Informative Presupposition in Inquisitive Logic

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Abstract

In this paper, we explore a logic of the modality of informative presupposition in the context of propositional intuitionistic inquisitive logic. We present a natural deduction calculus for this logic and show that it is sound and complete with respect to a Kripke semantics based on inquisitive Kripke models and an algebraic semantics based on the notion of inquisitive nucleus. We argue that our setting sheds some light on the relation between different approaches to intuitionistic inquisitive logic, namely, the approach developed by Wesley Holliday, on one side, and approaches based on the tensor disjunction, on the other side. Furthermore, we assess the connections of the presupposition modality to other related notions such as lax modality.

Keywords: Inquisitive logic, Lax logic, Information-based semantics, presupposition, informative content, nucleus.

1 Introduction

Inquisitive semantics and logic [6,7] is a framework for a logical analysis of questions. Standard inquisitive logic, as developed in [2,9], is based on classical logic for statements. In intuitionistic inquisitive logic the background logic of statements is intuitionistic. Various versions of propositional intuitionistic inquisitive logic have been considered in the literature.

The most simple version, that we call InqIL, replaces the intuitionistic disjunction with inquisitive disjunction (and thus in this version the background logic of statements is not the full intuitionistic logic but only its disjunction-free fragment). This logic was considered already in [15], though not in the context of modern inquisitive semantics. A generalization of the basic inquisitive semantics suitable for InqIL was introduced in [16]; an algebraic semantics

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for InqIL is developed in [18]; [21] proves that InqIL is the logic determined by proof-theoretic semantics; [11] shows that there is an intimate connection between InqIL and the notion of structural completeness; it follows from the main result of [12] that the schematic fragment of InqIL is the Medvedev logic of finite problems (a result that was stated without proof already in [15]).

In [8,17], the language of InqIL was expanded with an additional declarative disjunction (so that the background logic of statements is the full propositional intuitionistic logic). This disjunction was characterized by a semantic clause that is commonly used in the context of team semantics for dependence logic where the connective is called *tensor disjunction* (see, e.g., [23]). We call the resulting logic $\mathsf{InqIL}^{\otimes}$. A different treatment of declarative disjunction is developed by Holliday in [13] where inquisitive logic is construed in the framework of Beth semantics and declarative disjunction is captured through the notion of Beth nucleus. We denote Holliday's logic as HInqIL. Further work on intuitionistic inquisitive logic includes [19,20].

In this paper, we will add to InqIL a modality of informative presupposition $\circ$, by which we obtain the logic InqIL°. We will formulate a system of natural deduction for this logic, prove completeness, compactness, decidablity and several other results. Moreover, we will show that the presence of the presupposition modality leads to a treatment of declarative disjunction that will help us to clarify the relationship between Holliday's approach (related to HInqIL) and the approaches based on the tensor disjunction (related to $\mathsf{InqIL}^{\otimes}$).

2 Inquisitive Logics

The basic language of propositional inquisitive logic L is just like the language of intuitionistic logic with an unusual symbol for disjunction ($\veebar$). So, L is generated from a fixed set of atomic formulas with the following logical symbols: $\bot, \to, \wedge, \veebar$. Negation ($\neg$) and equivalence ($\leftrightarrow$) will be regarded as defined symbols: $\neg\varphi =_{def} \varphi \to \bot$, $\varphi \leftrightarrow \psi =_{def} (\varphi \to \psi) \wedge (\psi \to \varphi)$. In the context of L, as well as in the contexts of other languages employed in this paper, we will be distinguishing two categories of formulas: *declarative* formulas and *arbitrary* formulas. Declarative formulas, which form a particular subset of arbitrary formulas, correspond to statements. Arbitrary formulas encompass statements as well as questions. In the context of any of the considered languages, we will be using systematically the variables $\varphi, \psi, \chi, \vartheta$ for arbitrary formulas and α, β, γ for declarative formulas. In the language L, declarative formulas are defined as the $\veebar$-free fragment of L.

Let us introduce the usual intuitionistic Kripke semantics for L. A *Kripke frame* is a partially ordered set $\mathcal{F} = \langle S, \leq \rangle$. The elements of S will be called *states* in $\mathcal{F}$. A set $P \subseteq S$ is *upward closed* (in $\mathcal{F}$) if it satisfies the following condition: if $w \in P$ and $w \leq v$ then $v \in P$. The upward closed sets will be called *propositions*.

A *Kripke model* is a Kripke frame $\mathcal{F}$ equipped with a valuation V defined as a function that assigns to each atomic formula a proposition in $\mathcal{F}$. Given any Kripke model $\mathcal{M} = \langle S, \leq, V \rangle$, the relation $\Vdash$ between states of the model

and formulas is defined in the usual recursive way:

- $w \Vdash p$ iff $w \in V(p)$; $w \nVdash \bot$; $w \Vdash \varphi \to \psi$ iff for any $v \geq w$, if $v \Vdash \varphi$, $v \Vdash \psi$;
- $w \Vdash \varphi \wedge \psi$ iff $w \Vdash \varphi$ and $w \Vdash \psi$; $w \Vdash \varphi \curlyvee \psi$ iff $w \Vdash \varphi$ or $w \Vdash \psi$.

We define $||\varphi||_{\mathcal{M}} = \{w \in S \mid w \Vdash \varphi \text{ in } \mathcal{M}\}$. $||\varphi||_{\mathcal{M}}$ is called *the proposition expressed by* φ *in* $\mathcal{M}$. This is justified by the fact that the relation $\Vdash$ is persistent: if $w \Vdash \varphi$ and $w \leq v$ then $v \Vdash \varphi$, and thus $||\varphi||_{\mathcal{M}}$ is upward closed. We will be omitting the subscript $\mathcal{M}$ if clear from the context.

We say that φ is *valid in* $\mathcal{M}$ if $w \Vdash \varphi$ holds in $\mathcal{M}$ for every $w \in S$, i.e. if $||\varphi|| = S$. It is well-known that a formula is intuitionistically valid if and only if it is valid in all Kripke models [14].

In accordance with [16], classical and intuitionistic *inquisitive* logics are defined as the logics determined by particular classes of Kripke models. Let $\mathcal{M} = \langle S, \leq, V\rangle$ be a Kripke model. The *inquisitive variant of* $\mathcal{M}$ is defined as the Kripke model $inq(\mathcal{M}) = \langle Up_{\emptyset}S, \supseteq, V^*\rangle$, where $Up_{\emptyset}S$ denotes the set of all *nonempty* upward closed sets of states in $\mathcal{M}$, $\supseteq$ is the superset relation and $V^*(p) = \{s \in Up_{\emptyset}S \mid s \subseteq V(p)\}$, for every atomic formula p.

The idea behind the construction of $inq(\mathcal{M})$ is that the states in $inq(\mathcal{M})$ correspond to consistent propositions or consistent pieces of information in $\mathcal{M}$. Hence, in the model $inq(\mathcal{M})$, formulas are evaluated relative to propositions of $\mathcal{M}$ rather than relative to single states of $\mathcal{M}$. If s is such a proposition (i.e. an upward closed set in $\mathcal{M}$) and α is a declarative formula then $s \Vdash \alpha$ in $inq(\mathcal{M})$ can be interpreted as stating that the information s supports the statement α. This interpretation aligns with the following crucial feature of this construction (for a proof, see [16]).

Claim 2.1 *Let $\mathcal{M}$ be a Kripke model, s a state in $inq(\mathcal{M})$ and α an L-formula. Then $s \Vdash \alpha$ in $inq(\mathcal{M})$ iff for all $w \in s$, $w \Vdash \alpha$ in $\mathcal{M}$. As a consequence, α is valid in $inq(\mathcal{M})$ if and only if α is valid in $\mathcal{M}$.*

This claim cannot be expanded to the full language L. The disjunction $\curlyvee$ becomes inquisitive in $inq(\mathcal{M})$. For instance, the formula $p \curlyvee q$ does not express the statement *that p or q* in $inq(\mathcal{M})$ but rather the question *whether p or q*. If s is a consistent piece of information (a nonempty upward closed set) in $\mathcal{M}$ then $s \Vdash p \curlyvee q$ in $inq(\mathcal{M})$ means that s either supports p, or it supports q, and thus provides an answer to the question whether p or q.

A Kripke model $\mathcal{N}$ is called *inquisitive* if it is the inquisitive variant of a Kripke model, i.e. $\mathcal{N} = inq(\mathcal{M})$, for some Kripke model $\mathcal{M}$. An inquisitive Kripke model is called *classical* if it is the inquisitive variant of a Kripke model where the ordering is identity. The states in such an underlying Kripke model $\mathcal{M} = \langle W, =, V\rangle$ can be viewed as *possible worlds* and the states in $\mathcal{N} = inq(\mathcal{M})$ are nonempty sets of possible worlds. So, classical inquisitive Kripke models are exactly those of the shape $\langle \mathcal{P}(W)\backslash\{\emptyset\}, \supseteq, V^*\rangle$, where for each p there is $U \subseteq W$ such that $V^*(p) = \mathcal{P}(U)\backslash\{\emptyset\}$.

