A Modal Logic for Reasoning in Contexts

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Abstract

We propose a new semantics for indicatives and epistemic modals as well as a new notion of validity, according to which an inference is not valid or invalid per se, but valid or invalid under a set of assumptions. In the new semantics, a formula is not evaluated at a pointed model, but at a pointed model together with a context, provided either by the assumptive premises or by the antecedents of indicatives. Armed with the new semantic framework, we differentiate three forms of inferences, two notions of equivalence, and two notions of inconsistency, which can be used to solve some famous puzzles concerning indicatives and epistemic modals in a unified way. Our semantics may shed some light on natural language reasoning in contexts.

Keywords: indicatives, epistemic modality, validity, modus ponens, import-export, Moore sentences, Wittgenstein sentences, strict-tolerant logic

1 Three Puzzles

1.1 The barbershop paradox

In 1894, Lewis Carroll proposed a famous logical paradox [5], whose reformulation amounts to a counterexample to Modus Tollens. Though Modus Tollens seems equally valid as Modus Ponens, unlike Modus Ponens, counterexamples to Modus Tollens have been frequently discovered in the literature. Some involve nested conditionals [33], like Carroll's example. Others involve deontic, epistemic, or probabilistic operators [16,37].

Carroll's logical paradox invites us to consider the following scenario. Three barbers Allen, Brown, and Carr runs a barbershop. The barbershop is always open, which means at least one of the barbers must be in. Allen has been very nervous after a fever so that he never leaves the shop unless Brown goes out with him. Can we conclude that Carr is in the shop? Obviously not, for when Brown is in and Carr is out, both the constraints about the barbershop can be satisfied. On the other hand, we also have the following inference.

(1) A counterexample to Modus Tollens.

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(a) If Carr is out then if Allen is out then Brown is in.
(b) It is not the case that if Allen is out then Brown is in.
(c) ? Therefore, it is not the case that Carr is out.

The first premise (1 a) rings true, because at least one barber must be in. The second premise (1 b) is also true, because when Allen is out Brown must be out with him. Using Modus Tollens, we obtain (1 c), which is false. [2]

There are basically three approaches to solving the paradox. One is denying that both premises are true. The other is denying that the inference is an instance of Modus Tollens. The third is denying that Modus Tollens is universally valid. The first two approaches seem ad hoc. The third approach demands a new semantics and logic for indicative conditionals, which invalidates Modus Tollens but can also explain why in most cases Modus Tollens is innocuous.

1.2 Might

George Moore noticed that sentences like (2) sounds strange.

(2) It's raining and I don't know it's raining.

On one hand, it can be true. On the other hand, it seems that I cannot consistently assert it. One solution to dispel the tension is appealing to pragmatics. According to this view, the sentence is consistent in semantics but inconsistent in pragmatics. One evidence supporting this solution is that if we replace 'I' by my name in the sentence and the modified sentence is asserted by others, there will be no inconsistency. Another evidence supporting the solution is that if we put the sentence into an assumption rather than asserting it, there will be no inconsistency either. For example, the following sentences sound felicitous.

(3) (a) Suppose it's raining and I don't know it.
(b) If it's raining and I don't know it, then...

Wittgenstein noticed that sentences like (4) is also weird.

(4)# It's raining outside and it might not be raining outside.

On one hand, the sentence which can be formulated by $p \wedge \Diamond\neg p$ is consistent in standard semantics of modal logic. On the other hand, there seems to be some inconsistency in the sentence too. However, we cannot use the same strategy to dispel the tension. Even if the sentence is assert by someone else, there is still inconsistency. And we cannot put it into an assumption or use it as the antecedent of a conditional. The following sentences are still weird, which had been noticed by Yacin [36].

(5) (a) #Suppose it's raining outside and it might not be raining outside.
(b) #If it's raining outside and it might not be raining outside, then...

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[2] Carroll's original formulation of the puzzle used Reduction to Absurdity rather than Modus Tollens to obtain (1 c). The argument roughly goes as follows. Suppose Carr is out (C). Then if Allen is out then Brown is in (A), since at least one must be in. On the other hand, we also have if Allen is out then Brown is out (B), which contradicts A. Therefore, $\neg C$. To validate the argument in formal logic, however, we still need some other assumptions to make A and B contradictory. Our formulation avoids the trouble by simply taking A to be false.

Sentences like (4) are sometimes called Wittgenstein sentences, distinguished from Moore sentences like (2). The above comparison indicates that unlike Moore sentences, the inconsistency in Wittgenstein sentences lies in semantics rather than in pragmatics, which demands a new semantics for modality.

1.3 Three barbers $\times$ Might

The third puzzle involves both indicatives and epistemic modality. Consider the same scenario as the barbershop paradox and the following inference.

(6) A counterexample to Hypothetical Syllogism.
- (a) If Allen is out then Brown is out.
- (b) If Brown is out, Allen might be in.
- (c) ? Therefore, if Allen is out he might be in.

Both (6 a) and (6 b) are true in the scenario. And the inference is an instance of Hypothetical Syllogism, which is valid in classical logic. But the conclusion rings false.

This example appears to provide a counterexample to Hypothetical Syllogism, which had already been challenged by Lewis [17] for inferences involving counterfactuals. For indicatives, however, whether Hypothetical Syllogism is valid is controversial. If it is not valid, then we need a logical theory that can cope with both indicatives and epistemic modals.

1.4 Main idea

This paper aims to solve the above puzzles, among others. To solve the puzzles, we provide a new semantics for indicatives and epistemic modals, together with a new notion of validity.

According to the new notion of validity, an inference is not valid or invalid per se, but valid or invalid under a set of assumptions. An inference is thus tripart: a set of actual premises, which are true at the actual world, a set of assumptive premises, which are assumed to be true at all epistemically possible worlds, and a conclusion. An inference rule now has three different forms: the standard form, in which all premises are supposed to be actual, the assumptive form, in which all premises are assumed to be true in all epistemically possible worlds, and the hybrid form, in which some premises are supposed to be actual, some are assumptive. The same inference in different forms may have different results of validity.

In the new semantics, a formula is not evaluated at a pointed model, but at a pointed model together with a context, namely a set of worlds in the model. Contexts are provided either by the assumptive premises or by the antecedents of indicatives.

By differentiating three forms of inferences, the new semantics together with the new notion of validity predicts our intuitions of the validity and invalidity of the inference rules involving indicatives and epistemic modals. Not only the three puzzles above are resolved, but also some related puzzles can be solved in a unified way, which may shed some light on contextual reasoning, as well as the fundamental concepts of inferences and validity in logic.

The remaining part of the paper is organized as follows. Section 2 proposes our new semantics for indicatives and epistemic modals, with some properties of the semantics prepared. Section 3 contends the new notion of validity, distinguishing three forms of inferences, two notions of equivalence, and two notions of inconsistency. Section 4 applies the new semantic framework to the three puzzles above, among others, showing how linguistic data are predicted by our theory. Section 5 concludes the paper and suggests some future work. All proofs are put in the Appendix.

2 A Semantics for Indicatives and Epistemic Modals

Our formal language $\mathcal{L}$ is generated from a given set At of atomic formulas by the usual Boolean connectives $\neg, \wedge, \vee$ together with a binary connective $>$ for indicatives and a unary operator $\Diamond$ for epistemic modality. The modal operator $\Box$ is defined as the dual of $\Diamond$, as usual. Formulas without $>$ and $\Diamond$ are called Boolean. We abbreviate $p_0 \wedge \neg p_0$ by $\bot$ for the first atom p_0 in At. We stipulate that $\wedge$ and $\vee$ have stronger power of combination than $>$, so that $p \wedge q > r$ means $(p \wedge q) > r$.

We use selection models for our semantics. But the same idea can also be realized in other models for conditionals, including Lewis' sphere systems [17]. To avoid using an absurd world as in Stalnaker's semantics for conditionals [28], we treat selection functions as partial.

Definition 2.1 (Models) *A partial selection model is a triple $\mathfrak{M} = (W, f, V)$, where*

- *$W \neq \emptyset$, is a set of possible worlds;*
- *$f : W \times \wp(W) \to W$, is a partial selection function, satisfying the following conditions:*
  (id) *if $f(w, X)$ is defined then $f(w, X) \in X$,*
  (cs) *if $w \in X$ then $f(w, X) = w$;*
- *$V : At \to \wp(W)$, is a valuation function, mapping each atom to the set of worlds in which it is true.*

Unlike most semantics in modal logic, we evaluate a formula not simply at a pointed model but at a pointed model together with a set of worlds in the model, which is intended to represent a context. As selection functions are partial, a formula can be neither true nor false. Hence we propose both truth and falsity conditions. An atomic p is true (false, respectively) at w in X iff p is true (false, respectively) at the closest world in X to w. Boolean connectives are defined as usual. An indicative $\varphi > \psi$ is true (false, respectively) at w in X iff ψ is true (false, respectively) at w in the context updated by φ, which is the intersection of X and the truth set of φ in the context X. An epistemic modal $\Diamond\varphi$ is true at w in X iff φ is true at some world in X. More precisely, we have the following definition.

