then, in section 3, we offer the minimal logic of $\Rrightarrow$ and prove the corresponding completeness theorem; in section 4, we consider axiomatizations on symmetric models; and in section 5, we consider axiomatizations on transitive models with a special property we call the "continual existence property". Finally, in section 6, we conclude the results of the paper, and consider directions for future work.

[6] It should be noted, however, that our work is essentially a work on first-order modal logic, rather than a work on *conditional logic*. We are interested in the strict implication operator $\Rrightarrow$ mainly because it has very interesting expressive power on varying-domain models, and we can axiomatize the FOML of $\Rrightarrow$ in an elegant way, but *not* because we think it constitutes a reasonable semantics for conditionals or counterfactuals - as a candidate for the semantics of conditionals, $\Rrightarrow$ is basically as problematic as standard strict implication.

2 Language and semantics

We fix an at most countable set $\mathcal{P}$ of predicates and a countable set $\mathbf{X}$ of variables. In this paper, we restrict ourselves to the *countable* case.[7]

Definition 2.1 ($\mathcal{L}^{\Rrightarrow}$) $\mathcal{L}^{\Rrightarrow}$-formulas are defined recursively as follows:

$$\phi ::= Px_1, ..., x_n \mid \mathsf{E}x \mid \neg\phi \mid (\phi \wedge \phi) \mid \forall x\phi \mid (\phi \Rrightarrow \phi)$$

where $x, x_1, ..., x_n \in \mathbf{X}$, and $P \in \mathcal{P}$ is n-ary.

Other Boolean connectives and $\exists x\phi$ are defined in the usual way.[8]

We may abbreviate a sequence of variables $x_1, ..., x_n$ as $\vec{x}$, and a sequence of quantifiers $\forall x_1 \cdots \forall x_n$ (or $\exists x_1 \cdots \exists x_n$) as $\forall\vec{x}$ (or $\exists\vec{x}$).

$\phi[y/x]$ denotes the formula obtained by replacing every free occurrences of x in ϕ with a y. We say $\phi[y/x]$ is *admissible* if in ϕ, x does not occur free within the scope of $\exists y$.

$FV(\phi)$ denotes the free variables of ϕ.

Remark 2.2 Note that the *existential predicate* E is also introduced to our language. But this will *not* lead us to free logic, since, as we will soon see, $\mathsf{E}x$ is always trivially true in our logic (in fact, we can even define $\mathsf{E}x$ as the abbreviation of $(Px \vee \neg Px)$ for some random unary predicate P). Its behavior only becomes interesting within the scope of $\Rrightarrow$. This is also an interesting feature of our logic: we can do non-trivial reasoning using a trivial existential predicate.

Now, we give the semantics of the language. First, first-order Kripke models are defined as follows:

Definition 2.3 (First-order Kripke model) A *first-order Kripke model* is a 5-tuple $\mathcal{M} = (W, D, \delta, R, \rho)$, where

- $W \neq \emptyset$ is the set of possible worlds, and $D \neq \emptyset$ is the set of objects;
- $\delta : W \to \wp(D)$ assigns each $w \in W$ a non-empty set of objects, called the local domain on w (we will often denote $\delta(w)$ as D_w);
- $R \subseteq W \times W$ is the accessibility relation on worlds;
- $\rho : \mathcal{P} \times W \to \wp(D_w^{<\omega})$ is the interpretation of the model.

As noted in the introduction, we do not want to allow all talk about non-existent objects, as is the case of free logic. Thus, on the formal level, the truth value of a formula on a world is defined only when the corresponding assignment is a *relevant assignment* in the following sense:

Definition 2.4 (Relevant assignment) Given a model $\mathcal{M} = (W, D, \delta, R, \rho)$, a $w \in W$ and $\phi \in \mathcal{L}^{\Rrightarrow}$, a *relevant assignment* σ w.r.t. $\mathcal{M}, w$ and ϕ is a function from $\mathbf{X}$ to D s.t. for all $x \in FV(\phi)$, $\sigma(x) \in D_w$ (i.e. $\sigma[FV(\phi)] \subseteq D_w$).

---

[7] In this paper, by "countable", we always mean *countably infinite*.

[8] We do not include the identity relation in our language; nor do we include any constant symbol or function symbol. It seems better to start considering them in our language after we become more familiar with the behavior of the $\Rrightarrow$-operator.

For a set Δ of $\mathcal{L}^{\Rrightarrow}$-formulas, We also say σ is a relevant assignment w.r.t. $\mathcal{M}, w$ and Δ if for all $\phi \in \Delta$, σ is a relevant assignment w.r.t $\mathcal{M}, w$ and ϕ.

Then, we define the following semantics with the help of the above notions:

Definition 2.5 ($\vDash$) Given $\mathcal{M} = (W, D, \delta, R, \rho)$, $w \in W$, $\phi \in \mathcal{L}^{\Rrightarrow}$ and an assignment σ relevant to $\mathcal{M}, w$ and ϕ, the satisfaction relation $\vDash$ is defined recursively as follows: [9]

$\mathcal{M}, w, \sigma \vDash Px_1, ..., x_n$	$\iff$	$(\sigma(x_1), ..., \sigma(x_n)) \in \rho(P, w)$
$\mathcal{M}, w, \sigma \vDash \mathsf{E}x$	$\iff$	$\sigma(x) \in D_w$
$\mathcal{M}, w, \sigma \vDash \neg\phi$	$\iff$	$\mathcal{M}, w, \sigma \nvDash \phi$
$\mathcal{M}, w, \sigma \vDash \phi \wedge \psi$	$\iff$	$\mathcal{M}, w, \sigma \vDash \phi$ and $\mathcal{M}, w, \sigma \vDash \psi$
$\mathcal{M}, w, \sigma \vDash \forall x\phi$	$\iff$	For all $a \in D_w$, $\mathcal{M}, w, \sigma[x \mapsto a] \vDash \phi$
$\mathcal{M}, w, \sigma \vDash \phi \Rrightarrow \psi$	$\iff$	For all $v \in W$ s.t. wRv, if $\sigma[FV(\phi)] \subseteq D_v$ and $\mathcal{M}, v, \sigma \vDash \phi$, then $\sigma[FV(\psi)] \subseteq D_v$ and $\mathcal{M}, v, \sigma \vDash \psi$

For the *semantic consequence* relation, we write $\Delta \vDash \phi$ if for all pointed model $\mathcal{M}, w$ and all assignment σ which is relevant to $\mathcal{M}, w$ and $\Delta \cup \{\phi\}$, if $\mathcal{M}, w \vDash \Delta$, then $\mathcal{M}, w \vDash \phi$. As usual, ϕ is *valid* if $\emptyset \vDash \phi$.

Note that we consider only relevant assignments in our semantics - in other words, if σ is not a relevant assignment w.r.t. $\mathcal{M}, w$ and ϕ, then the satisfaction relation between $\mathcal{M}, w, \sigma$ and ϕ is left undefined. As a result, $\mathsf{E}x$ is true so long as its truth value is defined.

Also note that even when σ is relevant w.r.t. $\mathcal{M}, w$ and ϕ, it does not follow that for all successor v of w, σ is also relevant w.r.t. $\mathcal{M}, v$ and ϕ - In fact, this is exactly what makes the semantics for FOML on varying-domain models difficult to define. As a response, our semantics for $\Rrightarrow$ guarantees that the truth value of $\phi \Rrightarrow \psi$ is defined for $\mathcal{M}, w, \sigma$ as long as σ is relevant w.r.t. $\mathcal{M}, w$ and ϕ, ψ, since whether the truth values of ϕ and ψ are defined on a successor of w also constitutes a part of the truth condition of $\phi \Rrightarrow \psi$ on w. This should be viewed as a generalization of the idea behind the common sense $\Box$.

As we have noted in the introduction, according to the semantics of $\Rrightarrow$, we can easily formalize the sentence we discussed in the introduction, "I will not live longer than you", with a formula of the form Alive(x) $\Rrightarrow$ Alive(y) (or simply $\mathsf{E}x \Rrightarrow \mathsf{E}y$). Moreover, as noted in the introduction, using $\Rrightarrow$, we can express both the common sense $\Box$ and the operator $\boxdot$. Their definitions in $\mathcal{L}^{\Rrightarrow}$ are as follows: [10]

$$\Box\phi := \neg\phi \Rrightarrow \bot$$
$$\boxdot\phi := \top \Rrightarrow \phi$$

[9] $\sigma[x \mapsto a]$ is the assignment which is identical with σ except that $\sigma[x \mapsto a](x) = a$.
It is not hard to check that the semantics for $\forall x\phi$ and $\phi \Rrightarrow \psi$ are indeed well-defined: though their semantics involves changes in assignments or points, in the end, we are still dealing with assignments relevant to the point and formula in question.

[10] $\bot$ and $\top$ stand for logical contradiction and tautology with no free variable, respectively. For example, we may define $\top$ as the abbreviation for $\exists x\mathsf{E}x$, and $\bot$ as $\neg\top$.

It is easy to check that the above definitions work in the expected way. In the following discussions, we take these definitions to be our "official" definitions of $\Box\phi$ and $\boxdot\phi$ in the language $\mathcal{L}^{\Rrightarrow}$ (accordingly, $\Diamond\phi$ is viewed as the abbreviation for $\neg(\phi \Rrightarrow \bot)$).[11]

Conversely, there clearly exist formulas in $\mathcal{L}^{\Rrightarrow}$ that cannot be expressed even when we are allowed to use both $\Box$ and $\boxdot$. For an example, see Example A.1 in the appendix. Thus, $\mathcal{L}^{\Rrightarrow}$ is strictly more expressive than the first-order modal language with only $\Box$ and $\boxdot$ (we will not offer a precise characterization of the expressivity of $\mathcal{L}^{\Rrightarrow}$ in this paper, though).