Let us illustrate the semantics with a simple classical inquisitive Kripke model. Let $\mathcal{M} = \langle W, =, V\rangle$, where $W = \{w_1, w_2, w_3, w_4\}$, $V(p) = \{w_1, w_2\}$,

Table 1
The system $S(\mathsf{InqIL})$

ex falso	$\bot/\varphi$	
implication	$[\varphi/\psi]/\varphi \to \psi$	$\varphi \to \psi, \varphi/\psi$
conjunction	$\varphi, \psi/\varphi \wedge \psi$	(i) $\varphi \wedge \psi/\varphi$ (ii) $\varphi \wedge \psi/\psi$
disjunction	(i) $\varphi/\varphi \mathbin{\mathbb{V}} \psi$ (ii) $\psi/\varphi \mathbin{\mathbb{V}} \psi$	$\varphi \mathbin{\mathbb{V}} \psi, [\varphi/\chi], [\psi/\chi]/\chi$
split	$\alpha \to (\varphi \mathbin{\mathbb{V}} \psi)/(\alpha \to \varphi) \mathbin{\mathbb{V}} (\alpha \to \psi)$	

and $V(q) = \{w_1, w_3\}$. The propositions in $\mathcal{M}$ are just subsets of W. For example, $||p|| = \{w_1, w_2\}$ in $\mathcal{M}$. In contrast, the propositions in $inq(\mathcal{M})$ are downward closed sets of nonempty subsets of W. Nevertheless, Claim 2.1 guarantees that for declarative formulas these sets of subsets are generated by the corresponding propositions in $\mathcal{M}$. For a simple example, we have $||p|| = \{\{w_1, w_2\}, \{w_1\}, \{w_2\}\}$ in $inq(\mathcal{M})$. In general, if inquisitive disjunction is involved, the corresponding proposition in $inq(\mathcal{M})$ is not generated by a single proposition in $\mathcal{M}$ but rather by a set of such propositions (representing the possible answers to a question). For example, $||p \mathbin{\mathbb{V}} q|| = \{\{w_1, w_2\}, \{w_1, w_3\}, \{w_1\}, \{w_2\}, \{w_3\}\}$ in $inq(\mathcal{M})$.

In general, we say that a proposition P in $inq(\mathcal{M})$ is *declarative* if it is closed under arbitrary unions, i.e. whenever P contains some sets s_i, $i \in I$, then it contains also the set $\bigcup_{i \in I} s_i$. This means that declarative propositions are those that are generated by a single proposition in $\mathcal{M}$. That is, an upward closed set of states P in $inq(\mathcal{M})$ is declarative iff there is an upward closed set Q in $\mathcal{M}$ such that P is the set of all nonempty upward closed subsets of Q. We say that P is *inquisitive* if it is not declarative.

Note that according to our definition, the empty set is always declarative. Furthermore, if $inq(\mathcal{M})$ is classical then every declarative proposition in $inq(\mathcal{M})$ is the set of nonempty subsets of a set of possible worlds from $\mathcal{M}$.

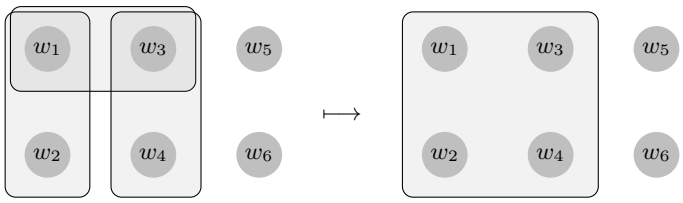
The intuitionistic inquisitive logic InqIL is defined as the set of L-formulas valid in all inquisitive Kripke models. The classical (or basic) inquisitive logic InqB is the set of L-formulas valid in all classical inquisitive Kripke models.

It follows from Claim 2.1 that in the declarative fragment of L, InqIL coincides with the (disjunction-free fragment of) intuitionistic logic and InqB with classical logic. However, the presence of inquisitive disjunction makes a difference. The logic InqIL can be characterized by a system of natural deduction consisting of the rules for intuitionistic logic plus one additional rule called *split*: $\alpha \to (\varphi \mathbin{\mathbb{V}} \psi)/(\alpha \to \varphi) \mathbin{\mathbb{V}} (\alpha \to \psi)$, where the antecedent α is restricted to declarative formulas (see [16]). Using a nonstandard but clear and compact notation we summarize the whole system, which we call $S(\mathsf{InqIL})$, in Table 1. The logic is not closed under uniform substitution. This is related to the fact that it is determined by a class of Kripke models (not frames) with specific valuations.

A system for the logic InqB, which is also not closed under uniform substitution, is obtained by adding to the system $S(\mathsf{InqIL})$ the double negation law restricted to declarative formulas: $\neg\neg\alpha/\alpha$. Both these logics, InqIL and InqB, have the disjunction property and the finite model property.

3 The modality of informative presupposition

Consider a Kripke model $\mathcal{M} = \langle W, =, V\rangle$, where $W = \{w_1, \ldots, w_6\}$, $V(p) = \{w_1, w_2\}$, $V(q) = \{w_1, w_3\}$, and $V(r) = \{w_3, w_4\}$. The formula $p \mathbin{\mathbb{\vee}} q \mathbin{\mathbb{\vee}} r$ represents the question *whether p or q or r*. This question presupposes the information that at least one of the three alternatives p, q, r is true and thus excludes the worlds w_5 and w_6. We want to capture such an informative presupposition as a modality. This modality takes a possibly inquisitive proposition P and returns the corresponding declarative proposition iP respecting the boundaries of P but destroying its internal structure. We can depict the situation as follows:

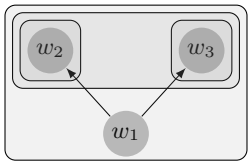


Recall that P and iP are closed under nonempty subsets and so only the maximal sets are depicted (so that the operation does not coincide with simple union). In general, we are concerned with the following operation. If $\mathcal{M} = \langle S, \leq, V\rangle$ is a Kripke model and P is a proposition in $inq(\mathcal{M})$ then

$$iP = \{s \in Up_{\emptyset}S \mid s \subseteq \bigcup P\}.$$

Note that the application of this operation always returns a proposition in $inq(\mathcal{M})$. In particular, iP is the strongest declarative proposition implied by P. This operation, also known as "non-inquisitive closure", has been considered and used in the context of InqB and its semantics based on classical inquisitive models. It is well-known that in this context the operation coincides with double negation and the exclamation mark has been used as a defined symbol for its syntactic representation: $!\varphi =_{def} \neg\neg\varphi$.[2]

However, in general, the presupposition modality is not identical with double negation. Consider the following Kripke model $\mathcal{M}$:



[2] Instead of the exclamation mark, we will use the symbol $\circ$ to indicate the connection to the lax modality [10].

Let $||p|| = \{w_2\}$ and $||q|| = \{w_3\}$ in $\mathcal{M}$. Then in $inq(\mathcal{M})$ we obtain: $||p \mathbin{\mathbb{V}} q|| = \{\{w_2\}, \{w_3\}\}$, $i||p \mathbin{\mathbb{V}} q|| = \{\{w_2, w_3\}, \{w_2\}, \{w_3\}\}$, $||\neg\neg(p \mathbin{\mathbb{V}} q)|| = \{\{w_1, w_2, w_3\}, \{w_2, w_3\}, \{w_2\}, \{w_3\}\}$.

Our main goal is to characterize the logic of informative presupposition in the context of intuitionistic inquisitive logic. Let us first make the following observation that shows that i is an operation of an important kind, namely a dense nucleus on the algebra of propositions.

Claim 3.1 *Let $\mathcal{M}$ be a Kripke model and P, Q propositions in $inq(\mathcal{M})$. Then: (a) $P \subseteq iP$, (b) $iiP = iP$, (c) $i(P \cap Q) = iP \cap iQ$, (d) $i\emptyset = \emptyset$.*

For a proof (of this and other results in this paper), see the appendix. Claim 3.1 suggests that there is a connection between the operation i and the lax modality [10] which is a logical counterpart of the notion of nucleus. We will study this connection in more detail in the next section. Inspired by the relation to the lax logic, we will be using the symbol $\circ$ for the presupposition modality. We denote the language L enriched with this modality as L°. Since $\circ\varphi$ expresses the information presupposed by φ, we can define the declarative disjunction as the presupposition of the corresponding inquisitive disjunction: $\varphi \vee \psi =_{def} \circ(\varphi \mathbin{\mathbb{V}} \psi)$.