Definition 2.2 (Truth and falsity conditions) *Given a partial selection model $\mathfrak{M} = (W, f, V)$, that φ is true (false, respectively) at (w, X) in $\mathfrak{M}$,* *denoted* $\mathfrak{M}, w, X \Vdash \varphi$ *(*$\mathfrak{M}, w, X \dashv\!\mid \varphi$*, respectively), is inductively defined as follows.*

- $\mathfrak{M}, w, X \Vdash p$ *iff* $f(w, X)$ *is defined and* $f(w, X) \in V(p)$*;*
- $\mathfrak{M}, w, X \dashv\!\mid p$ *iff* $f(w, X)$ *is defined and* $f(w, X) \notin V(p)$*;*
- $\mathfrak{M}, w, X \Vdash \neg\varphi$ *iff* $\mathfrak{M}, w, X \dashv\!\mid \varphi$*;*
- $\mathfrak{M}, w, X \dashv\!\mid \neg\varphi$ *iff* $\mathfrak{M}, w, X \Vdash \varphi$*;*
- $\mathfrak{M}, w, X \Vdash \varphi \wedge \psi$ *iff* $\mathfrak{M}, w, X \Vdash \varphi$ *and* $\mathfrak{M}, w, X \Vdash \psi$*;*
- $\mathfrak{M}, w, X \dashv\!\mid \varphi \wedge \psi$ *iff* $\mathfrak{M}, w, X \dashv\!\mid \varphi$ *or* $\mathfrak{M}, w, X \dashv\!\mid \psi$*;*
- $\mathfrak{M}, w, X \Vdash \varphi \vee \psi$ *iff* $\mathfrak{M}, w, X \Vdash \varphi$ *or* $\mathfrak{M}, w, X \Vdash \psi$*;*
- $\mathfrak{M}, w, X \dashv\!\mid \varphi \vee \psi$ *iff* $\mathfrak{M}, w, X \dashv\!\mid \varphi$ *and* $\mathfrak{M}, w, X \dashv\!\mid \psi$*;*
- $\mathfrak{M}, w, X \Vdash \Diamond\varphi$ *iff* $\mathfrak{M}, u, X \Vdash \varphi$ *for some* $u \in X$*;*
- $\mathfrak{M}, w, X \dashv\!\mid \Diamond\varphi$ *iff* $\mathfrak{M}, u, X \dashv\!\mid \varphi$ *for all* $u \in X$*;*
- $\mathfrak{M}, w, X \Vdash \varphi > \psi$ *iff* $\mathfrak{M}, w, X \cap [\![\varphi]\!]^X_+ \Vdash \psi$*, where*

$$[\![\varphi]\!]^X_+ = \{u \in W \mid \mathfrak{M}, u, X \Vdash \varphi\};$$

- $\mathfrak{M}, w, X \dashv\!\mid \varphi > \psi$ *iff* $\mathfrak{M}, w, X \cap [\![\varphi]\!]^X_+ \dashv\!\mid \psi$*.*

We write $\mathfrak{M}, w, X \Vdash \Gamma$, if $\mathfrak{M}, w, X \Vdash \varphi$ for all $\varphi \in \Gamma$. Given a partial selection model $\mathfrak{M} = (W, f, V)$, we denote by $[\![\varphi]\!]^X_-$ the set $\{u \in W \mid \mathfrak{M}, u, X \dashv\!\mid \varphi\}$. For brevity, we write $[\![\Delta]\!]^X_+$ for $\bigcap_{\delta\in\Delta}[\![\delta]\!]^X_+$. We often omit the brackets $\{\ \}$ to denote a set $\{\varphi_1, \ldots, \varphi_n\}$.

The following lemma says that a formula cannot be both true and false, which is straightforward by induction.

Lemma 2.3 *For any partial selection model* $\mathfrak{M}$ *and* X *in* $\mathfrak{M}$*, for any* $\varphi \in \mathcal{L}$*, we have* $[\![\varphi]\!]^X_+ \cap [\![\varphi]\!]^X_- = \emptyset$*.*

The following lemma indicates that Boolean formulas have the same truth and falsity conditions as atomic formulas.

Lemma 2.4 *If* φ *is Boolean, then*

(i) $\mathfrak{M}, w, X \Vdash \varphi$ *iff* $\mathfrak{M}, f(w, X), W \Vdash \varphi$

(ii) $\mathfrak{M}, w, X \dashv\!\mid \varphi$ *iff* $\mathfrak{M}, f(w, X), W \dashv\!\mid \varphi$

The following lemma says that a context X updated first by φ and then by ψ is the same as that updated by $\varphi \wedge \psi$, provided that ψ is Boolean.

Lemma 2.5 *Let* $\mathfrak{M} = (W, f, V)$ *be a normal model. If* ψ *is Boolean, then*

$$X \cap [\![\varphi]\!]^X_+ \cap [\![\psi]\!]^{X\cap[\![\varphi]\!]^X_+}_+ = X \cap [\![\varphi \wedge \psi]\!]^X_+.$$

The following lemma says that if w is in the context, then Boolean formulas at w are context insensitive.

Lemma 2.6 *Let* φ *be Boolean. If* $w \in X \cap Y$*, then*

(i) $\mathfrak{M}, w, X \Vdash \varphi$ *iff* $\mathfrak{M}, w, Y \Vdash \varphi$*;*

(ii) $\mathfrak{M}, w, X \dashv\!| \varphi$ *iff* $\mathfrak{M}, w, Y \dashv\!| \varphi$.

3 A New Notion of Validity

3.1 Three forms of inferences

The following are typical inferences in natural language.

(i) Either Carr is in or out. He is not in. Therefore, he is out.

(ii) Suppose Carr is out. Then Allen or Brown is in. (So, if Carr is out then Allen or Brown is in.)

(iii) Suppose Carr is out. Then Allen or Brown is in, since at least one is in.

The three inferences have the following forms, respectively.

Standard Form: A. Therefore, C. (Or, C, since A.)

Assumptive Form: Suppose A. Then C.

Hybrid Form: Suppose A. Then C, since B.

The difference between the standard form and the assumptive form is that the premise A in the former is a fact whereas it is an assumption in the latter. The assumptive form is usually followed by the conditional "if A then C", cancelling the assumption, which is called Conditional Proof. In formal logic, the standard form and the assumptive form are usually not distinguished. Both are formalized by $A \vDash C$. In particular, the conditional proof is formalized by inferring $\vDash A > C$ from $A \vDash C$. For the same reason, neither are the two premises A and B distinguished in the hybrid form, which is formalized by $A, B \vDash C$.

In natural language reasoning, however, factual premises and assumptive ones are different. The logical effect of the former is only in the actual world, whereas the logical effect of the latter is in all epistemically possible worlds. In mathematical reasoning, the difference is unimportant, as a mathematical fact is a fact true in all epistemically possible worlds. This is why the three forms are not distinguished in standard formal logic. Nevertheless, if we want to characterize reasoning in natural language better, we should distinguish the three forms in our formalization.

3.2 Formalizing the three forms of inferences

To formalize the three forms of inferences, we define a new notion of validity, according to which an inference is not valid or invalid per se, but valid or invalid under a set of assumptions.

Definition 3.1 (Validity) *The inference from* Γ *to* φ *under* Δ *is valid, denoted* $\Gamma \vDash^{\Delta} \varphi$*, if there is no partial selection model* $\mathfrak{M} = (W, f, V)$ *and* $w \in W$ *such that* $\mathfrak{M}, w, [\![\Delta]\!]^W_+ \Vdash \Gamma$ *and* $\mathfrak{M}, w, [\![\Delta]\!]^W_+ \dashv\!| \varphi$.

According to the definition, factual premises and conclusions are evaluated in the contexts specified by assumptive premises. Apart from generalizing logical consequence from a binary relation to a ternary one, we adopt the *st*-validity proposed in Strict-Tolerant logic [6,26] rather than the standard notion of truth preserving. The *st*-validity can be regarded as a formalization of Strawson entailment [32, pp. 176–177], according to which the inference from A to B is valid if and only if B is true whenever A is true and all presuppositions involved are satisfied. If we assume that a statement is neither true nor false if its presuppositions are not satisfied, and a statement cannot be both true and false, then it turns out that Strawson entailment and *st*-validity coincide. One merit of *st*-validity as shown by Strict-Tolerant logics is that when restricted to Boolean formulas, the semantics yields the same logical consequence as classical logic, unlike most three-valued logics, which are sublogics of classical logic.

In terms of the new notion of validity, the three forms of inferences above can be formulated as follows.

Standard Form: $A \vDash^{\emptyset} C$

Assumptive Form: $\emptyset \vDash^{A} C$

Hybrid Form: $B \vDash^{A} C$

For brevity, we will write $\Gamma \vDash \varphi$ for $\Gamma \vDash^{\emptyset} \varphi$ and $\vDash^{\Delta} \varphi$ for $\emptyset \vDash^{\Delta} \varphi$, respectively. Within the new semantic framework, we can define two notions of equivalence as follows.

Definition 3.2 (Equivalence) *We say that φ and ψ are*

(i) *semantically equivalent, denoted $\varphi \equiv \psi$, if for all partial selection models $\mathfrak{M}$ and X in $\mathfrak{M}$, we have $[\![\varphi]\!]^X_+ = [\![\psi]\!]^X_+$ and $[\![\varphi]\!]^X_- = [\![\psi]\!]^X_-$;*

(ii) *hypothetically equivalent, denoted $\varphi \asymp \psi$, if $\vDash (\varphi > \psi) \wedge (\psi > \varphi)$, or equivalently, $\vDash^{\varphi} \psi$ and $\vDash^{\psi} \varphi$.*

We can also define two notions of inconsistency as follows.

