Modal Logics in Dynamical Systems

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Abstract

Dynamic topological logic and its variant, intuitionistic temporal logic, combine topological semantics with linear temporal logic in order to reason about topological dynamics. I will provide a historical overview of the field, from its inception to recent results and open questions.

Keywords: Dymamical systems, topological semantics, linear temporal logic, intuitionistic logic

A common theme cutting across the natural and social sciences is the need to model the behaviour over time of a system whose evolution follow fixed laws. Mathematically, such phenomena may be represented as the continuous action of a semigroup Θ (e.g. the real or natural numbers) on a topological space X. The case where $\Theta = \mathbb{N}$ is particularly compelling from a modal perspective, as it lends itself to formalisation using the well-understood linear temporal logic (LTL) [8]. Such actions may be represented simply by a function S sending each point to its temporal successor, and for our purposes, a *dynamical system* consists of a pair (X, S), where X is a topological space and $S\colon X \to X$ is continuous.

The function S yields semantics for LTL. Meanwhile, it was already known by McKinsey and Tarski in the 1940s [12] that the topological structure of X may be used to interpret the modal logic $\mathsf{S4}$. These observations led Artemov et al. [1] to introduce the logic $\mathsf{S4C}$, combining the topological interior modality ($\blacksquare$) with the 'next' modality ($\circ$) from LTL. Here C stands for 'continuous', and Artemov et al. showed that the resulting logic is sound and complete for the class of dynamical systems, moreover enjoying the finite model property and hence decidability.

Kremer and Mints [11] established analogous results for the variant $\mathsf{S4H}$, interpreted over dynamical systems where S is a homeomorphism. They moreover noted that by adding the 'henceforth' modality ($\Box$), non-trivial recurrence

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phenomena such as the Poincaré recurrence theorem could be formalised in a modal setting. Kremer and Mints dubbed the resulting tri-modal logic *dynamic topological logic* (DTL). They also showed that, unlike the logic S4C, DTL can distinguish between its topological and Kripke semantics, from which in particular it follows that it cannot have the finite model property.

Unfortunately, DTL was soon shown to be undecidable by Konev et al. [9], and even non-axiomatisable when interpreted over systems with a homeomorphism [10]. This spurred a search for decidable variants of DTL, with Gabelaia et al. [7] proposing DTL with finite iterations and myself [4], following a suggestion of Mints, interpretations over *minimal systems,* characterised by the orbit of every point being dense. Nevertheless, in both of these cases, the complexity is non-primitive recursive. I also showed that DTL does enjoy a natural axiomatisation, albeit over an extended language [5]. In contrast, the Kripke variant of DTL has not been axiomatised or even proven to be computably enumerable.

Despite these advances, the quest for a decidable variant of dynamical topological logic interpretable over arbitrary dynamical systems with unbounded time remained elusive. One proposal that seemed to slip under the radar was Kremer's unpublished note (but see [3]), where he proposed an intuitionistic variant of DTL. Kremer showed that this variant does not validate key LTL axioms with $\Box$. Over a decade later, I realised that it *does* validate all axioms involving only 'eventually' ($\Diamond$) and indeed this variant of the logic, now dubbed intuitionistic (linear) temporal logic (ITL), is indeed decidable in primitive recursive time [6] (with the precise complexity not yet known). Moreover, both minimality and Poincaré recurrence are representable in this framework, thus finally obtaining a decidable framework for reasoning about arbitrary dynamic topological spaces as envisioned by Kremer and Mints.

Since then, Boudou, Diéguez and I have axiomatised $\mathsf{ITL}_{\Diamond\forall}$ [2] and, in forthcoming work, McLean, Zenger and I showed that by incorporating *co-implication,* $\mathsf{ITL}_{\Diamond\Box}$ becomes naturally axiomatisable, the first axiomatisation for any DTL-style Kripke-based logic where that does not coincide with its topological variant. The techniques developed here may lead to axiomatising Kripke DTL, as well as expanding products of modal logics in a more general setting [7].

In this talk we will overview these and other related results, highlighting notable proof ideas which are characteristic of the field and are likely to find greater applicability in modal logics in general and, specifically, other non-classical dynamic logics. I will also discuss future directions and open questions, both in the field and in potential applications to other non-classical dynamic logics.

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