The set of declarative formulas in L° is the smallest set of L°-formulas which is closed under $\rightarrow$ and $\wedge$ and contains $\bot$, all atomic formulas and all formulas of the form $\circ\varphi$. In contrast to L, declarative formulas in L° may involve inquisitive disjunction, but only in the scope of $\circ$. This definition of declarative formulas aligns with the general definition of declaratives in inquisitive modal logic [4].

If $\langle S, \leq\rangle$ is a partial order and $w \in S$ then $\uparrow w$ denotes the upward closed set generated by w, i.e. $\uparrow w = \{v \in S \mid w \leq v\}$. Relative to any inquisitive model $inq(\mathcal{M})$ and state s in $inq(\mathcal{M})$ we define the following support condition for $\circ$:

- $s \Vdash \circ\varphi$ iff for all $w \in s$, $\uparrow w \Vdash \varphi$.

With this support condition we obtain that $||\circ\varphi|| = i||\varphi||$. Now we would like to expand Claim 2.1 to the language L° but the semantic clause for $\circ$ is defined only for inquisitive models and so the formulation of Claim 2.1 would not make sense for declarative formulas of the shape $\circ\varphi$. Nevertheless, we can reformulate the claim in the following meaningful way that applies to L°.

Claim 3.2 *Let $\mathcal{M}$ be a Kripke model, s a state in $inq(\mathcal{M})$ and α a declarative L°-formula. Then $s \Vdash \alpha$ in $inq(\mathcal{M})$ iff for all $w \in s$, $\uparrow w \Vdash \alpha$ in $inq(\mathcal{M})$.*

The set of L°-formulas that are valid in all inquisitive Kripke models will be denoted as InqIL°. We also define the corresponding consequence relation. For any set of L°-formulas $\Delta \cup \{\varphi\}$ we define $\Delta \vDash_{(\mathsf{InqIL}^\circ)} \varphi$ as meaning that for every state s of any inquisitive Kripke model, if s supports all formulas from Δ then s supports φ.

Claim 3.2 can be reformulated as saying that $\alpha \leftrightarrow \circ\alpha \in \mathsf{InqIL}^\circ$, for every declarative α. The intuitive meaning of this fact is that the information presupposed by a statement is equivalent to that statement, which makes perfect sense because statements are (as is usual in logical semantics) reduced to their informative content.

Table 2
The system $S(\mathsf{InqIL}^\circ)$ consists of the rules of $S(\mathsf{InqIL})$ plus the following rules

$\circ$-*intro*: $\varphi/\circ\varphi$	$\circ$-*elim*: $\circ\varphi, [\varphi/\alpha]/\alpha$

The following observation demonstrates that in the intuitionistic inquisitive logic the presupposition modality is not expressible in the language L and so its addition to the language increases the expressive power.

Claim 3.3 *There is no L-formula χ such that $\circ(p \mathbin{\mathbb{V}} q) \leftrightarrow \chi \in \mathsf{InqIL}^\circ$.*

An important feature of the logic InqIL° that can be easily established is the disjunction property for $\mathbb{V}$.

Claim 3.4 *If $\varphi \mathbin{\mathbb{V}} \psi \in \mathsf{InqIL}^\circ$ then $\varphi \in \mathsf{InqIL}^\circ$ or $\psi \in \mathsf{InqIL}^\circ$.*

We will prove that the modality $\circ$ can be completely characterized by the introduction and elimination rules presented in Table 2. In $\circ$-*elim* we assume, as usual, that α is declarative. Note that in the context of the other rules *split* could be equivalently formulated purely schematically:

$$\textit{split}^*: \quad \circ\chi \to (\varphi \mathbin{\mathbb{V}} \psi)/(\circ\chi \to \varphi) \mathbin{\mathbb{V}} (\circ\chi \to \psi)$$

In this reformulation, the only rule responsible for the failure of substitution would be $\circ$-*elim*. We will write $\vdash_{S(\mathsf{InqIL}^\circ)} \varphi$ to express that φ is derivable in the system $S(\mathsf{InqIL}^\circ)$ from Table 2. This will be identified with $\emptyset \vdash_{S(\mathsf{InqIL}^\circ)} \varphi$. We further write $\Delta \vdash_{S(\mathsf{InqIL}^\circ)} \varphi$ if φ is derivable in $S(\mathsf{InqIL}^\circ)$ from (a finite subset of) formulas in Δ.

The rules $\circ$-*intro* and $\circ$-*elim* illustrate that $\circ\varphi$ can be viewed as a degenerate disjunction with only one disjunct. From this perspective, $\circ$-*intro* is like disjunction introduction (the disjunction $\circ\varphi$ can be derived from every, i.e. the only disjunct φ) and $\circ$-*elim* resembles disjunction elimination (if we have the disjunction $\circ\varphi$ and are able to derive α from all its disjuncts, i.e. from φ, then we can infer α).

One can observe that, in the context of the other rules, $\circ$-*intro* and $\circ$-*elim* are respectively interderivable with the following rules:

$$\circ\textit{-intro}^*: \circ\varphi \to \alpha/\varphi \to \alpha \qquad \circ\textit{-elim}^*: \varphi \to \alpha/\circ\varphi \to \alpha$$

If we put these two facts together we obtain that the introduction and elimination rules for $\circ$ could be replaced with a single axiom: $(\varphi \to \alpha) \leftrightarrow (\circ\varphi \to \alpha)$. The following lemma then amounts to soundness of the system with respect to inquisitive semantics.

Claim 3.5 *$(\varphi \to \alpha) \leftrightarrow (\circ\varphi \to \alpha) \in \mathsf{InqIL}^\circ$, for all φ, ψ and all declarative α.*

In order to prove completeness let us define the sublanguage of L° generated from atomic formulas by $\bot, \to, \wedge, \vee$. This is indeed a sublanguage of L° because $\vee$ is a symbol defined in L°. The formulas of this sublanguage will be called *simple*. Note that every simple formula is declarative. Simple formulas simulate the declarative language of intuitionistic logic. Importantly, we can simulate the usual introduction and elimination rules for $\vee$ in $S(\mathsf{InqIL}^\circ)$. The introduction

rules amount to $\alpha/\circ(\alpha \mathbin{\mathbb{\vee}} \beta)$ and $\beta/\circ(\alpha \mathbin{\mathbb{\vee}} \beta)$, which can be obtained as follows:

$$\dfrac{\dfrac{\alpha}{\alpha \mathbin{\mathbb{\vee}} \beta}\,\mathbb{\vee}\text{-}intro}{\circ(\alpha \mathbin{\mathbb{\vee}} \beta)}\,\circ\text{-}intro \qquad \dfrac{\dfrac{\beta}{\alpha \mathbin{\mathbb{\vee}} \beta}\,\mathbb{\vee}\text{-}intro}{\circ(\alpha \mathbin{\mathbb{\vee}} \beta)}\,\circ\text{-}intro$$

The elimination rule amounts to $\circ(\alpha \mathbin{\mathbb{\vee}} \beta), [\alpha/\gamma], [\beta/\gamma]/\gamma$, which is obtained in this way:

$$\dfrac{\circ(\alpha \mathbin{\mathbb{\vee}} \beta) \qquad \dfrac{[\alpha \mathbin{\mathbb{\vee}} \beta]^{(1)} \qquad \dfrac{[\alpha]^{(2)}}{\gamma} \qquad \dfrac{[\beta]^{(3)}}{\gamma}}{\gamma}\,(2),\,(3),\,\mathbb{\vee}\text{-}elim}{\gamma}\,(1),\,\circ\text{-}elim$$

As a consequence, the deductive system for InqIL° has all the resources of full intuitionistic logic on the level of simple formulas. That is, when we restrict ourselves to simple formulas all the intuitionistic introduction and elimination rules for $\bot, \to, \wedge, \vee$ are either directly contained or can be simulated in $S(\mathsf{InqIL}^\circ)$.

Claim 3.6 *If α is an intuitionistically valid simple formula (with $\vee$ in the role of intuitionistic disjunction) then $\vdash_{S(\mathsf{InqIL}^\circ)} \alpha$.*

This observation allows us to prove strong completeness of the system for InqIL° in a way that is standard in inquisitive logic. Before we formulate this result we will need one more characteristic property of propositional inquisitive logic adapted to our setting. We define recursively a function $\mathcal{R}$ assigning to every formula φ a finite set of simple formulas, called the resolutions of φ:

$\mathcal{R}(p) = \{p\}$, $\mathcal{R}(\bot) = \{\bot\}$,
$\mathcal{R}(\varphi \to \psi) = \{\bigwedge_{\alpha \in \mathcal{R}(\varphi)}(\alpha \to f(\alpha)) \mid f : \mathcal{R}(\varphi) \to \mathcal{R}(\psi)\}$,
$\mathcal{R}(\varphi \wedge \psi) = \{\alpha \wedge \beta \mid \alpha \in \mathcal{R}(\varphi), \beta \in \mathcal{R}(\psi)\}$,
$\mathcal{R}(\varphi \mathbin{\mathbb{\vee}} \psi) = \mathcal{R}(\varphi) \cup \mathcal{R}(\psi)$,
$\mathcal{R}(\circ\varphi) = \{\bigvee \mathcal{R}(\varphi)\}$.