Definition 3.3 (Inconsistency) *We say that φ is*

(i) *inconsistent, if $\varphi \vDash \bot$;*

(ii) *self-refuting, if $\vDash^{\varphi} \neg\varphi$.*

It is desirable that if an inference only involves Boolean formulas, it should be characterized by classical logic. After all, most challenges to classical logic are centering around conditionals and modals rather than Boolean operators. The following theorem says that confined to Boolean formulas, there is no difference between the three forms of inferences in our framework, which are indeed all equivalent to the inferences in classical logic.

Theorem 3.4 *If all the formulas in $\Gamma \cup \{\varphi\}$ are Boolean, then $\Sigma \vDash^{\Delta} \varphi$ iff $\Gamma \vDash_{cl} \varphi$, where $\Sigma \cup \Delta = \Gamma$ and $\vDash_{cl}$ is the logical consequence of classical logic.*

3.3 Import-Export

The following lemma shows that a restricted version of Import-Export, which had been proposed by Mandelkern [21], is valid in our semantics.

Lemma 3.5 *If ψ is Boolean then $\varphi > (\psi > \chi) \equiv \varphi \wedge \psi > \chi$.*

Validating Import-Export, however, is not without price. As Gibbard [10] noted, there is a tension between Import-Export and Modus Ponens. Together with the innocuous rule $\varphi \vDash \psi \implies \vDash \varphi > \psi$, validating both Modus Ponens and Import-Export leads to the collapse of conditionals, i.e., conditionals are logically equivalent to material implication, which is undesirable.

Noting that Modus Ponens does have counterexamples, McGee [22] proposed a semantics for conditionals that validates Import-Export and invalidates Modus Ponens. One most famous counterexample given by McGee invites us to consider the 1980 US presidential election. Opinion polls showed that the republican Ronald Reagan was decisively ahead of the democrat Jimmy Carter, who was in turn decisively ahead of the other republican John Anderson. Then the premises (7 a) and (7 b) below ring true, while the conclusion (7 c) does not, for if Reagan does not win, Carter will.

(7) A counterexample to Modus Ponens.
- (a) If a republican wins the election, then if it's not Reagan who wins it will be Anderson.
- (b) A republican will win the election.
- (c) Therefore, if it's not Reagan who wins, it will be Anderson.

Many don't think that this is a counterexample to Modus Ponens. Before discussing the (in)validity of Modus Ponens, we'd like to point out why Import-Export is only restrictedly valid.

According to Mandelkern [19,21], when evaluating a nested conditional $\varphi > (\psi > \chi)$, the nested antecedent ψ should not be evaluated only in the same context as that of the whole conditional $\varphi > (\psi > \chi)$, namely the global context, but should also be evaluated in the context of φ, which is the local context of ψ. When ψ is Boolean, φ will not have any effect on ψ. So we can ignore the local context provided by φ. But if ψ itself is a conditional, then it will be affected by φ. This means that when ψ is a conditional, $\varphi \wedge \psi > \chi$ is not equivalent to $\varphi > (\psi > \chi)$, as ψ is evaluated independently of φ in the former, whereas ψ is affected by the local context φ in the latter. Thus, both Importation and Exportation are invalid when ψ contains conditionals, as instantiated by Manderlkern [21] below.

Consider the same scenario as in McGee's counterexample, except that we only know that Reagan is well ahead of Carter and Anderson without knowing the ordering between Carter and Anderson.

(8) A counterexample to Exportation.
- (a) If a Republican will win the election, and Anderson will win if Reagan doesn't win, then both Republicans are currently in a stronger position to win than Carter.
- (b) If a Republican will win the election, then if Anderson will win if Reagan doesn't, then both Republicans are currently in a stronger position to win than Carter.

Intuitively, (8 a) is true whereas (8 b) is false. In formulas, we have $\varphi \wedge \psi > \chi$ but not $\varphi > (\psi > \chi)$.

The following counterexample is based on the same scenario as McGee's.

(9) A counterexample to Importation.
  (a) If a Republican will win the election, then if Anderson will win if Reagan doesn't, then the polling data we've just received are correct.
  (b) If a Republican will win the election, and Anderson will win if Reagan doesn't, then the polling data we've just received are correct.

Intuitively, (9 a) is true whereas (9 b) is false. In formulas, we have $\varphi > (\psi > \chi)$ but not $\varphi \wedge \psi > \chi$. Our semantics predicts Manderkern's counterexamples.

Proposition 3.6 *$p > ((q > r) > s)$ and $p \wedge (q > r) > s$ are not equivalent: either can be true with the other being false.*

4 Solving the Puzzles

Before solving the three puzzles in Section 1, we first discuss the (in)validity of Modus Ponens.

4.1 Modus Ponens

Proposition 4.1 *Modus Ponens is (in)valid in the following forms.*

(i) $p, p > (q > r) \nvDash q > r$

(ii) $p, p > \Box q \nvDash \Box q$

(iii) $\vDash^{p,p>(q>r)} q > r$

(iv) $\vDash^{p,p>\Box q} \Box q$

(v) $\varphi, \varphi > \psi \vDash \psi$, *where ψ is Boolean*

(vi) $\vDash^{\varphi,\varphi>\psi} \psi$, *where ψ is Boolean*

(vii) $\varphi > \psi \vDash^{\varphi} \psi$, *where φ is Boolean*

In our semantics, the standard form of Modus Ponens is invalid, if it involves nested conditionals, as in McGee's semantics. Our semantics also predicts that if epistemic modals are involved, Modus Ponens is also invalid, as exemplified by the following example, assuming that in the 1980 US election, there were only two candidates, Reagan and Carter.

(10) A counterexample to Modus Ponens involving 'must'.
  (a) If a republican wins the election, then the winner must be Reagan.
  (b) A republican will win the election.
  (c) Therefore, the winner must be Reagan.

Over [25], among others, objected to McGee's counterexamples. He argued that if the premises of Modus Ponens have been certainly assumed, the conclusion cannot be false. This intuition can be formulated by the assumptive form of Modus Ponens in our semantics. Though it is not generally valid, its restricted form for the two examples above are valid, predicted by (iii) and (iv) of Proposition 4.1.

Now in the literature concerning the validity of Modus Ponens, often two notions of validity are distinguished (cf. [31]). Mandelkern [20] categorized them as truth-preserving validity, which is the standard one, and informational validity, which was defended in [2], based on the semantics initiated by Veltman [33] and developed by Gillies [11] and Yalcin [36], among others. The two notions of validity yield two forms of Modus Ponens. The truth-preserving form reads: if φ and $\varphi > \psi$ are true, then ψ is true. The informational form reads: if φ and $\varphi > \psi$ are fully accepted, then rationality requires ψ to be fully accepted.

Bledin [3] argues that, though truth-preserving Modus Ponens may not be valid, informational Modus Ponens is still valid. Consider McGee's example again. If we fully accept that a republican will win, and that if a republican wins then if it is not Reagan who wins it will be Anderson, then we must accept that if Reagan does not win, then Anderson will win. The intuition is similar to Over's, which could also be formulated by the assumptive Modus Ponens.

But to dispel the counterexample does not require us to fully accept both premises. Fully accepting that a republican will win is enough for us to infer from "if a republican wins then if it is not Reagan who wins it will be Anderson" to its consequent, which is predicted by (vii) of Proposition 4.1. Note that fully accepting the other premise only does not make the inference valid.

4.2 Modus Tollens

Proposition 4.2 *Modus Tollens is (in)valid in the following forms.*

(i) $p > (q > r), \neg(q > r) \not\models \neg p$

(ii) $p > \Box q, \neg\Box q \not\models \neg p$

(iii) $\not\models^{p>(q>r),\neg(q>r)} \neg p$

(iv) $\not\models^{p>\Box q,\neg\Box q} \neg p$

(v) $\varphi > \psi, \neg\psi \models \neg\varphi$*, where* ψ *is Boolean*

(vi) $\models^{\varphi>\psi,\neg\psi} \neg\varphi$*, where* φ *and* ψ *are Boolean*

According to our semantics, the inference in the barbershop puzzle is indeed invalid. Unlike McGee's counterexamples to Modus Ponens, even if the premises of the inference in the barbershop puzzle are assumed, the inference is still invalid, which is predicted by (iii) of Proposition 4.2. This may explain why counterexamples to Modus Tollens were found much earlier and more often than those to Modus Ponens. Though Modus Ponens and Modus Tollens seem two symmetric rules, the latter is not as valid as the former.

4.3 Hypothetical Syllogism

Proposition 4.3 *Hypothetical syllogism is (in)valid in the following forms.*

(i) $p > q, q > r \not\models p > r$

(ii) $q > r \not\models^{p>q} p > r$

(iii) $\models^{\varphi>\psi,\psi>\chi} \varphi > \chi$*, where* φ, ψ, χ *are Boolean*

(iv) $\varphi > \psi \models^{\psi>\chi} \varphi > \chi$*, where* ψ *and* χ *are Boolean*

(v) $\not\models^{p>(q>r),(q>r)>s} p > s$

(vi) $p > (q > r) \not\models^{(q>r)>s} p > s$

(vii) $\not\models^{p>q, q>\Diamond r} p > \Diamond r$

(viii) $p > q \not\models^{q>\Diamond r} p > \Diamond r$

It has been well known since Lewis [17] that Hypothetical Syllogism is invalid for counterfactuals. Alleged counterexamples for indicatives were also found. The following is a typical one.