For convenience, we also introduce the following abbreviation:

$$\Diamond^{-\psi}\phi := \neg(\phi \Rrightarrow \psi)$$

Intuitively, $\Diamond^{-\psi}\phi$ says that the present point can see a successor where ϕ is defined and true, and ψ is either undefined or false. To put it more strictly:

$$\mathcal{M}, w, \sigma \vDash \Diamond^{-\psi}\phi \quad \Longleftrightarrow \quad \begin{array}{l}\text{There is } v \in W \text{ s.t. } wRv,\ \sigma[FV(\phi)] \subseteq D_v,\\ \mathcal{M}, v, \sigma \vDash \phi, \text{ and either } \sigma[FV(\psi)] \nsubseteq D_v \text{ or}\\ \sigma[FV(\psi)] \subseteq D_v \text{ but } \mathcal{M}, v, \sigma \nvDash \psi\end{array}$$

In particular, when the formula ψ in $\Diamond^{-\psi}\phi$ is of the form $\mathsf{E}x$, $\Diamond^{-\mathsf{E}x}\phi$ says that the present point can see a successor where ϕ is defined and true, and the object denoted by x does not exist (since $\mathsf{E}x$ cannot be false when its truth value is defined, as we have noted above). Thus, we may further abbreviate $\Diamond^{-\mathsf{E}x}\phi$ as $\Diamond^{-x}\phi$. Formally, the semantics of $\Diamond^{-x}\phi$ is the following one:

$$\mathcal{M}, w, \sigma \vDash \Diamond^{-x}\phi \quad \Longleftrightarrow \quad \begin{array}{l}\text{There is } v \in W \text{ s.t. } wRv,\ \sigma[FV(\phi)] \subseteq D_v,\\ \mathcal{M}, v, \sigma \vDash \phi, \text{ and } \sigma(x) \notin D_v\end{array}$$

In fact, we can also treat $\Diamond^{-x}\phi$ as a primitive operator instead, and add it to the language of common sense FOML. The resulting language have exactly the same expressivity as $\mathcal{L}^{\Rrightarrow}$, since $\phi \Rrightarrow \psi$ can be expressed by $\neg\Diamond^{-\psi}\phi$, and $\Diamond^{-\psi}\phi$ can be defined using $\Diamond^{-x}$ and the common sense $\Diamond$ as follows:

$$\Diamond^{-\psi}\phi := \Diamond(\phi \wedge \neg\psi) \vee \bigvee_{x \in FV(\psi)} \Diamond^{-x}\phi$$

In this sense, $\mathcal{L}^{\Rrightarrow}$ is more expressive than language of common sense FOML exactly because it has formulas of the form $\Diamond^{-x}\phi$, so it is essentially a language that extends common sense FOML by adding the $\Diamond^{-x}$ operator. That being said, we still choose to take the binary operator $\Rrightarrow$ (rather than $\Diamond$ and $\Diamond^{-x}$) as the primitive operator of our language, since the semantics of $\phi \Rrightarrow \psi$ is rather natural, and when $\Rrightarrow$ is the primitive operator of the language, the logics of $\Rrightarrow$ can be axiomatized in a very intuitive way, as we will soon see in the next section.

[11] Of course, a probably more straightforward way to define $\Box\phi$ is to use the $\mathcal{L}^{\Rrightarrow}$-formula $(\bigwedge_{x \in FV(\phi)} \mathsf{E}x) \Rrightarrow \phi$. But we choose to define $\Box\phi$ with $\neg\phi \Rrightarrow \bot$ because this definition seems more elegant, and it is also easier to prove some of the theorems involving $\Box\phi$ in the axiomatic systems when $\Box\phi$ is defined this way.

3 The minimal logic

3.1 The logic $\mathbf{QK}^{\Rrightarrow}$

We now introduce the minimal logic of $\Rrightarrow$, which corresponds to the class of all varying-domain first-order Kripke models.

The logic $\mathbf{QK}^{\Rrightarrow}$

Axioms		**Rules**	
`TAUT`	All propositional tautologies	`MP`	$\phi, \phi \to \psi \vdash \psi$
`UI`	$\forall x\phi \to \phi[y/x]$ ($\phi[y/x]$ is admissible)	`UG`	$\dfrac{\vdash \psi \to \phi}{\vdash \psi \to \forall x\phi}$
`E`	$\mathsf{E}x$		$(x \notin FV(\psi))$
`CH`$^{\Rrightarrow}$	$((\phi \Rrightarrow \psi) \wedge (\psi \Rrightarrow \chi)) \to (\phi \Rrightarrow \chi)$		
`AND`$^{\Rrightarrow}$	$((\phi \Rrightarrow \psi) \wedge (\phi \Rrightarrow \chi)) \to (\phi \Rrightarrow (\psi \wedge \chi))$	`NEC`$^{\Rrightarrow}$	$\dfrac{\vdash \phi \to \psi}{\vdash \phi \Rrightarrow \psi}$
`OR`$^{\Rrightarrow}$	$((\phi \Rrightarrow \chi) \wedge (\psi \Rrightarrow \chi)) \to ((\phi \vee \psi) \Rrightarrow \chi)$		
`EXP`$^{\Rrightarrow}$	$\bot \Rrightarrow \phi$		$(FV(\psi) \subseteq FV(\phi))$

Proposition 3.1 *(Soundness of* $\mathbf{QK}^{\Rrightarrow}$*)* $\mathbf{QK}^{\Rrightarrow}$ *is sound w.r.t. the class of all varying-domain first-order Kripke models.*

As we can see, within $\mathbf{QK}^{\Rrightarrow}$, we still have an intact classical first-order logic - unlike the case of free logic, we do not need the existential predicate when writing the first-order axioms and rules. In fact, the only axiom that mentions the existential predicate here is `E`, which says that $\mathsf{E}x$ always holds trivially.

As for the modal part of the logic, the axioms for $\Rrightarrow$ are just what one would naturally expect in a logic of strict implication; the rule `NEC`$^{\Rrightarrow}$, on the other hand, distinguishes $\Rrightarrow$ from standard strict implication, since it sets an requirement on the pattern of *free variables* of the formula to which it can be applied. Such a rule should remind us of Seligman's axiom $\mathtt{K}_{inv}$ used in the axiomatizations of common sense FOML (which can be found in [4] and [5]):

$$\mathtt{K}_{inv} \quad \Box(\phi \to \psi) \to (\Box\phi \to \Box\psi) \quad (FV(\phi) \subseteq FV(\psi))$$

And $\mathtt{K}_{inv}$ can indeed be deduced as a theorem in $\mathbf{QK}^{\Rrightarrow}$, when $\Box\phi$ is viewed as the abbreviation of $\neg\phi \Rrightarrow \bot$.

It is also worth noting that no interaction axiom or rule between the first-order part and the modal part is needed in this minimal logic. This is not always true for axiomatizations of FOML under varying-domain semantics. [12]

Below is a list of theorems and derivable rules of $\mathbf{QK}^{\Rrightarrow}$. The way to deduce them is shown in the proof of Proposition A.2 in the appendix.

---

[12] For example, we sometimes need rules of the following form in axiomatizations of varying-domain FOML, which involves both quantifier and modal operator (see, for example, Chapter 16 of [1]):

$$\frac{\vdash \psi_0 \to \Box(\psi_1 \to \Box(\psi_2 \to \cdots \Box(\psi_n \to \Box\phi)\cdots))}{\vdash \psi_0 \to \Box(\psi_1 \to \Box(\psi_2 \to \cdots \Box(\psi_n \to \Box\forall x\phi)\cdots))} \quad (x \notin \bigcup_{i \leq n} FV(\psi_i))$$

$\mathtt{ID}^{\Rrightarrow}$	$\phi \Rrightarrow \phi$
$\mathtt{ANT}^{\Rrightarrow}$	$(\phi \Rrightarrow \psi) \to (\phi \Rrightarrow (\phi \wedge \psi))$
$\mathtt{AS}^{\Rrightarrow}$	$(\phi \Rrightarrow \psi) \to ((\phi \wedge \chi) \Rrightarrow \psi)$
$\mathtt{MONO}^{\Rrightarrow}$	$\dfrac{\vdash \phi \to \psi}{\vdash (\psi \Rrightarrow \chi) \to (\phi \Rrightarrow \chi)}\ (FV(\psi) \subseteq FV(\phi))$
$\mathtt{SUB}^{\Rrightarrow}$	$\dfrac{\vdash \phi \leftrightarrow \psi}{\vdash (\phi \Rrightarrow \chi) \leftrightarrow (\psi \Rrightarrow \chi)}\ (FV(\psi) = FV(\phi))$
$\mathtt{DIV}^{\Rrightarrow}$	$\Diamond^{-\psi}\exists\vec{x}\phi \to (\Diamond^{-\psi}\exists\vec{x}(\phi \wedge \chi) \vee \Diamond^{-\psi}\exists\vec{x}(\phi \wedge \neg\chi))$ $(FV(\chi) \subseteq FV(\phi))$
$\mathtt{K}^{\Rrightarrow}$	$(\Diamond^{-\psi}(\phi \wedge \alpha) \wedge (\alpha \Rrightarrow \beta)) \to \Diamond^{-\psi}(\phi \wedge \beta)$
$\mathtt{K}_{inv}$	$\Box(\phi \to \psi) \to (\Box\phi \to \Box\psi)\ (FV(\phi) \subseteq FV(\psi))$
$\Box\text{-}\mathtt{Alt}$	$\Box\phi \leftrightarrow ((\bigwedge_{x \in FV(\phi)} \mathsf{E}x) \Rrightarrow \phi)$
$\Rrightarrow\mathtt{to}\Box$	$(\phi \Rrightarrow \psi) \to \Box(\phi \to \psi)$
$\Rrightarrow\text{-}\mathtt{Rdc}$	$\Diamond^{-\psi}\phi \leftrightarrow (\Diamond(\phi \wedge \neg\psi) \vee \bigvee_{x \in FV(\psi)} \Diamond^{-x}\phi)$

3.2 Completeness

Now, we show that the logic $\mathbf{QK}^{\Rrightarrow}$ is strongly complete w.r.t. the class of all variable-domain models. We will construct a kind of varying-domain canonical model in accordance to the special semantics of the $\Rrightarrow$-operator.

In order to define the canonical model, there are some preparations that should be done in advance. First, we introduce a larger set of variables $\mathbf{X}^+$ satisfying that $\mathbf{X} \subseteq \mathbf{X}^+$ and $\mathbf{X}^+ \setminus \mathbf{X}$ is countable; we use $\mathcal{L}_+^{\Rrightarrow}$ to denote the $\Rrightarrow$-language which takes $\mathbf{X}^+$ as its set of variables. $\mathbf{X}^+$ would provide us with enough witnesses for the existential formulas.