We obtain the inquisitive disjunctive normal form and consequently completeness, compactness, finite model property and decidability of InqIL°.

Claim 3.7 *If $\mathcal{R}(\varphi) = \{\alpha_1, \ldots, \alpha_n\}$, then $\varphi \leftrightarrow (\alpha_1 \mathbin{\mathbb{\vee}} \ldots \mathbin{\mathbb{\vee}} \alpha_n) \in \mathsf{InqIL}^\circ$.*

Theorem 3.8 *$\Delta \vdash_{S(\mathsf{InqIL}^\circ)} \varphi$ iff $\Delta \vDash_{(\mathsf{InqIL}^\circ)} \varphi$.*

Corollary 3.9 *$\Delta \vDash_{(\mathsf{InqIL}^\circ)} \varphi$ iff $\Delta' \vDash_{(\mathsf{InqIL}^\circ)} \varphi$, for some finite $\Delta' \subseteq \Delta$.*

Theorem 3.10 *If $\varphi \notin \mathsf{InqIL}^\circ$ then there is a finite Kripke model $\mathcal{M}$ such that φ is not valid in $inq(\mathcal{M})$.*

Corollary 3.11 *InqIL° is decidable.*

4 Propositional lax logic and algebraic semantics

The previous section suggests that there is a close connection between InqIL° and propositional lax logic [10]. In this section, we focus on this connection.

Table 3
The system $S(\mathsf{PLL})$ consists of the rules of intuitionistic logic (in the language L°) plus the following rules

L1: $\varphi/\circ\varphi$	L2: $\circ\varphi, [\varphi/\circ\psi]/\circ\psi$

A system $S(\mathsf{PLL})$ of natural deduction for propositional lax logic formulated in the language L° is presented in Table 3. The system $S(\mathsf{DPLL})$ for dense propositional lax logic is obtained from $S(\mathsf{PLL})$ by adding the rule $\circ\bot/\bot$. Now, we introduce the system $S^{*}(\mathsf{InqIL}^{\circ})$ as an extension of $S(\mathsf{DPLL})$ with the rules of schematic *split* (*split**) and the rule of declarative atoms (*da*):

$$\begin{array}{ll} \textit{split}^{*}: & \circ\chi \to (\varphi \mathbin{\mathbb{V}} \psi)/(\circ\chi \to \varphi) \mathbin{\mathbb{V}} (\circ\chi \to \psi) \\ \textit{da}: & \circ p/p, \text{ for every atomic formula } p \end{array}$$

Since $S^{*}(\mathsf{InqIL}^{\circ})$ directly extends a system for propositional lax logic, the following result clarifies the connection of propositional lax logic and InqIL°.

Theorem 4.1 $\Delta \vdash_{S(\mathsf{InqIL}^{\circ})} \varphi$ *iff* $\Delta \vdash_{S^{*}(\mathsf{InqIL}^{\circ})} \varphi$

So, the system $S^{*}(\mathsf{InqIL}^{\circ})$ gives us an alternative characterization of InqIL°. This characterization is closely connected to a common alternative characterization of basic inquisitive logic InqB in which *split* is replaced by the Kreisel-Putnam rule $\neg\chi \to (\varphi \mathbin{\mathbb{V}} \psi)/(\neg\chi \to \varphi) \mathbin{\mathbb{V}} (\neg\chi \to \psi)$ and the double negation law is stated just for atomic formulas. This presentation reflects the original Hilbert style axiomatization of InqB in [2]. It is instructive to observe that we can replace $\neg\chi$ with $\neg\neg\chi$ in the schematic formulation of the Kreisel-Putnam rule and the result, as a schema, is in the context of intuitionistic logic equivalent to the original formulation. So, a system of natural deduction for InqB can be equivalently presented as the system for intuitionistic logic enriched with the modified Kreisel-Putnam rule (*mkp*) and double negation restricted to atomic formulas (*dna*):

$$\begin{array}{ll} \textit{mkp}: & \neg\neg\chi \to (\varphi \mathbin{\mathbb{V}} \psi)/(\neg\neg\chi \to \varphi) \mathbin{\mathbb{V}} (\neg\neg\chi \to \psi) \\ \textit{dna}: & \neg\neg p/p, \text{ for every atomic formula } p \end{array}$$

There is an obvious connection between this system for InqB and the system $S^{*}(\mathsf{InqIL}^{\circ})$: the rules *mkp* and *dna* are respectively obtained from *split** and *da* by replacing $\circ$ with double negation. Note that in $S(\mathsf{DPLL})$ one can derive $(\neg\neg\varphi \to \varphi) \to (\varphi \leftrightarrow \circ\varphi)$. Hence, adding the rule of full double negation $(\neg\neg\varphi/\varphi)$ to $S(\mathsf{DPLL})$ leads to the trivialization of $\circ$ (in the sense that $\varphi \leftrightarrow \circ\varphi$ becomes derivable). In $S^{*}(\mathsf{InqIL}^{\circ})$ we can moreover derive $(\neg\neg\alpha \to \alpha) \leftrightarrow (\neg\neg\alpha \leftrightarrow \circ\alpha)$, for every declarative α. This observation is related to the fact that adding *dna* to $S^{*}(\mathsf{InqIL}^{\circ})$ has the same effect as (is equivalent to) adding the fully schematic axiom identifying $\circ$ with double negation: $\neg\neg\varphi \leftrightarrow \circ\varphi$.

The connection of InqIL° to propositional lax logic also suggests a straightforward algebraic semantics for InqIL° that we briefly outline. A *nucleus* on a Heyting algebra $\mathcal{H} = \langle H, \sqcap, \sqcup, \Rightarrow, 0\rangle$ is a function $j : H \to H$ such that for each $x, y \in H$: (a) $x \leq j(x)$; (b) $j(j(x)) = j(x)$; (c) $j(x \sqcap y) = j(x) \sqcap j(y)$. A nucleus is *dense* if $j(0) = 0$. We say that a nucleus is *inquisitive* if it is dense

and satisfies $j(x) \Rightarrow (y \sqcup z) = (j(x) \Rightarrow y) \sqcup (j(x) \Rightarrow z)$. A *nuclear algebra* is a Heyting algebra equipped with a nucleus. Note that every nucleus is a closure operator (but not the other way around). Moreover, for every nuclear algebra $\langle \mathcal{H}, j \rangle$, we can define the algebra $j\mathcal{H} = \langle jH, \sqcap_j, \sqcup_j, \Rightarrow_j, j(0) \rangle$, where jH is the set of j-fixed points in H; $\sqcap_j$ and $\Rightarrow_j$ coincide with $\sqcap$ and $\Rightarrow$ restricted to jH (using the fact that jH is always closed under $\sqcap$ and $\Rightarrow$); and $x \sqcup_j y = j(x \sqcup y)$. A crucial feature of nuclear algebras is that $j\mathcal{H}$ is always a Heyting algebra.

A *valuation* in a nuclear algebra $\langle \mathcal{H}, j \rangle$ is a function assigning to every atomic formula an element of H. A nuclear algebra equipped with a valuation will be called a *nuclear model*. We say that a valuation v in $\langle \mathcal{H}, j \rangle$ is *declarative* if $v(p) \in jH$, for every atom p. A nuclear model $\langle \mathcal{H}, j, v \rangle$ will be called *inquisitive* if j is an inquisitive nucleus and v is a declarative valuation.

Given a nuclear model $\langle \mathcal{H}, j, v \rangle$ we can assign to each L°-formula φ an element of H that will be denoted as $[\varphi]$. This assignment is defined recursively: $[p] = v(p)$, $[\bot] = 0$, $[\varphi \to \psi] = [\varphi] \Rightarrow [\psi]$, $[\varphi \wedge \psi] = [\varphi] \sqcap [\psi]$, $[\varphi \mathbin{\mathbb{\vee}} \psi] = [\varphi] \sqcup [\psi]$, $[\circ\varphi] = j([\varphi])$. We say that φ is *valid* in the nuclear model if $[\varphi]$ is the top element of $\mathcal{H}$.

It is well-known that an L°-formula is valid in (dense) propositional lax logic if and only if it is valid in all (dense) nuclear models. With respect to InqIL° we obtain the following algebraic completeness.