(11) An alleged counterexample to Hypothetical Syllogism.
- (a) If Ann drops out, then Bill will win.
- (b) If Bill wins, Ann will finish second.
- (c) Therefore, if Ann drops out, she will finish second.

Intuitively, we can think of a scenario in which both (11 a) and (11 b) are true, whereas (11 c) is definitely false. Like Stalnaker-Lewisian conditional logics and McGee's semantics, our semantics predicts (11). Note that even if (11 a) is assumed, the inference is still invalid, which is predicted by (iv) and (v) of Proposition 4.3.

Why do some argue that 11 is not a counterexample to Hypothetical Syllogism? If (11 b) is assumed, then it excludes the possibility that Ann drops out. So under this assumption, the antecedent of the conclusion (11 c) cannot be true, and thus the whole conditional cannot be false. Indeed, whenever the second premise of Hypothetical Syllogism is assumed, the inference is valid, provided that the formulas are Boolean, which is predicted by (iii) and (iv) of of Proposition 4.3.

Hypothetical Syllogism is not valid even in its assumptive form, if it involves nested conditionals. The following example is adapted from Mandelkern's counterexample to Import-Export, using the same scenario as McGee's.

(12) A counterexample to Hypothetical Syllogism involving nested conditionals.
- (a) If a republican wins the election, then if it's not Reagan who wins it will be Anderson.
- (b) If Anderson will win if Reagan doesn't, then the polling data we've just received are incorrect.
- (c) ? Therefore, if a republican wins the election, then the polling data we've just received are incorrect.

According to our semantics, the third puzzle in Section 4.3 does provide a counterexample to Hypothetical Syllogism, as even if the second premise in Hypothetical Syllogism is assumed, it is still invalid. The following is another counterexample in the same form.

(13) A counterexample to Hypothetical Syllogism involving epistemic modals.
- (a) If the flower is red, then it is not blue.
- (b) If the flower is not blue, it might be purple.
- (c) ? Therefore, if the flower is red, it might be purple.

Both (13 a) and (13 b) are true, whereas (13 c) rings false, even if (13 b) is assumed, or both (13 a) and (13 b) are assumed.

The counterexamples 12 and 13 are predicted by (v)–(viii) of Proposition 4.3. [3]

4.4 Simplification of Disjunctive Antecedents

The inference from $\varphi \vee \psi > \chi$ to $(\varphi > \chi) \wedge (\psi > \chi)$ is called Simplification of Disjunctive Antecedents (SDA, henceforth). It sounds a valid inference. If I'll be happy if Ann or Bill comes to my party, then I'll be happy if Ann comes to my party and I'll be happy if Bill comes to my party. However, the standard form of SDA together with Left Logical Equivalents (LLE), namely, equating $\varphi > \chi$ with $\psi > \chi$ from the equivalence of φ and ψ, which seems innocuous, will recover the standard form of Antecedent Strengthening, which is undesirable.

To circumvent the dilemma, one solution is to accept SDA and abandon LLE by constructing hyperintensional logics (e.g. [24,8]). Another solution is to reject SDA and account for its intuitive validity by translation lore (e.g. [18]). Both approaches appear ad hoc. In our framework, SDA is invalid in its standard form but valid in its assumptive form, if the formulas involved are Boolean.

Proposition 4.4 *SDA is (in)valid in the following forms.*

(i) $p \vee q > r \not\vDash (p > r) \wedge (q > r)$;

(ii) $\vDash^{\varphi \vee \psi > \chi} (\varphi > \chi) \wedge (\psi > \chi)$, *where* φ, ψ, χ *are Boolean;*

(iii) $\not\vDash^{p \vee q > \Diamond r} (p > \Diamond r) \wedge (q > \Diamond r)$.

On the other hand, SDA does have counterexamples, even in its assumptive form, if the consequent involves epistemic modals, as exemplified below.

(14) A counterexample to SDA involving 'might'.
  (a) If Ann or Bill goes to the party, there might be a woman in the party.
  (b)? Therefore, if Bill goes to the party, there might be a woman in the party.

Even if (14 a) is assumed, (14 b) can still be false. The counterexample is predicted by (iii) of Proposition 4.4.

Counterexamples to SDA without involving modality were also found in the literature (e.g. [4]), as given below.

(15) An alledged counterexample to SDA.
  (a) If Ivan is playing tennis or playing baseball, then he is playing baseball.
  (b) Therefore, if Ivan is playing tennis, he is playing baseball.

If Ivan plays baseball and nearly does not play tennis then (15 a) rings true but (15 b) does not. According to our analysis, (15) is only a counterexample to the standard form of SDA. It is not to the assumptive form. If we have

[3] The same analysis can be applied to Antecedent Strengthening, the rule inferring $\varphi \wedge \psi > \chi$ from $\varphi > \chi$, and Contraposition, the rule inferring $\neg\psi > \neg\varphi$ from $\varphi > \psi$. For lack of space, we omit it here.

assumed that (15 a), then whenever Ivan is playing one of the two games, he is playing baseball. It follows that Ivan never plays tennis (if we assume that one cannot play the two games at the same time). Hence, the conclusion is not false (though not true either). Our intuition of rejecting (15 b) is because we presuppose the possibility of its antecedent. But the possibility is already excluded by assuming (15 a). This analysis is similar to that given by Fine [8], except that his analysis is about the counterexample for counterfactuals given in [23].

4.5 Or-to-If

The inference from $\varphi \vee \psi$ to $\neg\varphi > \psi$ is called Or-to-If (a.k.a., the Direct Argument), which is invalid in Stalnaker-Lewisian conditional logics. This has to be so, as their logics validate Modus Ponens, together with which, Or-to-If leads to collapse: $\varphi > \psi$ will be equivalent to $\neg\varphi \vee \psi$. Or-to-If is not valid in our semantics either. Otherwise, the restricted Modus Ponens (which is valid in our semantics) also leads to collapse.

Nevertheless, Or-to-If seems pervasive and innocuous in natural language reasoning. If there are only two cases, then ruling out one surly leads us to the other. To explain the plausibility of this inference, Stalnaker [29] distinguished reasonable inferences from valid ones. An inference is reasonable if in every context in which the premises could appropriately be asserted or supposed, it is impossible for anyone to accept the premises without accepting the conclusion. This idea is in line with informational validity mentioned in Section 4.1, which can be formulated in the assumptive form of Or-to-If. Moreover, it seems that we can not only infer $\neg\varphi > \psi$ from $\varphi \vee \psi$, but also $\neg\varphi > \Box\psi$. The following is a common example.

(16) A valid inference of modal Or-to-If.
  (a) Either the butler or the gardener did it.
  (b) Therefore, if the butler didn't do it, it must be the gardener.

We call it modal Or-to-If. Besides, Or-to-If also has a conditional reading: If $\varphi \vee \psi$, then if $\neg\varphi$ then ψ. All the forms of Or-to-If can be formulated in our framework.

Proposition 4.5 *Or-to-If is (in)valid in the following forms, where φ and ψ are Boolean.*

(i) $p \vee q \not\models \neg p > q$

(ii) $\models^{\varphi \vee \psi} \neg\varphi > \psi$

(iii) $\models^{\varphi \vee \psi} \neg\varphi > \Box\psi$

(iv) $\models (\varphi \vee \psi) > (\neg\varphi > \psi)$

Clause (ii) can be regarded as a formalization of Stalnaker's reasonable inference. Clause (iii) predicts (16) above. Note that Clause (iv) does not hold in Stalnaker-Lewisian conditional logics, since by Modus Ponens it leads to collapse.

Some authors advocate that indicative implication is no more than material implication in semantics (e.g. [14,12]). Our semantics inherits some basic ideas of Stalnaker's theory of conditionals. Can we also incorporate the theory of material implication for conditionals? Yes. According to our semantics, $\varphi > \psi$ and $\neg\varphi \vee \psi$ are indeed 'equivalent'. The equivalence, however, is not in the sense that if one is *true* so is the other. It is in the sense that if one is *assumed* then the other cannot be false. In our terminology, they are not semantically equivalent, but hypothetically equivalent, as given by the following theorem.

Theorem 4.6 *If φ and ψ are Boolean, then $\varphi > \psi \asymp \neg\varphi \vee \psi$, i.e., $\varphi > \psi$ and $\neg\varphi \vee \psi$ are hypothetically equivalent.*

4.6 Wittgenstein Sentences

Armed with two notions of inconsistency, the puzzle about Wittgenstein sentences can be resolved.

Proposition 4.7 *Let $\omega = p \wedge \Diamond\neg p$ or $\omega = \neg p \wedge \Diamond p$. Then*

(i) $\omega \not\models \bot$

(ii) $\models^{\omega} \neg\omega$

Clause 1 says that the Wittgenstein sentence ω is consistent, which explains why it can be true. Clause 2 says that ω is self-refuting, which explains why it cannot be consistently asserted or assumed. By assuming $p \wedge \Diamond\neg p$, we exclude all worlds in which p is false in the context. Then $\Diamond\neg p$ can no longer be true in the context and thus $p \wedge \Diamond\neg p$ can no longer be true either. The existence of such self-refuting formulas also makes the semantic equivalence between $\Diamond\varphi$ and $\neg(\varphi > \neg\varphi)$, which holds in most conditional logics, no longer hold. It can be verified that $\Diamond\omega$ can be true, whereas $\neg(\omega > \neg\omega)$ cannot be true, since $\omega > \neg\omega$ is valid and hence cannot be false.