We also need the following notions: for a set Δ of $\mathcal{L}_+^{\Rrightarrow}$-formulas, we say that Δ has the *FV-$\exists$-property*, iff for all $x \in \mathbf{X}^+$ and $\phi \in \mathcal{L}_+^{\Rrightarrow}$ s.t. $FV(\exists x\phi) \subseteq FV(\Delta)$, there is $y \in \mathbf{X}^+$ s.t. $\exists x\phi \to \phi[y/x] \in \Delta$ (where $\phi[y/x]$ is admissible); and given a logic $\mathbf{L}$, we say that Δ is an *$\mathbf{L}$-FV-maximal consistent set* (*$\mathbf{L}$-FV-MCS* for short), iff Δ is $\mathbf{L}$-consistent, and for all $\phi \in \mathcal{L}_+^{\Rrightarrow}$ s.t. $FV(\phi) \subseteq FV(\Delta)$, either $\phi \in \Delta$ or $\neg\phi \in \Delta$.[13]

Unlike standard MCSs, different FV-MCSs can have different free variables. Thus, we can use them to construct a *varying-domain* canonical model in a natural way (by taking the sets of free variables as the local domains).

It is routine to prove the following lemma:

Lemma 3.2 *Let $\mathbf{L}$ be a logic at least as strong as $\mathbf{QK}^{\Rrightarrow}$, and let Δ_0 be an $\mathbf{L}$-consistent set of $\mathcal{L}_+^{\Rrightarrow}$-formulas. Then, the following hold:*

- *If $\mathbf{Y} \subseteq \mathbf{X}^+ \setminus FV(\Delta_0)$ is countable, then there is $\Delta \subseteq \mathcal{L}_+^{\Rrightarrow}$ s.t. $\Delta_0 \subseteq \Delta$, $FV(\Delta) = FV(\Delta_0) \cup \mathbf{Y}$, Δ is $\mathbf{L}$-consistent and has the FV-$\exists$-property;*
- *There is $\Delta \subseteq \mathcal{L}_+^{\Rrightarrow}$ s.t. $\Delta_0 \subseteq \Delta$, $FV(\Delta) = FV(\Delta_0)$, and Δ is an $\mathbf{L}$-FV-MCS.*

Now, we are ready to offer our definition of the canonical model.

[13] It is not hard to check that FV-MCSs still have properties that resemble standard MCSs (e.g. if Δ is an FV-MCS, $\Delta \vdash \phi$ and $FV(\phi) \subseteq FV(\Delta)$, then $\phi \in \Delta$).
In fact, the notions of MCS and witness property introduced in [4] are very similar to the ones we introduced here.

Definition 3.3 (Canonical model) Given a logic $\mathbf{L}$, the corresponding *canonical model* $\mathcal{M}^c_{\mathbf{L}} = (W^c_{\mathbf{L}}, D^c_{\mathbf{L}}, \delta^c_{\mathbf{L}}, R^c_{\mathbf{L}}, \rho^c_{\mathbf{L}})$ is defined as follows:

- $W^c_{\mathbf{L}} = \{\Delta \subseteq \mathcal{L}^{\Rrightarrow}_{+} \mid \Delta$ is an $\mathbf{L}$-FV-MCS with the FV-$\exists$-property and $\mathbf{X}^+ \setminus FV(\Delta)$ is countable$\}$, and $D^c_{\mathbf{L}} = \mathbf{X}^+$;
- $\delta^c_{\mathbf{L}}(\Delta) = FV(\Delta)$ for all $\Delta \in W^c_{\mathbf{L}}$;
- $R^c_{\mathbf{L}}$ satisfies that for all $\Delta, \Theta \in W^c$, $\Delta R^c_{\mathbf{L}} \Theta$ iff for all $\phi, \psi \in \mathcal{L}^{\Rrightarrow}_{+}$, if $\phi \Rrightarrow \psi \in \Delta$ and $\phi \in \Theta$, then $\psi \in \Theta$;
- $\rho^c_{\mathbf{L}}$ satisfies that for all $\Delta \in W^c$, $P \in \mathcal{P}$ and $\vec{x} \in (FV(\Delta))^{<\omega}$, $\vec{x} \in \rho^c_{\mathbf{L}}(P, \Delta)$ iff $P\vec{x} \in \Delta$.

The canonical assignment σ^c is the assignment s.t. $\sigma^c(x) = x$ for any $x \in \mathbf{X}^+$.

For simplicity, we now fix a logic $\mathbf{L}$ no weaker than $\mathbf{QK}^{\Rrightarrow}$, and omit all the prefix and subscript "$\mathbf{L}$" in the following proofs.

Note that for any $\Delta \in W^c$, we have $\exists x \mathrm{E}x \in \Delta$, so $FV(\Delta) \neq \emptyset$ by the FV-$\exists$-property of Δ. Hence, the local domains in the canonical model are all non-empty.

We then prove the most non-trivial step in the completeness proof: the *existence lemma*.

Lemma 3.4 *(Existence) For any $\Delta \in W^c$ and $\phi, \psi \in \mathcal{L}^{\Rrightarrow}_{+}$, if $\Diamond^{-\psi}\phi \in \Delta$ and $\mathbf{Y} \subseteq \mathbf{X}^+ \setminus FV(\Delta)$ is countable, then there is some $\Theta \in W^c$ s.t. $FV(\Theta) \subseteq FV(\Delta) \cup \mathbf{Y}$, $\Delta R^c \Theta$, $\phi \in \Theta$ and $\psi \notin \Theta$.*[14]

Proof Let $\Delta \in W^c$ and $\phi, \psi \in \mathcal{L}^{\Rrightarrow}_{+}$ be arbitrary, and assume that $\Diamond^{-\psi}\phi \in \Delta$. Also let $\mathbf{Y}$ be an arbitrary countable subset of $\mathbf{X}^+ \setminus FV(\Delta)$.

We first construct a countable sequence $(\Phi_k)_{k\in\omega}$ of finite formula sets s.t. for all $k \in \omega$, $\Diamond^{-\psi} \bigwedge \Phi_k \in \Delta$ (for simplicity, we use ϕ_k to denote $\bigwedge \Phi_k$). The union of all such Φ_k's will form a "basis" of the successor Θ we want to construct, which guarantees that $\Delta R^c \Theta$.

As preparation, we first enumerate all formulas in Δ of the form $\alpha \Rrightarrow \beta$ as $\alpha_0 \Rrightarrow \beta_0$, $\alpha_1 \Rrightarrow \beta_1$, $\alpha_2 \Rrightarrow \beta_2$, ...

As the starting point, let $\Phi_0 = \{\phi\}$. Clearly we have $\Diamond^{-\psi}\phi \in \Delta$ by assumption.

Then, given Φ_k s.t. $\Diamond^{-\psi}\phi_k \in \Delta$, we construct Φ_{k+1} in the following way. Let l be the least natural number s.t. $FV(\alpha_l) \subseteq FV(\Phi_k)$, but $\neg\alpha_l \notin \Phi_k$ and $\beta_l \notin \Phi_k$. (If there does not exist such an l, simply let $\Phi_{k+1} = \Phi_k$.) Then, since $\Diamond^{-\psi}\phi_k \in \Delta$ and $FV(\alpha_l) \subseteq FV(\phi_k)$, by $\mathtt{DIV}^{\Rrightarrow}$, we have $\Diamond^{-\psi}(\phi_k \wedge \alpha_l) \in \Delta$ or $\Diamond^{-\psi}(\phi_k \wedge \neg\alpha_l) \in \Delta$. If the former holds, then since we also have $\alpha_l \Rrightarrow \beta_l \in \Delta$, by $\mathtt{K}^{\Rrightarrow}$, $\Diamond^{-\psi}(\phi_k \wedge \beta_l) \in \Delta$; so in this case, let $\Phi_{k+1} = \Phi_k \cup \{\beta_l\}$. Otherwise, we have $\Diamond^{-\psi}(\phi_k \wedge \neg\alpha_l) \in \Delta$, so we simply let $\Phi_{k+1} = \Phi_k \cup \{\neg\alpha_l\}$. In both cases, we clearly have $\Diamond^{-\psi}\phi_{k+1} \in \Delta$.

[14] Note that the lemma allows us to choose the variable set $\mathbf{Y}$ which provides witnesses for the existential formulas. The power to make such a choice is necessary for the proof of Lemma 5.2 below.

Finally, let $\Theta_0 = \bigcup_{k\in\omega} \Phi_k$. Clearly Θ_0 is consistent: if not, then there is some $k \in \omega$ s.t. $\vdash \phi_k \to \bot$, so $\phi_k \Rrightarrow \bot \in \Delta$ by $\texttt{NEC}^{\Rrightarrow}$; then, by $\texttt{EXP}^{\Rrightarrow}$, we also have $\phi_k \Rrightarrow \psi \in \Delta$, contradicting that $\Diamond^{-\psi}\phi_k \in \Delta$.

Then, let $\mathbf{Z}$ be a random countable subset of $\mathbf{Y}$ s.t. $\mathbf{Y} \setminus \mathbf{Z}$ is countable. By Lemma 3.2, we can extend Θ_0 into an FV-MCS Θ with the FV-$\exists$-property s.t. $FV(\Theta) = FV(\Theta_0) \cup \mathbf{Z}$. Clearly, $\Theta \in W^c$ and $\phi \in \Theta$.

Then, we check that $\Delta R^c \Theta$. Assume that $\alpha \Rrightarrow \beta \in \Delta$, $\alpha \in \Theta$, and suppose (towards a contradiction) that $\beta \notin \Theta$. Say the index number of the pair $\alpha \Rrightarrow \beta$ is i, i.e. $\alpha \Rrightarrow \beta = \alpha_i \Rrightarrow \beta_i$. Then, since $FV(\Theta) = FV(\Theta_0) \cup \mathbf{Z}$ and $\mathbf{Z} \cap FV(\Delta) = \emptyset$, we have $FV(\alpha_i) \subseteq FV(\Theta_0)$, so there is some $k \in \omega$ s.t. $FV(\alpha_i) \subseteq FV(\Phi_k)$. Then, there must exist some $m \geq k$, for which i becomes the least natural number s.t. $FV(\alpha_i) \subseteq FV(\Phi_m)$, $\neg\alpha_i \notin \Phi_k$ and $\beta_i \notin \Phi_k$. Hence, $\Phi_{m+1} = \Phi_m \cup \{\beta_i\}$ or $\Phi_{m+1} = \Phi_m \cup \{\neg\alpha_i\}$; and in both cases, a contradiction can be reached. Therefore, $\beta_i \in \Theta$, i.e. $\beta \in \Theta$.