Theorem 4.2 *$\varphi \in \mathsf{InqIL}^\circ$ iff φ is valid in all inquisitive nuclear models.*

Let us conclude this section with a suggestion for future research. The operation of adding *dna* to superintuitionistic intuitionistic logics was studied in depth in [1]. The logic obtained by adding *dna* to a logic L is called the negative variant of L. This study was motivated mainly by the fact that InqB can be obtained as the negative variant of several interesting superintuitionistic logics, e.g. the Maksimova logic ND, Kreisel-Putnam logic KP, or the Medvedev logic of finite problems ML.[3]

The observations that we have made in this section suggest that a natural generalization of this research would be the study of the operation of adding the rule of declarative atoms *da* to the extensions of PLL. We expect that many results of [1] would generalize to that setting.

5 Declarative disjunction and tensor

We have shown that in the language L°, we can define declarative disjunction $\varphi \vee \psi$ as $\circ(\varphi \mathbin{\mathbb{\vee}} \psi)$. We will observe that one could alternatively take $\vee$ as primitive and define $\circ\varphi$ as $\varphi \vee \varphi$.

The disjunction $\vee$ differs from the tensor disjunction $\otimes$ used in the context of dependence logic. To make this difference clear we will consider the following languages. The language $L^\vee$ is built up from atomic formulas with the following logical symbols: $\bot, \to \wedge, \mathbin{\mathbb{\vee}}, \vee$. The language $L^\otimes$ is like $L^\vee$ but

[3] It was proved in [9] that the negative variant of a superintuitionistic logic L coincides with InqB if and only if L is in the interval between ND and ML.

the connective $\vee$ is replaced with $\otimes$. Finally, the language $L^{\vee\otimes}$ contains both $\vee$ and $\otimes$ as primitive connectives. *Declarative formulas* in $L^{\vee\otimes}$ is the smallest set of formulas that includes $\bot$, all atomic formulas, all formulas of the form $\varphi \vee \psi$, and is closed under $\rightarrow$, $\wedge$ and $\otimes$. The declarative formulas of $L^{\vee}$ are the declarative $\otimes$-free formulas of $L^{\vee\otimes}$ and the declarative formulas of $L^{\otimes}$ are the declarative $\vee$-free formulas of $L^{\vee\otimes}$ (i.e. $\mathbin{\!\!\diagup\!\!\!\!\diagup}$-free formulas of $L^{\otimes}$).

Semantically, the connectives $\vee$ and $\otimes$ are characterized by the following clauses defined for any state s of any inquisitive model $inq(\mathcal{M})$ and for any $L^{\vee\otimes}$-formulas φ, ψ:

- $s \Vdash \varphi \vee \psi$ iff for all $w \in s$, $\uparrow w \Vdash \varphi$ or $\uparrow w \Vdash \psi$,
- $s \Vdash \varphi \otimes \psi$ iff $s \Vdash \varphi$ or $s \Vdash \psi$ or there are states t, u in $inq(\mathcal{M})$ such that $t \Vdash \varphi$, $u \Vdash \psi$, and $t \cup u = s$.

The semantic clause for $\vee$ is in accordance with our previous interpretation according to which the declarative disjunction is the presupposition of the corresponding inquisitive disjunction. The clause for $\otimes$ is equivalent to the one used in team semantics [23]. [4] In the presence of both these conditions persistence of $\Vdash$ is guaranteed.

Note that $||\varphi \vee \psi||$ is always a declarative proposition (and thus every declarative formula of $L^{\vee\otimes}$ expresses a declarative proposition). This does not hold for $\otimes$. We can illustrate the difference with the following example. Consider the classical Kripke model $\mathcal{M}$ consisting of three worlds w_p, w_q, w_r, and a valuation V such that $V(p) = \{w_p\}$ and $V(q) = \{w_q\}$ and $V(r) = \{w_r\}$. Then, we obtain the following difference:

- $||(p \mathbin{\!\!\diagup\!\!\!\!\diagup} q) \vee r|| = ||p \vee q \vee r|| = \uparrow\{w_p, w_q, w_r\}$,
- $||(p \mathbin{\!\!\diagup\!\!\!\!\diagup} q) \otimes r|| = ||(p \otimes r) \mathbin{\!\!\diagup\!\!\!\!\diagup} (q \otimes r)|| = \uparrow\{w_p, w_r\} \cup \uparrow\{w_q, w_r\}$.

However, the connectives $\vee$, $\otimes$ have the same effect when applied to declarative formulas.

Claim 5.1 *$||\alpha \vee \beta|| = ||\alpha \otimes \beta||$, for any declarative $L^{\vee\otimes}$-formulas α, β and in any inquisitive model.*

The logic $\mathsf{InqIL}^{\vee}$ ($\mathsf{InqIL}^{\otimes}$, $\mathsf{InqIL}^{\vee\otimes}$) is the set of $L^{\vee}$-formulas ($L^{\otimes}$-formulas, $L^{\vee\otimes}$-formulas) valid in all inquisitive models. The corresponding consequence relations $\models_{(\mathsf{InqIL}^{\vee})}$, $\models_{(\mathsf{InqIL}^{\otimes})}$, $\models_{(\mathsf{InqIL}^{\vee\otimes})}$ are defined by preservation of support in inquisitive models. We will show that $\mathsf{InqIL}^{\vee}$ and $\mathsf{InqIL}^{\otimes}$ can be characterized by the systems $S(\mathsf{InqIL}^{\vee})$ and $S(\mathsf{InqIL}^{\otimes})$ presented in Tables 4 and 5. The system

[4] The usual clause for tensor is slightly different and a bit simpler: $s \Vdash \varphi \otimes \psi$ iff there are states t, u such that $t \Vdash \varphi$, $u \Vdash \psi$, and $t \cup u = s$. This is because tensor is usually considered in the context of team semantics or information state-based semantics for inquisitive logic where also the empty set (empty team or empty information state) is considered. However, in this paper, we formulate the framework in the context of Kripke semantics for intuitionistic logic where the empty set is not considered as a separate state. In the presence of the empty state our clause for $\otimes$ is equivalent to the usual one and so the disjuncts "$s \Vdash \varphi$ or $s \Vdash \psi$" can be omitted. Without the empty state we have to include them to guarantee persistence of $\Vdash$ and the soundness of $\varphi / \varphi \otimes \psi$ and $\psi / \varphi \otimes \psi$.

Table 4
The system $S(\mathsf{InqIL}^{\vee})$ consists of the rules of $S(\mathsf{InqIL})$ plus the following rules

$\vee$-*intro*: (i) $\varphi/\varphi \vee \psi$, (ii) $\psi/\varphi \vee \psi$	$\vee$-*elim*: $\varphi \vee \psi, [\varphi/\alpha], [\psi/\alpha]/\alpha$

Table 5
The system $S(\mathsf{InqIL}^{\otimes})$ consists of the rules of $S(\mathsf{InqIL})$ plus the following rules

T1: $\alpha \otimes \alpha/\alpha$ T2: $\varphi \otimes \psi/\psi \otimes \varphi$ T3: $\varphi/\varphi \otimes \psi$
T4: $\varphi \otimes \psi, [\varphi/\chi], [\psi/\vartheta]/\chi \otimes \vartheta$ T5: $\varphi \otimes (\psi \mathbin{\mathbb{V}} \chi)/(\varphi \otimes \psi) \mathbin{\mathbb{V}} (\varphi \otimes \chi)$

$S(\mathsf{InqIL}^{\vee\otimes})$ is obtained by combining the rules of $S(\mathsf{InqIL}^{\vee})$ and $S(\mathsf{InqIL}^{\otimes})$.

Theorem 5.2 *(a)* $\Delta \vdash_{S(\mathsf{InqIL}^{\vee})} \varphi$ *iff* $\Delta \vDash_{(\mathsf{InqIL}^{\vee})} \varphi$*, (b)* $\Delta \vdash_{S(\mathsf{InqIL}^{\otimes})} \varphi$ *iff* $\Delta \vDash_{(\mathsf{InqIL}^{\otimes})} \varphi$*, (c)* $\Delta \vdash_{S(\mathsf{InqIL}^{\vee\otimes})} \varphi$ *iff* $\Delta \vDash_{(\mathsf{InqIL}^{\vee\otimes})} \varphi$*.*

Theorem 5.3 *All the logics* $\mathsf{InqIL}^{\vee}$*,* $\mathsf{InqIL}^{\otimes}$*,* $\mathsf{InqIL}^{\vee\otimes}$ *are compact, have the disjunction property, the finite model property and are decidable.*

The logic $\mathsf{InqIL}^{\otimes}$ coincides with the intuitionistic inquisitive logic from [8,17] (though there are some slight differences in the formulation of the corresponding deductive systems). The logic $\mathsf{InqIL}^{\vee}$ is closely connected to the logic InqIL°. The connection can be described as follows: We can introduce translation k from the language L° to the language $L^{\vee}$ and l from the language L° to the language $L^{\vee}$.