Moreover, our semantics also predicts the order sensitivity between p and $\Diamond p$, which cannot be explained by Yalcin's semantics. As noted by Veltman [33], there are differences between the following two sequences of statements.

(17) Order sensitivity involving 'might'.
- (a) Someone is knocking at the door. Maybe it's Bill (and thus not Ann). It's Ann.
- (b) #Someone is knocking at the door. It's Ann (and thus not Bill). Maybe it's Bill.

(17 a) is felicitous, whereas (17 b) sounds odd. The moral is that after asserting $\Diamond p$ you can still assert $\neg p$. But after asserting $\neg p$ you can no longer assert $\Diamond p$. The difference is predicted by the following result.

Proposition 4.8 *For any $p \in At$,*

(i) $\neg p \not\models^{\Diamond p} \bot$

(ii) $\Diamond p \models^{\neg p} \bot$

Clause 1 says that $\neg p$ is consistent under $\Diamond p$. Assuming or asserting $\Diamond p$

does not rule out the possibility of $\neg p$, and thus $\neg p$ can be true and asserted. Clause 2 says that $\Diamond p$ is inconsistent under $\neg p$. Assuming or asserting $\neg p$ has ruled out the possibility of p, and thus $\Diamond p$ cannot be true or asserted.

4.7 Conditional Excluded Middle

Conditional Excluded Middle (CEM, henceforth) refers to the scheme $(\varphi > \psi) \vee (\varphi > \neg\psi)$, which is valid in Stalnaker's conditional logic. Though admitting the plausibility of CEM, Lewis [17] gave an If-Might argument against CEM, which was further developed by Bennett [1]. The argument roughly goes as follows. First, there is strong intuition that $\varphi > \neg\psi$ is inconsistent with $\varphi > \Diamond\psi$, which can be formulated as follows. [4]

If-Might Contradiction (IMC) $(\varphi > \neg\psi) \wedge (\varphi > \Diamond\psi) \vDash \bot$

From IMC, using reduction to absurdity, it follows that $\varphi > \Diamond\psi \vDash \neg(\varphi > \neg\psi)$. By CEM, using disjunctive syllogism, we have $\neg(\varphi > \neg\psi) \vDash \varphi > \psi$. Then by the transitivity of $\vDash$, it follows that $\varphi > \Diamond\psi \vDash \varphi > \psi$. Analogously, we obtain $\varphi > \psi \vDash \varphi > \Diamond\psi$. But it is absurd that $\varphi > \Diamond\psi$ and $\varphi > \psi$ are logically equivalent. So if $\vDash$ is a conservative extension of classical logic, CEM is incompatible with IMC.

We argue that the If-Might argument against CEM is flawed. The problem of the If-Might argument lies in the formulation of the If-Might contradiction. The inconsistency between $\varphi > \neg\psi$ and $\varphi > \Diamond\psi$ is not that they cannot both be true. It is that they cannot both be asserted or assumed, as the following examples shows.

(18) #If it rains, the game will not be held. But even if it rains, the game might be held.

(19) #Suppose that if it rains, the game will not be held, and that if it rains, the game might be held.

Analogous to the analysis of Wittgenstein sentences, in our terminology, $(\varphi > \neg\psi) \wedge (\varphi > \Diamond\psi)$ is consistent but self-refuting, demonstrated below.

Proposition 4.9 *Let* $\lambda = (p > \neg q) \wedge (p > \Diamond q)$. *Then*

(i) $\lambda \nvDash \bot$

(ii) $\vDash^{\lambda} \neg\lambda$

In fact, according to our semantics, $(p > \neg q) \wedge (p > \Diamond q)$ is equivalent to a conditional Wittgenstein sentence, namely, $p > (\neg q \wedge \Diamond q)$, which can be regarded as the Wittgenstein sentence $\neg q \wedge \Diamond q$ in the context of p. If we agree that $\neg q \wedge \Diamond q$ is consistent (in the sense of possibly being true), then it should also be consistent in the context of p. If we think that $\neg q \wedge \Diamond q$ is inconsistent (in the sense of impossibly being assumed), then so is it in the context of p.

Since the correct If-Might contradiction should be formulated by Clause

[4] Lewis used a single might-conditional $\Diamond\!\!\rightarrow$ rather than combining two logical operators as we formulate here. Our formulation follows the tradition in formal semantics about indicatives and epistemic modals. See, e.g., [27].

(ii) rather than (i) of Proposition 4.9 above, the reasoning from IMC to the equivalence of $\varphi > \psi$ and $\varphi > \Diamond\psi$ is unjustified. In fact, more and more authors are defending CEM now (e.g., [30,7,35,15,27]). It is also valid in our semantics.

Proposition 4.10 $\vDash (\varphi > \psi) \vee (\varphi > \neg\psi)$.

5 Conclusion

We define a new semantics for indicatives and epistemic modals, together with a ternary notion of validity, by which an inference is not (in)valid per se, but (in)valid under a set of assumptions. Thus, an inference has three different forms: standard form (no premises are assumed), assumptive form (all premises are assumed), and hybrid form (some premises are assumed), which are not equivalent to each other. We also define two kinds of equivalence and two kinds of inconsistency. Armed with these differentiations, we give a unified solution to several puzzles concerning indicatives and epistemic modals, by predicting linguistic data in natural language reasoning. This is a preliminary attempt to unify two kinds of validity proposed in the literature. Future work includes a full comparison to existing works, including Fitting's work on ternary consequence for modal logic [9] [5] and other logics for conditionals and modals (e.g., [34]). Our logic also demands a proof theory suitable for its ternary notion of validity, so that the logic can be better understood and compared to other logics.

Note that our semantics in the current form cannot deal with Wittgenstein disjunctions (disjunction of two Wittgenstein sentences), which were argued by Mandelkern and Holliday in [19,13] to be as inconsistent as Wittgenstein sentences. In fact, they take Wittgenstein sentences to be semantically equivalent to a contradiction. They would also take $p > \neg q \wedge \Diamond q$ to be a contradiction, which we only regard as self-refuting. We argue that both $\neg q \wedge \Diamond q$ and $p > \neg q \wedge \Diamond q$ can be true. Suppose I will take a flight (p). In the most plausible worlds for me, there will not be an accident of my flight ($\neg q$). So I believe $\neg q$. Meanwhile, I do not exclude the possibility of an accident. That's why I will take out an insurance for my flight. So under the assumption of p, I believe both $\neg q$ and $\Diamond q$. On the other hand, our semantic framework leaves room for treating Wittgenstein sentences as real contradictions. One strategy is to apply local contexts also to conjunctions and disjunctions, as in [19]. We leave the investigation of whether this constitutes a better semantics for future work.

[5] Using Fitting's logical consequence and our notation, $\Gamma \vDash^{\Delta} \varphi$ iff for all models $\mathfrak{M}$ in which all formulas in Δ are globally true (i.e., true at all worlds in $\mathfrak{M}$), if all formulas in Γ are (locally) true at some world in $\mathfrak{M}$ then φ is also (locally) true at the world. The formulas in Δ are called global assumptions, and those in Γ are called local assumptions, paralleling our differentiation between assumptive premises and actual premises. Nevertheless, since a Wittgenstein sentence can never be globally true, in Fitting's logical consequence, any inference from Γ to φ under a Wittgenstein sentence is (vacuously) valid , which does not hold in our semantics. The set Δ in Fitting's logical consequence is used to choose models for Γ, whereas in ours it is used to modify contexts.

Appendix: Proofs

Proof of Lemma 2.4. By induction on φ.

Case 1: $\varphi = p$. Then $\mathfrak{M}, w, X \Vdash \varphi$ iff $f(w,X)$ is defined and $f(w,X) \in V(p)$ iff $f(w,X)$ is defined and $f(f(w,X),W) \in V(p)$ iff $\mathfrak{M}, f(w,X), W \Vdash \varphi$, where the second 'iff' is by (cs). Analogously, we have $\mathfrak{M}, w, X \dashV \varphi$ iff $\mathfrak{M}, f(w,X), W \dashV \varphi$.

Case 2: $\varphi = \neg\psi, \psi \wedge \chi, \psi \vee \chi$. Immediate from the inductive hypothesis.