Finally, we check that $\psi \notin \Theta$. If $FV(\psi) \not\subseteq FV(\Theta_0)$, then clearly $FV(\psi) \not\subseteq FV(\Theta)$, so $\psi \notin \Theta$. Thus, we assume that $FV(\psi) \subseteq FV(\Theta_0)$. Then, there is some $k \in \omega$ s.t. $FV(\psi) \subseteq FV(\Phi_k)$. Again, since $\psi \Rrightarrow \psi \in \Delta$ (by $\texttt{ID}^{\Rrightarrow}$), there is some $m \geq k$ s.t. $\Phi_{m+1} = \Phi_m \cup \{\psi\}$ or $\Phi_{m+1} = \Phi_m \cup \{\neg\psi\}$; and since $\Diamond^{-\psi}\phi_{m+1} \in \Delta$, clearly Φ_{m+1} can only be $\Phi_m \cup \{\neg\psi\}$, so $\neg\psi \in \Phi_{m+1} \subseteq \Theta$. Thus, in this case, we also have $\psi \notin \Theta$. □

Note that for all $\Delta \in W^c$ and $\phi \in \mathcal{L}^{\Rrightarrow}$, the canonical assignment σ^c is relevant w.r.t. $\mathcal{M}^c, \Delta$ and ϕ iff $FV(\phi) \subseteq FV(\Delta)$. Thus, with the existence lemma in hand, it is now routine to prove the following *truth lemma*:

Lemma 3.5 *(Truth) For all $\Delta \in W^c$ and all $\phi \in \mathcal{L}^{\Rrightarrow}_{+}$ s.t. $FV(\phi) \subseteq FV(\Delta)$, we have*

$$\mathcal{M}^c, \Delta, \sigma^c \vDash \phi \iff \phi \in \Delta$$

Then, since any $\mathbf{L}$-consistent set Δ_0 of $\mathcal{L}^{\Rrightarrow}$-formulas can be extended into an $\mathbf{L}$-FV-MCS Δ with the FV-$\exists$-property (using Lemma 3.2) which exists in $W^c_{\mathbf{L}}$, Δ_0 can be satisfied in $\mathcal{M}^c_{\mathbf{L}}$ according to the truth lemma.

Therefore, when the logic $\mathbf{L}$ in question is just $\mathbf{QK}^{\Rrightarrow}$, we have the following completeness theorem:

Theorem 3.6 *(Completeness of $\mathbf{QK}^{\Rrightarrow}$) $\mathbf{QK}^{\Rrightarrow}$ is sound and strongly complete w.r.t. the class of all varying-domain first-order Kripke models.*

It is also routine to check that if $\texttt{D}^{\Box} : \Diamond\top$ is an axiom of $\mathbf{L}$, then $\mathcal{M}^c_{\mathbf{L}}$ is serial, and if $\texttt{T}^{\Box} : \Box\phi \to \phi$ is an axiom of $\mathbf{L}$, then $\mathcal{M}^c_{\mathbf{L}}$ is reflexive. Thus, the following completeness results are also direct:

Theorem 3.7 *(Completeness of $\mathbf{QD}^{\Rrightarrow}$ and $\mathbf{QT}^{\Rrightarrow}$)*

- $\mathbf{QD}^{\Rrightarrow} := \mathbf{QK}^{\Rrightarrow} \oplus \texttt{D}^{\Box}$ *is sound and strongly complete w.r.t. the class of all serial models;*
- $\mathbf{QT}^{\Rrightarrow} := \mathbf{QK}^{\Rrightarrow} \oplus \texttt{T}^{\Box}$ *is sound and strongly complete w.r.t. the class of all reflexive models.*

4 Symmetric models

In this section, we consider axiomatizations on symmetric models, which is more complex than the basic case.[15] The complexity mainly comes from the fact that on symmetric models, we can express a "conditional version" of the Barcan formula in $\mathcal{L}^{\Rrightarrow}$.

4.1 The axiom $\mathtt{Bf}^{\Rrightarrow}_{\mathtt{Cond}}$

In axiomaitzations on symmetric models, besides the axiom $\mathtt{B}^{\Box} : \phi \to \Box\Diamond\phi$, we also need the following Barcan-like axiom:

$$\mathtt{Bf}^{\Rrightarrow}_{\mathtt{Cond}} \quad \Diamond^{-\psi}\exists x(\phi \wedge (\chi \Rrightarrow \mathrm{E}x)) \to (\chi \to \exists x\Diamond^{-\psi}\phi) \quad (x \notin FV(\psi) \cup FV(\chi))$$

Proposition 4.1 $\mathtt{Bf}^{\Rrightarrow}_{\mathtt{Cond}}$ *is valid on all symmetric models.*

Similar to the standard Barcan formula $\Diamond\exists x\phi \to \exists x\Diamond\phi$[16], the axiom $\mathtt{Bf}^{\Rrightarrow}_{\mathtt{Cond}}$ also tells us for wRv, which object in D_v should also exist in D_w. But unlike the Barcan formula, which says that *every* object in D_v also exists in D_w unconditionally, $\mathtt{Bf}^{\Rrightarrow}_{\mathtt{Cond}}$ only says that an object a in D_v also exists in D_w when some χ is satisfied on w, and $\chi \Rrightarrow \mathrm{E}x$ is satisfied on v, with x taking value a - such a condition forces a to exist in D_w, since the models we are now considering are symmetric. (In contrast, we do not need rules like $\mathtt{Bf}^{\Rrightarrow}_{\mathtt{Cond}}$ in the minimal logic $\mathbf{QK}^{\Rrightarrow}$, because in that case there does not exist a kind of condition that forces an object in D_v to also exist in D_w for all wRv).

In fact, we can also view $\mathtt{Bf}^{\Rrightarrow}_{\mathtt{Cond}}$ as a kind of $\mathtt{B}$ axiom. We can view $\neg\phi \Rrightarrow \psi$ as a special kind of $\Box$-operator $\Box^{-\psi}\phi$ (just like we treat $\neg(\phi \Rrightarrow \psi)$ as a special kind of $\Diamond$-operator $\Diamond^{-\psi}\phi$), and write $\mathtt{Bf}^{\Rrightarrow}_{\mathtt{Cond}}$ in the following equivalent form:

$$\forall x\Box^{-\psi}\neg\chi \to (\phi \to \Box^{-\psi}\forall x(\chi \to \Diamond^{-x}\phi))$$

It is not hard to see its connection with $\mathtt{B}^{\Box} : \phi \to \Box\Diamond\phi$ - we may say that it is essentially the $\Diamond^{-x}$ version of the $\mathtt{B}$ axiom.

4.2 Completeness

Unfortunately, even for a logic which includes both $\mathtt{B}^{\Box}$ and $\mathtt{BF}^{\Rrightarrow}_{\mathtt{Cond}}$ and is no weaker than $\mathbf{QK}^{\Rrightarrow}$, the kind of canonical model we defined previously is not automatically symmetric. Thus, in order to prove the completeness theorem for the symmetric case, we introduce the following notion of symmetric canonical models, which are obtained by deleting all the asymmetric edge in the original canonical model.

Definition 4.2 (Symmetric canonical model) Given an logic $\mathbf{L}$, the corresponding *symmetric canonical model* is $\mathcal{M}^s_{\mathbf{L}} = (W^c_{\mathbf{L}}, D^c_{\mathbf{L}}, \delta^c_{\mathbf{L}}, R^s_{\mathbf{L}}, \rho^c_{\mathbf{L}})$, where $\mathbf{R}^s_{\mathbf{L}} = \{(\Delta, \Theta) \in W^c_{\mathbf{L}} \times W^c_{\mathbf{L}} \mid \Delta R^c_{\mathbf{L}}\Theta \text{ and } \Theta R^c_{\mathbf{L}}\Delta\}$.

---

[15] A model $\mathcal{M} = (W, D, \delta, R, \rho)$ is symmetric, if for all $w, v \in W$, if wRv, then vRw.

[16] Note that when $\Diamond$ takes the common sense semantics, the Barcan formula still corresponds to the decreasing-domain property, as shown in [6].

For simplicity, we now fix a logic $\mathbf{L}$ which is at least as strong as $\mathbf{QK}^{\Rrightarrow}$ and contains both $\mathtt{B}^{\Box}$ and $\mathtt{BF}^{\Rrightarrow}_{\mathtt{Cond}}$, and omit all prefix and subscript "$\mathbf{L}$" in the following discussion.

Since we now have fewer edges in the symmetric canonical model, we need to prove a stronger version of existence lemma. The proof of such a lemma is more complex than the one for Lemma 3.4, since we can no longer handle the modal part and the first-order part separately here.

Lemma 4.3 *(Symmetric existence) For any $\Delta \in W^c$ and $\phi, \psi \in \mathcal{L}^{\Rrightarrow}_{+}$, if $\Diamond^{-\psi}\phi \in \Delta$ and $\mathbf{Y} \subseteq \mathbf{X}^+ \setminus FV(\Delta)$ is countable, then there is some $\Theta \in W^c$ s.t. $FV(\Theta) \subseteq FV(\Delta) \cup \mathbf{Y}$, $\Delta R^s \Theta$, $\phi \in \Theta$ and $\psi \notin \Theta$.*

Proof Let $\Delta \in W^c$ and $\phi, \psi \in \mathcal{L}^{\Rrightarrow}_{+}$ be arbitrary, and assume that $\Diamond^{-\psi}\phi \in \Delta$. Also let $\mathbf{Y}$ be an arbitrary countable subset of $\mathbf{X}^+ \setminus FV(\Delta)$, and let $\mathbf{Z}$ be some countable subset of $\mathbf{Y}$ s.t. $\mathbf{Y} \setminus \mathbf{Z}$ is countable.

We now construct a countable sequence $(\Phi_k)_{k\in\omega}$ s.t. for all $k \in \omega$, $\Diamond^{-\psi}\exists\overrightarrow{y_k}\bigwedge\Phi_k \in \Delta$, where $\overrightarrow{y_k}$ collects exactly all free variables in Φ_k that is not free in Δ (again, for simplicity, we use ϕ_k to denote $\bigwedge\Phi_k$).