The translation k is defined as follows: $k\bot = \bot$, $kp = p$, for every atom p, and for every L°-formulas φ, ψ, $k(\varphi * \psi) = k\varphi * k\psi$, for each $* \in \{\to, \wedge, \mathbb{V}\}$, and $k(\circ\varphi) = k\varphi \vee k\varphi$. The translation l is defined as follows: $l\bot = \bot$, $lp = p$, for every atom p, and for every $L^{\vee}$-formulas φ, ψ, $l(\varphi * \psi) = l\varphi * l\psi$, for each $* \in \{\to, \wedge, \mathbb{V}\}$, and $l(\varphi \vee \psi) = \circ(l\varphi \mathbin{\mathbb{V}} l\psi)$.

The mutual definability of $\circ$ and $\vee$ then leads to the following connection between InqIL° and $\mathsf{InqIL}^{\vee}$.

Theorem 5.4 *Let* φ *be an* L°*-formula and* ψ *an* $L^{\vee}$*-formula. Then (a)* $\varphi \in \mathsf{InqIL}^{\circ}$ *iff* $k\varphi \in \mathsf{InqIL}^{\vee}$*, (b)* $\psi \in \mathsf{InqIL}^{\vee}$ *iff* $l\psi \in \mathsf{InqIL}^{\circ}$*, (c)* $\varphi \leftrightarrow lk\varphi \in \mathsf{InqIL}^{\circ}$*, (d)* $\psi \leftrightarrow kl\psi \in \mathsf{InqIL}^{\vee}$*.*

To make a comparison between the logics $\mathsf{InqIL}^{\vee}$ and $\mathsf{InqIL}^{\otimes}$ we note that $\mathsf{InqIL}^{\vee}$ could be equivalently characterized by the system consisting of the rules of $S(\mathsf{InqIL})$ plus the following four rules:

D1: $\alpha \vee \alpha/\alpha$ D2: $\varphi \vee \psi/\psi \vee \varphi$ D3: $\varphi/\varphi \vee \psi$ D4: $\varphi \vee \psi, [\varphi/\chi], [\psi/\vartheta]/\chi \vee \vartheta$

These rules are mirroring the rules T1–T4 from Table 5. Nevertheless, that does not mean that, if formulated in the same language, the logic $\mathsf{InqIL}^{\vee}$ would be weaker than $\mathsf{InqIL}^{\otimes}$. The rule D1 is actually much stronger than T1 because the languages $L^{\vee}$ and $L^{\otimes}$ have different definitions of declarative formulas ($\varphi \vee \psi$ is always declarative but $\varphi \otimes \psi$ is not). In fact, the logic $\mathsf{InqIL}^{\vee}$ is neither weaker nor stronger than $\mathsf{InqIL}^{\otimes}$.[5]

[5] E.g. $(p \vee (q \vee r)) \to (p \vee (q \mathbin{\mathbb{V}} r))$ is valid in $\mathsf{InqIL}^{\vee}$ but $(p \otimes (q \otimes r)) \to (p \otimes (q \mathbin{\mathbb{V}} r))$ is not valid in

Table 6
The system $S(\mathsf{HInqIL})$ consists of the rules for intuitionistic logic (with $\Vbar$ in the role of intuitionistic disjunction) plus the following rules for $\vee$

H1: $\alpha \vee \alpha/\alpha$, for $\Vbar$-free α	H2: $\varphi/\varphi \vee \varphi$	H3: $(\varphi \vee \varphi) \vee (\varphi \vee \varphi)/\varphi \vee \varphi$
H4: $(\varphi \wedge \psi) \vee (\varphi \wedge \psi)//(\varphi \vee \varphi) \wedge (\psi \vee \psi)$		H5: $(\varphi \Vbar \psi) \vee (\varphi \Vbar \psi)//\varphi \vee \psi$

6 Holliday's approach

In this section, we compare our approach to Holliday's inquisitive intuitionistic logic. Holliday's semantics is based on Beth semantics for intuitionistic logic which we will now briefly present.

A *Beth frame* is, just like Kripke frame, a partially ordered set $\mathcal{F} = \langle S, \leq \rangle$. A *chain* in $\mathcal{F}$ is a subset $C \subseteq S$ such that for all $s, t \in C$, $s \leq t$ or $t \leq s$. A *path* in $\mathcal{F}$ is a chain C that is closed under upper bounds, i.e. if $s \leq t$, for all $s \in C$, then $t \in C$. If C is a path and $s \in C$ then we say that C *is a path through* s.

The *Beth nucleus* is a particular nucleus j_B on the Heyting algebra of upward closed sets in $\mathcal{F}$. For every upward closed set X in $\mathcal{F}$ we define:

$$j_B(X) = \{s \in S \mid \text{ every path through } s \text{ intersects } X\}.$$

An upward closed set X in $\mathcal{F}$ is called *Beth closed* if $j_B(X) = X$. A *Beth valuation* in $\mathcal{F}$ is a function assigning to each atomic formula a Beth closed set. A *Beth model* is a Beth frame equipped with a Beth valuation. Relative to each Beth model $\langle \mathcal{F}, V \rangle$, Holliday defines a forcing relation between states in the model and $L^{\vee}$-formulas by the following clauses:

- $s \Vdash p$ iff $s \in V(p)$; not $s \Vdash \bot$; $s \Vdash \varphi \to \psi$ iff for any $t \geq s$, if $t \Vdash \varphi$, $t \Vdash \psi$;
- $s \Vdash \varphi \wedge \psi$ iff $s \Vdash \varphi$ and $s \Vdash \psi$; $s \Vdash \varphi \Vbar \psi$ iff $s \Vdash \varphi$ or $s \Vdash \psi$;
- $s \Vdash \varphi \vee \psi$ iff every path through s goes through some t s.t. $t \Vdash \varphi$ or $t \Vdash \psi$.

Let $|\varphi| = \{s \in S \mid s \Vdash \varphi\}$. With the clause for $\vee$ we obtain $|\varphi \vee \psi| = j_B(|\varphi| \cup |\psi|)$. Moreover, $|\alpha|$ is Beth closed, for every declarative α, but union of Beth closed sets is not generally Beth closed and thus, in analogy to our semantics for inquisitive logic, inquisitive disjunction $\Vbar$ generates propositions that are not Beth closed.

We say that φ is *valid* in a Beth model if φ is forced by every state of that model. Holliday's intuitionistic inquisitive logic, called $\mathsf{Inq}(\mathsf{IPC})$ in [13] and HInqIL in this paper, is defined as the set of $L^{\vee}$-formulas valid in all Beth models. Holliday's system for HInqIL (or rather its reformulation in natural deduction) is presented in Table 6. The double slash // in the formulation of H4 and H5 means that these are bi-directional rules allowing us to infer also the formula on the left side of // from the formula on the right side. The system directly reflects the nuclear character of $\vee$ and all the rules from Table 6 would remain to be sound with respect to the obvious generalized semantics with

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$\mathsf{InqIL}^{\otimes}$ and $(p \otimes (q \Vbar r)) \to ((p \otimes q) \Vbar (p \otimes r))$ is valid in $\mathsf{InqIL}^{\otimes}$ but $(p \vee (q \Vbar r)) \to ((p \vee q) \Vbar (p \vee r))$ is not valid in $\mathsf{InqIL}^{\vee}$.

Table 7
Relations among the nuclei i, j_B, j_{DN} in inquisitive models

	finite	infinite
classical	$i = j_B = j_{DN}$	$i = j_{DN}$, $j_B \neq j_{DN}$
non-classical	$i \neq j_{DN}$, $j_B = j_{DN}$	$i \neq j_{DN}$, $j_B \neq j_{DN}$, $i \neq j_B$

j_B replaced by an arbitrary nucleus. The main result of [13] is the following completeness theorem.

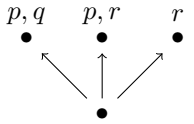
Theorem 6.1 *$\varphi \in \mathsf{HInqIL}$ if and only if φ is derivable in $S(\mathsf{HInqIL})$.*

There are some strong analogies between Holliday's approach determining the logic HInqIL and the approach based on inquisitive models which determines the logic $\mathsf{InqIL}^{\vee}$. Both these approaches employ some specific kinds of nuclei and use them in order to define declarative disjunction as the nuclear closure of the corresponding inquisitive disjunction. However, the nuclei they employ are very different. Only in very specific cases the nucleus i that captures our notion of informative presupposition coincides with the nucleus j_B on $\langle Up_{\emptyset}S, \supseteq\rangle$. In every finite partial order, j_B corresponds to the double negation operation and we showed in Section 3 a simple counterexample of a finite inquisitive model in which i is different from double negation. Nevertheless, we can combine the fact that in finite models j_B coincides with double negation with the fact that in classical inquisitive models i coincides with double negation. This clarifies in which sense Holliday's approach generalizes the usual semantics of InqB.