Proof of Lemma 2.5. By Lemma 2.4 and (cs), we have

$$\begin{aligned}
X \cap [\![\varphi]\!]^X_+ \cap [\![\psi]\!]^{X\cap[\![\varphi]\!]^X_+} &= X \cap [\![\varphi]\!]^X_+ \cap \{w \in W \mid \mathfrak{M}, f(w, X \cap [\![\varphi]\!]^X_+), W \Vdash \psi\} \\
&= \{w \in X \cap [\![\varphi]\!]^X_+ \mid \mathfrak{M}, f(w, X \cap [\![\varphi]\!]^X_+), W \Vdash \psi\} \\
&= \{w \in X \cap [\![\varphi]\!]^X_+ \mid \mathfrak{M}, w, W \Vdash \psi\} \\
&= [\![\varphi]\!]^X_+ \cap \{w \in X \mid \mathfrak{M}, w, W \Vdash \psi\} \\
&= [\![\varphi]\!]^X_+ \cap \{w \in X \mid \mathfrak{M}, f(w,X), W \Vdash \psi\} \\
&= [\![\varphi]\!]^X_+ \cap \{w \in X \mid \mathfrak{M}, w, X \Vdash \psi\} \\
&= X \cap [\![\varphi]\!]^X_+ \cap \{w \in W \mid \mathfrak{M}, w, X \Vdash \psi\} \\
&= X \cap [\![\varphi]\!]^X_+ \cap [\![\psi]\!]^X_+ \\
&= X \cap [\![\varphi \wedge \psi]\!]^X_+
\end{aligned}$$

Proof of Lemma 2.6. For (i), we have $\mathfrak{M}, w, X \Vdash \varphi$ iff $f(w,X)$ is defined and $\mathfrak{M}, f(w,X), W \Vdash \varphi$ iff $\mathfrak{M}, w, W \Vdash \varphi$, where the first 'iff' is by Lemma 2.4, and the second 'iff' is by (cs), since $w \in X$. Similarly, we have $\mathfrak{M}, w, Y \Vdash \varphi$ iff $\mathfrak{M}, w, W \Vdash \varphi$. Hence, $\mathfrak{M}, w, X \Vdash \varphi$ iff $\mathfrak{M}, w, Y \Vdash \varphi$. (ii) can be proved analogously.

Proof of Theorem 3.4. We prove the contrapositive. Suppose $\Gamma \nvDash_{cl} \varphi$. Then there is a valuation $V : At \to \{1, 0\}$ such that $V(\Gamma) = 1$ and $V(\varphi) = 0$. Let $\mathfrak{M} = (\{w\}, f, V')$, where $f(w, \{w\}) = w$, $V'(p) = \{w\}$ iff $V(p) = 1$ for all $p \in At$. It can be verified by induction that $\mathfrak{M}, w, \{w\} \Vdash \psi$ iff $V(\psi) = 1$ and $\mathfrak{M}, w, \{w\} \dashV \psi$ iff $V(\psi) = 0$ for all Boolean ψ. Hence, $\mathfrak{M}, w, [\![\Delta]\!]^{\{w\}}_+ \Vdash \Sigma$ and $\mathfrak{M}, w, [\![\Delta]\!]^{\{w\}}_+ \dashV \varphi$, whence $\Sigma \nvDash^\Delta \varphi$.

For the other direction, suppose $\Sigma \nvDash^\Delta \varphi$. Then there is a contextually pointed model $(\mathfrak{M}, w, X)$ with $\mathfrak{M} = (W, f, V)$ such that $\mathfrak{M}, w, [\![\Delta]\!]^W_+ \Vdash \Sigma$ and $\mathfrak{M}, w, [\![\Delta]\!]^W_+ \dashV \varphi$. Define $V' : At \to \{1, 0, u\}$ such that for all $p \in At$, $V'(p) = 1$ iff $f(w, [\![\Delta]\!]^W_+) \in V(p)$. By Lemma 2.4, V' is well-defined. It can be verified by induction that $\mathfrak{M}, w, [\![\Delta]\!]^W_+ \Vdash \psi$ iff $V'(\psi) = 1$ and $\mathfrak{M}, w, [\![\Delta]\!]^W_+ \dashV \psi$ iff $V'(\psi) = 0$. Moreover, since $\mathfrak{M}, w, [\![\Delta]\!]^W_+ \dashV \varphi$, by Lemma 2.4, $f(w, [\![\Delta]\!]^W_+)$ is defined. Then by (id), we have $\mathfrak{M}, f(w, [\![\Delta]\!]^W_+), W \Vdash \Delta$. By Lemma 2.4 again, we have $\mathfrak{M}, w, [\![\Delta]\!]^W_+ \Vdash \Delta$. Together with $\mathfrak{M}, w, [\![\Delta]\!]^W_+ \Vdash \Sigma$, it follows that $\mathfrak{M}, w, [\![\Delta]\!]^W_+ \Vdash \Gamma$. Hence, $V'(\psi) = 1$ for all $\psi \in \Gamma$. Note that we also have $V'(\varphi) = 0$. It follows that $\Gamma \nvDash_{cl} \varphi$.

Proof of Lemma 3.5. Given any partial selection model $\mathfrak{M} = (W, f, V)$, we have

$$\begin{aligned}
[\![\varphi > (\psi > \chi)]\!]_+^X &= [\![\psi > \chi]\!]_+^{X \cap [\![\varphi]\!]_+^X} \\
&= [\![\chi]\!]_+^{X \cap [\![\varphi]\!]_+^X \cap [\![\psi]\!]_+^{X \cap [\![\varphi]\!]_+^X}} \\
&= [\![\chi]\!]^{X \cap [\![\varphi \wedge \psi]\!]_+^X} \\
&= [\![\varphi \wedge \psi > \chi]\!]_+^X,
\end{aligned}$$

where the third '=' is by Lemma 2.5. Analogously, we have $[\![\varphi > (\psi > \chi)]\!]_-^X = [\![\varphi \wedge \psi > \chi]\!]_-^X$.

Proof of Proposition 3.6. Let $r = p$. Let $\mathfrak{M} = (W, f, V)$, where $W = \{w, u, v\}$, $f(w, \{u, v\}) = \{u\}$, $f(u, \{v\}) = \{v\}$, $V(p) = \{u, v\}$, $V(q) = \{v\}$, $V(s) = \{v\}$. Then it can be verified that $\mathfrak{M}, w, W \Vdash p > ((q > r) > s)$ but $\mathfrak{M}, w, W \dashv\!\vert\, p \wedge (q > r) > s$.

Let $\mathfrak{M}' = (W, f, V')$ be the same as $\mathfrak{M}$, except that $V'(s) = W - V(s)$. Then it can be verified that $\mathfrak{M}', w, W \Vdash p \wedge (q > r) > s$ but $\mathfrak{M}', w, W \dashv\!\vert\, p > ((q > r) > s)$.

Proof of Proposition 4.1. For (i), let $\mathfrak{M} = (W, f, V)$ where $W = \{w, u\}$, $V(p) = \{w\}$, $V(q) = \{w, u\}$, $V(r) = \{w\}$. Then $\mathfrak{M}, w, W \Vdash p$ and $\mathfrak{M}, w, W \Vdash p > (q > r)$ but $\mathfrak{M}, w, W \dashv\!\vert\, q > r$.

For (ii), let $\mathfrak{M} = (W, f, V)$, where $W = \{w, u\}$, $V(p) = \{w\}$, $V(q) = \{w\}$. Then $\mathfrak{M}, w, W \Vdash p$ and $\mathfrak{M}, w, W \Vdash p > \Box q$ but $\mathfrak{M}, w, W \dashv\!\vert\, \Box q$.

For (iii), suppose otherwise $\mathfrak{M}, w, [\![p, p > (q > r)]\!]_+^W \dashv\!\vert\, q > r$. Then $\mathfrak{M}, w, [\![p, p > (q > r)]\!]_+^W \cap [\![q]\!]_+^X \dashv\!\vert\, r$, where $X = [\![p, p > (q > r)]\!]_+^W$. Let $Y = [\![p, p > (q > r)]\!]_+^W \cap [\![q]\!]_+^X$. By Lemma 2.4, we have $\mathfrak{M}, f(w, Y), W \dashv\!\vert\, r$. Let $u = f(w, Y)$. By (id), we have $\mathfrak{M}, u, W \Vdash p$, $\mathfrak{M}, u, W \Vdash p > (q > r)$, and $\mathfrak{M}, u, X \Vdash q$. By $\mathfrak{M}, u, W \Vdash p > (q > r)$ and Lemma 3.5, we have $\mathfrak{M}, u, W \Vdash p \wedge q > r$. Note that $u \in X$ and $u \in Y$. By Lemma 2.6, we have $\mathfrak{M}, u, W \Vdash q$. Then $\mathfrak{M}, u, W \Vdash p \wedge q$. It follows by (iii) that $\mathfrak{M}, u, W \not\dashv\!\vert\, r$, contradicting $\mathfrak{M}, f(w, Z), W \dashv\!\vert\, r$.

For (iv), suppose otherwise $\mathfrak{M}, w, [\![p, p > \Box q]\!]_+^W \dashv\!\vert\, \Box q$. Then there exists $u \in [\![p, p > \Box q]\!]_+^W \dashv\!\vert\, q$ such that $\mathfrak{M}, u, [\![p, p > \Box q]\!]_+^W \dashv\!\vert\, q$. By Lemma 2.4, there exists $v \in W$ such that $f(u, [\![p, p > \Box q]\!]_+^W) = v$ and $\mathfrak{M}, v, W \dashv\!\vert\, q$. On the other hand, by (id), we have $\mathfrak{M}, v, W \Vdash p$ and $\mathfrak{M}, v, W \Vdash p > \Box q$. By the latter, we have $\mathfrak{M}, v, [\![p]\!]_+^W \Vdash \Box q$. Note that $v \in [\![p]\!]_+^W$. It follows that $\mathfrak{M}, v, [\![p]\!]_+^W \Vdash q$. Then by Lemma 2.6, we have $\mathfrak{M}, v, W \Vdash q$, contradiction.