As preparation, we first make the following enumerations:

- Enumerate all formulas as χ_0, χ_4, χ_8, ...
- Enumerate all formulas of the form $\exists x\xi$ as $\exists x_1\xi_1$, $\exists x_5\xi_5$, $\exists x_9\xi_9$, ...
- Enumerate all formulas of the form $\alpha \Rrightarrow \beta$ as $\alpha_2 \Rrightarrow \beta_2$, $\alpha_6 \Rrightarrow \beta_6$, $\alpha_{10} \Rrightarrow \beta_{10}$, ...
- Enumerate all formulas of the form $\gamma \Rrightarrow \mathsf{E}z$ as $\gamma_3 \Rrightarrow \mathsf{E}z_3$, $\gamma_7 \Rrightarrow \mathsf{E}z_7$, $\gamma_{11} \Rrightarrow \mathsf{E}z_{11}$, ...

As the starting point, let $\Phi_0 = \{\phi\}$. Clearly $\Diamond^{-\psi}\phi_0 \in \Delta$ by assumption.

Then, given Φ_k satisfying $\Diamond^{-\psi}\exists\overrightarrow{y_k}\phi_k$, we show how to construct Φ_{k+1}. Let l be the least natural number satisfying one of the followings:

(i) $FV(\chi_l) \subseteq FV(\Phi_k)$, but $\chi_l \notin \Phi_k$ and $\neg\chi_l \notin \Phi_k$;

(ii) $FV(\exists x_l\xi_l) \subseteq FV(\Phi_k)$, but $\exists x_l\xi_l \to \xi_l[y/x] \notin \Phi_k$ for all $y \in \mathbf{X}^+$;

(iii) $\alpha_l \Rrightarrow \beta_l \in \Delta$, $FV(\alpha_l) \subseteq FV(\Phi_k)$, but $\neg\alpha_l \notin \Phi_k$ and $\beta_l \notin \Phi_k$;

(iv) $\gamma_l \in \Delta$, $z_l \in \mathbf{Z}$, and $\gamma_l \Rrightarrow \mathsf{E}z_l \in \Phi_k$.

Correspondingly, we have four kinds of operations:

- If l satisfies (i), then we add either χ_l or $\neg\chi_l$ to Φ_k;
- If l satisfies (ii), then we add a witness formula $\exists x_l\xi_l \to \xi_l[y/x]$ to Φ_k, where y is a fresh variable in $\mathbf{Z}$ that we have *never* used before;[17]
- If l satisfies (iii), then we add either $\neg\alpha_l$ or β_l to Φ_k;

[17] Note that the variable y we choose here is not only a variable that does not occur freely in Φ_k, but a variable we have *never* used in the previous construction. Since we may also *remove* variables in $\mathbf{Z}$ from Φ_k's during the construction, we need this constraint to guarantee that for any index l, we need to deal with l for at most once.

- If l satisfies (iv), then we replace all free occurrences of z_l in Φ_k with a variable in $FV(\Delta)$, using $\mathtt{Bf}^{\Rrightarrow}_{\mathtt{Cond}}$ and the FV-$\exists$-property of Δ.

The first operation can be easily carried out using $\mathtt{DIV}^{\Rrightarrow}$, the second operation is basically only a matter of relettering, and the third operation is already demonstrated in the proof of Lemma 3.4. Thus, we only write down the fourth operation in detail here.

Assume that l satisfies (iv); for simplicity, also assume that $\overrightarrow{y_k} = z_l \overrightarrow{y}$. Since $\Diamond^{-\psi}\exists z_l \exists \overrightarrow{y} \phi_k \in \Delta$ and $\gamma_l \Rrightarrow \mathsf{E} z_l \in \Phi_k$, by $\mathtt{SUB}^{\Rrightarrow}$, we have

$$\Diamond^{-\psi}\exists z_l \exists \overrightarrow{y}(\phi_k \wedge (\gamma_l \Rrightarrow \mathsf{E} z_l)) \in \Delta$$

Since $\gamma_l \in \Delta$, we have $FV(\gamma_l) \subseteq FV(\Delta)$, so $\exists \overrightarrow{y}$ does not bound γ_l. Hence, we can push the quantifier $\exists \overrightarrow{y}$ inside (using $\mathtt{SUB}^{\Rrightarrow}$) to obtain the following:

$$\Diamond^{-\psi}\exists z_l (\exists \overrightarrow{y} \phi_k \wedge (\gamma_l \Rrightarrow \mathsf{E} z_l)) \in \Delta$$

Then, since we also have $\gamma_l \in \Delta$ and $z_l \notin FV(\gamma_l)$, by $\mathtt{Bf}^{\Rrightarrow}_{\mathtt{Cond}}$, it follows that $\exists z_l \Diamond^{-\psi} \exists \overrightarrow{y} \phi_k \in \Delta$. Thus, by the FV-$\exists$-property of Δ, there is some admissible $x \in FV(\Delta)$ s.t. $\Diamond^{-\psi}\exists \overrightarrow{y} \phi_k[x/z_l] \in \Delta$. Hence, let $\Phi_{k+1} = \Phi_k[x/z_l]$.[18]

Then, based on $(\Phi_k)_{k\in\omega}$, we define the following formula set:

$$\Theta = \{\theta \mid \text{ there is some } m \in \omega \text{ s.t. } \theta \in \bigcap_{k \geq m} \Phi_k\}$$

It is not hard to check that Θ is an FV-MCS with the FV-$\exists$-property, and satisfies that $\phi \in \Theta$, $\psi \notin \Theta$ and $\Delta R^c \Theta$. We only show in detail that $\Theta R^c \Delta$. It suffices to show that for all $\gamma \in \Delta$ and $z \in \mathbf{Z} \cap FV(\Theta)$, $\Diamond\gamma \in \Theta$ and $\Diamond^{-z}\gamma \in \Theta$.

For $\Diamond\gamma$: If $\Diamond\gamma \notin \Theta$, then $\Box\neg\gamma \in \Theta$, so there is some $k \in \omega$ s.t. $\Box\neg\gamma \in \Phi_k$. But then, we have $\Diamond^{-\psi}\Box\neg\gamma \in \Delta$, so $\Diamond\Box\neg\gamma \in \Delta$, and thus $\neg\gamma \in \Delta$ by $\mathtt{B}^{\Box}$, causing a contradiction. Thus, $\Diamond\gamma \in \Theta$.

For $\Diamond^{-z}\gamma$: If $\Diamond^{-z}\gamma \notin \Theta$, then $\gamma \Rrightarrow \mathsf{E} z \in \Theta$, so there is some $m \in \omega$ s.t. $\gamma \Rrightarrow \mathsf{E} z \in \bigcap_{k \geq m} \Phi_k$. But then, there must be some $n \geq m$ s.t. $\Phi_{n+1} = \Phi_n[x/z]$ for some $x \in FV(\Delta)$, contradicting that $z \in FV(\Theta)$. Thus, $\Diamond^{-z}\gamma \in \Theta$. $\Box$

With the help of the above existence lemma, the truth lemma can be easily proved. Thus, we have the following completeness theorem:

Theorem 4.4 *(Completeness of* $\mathbf{QKB}^{\Rrightarrow}$*,* $\mathbf{QDB}^{\Rrightarrow}$ *and* $\mathbf{QTB}^{\Rrightarrow}$*)*

- $\mathbf{QKB}^{\Rrightarrow} := \mathbf{QK}^{\Rrightarrow} \oplus \{\mathtt{B}^{\Box}, \mathtt{Bf}^{\Rrightarrow}_{\mathtt{Cond}}\}$ *is sound and strongly complete w.r.t. the class of all symmetric models;*
- $\mathbf{QDB}^{\Rrightarrow} := \mathbf{QD}^{\Rrightarrow} \oplus \{\mathtt{B}^{\Box}, \mathtt{Bf}^{\Rrightarrow}_{\mathtt{Cond}}\}$ *is sound and strongly complete w.r.t. the class of all serial and symmetric models;*
- $\mathbf{QTB}^{\Rrightarrow} := \mathbf{QT}^{\Rrightarrow} \oplus \{\mathtt{B}^{\Box}, \mathtt{Bf}^{\Rrightarrow}_{\mathtt{Cond}}\}$ *is sound and strongly complete w.r.t. the class of all reflexive and symmetric models.*

[18] To avoid using the axiom of choice, we may assume that $\mathbf{X}^+$ is indexed by natural numbers, and let the x here be the variable in $\mathbf{X}^+$ with the least index number s.t. $\Diamond^{-\psi}\exists \overrightarrow{y} \phi_k[x/z_l] \in \Delta$.

5 Transitivity and continual existence

In this section, we consider axiomatizations on transitive models.[19] Unfortunately, we still do not know how to axiomatize the logic of $\Rrightarrow$ on the class of *all* transitive models. But we can offer a rather simple axiomatization for a special kind of transitive models - namely, transitive models with the *continual existence property.*

5.1 The continual existence property

The continual existence property is defined as follows:

Definition 5.1 (CEP) We say a model $\mathcal{M} = (W, D, \delta, R, \rho)$ has the *continual existence property* (CEP for short), iff for all $w, v, u \in W$ s.t. wRv and vRu, we have $\delta(w) \cap \delta(u) \subseteq \delta(v)$.

Such a property seems very natural when we interpret the accessibility relation in the models in a *temporal* sense: interpreted this way, the property says that the existence of an object should always be *continual*: if an object exists at some time t_0, and also exists at some later time t_1, then it should also exist at any time between t_0 and t_1.[20]

On the technical level, it is also easier to offer axiomatizations for transitive models with CEP. A transitive model with such a property validates the following two axioms:

$$\begin{array}{ll} 4^{\Rrightarrow}_{\Box} & (\phi \Rrightarrow \psi) \to \Box(\phi \Rrightarrow \psi) \\ 4^{\Rrightarrow}_{\Diamond} & (\phi \Rrightarrow \psi) \to (\Diamond\phi \Rrightarrow \Diamond\psi) \end{array}$$

In contrast, on a transitive model without CEP, while $4^{\Rrightarrow}_{\Box}$ is still valid, $4^{\Rrightarrow}_{\Diamond}$ might not hold. But our completeness proof below would not work without the latter axiom.

5.2 Completeness

Again, even when a logic no weaker than $\mathbf{QK}^{\Rrightarrow}$ has both $4^{\Rrightarrow}_{\Box}$ and $4^{\Rrightarrow}_{\Diamond}$ as its axioms, the corresponding canonical model is still non-transitive, nor does it has CEP. Thus, we will find *sub-models* within the canonical model with all the properties we need.