Theorem 6.2 *In every finite classical inquisitive model the nucleus i is identical with the nucleus j_B. Hence, with a restriction to the language L, the logic of all finite classical inquisitive models in Beth semantics is InqB. Moreover, in these models $\varphi \vee \psi$ is according to the Beth semantics equivalent to $\neg\neg(\varphi \mathbin{\mathbb{V}} \psi)$ and thus also to $\neg(\neg\varphi \wedge \neg\psi)$.*

Nevertheless, Beth semantics diverges from the usual inquisitive semantics even in regards to classical models. Consider the following Kripke model $\mathcal{M} = \langle W, =, V\rangle$, where $W = \{w_n \mid n \in \mathbb{N}\}$, $V(p) = \{w_n \in W \mid n \text{ is even}\}$, and $V(q) = \{w_n \in W \mid n \text{ is odd}\}$. Take the proposition $P = ||p \mathbin{\mathbb{V}} q||$ in the classical inquisitive model $inq(\mathcal{M})$. It holds that $W \in iP$ but $W \notin j_B P$. To see the latter, consider the path $C = \{s_n \mid n \in \mathbb{N}\}$ where $s_n = \{w_m \mid m \geq n\}$. Then C is a path through W which does not intersect P. Table 7 summarizes the relationships between the nuclei i, j_B, and j_{DN} (standing for double negation) in finite and infinite, classical and non-classical inquisitive models.

Another crucial difference between Holliday's framework and the usual inquisitive semantics is that HInqIL does not validate *split*. As a Beth counterexample to *split* consider the following Beth model:



One can easily check that $|p|$, $|q|$, $|r|$ are all Beth closed (but note that without the state where only r holds the root state would have to be included into $|p|$ and the model would not work as a counterexample to *split*). In this model the root state forces $p \to (q \mathbin{\mathbb{V}} r)$ but not $(p \to q) \mathbin{\mathbb{V}} (p \to r)$.

In the light of these differences between Holliday's approach and the usual inquisitive semantics it is perhaps surprising that on the logical level there is a tight connection between Holliday's rules from $S(\mathsf{HInqIL})$ and the rules $\vee$-*intro* and $\vee$-*elim* of our system $S(\mathsf{InqIL}^{\vee})$. In fact, over intuitionistic logic $\vee$-*intro* and $\vee$-*elim* are interderivable with the collection of rules H1–H5. This insight leads to a more perspicuous reformulation of Holliday's logic and an intriguing observation that Holliday's disjunction can be completely characterized by the usual introduction and elimination rules where just a syntactic restriction on the conclusion is imposed in the elimination rule. Let $S^*(\mathsf{HInqIL})$ be the system $S(\mathsf{InqIL}^{\vee})$ without *split*. Then we obtain:

Theorem 6.3 $\Delta \vdash_{S(\mathsf{HInqIL})} \varphi$ *iff* $\Delta \vdash_{S^*(\mathsf{HInqIL})} \varphi$.

In the conclusion of his paper, Holliday formulates an open problem of characterizing *split* in the context of Beth semantics. Let us conclude this section with a related but a bit more specific open problem. We say that a path C_1 in a Beth frame *joins* a path C_2 if there is $s \in C_1 \cap C_2$ such that for all $t \geq s$, $t \in C_1$ iff $t \in C_2$. Consider the following constraint on Beth frames:

(C) If $s \leq t$ and $s \leq u$ then there is v such that $s \leq v$, $v \leq t$ and $v \leq u$, and every path through v joins a path that goes through t or through u.

Note that all the finite classical inquisitive Kripke frames have this property.

Claim 6.4 *If a Beth frame $\mathcal{F}$ satisfies (C) then* split *is valid in every Beth model on $\mathcal{F}$.*

So, $S(\mathsf{InqIL}^{\vee})$ is Beth sound with respect to the class of frames satisfying (C). We leave as an open problem for future research whether the system is also complete with respect to this class.

7 Final remarks

We conclude the paper by outlining a broader plan for future research. The modality of informative presupposition $\circ$ that we studied in this paper bears a striking similarity to propositional truncation modality $\triangle$. Propositional truncation is a crucial concept in many type theories utilizing the propositions-as-types correspondence, most notably in homotopy type theory [22]. It allows us to turn an arbitrary type A into a "mere proposition" $\triangle A$, i.e. a type that has at most one proof. More specifically, $\triangle A$ makes all proofs of a type A equal, thus suppressing all information contained in them other than their exis-

tence and erasing the computational content of A in the process. Interestingly, we can observe that this behavior of truncation is in many respects similar to the behavior of informative presupposition $\circ$. While presupposition turns arbitrary propositions into declarative propositions by hiding their inquisitive structure, truncation turns arbitrary types into mere propositions by hiding their computational structure.

This link between presupposition and truncation can also be observed in their respective inference rules. The truncation modality $\triangle$ is often specified by the following introduction and elimination rules and a single axiom (for simplicity, we omit the proof terms and focus only on the propositional side):

$$\triangle\text{-}intro\colon A/\triangle A \qquad \triangle\text{-}elim\colon A \to B/\triangle A \to B \qquad \triangle\text{-}ax\colon \triangle A \text{ is a mere proposition}$$

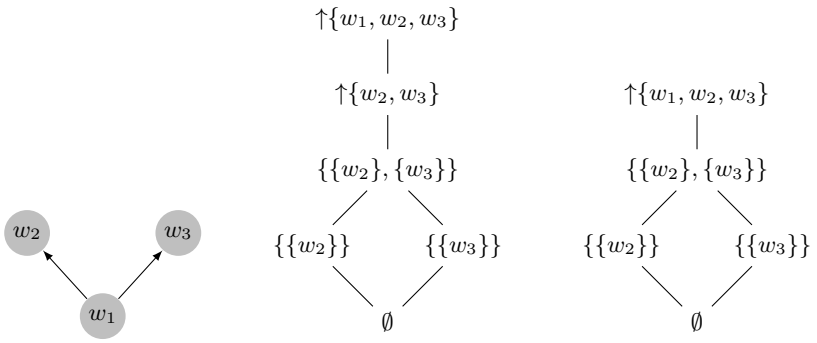
with the side condition that B is a mere proposition. We can observe that $\triangle$-*intro* and $\triangle$-*elim* correspond directly to our rules $\circ$-*intro* and $\circ$-*elim**, while $\triangle$-ax corresponds to our stipulation that $\circ\varphi$ is a declarative formula.

These initial observations seem to suggest that there is a link not only between the notions of presupposition and truncation but more generally between the ideas of inquisitive and computational content. This seems to be further strengthened by the fact that inquisitive proofs are known to have computational content in the sense of the propositions-as-types principle (see [3] and Section 4.2 in [6]). The connection to the truncation modality also seems to match the interpretation of non-inquisitive closure when we regard inquisitive propositions as information types as in [5]. We plan to focus on these intriguing connections in our future research.

Appendix

Proof of Claim 3.1: We will show just the least obvious part of this claim, namely that $iP \cap iQ \subseteq i(P \cap Q)$. Assume that $s \in iP \cap iQ$. Then $s \subseteq \bigcup P$ and $s \subseteq \bigcup Q$. So, for every $w \in s$ there is $t \in P$ such that $w \in t$ and there is $u \in Q$ such that $w \in u$. Then $w \in t \cap u$ and since P and Q are closed under nonempty subsets, $t \cap u \in P \cap Q$. So, $w \in \bigcup(P \cap Q)$. Since this holds for every $w \in s$, we obtain $s \subseteq \bigcup(P \cap Q)$, i.e. $s \in i(P \cap Q)$. Hence, $iP \cap iQ \subseteq i(P \cap Q)$.

Proof of Claim 3.3: Consider the Kripke model $\mathcal{M}$ depicted below on the left. The algebra of propositions in $inq(\mathcal{M})$ is depicted in the middle:



Assume that the valuation V in $\mathcal{M}$ is defined as follows: $V(p) = \{w_2\}$, $V(q) = \{w_3\}$, and $V(r) = \emptyset$, for every other atom r. Now we can observe that the algebra generated from the atomic propositions in $inq(\mathcal{M})$ by the logical operators from L is the one depicted on the right. In other words, no L-formula expresses in $inq(\mathcal{M})$ the proposition generated by $\{w_2, w_3\}$. But that is the proposition expressed by $\circ(p \mathbin{\mathbb{\vee}} q)$. Hence, no formula in L is equivalent to $\circ(p \mathbin{\mathbb{\vee}} q)$ over inquisitive Kripke models.

Proof of Claim 3.4: Assume that a model $inq(\mathcal{M}_1)$ contains a state s_1 that does not support φ and $inq(\mathcal{M}_2)$ a state s_2 that does not support ψ. Then $inq(\mathcal{M}_1 \uplus \mathcal{M}_2)$, where $\mathcal{M}_1 \uplus \mathcal{M}_2$ is the disjoint union of $\mathcal{M}_1$ and $\mathcal{M}_2$, contains the state $s_1' \cup s_2'$, where s_1' is the copy of s_1 and s_2' is the copy of s_2. The state $s_1' \cup s_2'$ does not support $\varphi \mathbin{\mathbb{\vee}} \psi$.