For (v), suppose $\mathfrak{M}, w, W \Vdash \varphi$ and $\mathfrak{M}, w, W \Vdash \varphi > \psi$. By the latter, we have $\mathfrak{M}, w, [\![\varphi]\!]_+^W \Vdash \psi$. By the former, we have $w \in [\![\varphi]\!]_+^W$. Then by Lemma 2.6, we have $\mathfrak{M}, w, W \Vdash \psi$. Hence, $\mathfrak{M}, w, W \not\dashv\!\vert\, \psi$.

For (vi), suppose otherwise $\mathfrak{M}, w, [\![\varphi, \varphi > \psi]\!]_+^W \dashv\!\vert\, \psi$. Then there exists $u \in W$ such that $f(w, [\![\varphi, \varphi > \psi]\!]_+^W) = u$ and $\mathfrak{M}, u, W \dashv\!\vert\, \psi$. By (id), $\mathfrak{M}, u, W \Vdash \varphi$ and $\mathfrak{M}, u, W \Vdash \varphi > \psi$. It follows by (iii) that $\mathfrak{M}, u, W \not\dashv\!\vert\, \psi$, contradiction.

For (vii), suppose $\mathfrak{M}, w, [\![\varphi]\!]_+^W \Vdash \varphi > \psi$. Then $\mathfrak{M}, w, [\![\varphi]\!]_+^W \cap [\![\varphi]\!]^{[\![\varphi]\!]_+^W} \Vdash$

ψ. Since φ is Boolean, by Lemma 2.6, we have $\mathfrak{M}, w, [\![\varphi \wedge \varphi]\!]_+^W \Vdash \psi$, i.e., $\mathfrak{M}, w, [\![\varphi]\!]_+^W \Vdash \psi$. By Lemma 2.3, we have $\mathfrak{M}, w, [\![\varphi]\!]_+^W \nVdash \psi$, as required.

Proof of Proposition 4.2. (i) and (iii) can be proved using the same counter-models as in the proof of (i) of Proposition 4.1. (ii) and (iv) can be proved using the same counter-models as in the proofs of (ii) of Proposition 4.1.

For (v), suppose otherwise there exist $\mathfrak{M} = (W, f, V)$ and $w \in W$ such that $\mathfrak{M}, w, W \Vdash \varphi > \psi$, $\mathfrak{M}, w, W \Vdash \neg\psi$, and $\mathfrak{M}, w, W \dashv\Vdash \neg\varphi$. Then $\mathfrak{M}, w, [\![\varphi]\!]_+^W \Vdash \psi$ and $\mathfrak{M}, w, W \Vdash \varphi$. It follows by Lemma 2.6 that $\mathfrak{M}, w, W \Vdash \psi$, contradicting $\mathfrak{M}, w, W \Vdash \neg\psi$.

For (vi), suppose otherwise there exists $\mathfrak{M} = (W, f, V)$ and $w \in W$ such that $\mathfrak{M}, w, [\![\varphi > \psi, \neg\psi]\!]_+^W \dashv\Vdash \neg\varphi$, i.e., $\mathfrak{M}, w, [\![\varphi > \psi, \neg\psi]\!]_+^W \Vdash \varphi$. Since φ is Boolean, there exits $u \in W$ such that $f(w, [\![\varphi > \psi, \neg\psi]\!]_+^W) = u$ and $\mathfrak{M}, u, W \Vdash \varphi$. By (id), we have $\mathfrak{M}, u, W \Vdash \varphi > \psi$ and $\mathfrak{M}, u, W \Vdash \neg\psi$. By (v), we have $\mathfrak{M}, u, W \Vdash \neg\varphi$, contradiction.

Proof of Proposition 4.3. For (i), consider $\mathfrak{M} = (W, f, V)$, where $W = \{w, u, v\}$, $f(w, \{u\}) = u$, $f(w, \{u, v\}) = v$, $V(p) = \{u\}$, $V(q) = \{u, v\}$, $V(r) = \{v\}$. Then $\mathfrak{M}, w, W \Vdash p > q$ and $\mathfrak{M}, w, W \Vdash q > r$ but $\mathfrak{M}, w, W \dashv\Vdash p > r$.

For (ii), consider $\mathfrak{M} = (W, f, V)$, where $W = \{w, u, v\}$, $f(w, \{u, v\}) = u$, $f(w, \{v\}) = f(u, \{v\}) = f(v, \{v\}) = v$, $V(p) = \{v\}$, $V(q) = \{u, v\}$, $V(r) = \{u\}$. Then $[\![p > q, q]\!]_+^W = \{w, u, v\} \cap \{u, v\} = \{u, v\}$, $[\![p > q, p]\!]_+^W = \{w, u, v\} \cap \{v\} = \{v\}$. It follows that $\mathfrak{M}, w, [\![p > q]\!]_+^W \Vdash q > r$ and $\mathfrak{M}, w, [\![p > q]\!]_+^W \dashv\Vdash p > r$. Hence, $q > r \nvDash^{p>q} p > r$.

For (iii), suppose $\mathfrak{M}, w, [\![\psi > \chi]\!]_+^W \Vdash \varphi > \psi$. Then $\mathfrak{M}, w, [\![\psi > \chi]\!]_+^W \cap [\![\varphi]\!]^X \Vdash \psi$, where $X = [\![\psi > \chi]\!]_+^X$. Let $Y = [\![\psi > \chi]\!]_+^W \cap [\![\varphi]\!]^X$. By Lemma 2.4, there exists $u \in W$ such that $f(w, Y) = u$ and $\mathfrak{M}, u, W \Vdash \psi$. By (id), we have $\mathfrak{M}, u, W \Vdash \psi > \chi$. It follows by (v) of Proposition 4.1 that $\mathfrak{M}, u, W \nVdash \chi$. By Lemma 2.4 again, we have $\mathfrak{M}, w, Y \nVdash \chi$. Hence, $\mathfrak{M}, w, [\![\psi > \chi]\!]_+^W \nVdash \varphi > \chi$, as required.

For (iv), suppose otherwise there exist $\mathfrak{M} = (W, f, V)$ and $w \in W$ such that $\mathfrak{M}, w, [\![\varphi > \psi, \psi > \chi]\!]_+^W \dashv\Vdash \varphi > \chi$. Then $\mathfrak{M}, w, [\![\varphi > \psi, \psi > \chi]\!]_+^W \cap [\![\varphi]\!]_+^X \dashv\Vdash \chi$, where $X = [\![\varphi > \psi, \psi > \chi]\!]_+^W$. By Lemma 2.5, we have $\mathfrak{M}, w, [\![\varphi > \psi, \psi > \chi, \varphi]\!]_+^W \dashv\Vdash \chi$. Let $Y = [\![\varphi > \psi, \psi > \chi, \varphi]\!]_+^W$. By Lemma 2.4, there exists $u \in W$ such that $f(w, Y) = u$ and $\mathfrak{M}, u, W \dashv\Vdash \chi$. On the other hand, by the proof of (v) of Proposition 4.1 and (id), we have $\mathfrak{M}, u, W \Vdash \chi$, contradiction.

For (v), consider $\mathfrak{M} = (W, f, V)$, where $W = \{w, u, v\}$, $f(w, \{u, v\}) = u$, $f(w, \{u\}) = u$, $V(p) = W$, $V(q) = \{u, v\}$, $V(r) = \{u\}$. Then it can be verified that $\mathfrak{M}, w, [\![(p > (q > r), (q > r) > s]\!]_+^W \dashv\Vdash p > s$.

(vi) can be proved using the same counter-model as in (v).

For (vii), consider $\mathfrak{M} = (W, f, V)$, where $W = \{w, u\}$, $f(w, \{w\}) = f(w, \{w, u\}) = w$, $f(u, \{w\}) = w$, $f(u, \{u\}) = f(u, \{w, u\}) = u$, $V(p) = \{w\}$, $V(q) = \{w, u\}$, $V(r) = \{u\}$. Then it can be verified that $\mathfrak{M}, w, [\![p > q, q > \Diamond r]\!]_+^W \dashv\Vdash p > \Diamond r$.

(viii) can be proves using the same counter-model as in (vii).

Proof of Proposition 4.4. For (i), consider $\mathfrak{M} = (W, f, V)$, where $W = \{w, u\}$, $f(w, \{w, u\}) = w$, $f(w, \{u\}) = u$, $V(p) = V(r) = \{w\}$, $V(q) = \{u\}$. Then $\mathfrak{M}, w, W \Vdash p \vee q > r$ but $\mathfrak{M}, w, W \dashv\mid q > r$.