Now, we fix a logic $\mathbf{L}$ which is at least as strong as $\mathbf{QK}^{\Rrightarrow}$ with both $4^{\Rrightarrow}_{\Box}$ and $4^{\Rrightarrow}_{\Diamond}$, and omit all the prefix and subscript "$\mathbf{L}$" in the following proofs.

The following lemma shows how to construct the sub-models we need for a FV-MCSs in the canonical model. Such a construction gives us a better control on the local domains of the points in the model, so that we can secure transitivity and CEP for the model.

---

[19] A model $\mathcal{M} = (W, D, \delta, R, \rho)$ is *transitive* if for all $w, v, u \in W$, wRv and vRu implies wRu.

[20] We have to admit that when the accessibility relation in the models is interpreted in terms of knowledge, belief or necessity, it becomes more difficult to make sense of CEP. In particular, note that if the accessibility relation is *symmetric*, then CEP implies *constant domain*.

Lemma 5.2 *Let $\Delta \in W^c$. Then, there is a sub-model $\mathcal{M}_\Delta = (W, D, \delta, R, \rho)$ of $\mathcal{M}^c$, s.t. $\Delta \in W$, and the followings hold:*

- *$\mathcal{M}_\Delta$ is transitive and has CEP;*
- *For all $\Theta \in W$ and $\Diamond^{-\psi}\phi \in \Theta$, there is $\Gamma \in W$ s.t. $\Theta R \Gamma$, $\phi \in \Gamma$ and $\psi \notin \Gamma$.*

Proof In order to construct the model $\mathcal{M}_\Delta$, we construct a countable sequence $(\Delta_k, R_k)_{k\in\omega}$, s.t. for each $k \in \omega$, we have $\Delta_k \in W^c$, $\mathbf{X}^+ \setminus \bigcup_{i\leq k} FV(\Delta_i)$ is countable, $R_k \subseteq R^c$ is a transitive relation on $\{\Delta_i \mid i \leq k\}$, and for all $i, m, j \leq k$, $\Delta_i R_k \Delta_m R_k \Delta_j$ implies $FV(\Delta_i) \cap FV(\Delta_j) \subseteq FV(\Delta_m)$. At each step of the construction, we add a witness for a $\Diamond^{-\psi}\phi$ formula in some MCS that has already been introduced.

As preparation, we first enumerate all pairs of natural number and formulas in $\mathcal{L}_+^{\Rrightarrow}$ of the form $\Diamond^{-\psi}\phi$ as $(n_0, \Diamond^{-\psi_0}\phi_0)$, $(n_1, \Diamond^{-\psi_1}\phi_1)$, $(n_2, \Diamond^{-\psi_2}\phi_2)$, ...

As the starting point of the construction, let $(\Delta_0, R_0) = (\Delta, \emptyset)$. Clearly it satisfies all conditions we require.

Then, given (Δ_0, R_0), ..., $(\Delta_k R_k)$, we show how to construct (Δ_{k+1}, R_{k+1}). Let l be the least natural number s.t. $n_l \leq k$, $\Diamond^{-\psi_l}\phi_l \in \Delta_{n_l}$, but there is no $i \leq k$ s.t. $\Delta_{n_l} R_k \Delta_i$, $\phi \in \Delta_i$ and $\psi \notin \Delta_i$ (if there is no such l, just let $\Delta_{k+1} = \Delta_k$ and $R_{k+1} = R_k$). Since $\mathbf{X}^+ \setminus \bigcup_{i\leq k} FV(\Delta_i)$ is countable, let $\mathbf{Y}$ be a countable subset of $\mathbf{X}^+ \setminus \bigcup_{i\leq k} FV(\Delta_i)$ s.t. $\mathbf{X}^+ \setminus (\bigcup_{i\leq k} FV(\Delta_i) \cup \mathbf{Y})$ is still countable. Then Lemma 3.4 gives us a $\Theta \in W^c$ s.t. $FV(\Theta) \subseteq FV(\Delta_{n_l}) \cup \mathbf{Y}$, $\Delta_{n_l} R^c \Theta$, $\phi \in \Theta$ and $\psi \notin \Theta$. Also note that according to our construction in Lemma 3.4, $FV(\Theta) \cap \mathbf{Y} \neq \emptyset$, so $\Theta \notin \{\Delta_i \mid i \leq k\}$.

Then, let $\Delta_{k+1} = \Theta$, and define R_{k+1} as follows:

$$R_{k+1} = R_k \cup \{(\Delta_{n_l}, \Theta)\} \cup \{(\Delta_i, \Theta) \mid i \leq k \text{ and } \Delta_i R_k \Delta_{n_l}\}$$

Clearly $\Delta_{k+1} = \Theta \in W^c$, $\mathbf{X}^+ \setminus \bigcup_{i\leq k+1} FV(\Delta_i)$ is countable and R_{k+1} is transitive. The only non-trivial part is to show that (i) $R_{k+1} \subseteq R^c$ and (ii) for all $i, m, j \leq k+1$, $\Delta_i R_{k+1} \Delta_m R_{k+1} \Delta_j$ implies $FV(\Delta_i) \cap FV(\Delta_j) \subseteq FV(\Delta_m)$.

For (i): We only need to show that for all $i \leq k$, $\Delta_i R_k \Delta_{n_l}$ implies $\Delta_i R^c \Theta$. Assume that $\Delta_i R_k \Delta_{n_l}$. Let $\alpha \Rrightarrow \beta \in \Delta_i$ be arbitrary, and also assume that $\alpha \in \Theta$. Then, $FV(\alpha) \subseteq FV(\Delta_i) \cap FV(\Theta)$, which implies $FV(\alpha) \subseteq FV(\Delta_{n_l})$, according to our construction of Θ. Thus, $\Diamond\alpha \in \Delta_{n_l}$. Then, since $\alpha \Rrightarrow \beta \in \Delta_i$, by $4^{\Rrightarrow}_{\Diamond}$, $\Diamond\alpha \Rrightarrow \Diamond\beta \in \Delta_i$; so since $\Delta_i R_k \Delta_{n_l}$ (and thus $\Delta_i R^c \Delta_{n_l}$), $\Diamond\beta \in \Delta_{n_l}$. Thus, $FV(\alpha \Rrightarrow \beta) \subseteq FV(\Delta_{n_l})$. Then, by $4^{\Rrightarrow}_{\Box}$, we also have $\Box(\alpha \Rrightarrow \beta) \in \Delta_i$, so $\alpha \Rrightarrow \beta \in \Delta_{n_l}$. Finally, since $\Delta_{n_l} R^c \Theta$, we have $\beta \in \Theta$. Thus, $\Delta_i R^c \Theta$.

For (ii): Since $\Theta \notin \{\Delta_i \mid i \leq k\}$, we only need to show that for all $i, j \leq k$, $\Delta_i R_k \Delta_j R_{k+1} \Theta$ implies $FV(\Delta_i) \cap FV(\Theta) \subseteq FV(\Delta_j)$. Assume that $\Delta_i R_k \Delta_j R_{k+1} \Theta$. As we have noted above, clearly $FV(\Delta_i) \cap FV(\Theta) \subseteq FV(\Delta_{n_l})$; and according to our construction of R_{k+1}, either $\Delta_j R_k \Delta_{n_l}$ or $\Delta_j = \Delta_{n_l}$. Thus, $FV(\Delta_i) \cap FV(\Theta) \subseteq FV(\Delta_i) \cap FV(\Delta_{n_l}) \subseteq FV(\Delta_j)$.

Finally, let $\mathcal{M}_\Delta = (W, D, \delta, R, \rho)$, where $W = \{\Delta_k \mid k \in \omega\}$, $D = D^c$, $\delta = \delta^c \restriction_W$, $R = \bigcup_{k\in\omega} R_k$, and $\rho = \rho^c \restriction_{\mathcal{P}\times W}$. It is not hard to check that Δ has all the properties we need. $\square$

Remark 5.3 Also notice that when $\mathtt{T}^{\Box}$ is in $\mathbf{L}$, the reflexive closure of the kind of sub-model constructed above also has all the properties we need.

Then, we can also easily prove a version of the truth lemma for these sub-models, so the following completeness results are direct:

Theorem 5.4 *(Completeness of $\mathbf{QK4}^{\Rrightarrow}_{CE}$, $\mathbf{QD4}^{\Rrightarrow}_{CE}$ and $\mathbf{QS4}^{\Rrightarrow}_{CE}$)*

- $\mathbf{QK4}^{\Rrightarrow}_{CE} := \mathbf{QK}^{\Rrightarrow} \oplus \{\mathtt{4}^{\Rrightarrow}_{\Box}, \mathtt{4}^{\Rrightarrow}_{\Diamond}\}$ *is sound and strongly complete w.r.t. the class of all transitive models with CEP;*
- $\mathbf{QD4}^{\Rrightarrow}_{CE} := \mathbf{QD}^{\Rrightarrow} \oplus \{\mathtt{4}^{\Rrightarrow}_{\Box}, \mathtt{4}^{\Rrightarrow}_{\Diamond}\}$ *is sound and strongly complete w.r.t. the class of all serial and transitive models with CEP;*
- $\mathbf{QS4}^{\Rrightarrow}_{CE} := \mathbf{QT}^{\Rrightarrow} \oplus \{\mathtt{4}^{\Rrightarrow}_{\Box}, \mathtt{4}^{\Rrightarrow}_{\Diamond}\}$ *is sound and strongly complete w.r.t. the class of all reflexive and transitive models with CEP.*

6 Conclusion

In this paper, we introduced a strict implication operator $\Rrightarrow$, which enables us to make reasoning involving the existence and non-existence of objects on varying-domain Kripke models, in a way that does not resort to free logic. We offered complete axiomatizations for the FOML of $\Rrightarrow$ on the class of all varying-domain models, as well as the class of symmetric models and the class of transitive models with the continual existence property.

Based on our work, there are a lot of interesting directions for future work.

For example, we may further study axiomatizations of the logics of $\Rrightarrow$. There are a lot of model classes that have not been touched upon in this paper: even the axiomatization on the class of transitive models without CEP is still unknown, not to mention the class of $S5$-models; and the techniques we used in this paper do not apply to these cases directly. We may also add constant symbols, functional symbols and the identity relation to our language, and consider how to characterize their behavior in the axiomatizations.

It also seems interesting to study the other aspects of the $\Rrightarrow$-operator, e.g. its proof theoretical properties or frame definability. In [4], a sequent calculi for common sense FOML is offered, and it is also shown that cut elimination holds for such a calculi; so it is natural to wonder whether similar results can also be shown for the FOML of $\Rrightarrow$. On the other hand, in [6] and [5], frame correspondences of the common sense $\Box$ is discussed; then, since $\Rrightarrow$ is more expressive than the common sense $\Box$, it also seems interesting to consider what frame classes can we define using $\Rrightarrow$.

We may also consider further generalizations of the $\Rrightarrow$-operator: for example, for each $n \in \omega$, we can introduce an $(n+1)$-ary operator $\phi \Rrightarrow (\psi_1, ..., \psi_n)$ with the following semantics:

$$\begin{array}{lcl} \mathcal{M}, w, \sigma \vDash \phi \Rrightarrow (\psi_1, ..., \psi_n) & \iff & \text{for all } v \in W \text{ s.t. } wRv, \\ & & \text{if } \sigma[FV(\phi)] \subseteq D_v \text{ and } \mathcal{M}, v, \sigma \vDash \phi, \\ & & \text{then there is } 1 \leq i \leq n \text{ s.t.} \\ & & \sigma[FV(\psi_i)] \subseteq D_v \text{ and } \mathcal{M}, v, \sigma \vDash \psi_i \end{array}$$

A language with such polyadic operators would clearly be more expressive than the language with only a binary $\Rrightarrow$-operator, but it should not be very difficult to generalize the axiomatizations presented in this paper to the polyadic case. It also seems interesting to consider whether the language with all such polyadic strict implications is in a sense "maximal" in terms of expressivity.

Acknowledgement

I would like to thank Johan van Benthem and Jeremy Seligman for offering me information about the common sense modal predicate logic. I would also like to thank Yanjing Wang for his advices on the writing of this paper. Finally, I would like to thank three anonymous reviewers from AiML 2024 for their critical advices and insightful comments on this work.

References

[1] Cresswell, M. J. and G. E. Hughes, "A New Introduction to Modal Logic," Routledge, New York, 1996.

[2] Fitting, M. and R. L. Mendelsohn, "First-Order Modal Logic," Kluwer Academic Publishers, Dordrecht, Netherland, 1998.

[3] Garson, J. W., *Quantification in modal logic*, in: D. M. Gabbay and F. Guenthner, editors, *Handbook of Philosophical Logic*, Springer Netherlands, Dordrecht, 2001 pp. 267–323.

[4] Sawasaki, T. and K. Sano, *Proof-theoretic results of common sense modal predicate calculi*, in: B. Liao and Y. N. Wáng, editors, *Context, Conflict and Reasoning* (2020), pp. 127–138.

[5] Sawasaki, T. and K. Sano, *Frame definability, canonicity and cut elimination in common sense modal predicate logics*, Journal of Logic and Computation **31** (2021), pp. 1933–1958.

[6] van Benthem, J., *Frame correspondences in modal predicate logic*, Learning and Motivation - LEARN MOTIV (2010).

[7] van Benthem, J., "Modal Logic for Open Minds," Center for the Study of Language and Information, Stanford, California, 2010.

A Appendix

Example A.1 Consider the sentence $\exists x \forall y(\mathsf{E}x \Rrightarrow \mathsf{E}y)$: we can find a pair of models that can be distinguished by this sentence, but cannot be distinguished in the first-order modal language with both $\Box$ and $\boxdot$ (and without identity).

Let $\mathcal{M} = (W, D, \delta, R, \rho)$ and $\mathcal{N} = (W', D, \delta', R', \rho')$, where

- $W = \{w_0, w_1, w_2, w_3\}$, $W' = \{v_0, v_1, v_2\}$, and $D = \{a, b\}$;
- $\delta = \{(w_0, \{a, b\}), (w_1, \{a, b\}), (w_2, \{a\}), (w_3, \{b\})\}$, and $\delta' = \{(v_0, \{a\}), (v_1, \{a\}), (v_2, \{b\})\}$;
- $R = \{(w_0, w_i) \mid i \in \{1, 2, 3\}\}$, and $R' = \{(v_0, w_i) \mid i \in \{1, 2\}\}$;
- For all $P \in \mathcal{P}$, ρ satisfies that $\rho(P, w) = \emptyset$ for all $w \in W$, and ρ' satisfies that $\rho(P, v) = \emptyset$ for all $v \in W'$.

Below is a simple diagram of $\mathcal{M}$ and $\mathcal{N}$:

$$\begin{tikzcd} & w_1\{a,b\} \\ \mathcal{M} \quad w_0\{a,b\} \arrow[ur] \arrow[r] \arrow[dr] & w_2\{a\} \\ & w_3\{b\} \end{tikzcd} \qquad \begin{tikzcd} & v_1\{a\} \\ \mathcal{N} \quad v_0\{a\} \arrow[ur] \arrow[dr] & \\ & v_2\{b\} \end{tikzcd}$$

It is easy to check that $\mathcal{M}, w_0 \nvDash \exists x \forall y(\mathsf{E}x \Rrightarrow \mathsf{E}y)$ and $\mathcal{N}, v_0 \vDash \exists x \forall y(\mathsf{E}x \Rrightarrow \mathsf{E}y)$. However, it is also not hard to check by induction that $\mathcal{M}$ and $\mathcal{N}$ cannot be distinguished in the first-order modal language with both $\Box$ and $\boxdot$ (and without identity).

Proposition A.2 $\mathtt{ID}^{\Rrightarrow}$, $\mathtt{ANT}^{\Rrightarrow}$, $\mathtt{AS}^{\Rrightarrow}$, $\mathtt{MONO}^{\Rrightarrow}$, $\mathtt{SUB}^{\Rrightarrow}$, $\mathtt{DIV}^{\Rrightarrow}$, $\mathtt{K}^{\Rrightarrow}$, $\mathtt{K}_{inv}$, $\Box$-$\mathtt{Alt}$, $\Rrightarrow\mathtt{to}\Box$ *and* $\Rrightarrow$-$\mathtt{Rdc}$ *are all theorems / derivable rules of* $\mathbf{QK}^{\Rrightarrow}$.

Proof (For simplicity, in the following proof, when we write $\vdash \phi$, we mean that ϕ is a theorem of $\mathbf{QK}^{\Rrightarrow}$.)

For $\mathtt{ID}^{\Rrightarrow}$: Clearly we have $\vdash \phi \to \phi$. Then, since $FV(\phi) \subseteq FV(\phi)$, we can apply $\mathtt{NEC}^{\Rrightarrow}$ to obtain $\vdash \phi \Rrightarrow \phi$.

For $\mathtt{ANT}^{\Rrightarrow}$: By $\mathtt{AND}^{\Rrightarrow}$, we have $\vdash ((\phi \Rrightarrow \phi) \wedge (\phi \Rrightarrow \psi)) \to (\phi \Rrightarrow (\phi \wedge \psi))$; and by $\mathtt{ID}^{\Rrightarrow}$, we have $\vdash \phi \Rrightarrow \phi$. So we can combine them to obtain $\vdash (\phi \Rrightarrow \psi) \to (\phi \Rrightarrow (\phi \wedge \psi))$.

For $\mathtt{AS}^{\Rrightarrow}$: Clearly, we have $\vdash (\phi \wedge \chi) \to \phi$. Then, since $FV(\phi) \subseteq FV(\phi \wedge \chi)$, we can apply $\mathtt{NEC}^{\Rrightarrow}$ to obtain $\vdash (\phi \wedge \chi) \Rrightarrow \phi$. Then, by $\mathtt{CH}^{\Rrightarrow}$, it follows that $\vdash (\phi \Rrightarrow \psi) \to ((\phi \wedge \chi) \Rrightarrow \psi)$.

For $\mathtt{MONO}^{\Rrightarrow}$: Assume that $\vdash \phi \to \psi$ and $FV(\psi) \subseteq FV(\phi)$. Then, by $\mathtt{NEC}^{\Rrightarrow}$, $\vdash \phi \Rrightarrow \psi$; so by $\mathtt{CH}^{\Rrightarrow}$, it follows that $\vdash (\psi \Rrightarrow \chi) \to (\phi \Rrightarrow \chi)$. For $\mathtt{SUB}^{\Rrightarrow}$, Just apply $\mathtt{MONO}^{\Rrightarrow}$ twice.

For $\mathtt{DIV}^{\Rrightarrow}$: Assume that $FV(\chi) \subseteq FV(\phi)$, we prove the following:

$$\vdash (\exists \vec{x}(\phi \wedge \chi) \Rrightarrow \psi) \wedge (\exists \vec{x}(\phi \wedge \neg\chi) \Rrightarrow \psi) \to (\exists \vec{x}\phi \Rrightarrow \psi)$$

Then, since $FV(\chi) \subseteq FV(\phi)$, by $\mathtt{SUB}^{\Rrightarrow}$, we have

$$\vdash ((\exists \vec{x}(\phi \wedge \chi) \vee \exists \vec{x}(\phi \wedge \neg\chi)) \Rrightarrow \psi) \to (\exists \vec{x}\phi \Rrightarrow \psi)$$

Moreover, by $\mathtt{OR}^{\Rrightarrow}$, we have

$$\vdash (\exists \vec{x}(\phi \wedge \chi) \Rrightarrow \psi) \wedge (\exists \vec{x}(\phi \wedge \neg\chi) \Rrightarrow \psi) \to ((\exists \vec{x}(\phi \wedge \chi) \vee \exists \vec{x}(\phi \wedge \neg\chi)) \Rrightarrow \psi)$$

Thus, the desired theorem can be obtained using $\mathtt{MP}$.

For $\mathtt{K}^{\Rrightarrow}$: Note that $\mathtt{K}^{\Rrightarrow}$ is equivalent to the following formula, according to our definition of $\Diamond^{-\psi}$:

$$(\alpha \Rrightarrow \beta) \wedge ((\phi \wedge \beta) \Rrightarrow \psi) \to ((\phi \wedge \alpha) \Rrightarrow \psi)$$

Hence, we only show that this formula is a theorem of $\mathbf{QK}^{\Rrightarrow}$. First, by $\mathtt{AS}^{\Rrightarrow}$ (and also $\mathtt{SUB}^{\Rrightarrow}$), we have $\vdash (\alpha \Rrightarrow \beta) \to ((\phi \wedge \alpha) \Rrightarrow \beta)$. Then, by $\mathtt{ANT}^{\Rrightarrow}$, it follows that $\vdash (\alpha \Rrightarrow \beta) \to ((\phi \wedge \alpha) \Rrightarrow (\phi \wedge \alpha \wedge \beta))$. We can also easily show $\vdash (\phi \wedge \alpha \wedge \beta) \Rrightarrow (\phi \wedge \beta)$ using $\mathtt{NEC}^{\Rrightarrow}$, so we have $\vdash (\alpha \Rrightarrow \beta) \to ((\phi \wedge \alpha) \Rrightarrow (\phi \wedge \beta))$ by $\mathtt{CH}^{\Rrightarrow}$. Finally, by $\mathtt{CH}^{\Rrightarrow}$ again, we can obtain $\vdash (\alpha \Rrightarrow \beta) \wedge ((\phi \wedge \beta) \Rrightarrow \psi) \to ((\phi \wedge \alpha) \Rrightarrow \psi)$, as intended.