For (ii), suppose otherwise there exist $\mathfrak{M} = (W, f, V)$ and $w \in W$ such that $\mathfrak{M}, w, [\![\varphi \vee \psi > \chi]\!]^W_+ \dashv\mid (\varphi > \chi) \wedge (\psi > \chi)$. Then $\mathfrak{M}, w, [\![\varphi \vee \psi > \chi]\!]^W_+ \dashv\mid \varphi > \chi$ or $\mathfrak{M}, w, [\![\varphi \vee \psi > \chi]\!]^W_+ \dashv\mid \psi > \chi$. Let $X = [\![\varphi \vee \psi > \chi]\!]^W_+$. Then $\mathfrak{M}, w, X \cap [\![\varphi]\!]^X \dashv\mid \chi$ and $\mathfrak{M}, w, X \cap [\![\psi]\!]^X \dashv\mid \chi$. By Lemma 2.5, it follows that $\mathfrak{M}, w, Y \dashv\mid \chi$ or $\mathfrak{M}, w, Z \dashv\mid \chi$, where $Y = [\![\varphi \vee \psi > \chi, \varphi]\!]^W_+$, $Z = [\![\varphi \vee \psi > \chi, \psi]\!]^W_+$. Suppose $\mathfrak{M}, w, Y \dashv\mid \chi$. By Lemma 2.5, there exists $u \in W$ such that $f(w, Y) = u$ and $\mathfrak{M}, u, W \dashv\mid \chi$. By (id), we have $\mathfrak{M}, u, W \Vdash \varphi \vee \psi > \chi$ and $\mathfrak{M}, u, W \Vdash \psi$. By the latter, we have $\mathfrak{M}, u, W \Vdash \varphi \vee \psi$. Then by (v) of Proposition 4.1, we have $\mathfrak{M}, u, W \not\dashv\mid \chi$, contradiction. A similar contradiction can be derived if $\mathfrak{M}, w, Z \dashv\mid \chi$. Hence, $\mathfrak{M}, w, [\![\varphi \vee \psi > \chi]\!]^W_+ \not\dashv\mid (\varphi > \chi) \wedge (\psi > \chi)$, as required.

For (iii), consider $\mathfrak{M} = (W, f, V)$, where $W = \{w, u\}$, $f(w, \{w, u\}) = f(w, \{w\}) = w$, $f(u, \{w, u\}) = u$, $f(u, \{w\}) = w$, $V(p) = \{w\}$, $V(q) = V(r) = \{u\}$. Then it can be verified that $\mathfrak{M}, w, [\![p \vee q > \Diamond r]\!]^W_+ \dashv\mid p > \Diamond r$.

Proof of Proposition 4.5. For (i), consider $\mathfrak{M} = (W, f, V)$, where $W = \{w, u\}$, $f(w, \{u\}) = u$, $V(p) = V(q) = \{w\}$. Then it is easily verified that $\mathfrak{M}, w, W \Vdash p \vee q$ but $\mathfrak{M}, w, W \dashv\mid \neg p > q$.

For (ii), suppose otherwise there exist $\mathfrak{M} = (W, f, V)$ and $w \in W$ such that $\mathfrak{M}, w, [\![\varphi \vee \psi]\!]^W_+ \dashv\mid \neg\varphi > \psi$. Then $\mathfrak{M}, w, [\![\varphi \vee \psi]\!]^W_+ \cap [\![\neg\varphi]\!]^{[\![\varphi \vee \psi]\!]^W_+} \dashv\mid \psi$. By Lemma 2.5, it follows that $\mathfrak{M}, w, [\![(\varphi \vee \psi) \wedge \neg\varphi]\!]^W_+ \dashv\mid \psi$. By Lemma 2.4, there exists $u \in W$ such that $f(w, [\![(\varphi \vee \psi) \wedge \neg\varphi]\!]^W_+) = u$ and $\mathfrak{M}, u, W \dashv\mid \psi$. On the other hand, we have $[\![(\varphi \vee \psi) \wedge \neg\varphi]\!]^W_+ \subseteq [\![\psi]\!]^W_+$. Then by (id), it follows that $\mathfrak{M}, u, W \Vdash \psi$, contradiction.

(iii) is straightforward by (ii), and (iv) is a reformulation of (ii).

Proof of Theorem 4.6. By (iv) of Proposition 4.5, it suffices to show that $\vDash (\neg\varphi > \psi) > (\varphi \vee \psi)$. Suppose otherwise there exist $\mathfrak{M} = (W, f, V)$ and $w \in W$ such that $\mathfrak{M}, w, W \dashv\mid (\neg\varphi > \psi) > (\varphi \vee \psi)$. Then $\mathfrak{M}, w, [\![\neg\varphi > \psi]\!]^W_+ \dashv\mid \varphi \vee \psi$, whence $\mathfrak{M}, w, [\![\neg\varphi > \psi]\!]^W_+ \dashv\mid \varphi$ and $\mathfrak{M}, w, [\![\neg\varphi > \psi]\!]^W_+ \dashv\mid \psi$. By Lemma 2.4, there exists $u \in W$ such that $f(w, [\![\neg\varphi > \psi]\!]^W_+ = u$ and $\mathfrak{M}, u, W \dashv\mid \varphi$ and $\mathfrak{M}, u, W \dashv\mid \psi$. By the former, we have $\mathfrak{M}, u, W \Vdash \neg\varphi$. By (id), we also have $\mathfrak{M}, u, W \Vdash \neg\varphi > \psi$. It follows from (v) of Proposition 4.1 that $\mathfrak{M}, u, W \not\dashv\mid \psi$, contradiction.

Proof of Proposition 4.7. We prove the case for $\omega = p \wedge \Diamond\neg p$. The other case can be prove analogously.

For (i), it is easily seen that ω can be true in a model with two worlds, one verifying p and the other falsifying p.

For (ii), suppose otherwise there exist $\mathfrak{M} = (W, f, V)$ and $w \in W$ such that $\mathfrak{M}, w, [\![p \wedge \Diamond\neg p]\!]^W_+ \dashv\mid \neg(p \wedge \Diamond\neg p)$. Then $\mathfrak{M}, w, [\![p \wedge \Diamond\neg p]\!]^W_+ \Vdash \Diamond\neg p$. Thus there exists $u \in [\![p \wedge \Diamond\neg p]\!]^W_+$ such that $\mathfrak{M}, u, [\![p \wedge \Diamond\neg p]\!]^W_+ \Vdash \neg p$. Then $f(u, [\![p \wedge \Diamond\neg p]\!]^W_+)$ is defined and $f(u, [\![p \wedge \Diamond\neg p]\!]^W_+) \notin V(p)$. On the other hand, by (id), we have $f(u, [\![p \wedge \Diamond\neg p]\!]^W_+) \in [\![p]\!]^W_+ = V(p)$, contradiction.

Proof of Proposition 4.8. For (i), consider $\mathfrak{M} = (W, f, V)$, where $W = \{w, u\}$, $f(w, W) = w$, $V(p) = \{u\}$. Then $\mathfrak{M}, w, [\![\Diamond p]\!]^W_+ \Vdash \neg p$, since $f(w, [\![\Diamond p]\!]^W_+) = f(w, W) = w$. And $\mathfrak{M}, w, [\![\Diamond p]\!]^W_+ \dashV \bot$. Hence, $\neg p \not\models^{\Diamond p} \bot$.

For (ii), suppose there exist $\mathfrak{M} = (W, f, V)$ and $w \in W$ such that $\mathfrak{M}, w, [\![\neg p]\!]^W_+ \Vdash \Diamond p$. Then $\mathfrak{M}, u, [\![\neg p]\!]^W_+ \Vdash p$ for some $u \in [\![\neg p]\!]^W_+$. By Lemma 2.6, we have $\mathfrak{M}, u, W \Vdash p$. On the other hand, by (id), we have $\mathfrak{M}, u, W \dashV p$, contradiction.

Proof of Proposition 4.9. For (i), consider $\mathfrak{M} = (W, f, V)$, where $W = \{w, u\}$, $f(w, W) = \{w\}$, $f(u, W) = \{u\}$, $V(p) = W$, $V(q) = \{u\}$. Then $\mathfrak{M}, w, W \Vdash p > \neg q$ and $\mathfrak{M}, u, [\![p]\!]^W_+ \Vdash q$ and hence $\mathfrak{M}, w, W \Vdash p > \Diamond q$.

For (ii), suppose otherwise there exist $\mathfrak{M} = (W, f, V)$ and $w \in W$ such that $\mathfrak{M}, w, [\![p > \neg q, p > \Diamond q]\!]^W_+ \dashV \neg((p > \neg q) \wedge (p > \Diamond q))$. Then $\mathfrak{M}, w, X \Vdash p > \Diamond q$, where $X = [\![p > \neg q, p > \Diamond q]\!]^W_+$. It follows that $\mathfrak{M}, w, X \cap [\![p]\!]^X_+ \Vdash \Diamond q$. Thus there exists $u \in X \cap [\![p]\!]^X_+$ such that $\mathfrak{M}, u, X \cap [\![p]\!]^X_+ \Vdash q$. By Lemma 2.6, we have $\mathfrak{M}, u, W \Vdash q$. On the other hand, by (id), we have $\mathfrak{M}, u, W \Vdash p > \neg q$ and $\mathfrak{M}, u, X \Vdash p$. Since $u \in X$, by Lemma 2.6 again, we have $\mathfrak{M}, u, W \Vdash p$. Then by (v) of Proposition 4.1, we have $\mathfrak{M}, u, W \nVdash q$, contradiction.

Proof of Proposition 4.10. Suppose otherwise there exist $\mathfrak{M} = (W, f, V)$ and $w \in W$ such that $\mathfrak{M}, w, W \dashV (\varphi > \psi) \vee (\varphi > \neg\psi)$. Then $\mathfrak{M}, w, [\![\varphi]\!]^W_+ \dashV \psi$ and $\mathfrak{M}, w, [\![\varphi]\!]^W_+ \dashV \neg\psi$. From the latter we have $\mathfrak{M}, w, [\![\varphi]\!]^W_+ \Vdash \psi$, contradicting the former by Lemma 2.3.

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