For $\mathtt{K}_{inv}$: Assume that $FV(\phi) \subseteq FV(\psi)$. We first show that in this case, we have $\vdash \Box(\phi \to \psi) \to (\neg\psi \Rrightarrow \neg\phi)$. Recall that $\Box(\phi \to \psi)$ is the abbreviation for $\neg(\phi \to \psi) \Rrightarrow \bot$, i.e. $(\phi \wedge \neg\psi) \Rrightarrow \bot$. Thus, by $\mathtt{EXP}^{\Rrightarrow}$ and $\mathtt{CH}^{\Rrightarrow}$, it is easy to see that $\vdash \Box(\phi \to \psi) \to (\phi \wedge \neg\psi) \Rrightarrow \neg\phi$. On the other hand, by $\mathtt{NEC}^{\Rrightarrow}$, we also have $\vdash (\neg\phi \wedge \neg\psi) \Rrightarrow \neg\phi$. Thus, by $\mathtt{OR}^{\Rrightarrow}$, it follows that

$$\vdash \Box(\phi \to \psi) \to (((\phi \wedge \neg\psi) \vee (\neg\phi \wedge \neg\psi)) \Rrightarrow \neg\phi)$$

Since $FV(\phi) \subseteq FV(\psi)$, by $\mathtt{SUB}^{\Rrightarrow}$, it follows that $\vdash \Box(\phi \to \psi) \to (\neg\psi \Rrightarrow \neg\phi)$. Finally, since $\Box\phi$ and $\Box\psi$ are abbreviations of $\neg\phi \Rrightarrow \bot$ and $\neg\psi \Rrightarrow \bot$ respectively, by $\mathtt{CH}^{\Rrightarrow}$, we have $\vdash \Box(\phi \to \psi) \to ((\neg\phi \Rrightarrow \bot) \to (\neg\psi \Rrightarrow \bot))$, i.e. $\vdash \Box(\phi \to \psi) \to (\Box\phi \to \Box\psi)$.

For $\Box$-$\mathtt{Alt}$: We first show that $\vdash \Box\phi \to ((\bigwedge_{x \in FV(\phi)} \mathsf{E}x) \Rrightarrow \phi)$. Since $\Box\phi$ is the abbreviation for $\neg\phi \Rrightarrow \bot$, by $\mathtt{EXP}^{\Rrightarrow}$ and $\mathtt{CH}^{\Rrightarrow}$, we have $\vdash \Box\phi \to (\neg\phi \Rrightarrow \phi)$. Thus, by $\mathtt{AS}^{\Rrightarrow}$, it follows that $\vdash \Box\phi \to ((\bigwedge_{x \in FV(\phi)} \mathsf{E}x \wedge \neg\phi) \Rrightarrow \phi)$. On the other hand, by $\mathtt{NEC}^{\Rrightarrow}$, we have $\vdash (\bigwedge_{x \in FV(\phi)} \mathsf{E}x \wedge \phi) \Rrightarrow \phi$. Thus, by $\mathtt{OR}^{\Rrightarrow}$, it follows that

$$\vdash \Box\phi \to (((\bigwedge_{x \in FV(\phi)} \mathsf{E}x \wedge \phi) \vee (\bigwedge_{x \in FV(\phi)} \mathsf{E}x \wedge \neg\phi)) \Rrightarrow \phi)$$

Thus, $\vdash \Box\phi \to ((\bigwedge_{x \in FV(\phi)} \mathsf{E}x) \Rrightarrow \phi)$ by $\mathtt{SUB}^{\Rrightarrow}$.

Next, we show that $\vdash ((\bigwedge_{x \in FV(\phi)} \mathsf{E}x) \Rrightarrow \phi) \to \Box\phi$. By $\mathtt{MONO}^{\Rrightarrow}$, we have $\vdash ((\bigwedge_{x \in FV(\phi)} \mathsf{E}x) \Rrightarrow \phi) \to (\neg\phi \Rrightarrow \phi)$; and by $\mathtt{ANT}^{\Rrightarrow}$, we also have $\vdash (\neg\phi \Rrightarrow \phi) \to (\neg\phi \Rrightarrow (\neg\phi \wedge \phi))$, so $\vdash (\neg\phi \Rrightarrow \phi) \to (\neg\phi \Rrightarrow \bot)$, i.e. $\vdash (\neg\phi \Rrightarrow \phi) \to \Box\phi$. Hence, by $\mathtt{MP}$, we obtain $\vdash ((\bigwedge_{x \in FV(\phi)} \mathsf{E}x) \Rrightarrow \phi) \to \Box\phi$.

For $\Rrightarrow\mathtt{to}\Box$: Recall that $\Box(\phi \to \psi)$ is the abbreviation for $\neg(\phi \to \psi) \Rrightarrow \bot$, which is equivalent to $(\phi \wedge \neg\psi) \Rrightarrow \bot$. Thus, we only need to show that $\vdash (\phi \Rrightarrow \psi) \to ((\phi \wedge \neg\psi) \Rrightarrow \bot)$. By $\mathtt{MONO}^{\Rrightarrow}$, we have $\vdash (\phi \Rrightarrow \psi) \to ((\phi \wedge \neg\psi) \Rrightarrow \psi)$; and by $\mathtt{NEC}^{\Rrightarrow}$, we also have $\vdash (\phi \wedge \neg\psi) \Rrightarrow \neg\psi$. Then, by $\mathtt{AND}^{\Rrightarrow}$, it follows that $\vdash (\phi \Rrightarrow \psi) \to ((\phi \wedge \neg\psi) \Rrightarrow (\psi \wedge \neg\psi))$, so $\vdash (\phi \Rrightarrow \psi) \to ((\phi \wedge \neg\psi) \Rrightarrow \bot)$. Hence, $\vdash (\phi \Rrightarrow \psi) \to \Box(\phi \to \psi)$.

For $\Rrightarrow$-`RDC`: We first show that $\vdash \Diamond^{-\psi}\phi \to (\Diamond(\phi \wedge \neg\psi) \vee \bigvee_{x \in FV(\psi)} \Diamond^{-x}\phi)$. We only need to show that we have the following theorem in $\mathbf{QK}^{\Rrightarrow}$:

$$(\Box(\phi \to \psi) \wedge \bigwedge_{x \in FV(\psi)} (\phi \Rrightarrow \mathsf{E}x)) \to (\phi \Rrightarrow \psi)$$

By $\mathtt{AND}^{\Rrightarrow}$, we have $\vdash \bigwedge_{x \in FV(\psi)}(\phi \Rrightarrow \mathsf{E}x) \to (\phi \Rrightarrow \bigwedge_{x \in FV(\psi)} \mathsf{E}x)$. Moreover, by $\mathtt{ANT}^{\Rrightarrow}$, we have $\vdash (\phi \Rrightarrow \bigwedge_{x \in FV(\psi)} \mathsf{E}x) \to (\phi \Rrightarrow (\phi \wedge \bigwedge_{x \in FV(\psi)} \mathsf{E}x))$; and by $\mathtt{NEC}^{\Rrightarrow}$, we also have $\vdash (\phi \wedge \bigwedge_{x \in FV(\psi)} \mathsf{E}x) \Rrightarrow \bigwedge_{x \in FV(\phi) \cup FV(\psi)} \mathsf{E}x$. Thus, by $\mathtt{CH}^{\Rrightarrow}$, we have

$$\vdash \bigwedge_{x \in FV(\psi)} (\phi \Rrightarrow \mathsf{E}x) \to (\phi \Rrightarrow \bigwedge_{x \in FV(\phi) \cup FV(\psi)} \mathsf{E}x)$$

Then, by $\Box$-`Alt`, $\vdash \Box(\phi \to \psi) \to ((\bigwedge_{x \in FV(\phi) \cup FV(\psi)} \mathsf{E}x) \Rrightarrow (\phi \to \psi))$. Thus, by $\mathtt{CH}^{\Rrightarrow}$, we obtain the following:

$$\vdash (\Box(\phi \to \psi) \wedge \bigwedge_{x \in FV(\psi)} (\phi \Rrightarrow \mathsf{E}x)) \to (\phi \Rrightarrow (\phi \to \psi))$$

Thus, we only need to check that $\vdash (\phi \Rrightarrow (\phi \to \psi)) \to (\phi \Rrightarrow \psi)$. This is easy: by $\mathtt{ANT}^{\Rrightarrow}$, we have $\vdash (\phi \Rrightarrow (\phi \to \psi)) \to (\phi \Rrightarrow (\phi \wedge (\phi \to \psi))$; and by $\mathtt{NEC}^{\Rrightarrow}$, we have $\vdash (\phi \wedge (\phi \to \psi)) \Rrightarrow \psi$. Thus, $\vdash (\phi \Rrightarrow (\phi \to \psi)) \to (\phi \Rrightarrow \psi)$ follows by $\mathtt{CH}^{\Rrightarrow}$, and we are done.

Then, we show that $\vdash (\Diamond(\phi \wedge \neg\psi) \vee \bigvee_{x \in FV(\psi)} \Diamond^{-x}\phi) \to \Diamond^{-\psi}\phi$. This is the relatively easy direction: we only need to show that $\vdash (\phi \Rrightarrow \psi) \to \Box(\phi \to \psi)$ and $\vdash (\phi \Rrightarrow \psi) \to (\phi \Rrightarrow \mathsf{E}x)$ for all $x \in FV(\psi)$. The former is just the theorem $\Rrightarrow$`to`$\Box$, and the latter follows directly from $\mathtt{NEC}^{\Rrightarrow}$ and $\mathtt{CH}^{\Rrightarrow}$. Thus, this direction of the implication is also proved. $\Box$