Proof of Claim 3.5: The right-to-left implication is equivalent to $\circ$-intro which holds by persistence. Let us show that the left-to-right implication also holds. Assume that s is a state of an inquisitive model $inq(\mathcal{M})$ such that $s \Vdash \varphi \to \alpha$. Take any nonempty $t \subseteq s$ such that $t \Vdash \circ\varphi$. Then for every $w \in t$, $\uparrow w \Vdash \varphi$. Since $\uparrow w \subseteq s$, we obtain $\uparrow w \Vdash \alpha$. Since this holds for every $w \in t$ we obtain $t \Vdash \alpha$, by Claim 3.2 and the assumption that α is declarative. So, $s \Vdash \circ\varphi \to \alpha$.

Proof of Claim 3.7: First, we need to make sure that equivalent formulas are interchangeable in $S(\mathsf{InqIL}^\circ)$. That is, if $\vdash_{S(\mathsf{InqIL}^\circ)} \varphi \leftrightarrow \psi$ and χ' is obtained from χ by replacing an occurrence of a subformula φ in χ with ψ then $\vdash_{S(\mathsf{InqIL}^\circ)} \chi' \leftrightarrow \chi$. To establish this claim we need to show that logical equivalence is preserved by application of all logical connectives. We show just the case of $\circ$. We need to show that if $\vdash_{S(\mathsf{InqIL}^\circ)} \varphi \leftrightarrow \psi$ then $\vdash_{S(\mathsf{InqIL}^\circ)} \circ\varphi \leftrightarrow \circ\psi$. This follows from the fact that if $\vdash_{S(\mathsf{InqIL}^\circ)} \varphi \to \psi$ then $\vdash_{S(\mathsf{InqIL}^\circ)} \circ\varphi \to \circ\psi$ which can be proved as follows:

$$\dfrac{\dfrac{\dfrac{\dfrac{\varphi \to \psi \qquad [\varphi]^{(1)}}{\psi}\,{\to}\text{-}elim}{\circ\psi}\,\circ\text{-}intro}{\varphi \to \circ\psi}\,(1),\ {\to}\text{-}intro}{\circ\varphi \to \circ\psi}\,\circ\text{-}elim^*$$

The rest is established by induction. All the steps are standard. See [17] for the inductive steps for $\to, \wedge, \mathbb{\vee}$ (*split* is needed in the step for implication). Note that the inductive step for $\circ$ is a direct application of interchangeability of equivalent formulas.

Proof of Theorem 3.8: To prove this result we will need two auxiliary claims.

Lemma A.1 *For every set of L°-formulas $\Delta \cup \{\varphi\}$, if $\Delta \nvdash_{S(\mathsf{InqIL}^\circ)} \varphi$ then there is a set of simple formulas Δ^φ such that $\Delta^\varphi \nvdash_{S(\mathsf{InqIL}^\circ)} \varphi$ and for every $\psi \in \Delta$ there is $\beta \in \Delta^\varphi$ such that $\beta \in \mathcal{R}(\psi)$.*

Proof. If Δ is finite then the proof is straightforward. We will assume that Δ is infinite. Let $\psi_1, \psi_2, \ldots$ be all the formulas in Δ. We define a sequence of sets of

formulas $\Delta_1, \Delta_2, \ldots$ such that $\Delta_1 = \Delta$ and $\Delta_{n+1} = \Delta_n \backslash \{\psi_n\}$. Then we define by induction a function f assigning to each $\psi \in \Delta$ a formula $\beta \in \mathcal{R}(\psi)$. Since $\{\psi_1\} \cup \Delta_2 \nvdash_{S(\mathsf{InqIL}^\circ)} \varphi$ there is $\beta_1 \in \mathcal{R}(\psi_1)$ such that $\{\beta_1\} \cup \Delta_2 \nvdash_{S(\mathsf{InqIL}^\circ)} \varphi$. Then we fix $f(\psi_1) = \beta_1$. Now assume that $f(\psi_1), \ldots, f(\psi_n)$ are already defined and $\{f(\psi_1), \ldots, f(\psi_n), \psi_{n+1}\} \cup \Delta_{n+2} \nvdash_{S(\mathsf{InqIL}^\circ)} \varphi$. Then there is $\beta_{n+1} \in \mathcal{R}(\psi_{n+1})$ such that $\{f(\psi_1), \ldots, f(\psi_n), \beta_{n+1}\} \cup \Delta_{n+2} \nvdash_{S(\mathsf{InqIL}^\circ)} \varphi$ and we fix $f(\psi_{n+1}) = \beta_{n+1}$. In this way the function f is defined for every $\psi \in \Delta$. Fix $\Delta^\varphi = \{f(\psi) \mid \psi \in \Delta\}$. Then $\Delta^\varphi \nvdash_{S(\mathsf{InqIL}^\circ)} \varphi$ and for every $\psi \in \Delta$ there is $\beta \in \Delta^\varphi$ such that $\beta \in \mathcal{R}(\psi)$. □

We say that a set of simple L°-formulas Δ is a *theory* if (a) Δ is nonempty; (b) $\bot \notin \Delta$; (c) Δ is closed under the rules of intuitionistic logic. We say that a theory Δ is *prime* if $\varphi \vee \psi \in \Delta$ only if $\varphi \in \Delta$ or $\psi \in \Delta$. The set of all prime theories will be denoted as $Prime(\mathsf{IL})$.

Now, let us consider the usual canonical model of propositional intuitionistic logic $\mathcal{M}^{\mathsf{IL}} = \langle Prime(\mathsf{IL}), \subseteq, V\rangle$, where $V(p) = \{\Gamma \in Prime(\mathsf{IL}) \mid p \in \Gamma\}$, for every atomic formula p. For this construction, we can prove the following form of a truth lemma.

Lemma A.2 *Let α be a simple formula and X a nonempty upward closed set of prime theories, i.e. a state in $inq(\mathcal{M}^{\mathsf{IL}})$. Then*

$$X \Vdash \alpha \text{ in } inq(\mathcal{M}^{\mathsf{IL}}) \text{ iff } \alpha \in \bigcap X.$$

Proof. We proceed by induction. We show just the inductive step for $\vee$. Assume that the claim holds for α and β. Then the following claims are equivalent:

- $X \Vdash \alpha \vee \beta$, i.e. $X \Vdash \circ(\alpha \curlyvee \beta)$,
- for all $\Gamma \in X$, $\uparrow\Gamma \Vdash \alpha$ or $\uparrow\Gamma \Vdash \beta$,
- for all $\Gamma \in X$, $\alpha \in \bigcap \uparrow\Gamma$ or $\beta \in \bigcap \uparrow\Gamma$,
- for all $\Gamma \in X$, $\alpha \in \Gamma$ or $\beta \in \Gamma$,
- for all $\Gamma \in X$, $\alpha \vee \beta \in \Gamma$,
- $\alpha \vee \beta \in \bigcap X$.

□

Using these lemmas, Theorem 3.8 can be established as follows. The left-to-right implication follows from Claim 3.5. We prove the right-to-left implication. Assume $\Delta \nvdash_{S(\mathsf{InqIL}^\circ)} \varphi$. Then, by Lemma A.1, there is a set of simple L°-formulas Δ^φ such that $\Delta^\varphi \nvdash_{S(\mathsf{InqIL}^\circ)} \varphi$ and it holds that for every $\psi \in \Delta$ there is $\beta \in \Delta^\varphi$ such that $\beta \in \mathcal{R}(\psi)$.

Take any $\alpha \in \mathcal{R}(\varphi)$. Then we have $\Delta^\varphi \nvdash_{S(\mathsf{InqIL}^\circ)} \alpha$ and by the standard construction, we can build a prime theory Γ^α such that $\Delta^\varphi \subseteq \Gamma^\alpha$ and $\alpha \notin \Gamma^\alpha$. Now we can define the following nonempty upward closed set of prime theories, i.e a state in $inq(\mathcal{M}^{\mathsf{IL}})$:

$$X^\Delta_\varphi = \{\Gamma \in Prime(\mathsf{IL}) \mid \Gamma^\alpha \subseteq \Gamma, \text{ for some } \alpha \in \mathcal{R}(\varphi)\}.$$

For each $\alpha \in \mathcal{R}(\varphi)$, $\alpha \notin \bigcap X^{\Delta}_{\varphi}$, and thus, by Lemma A.2, $X^{\Delta}_{\varphi} \nVdash \alpha$. Hence $X^{\Delta}_{\varphi} \nVdash \varphi$. Moreover, for every $\psi \in \Delta$ there is $\beta \in \mathcal{R}(\psi)$ such that $\beta \in \bigcap X^{\Delta}_{\varphi}$ and thus $X^{\Delta}_{\varphi} \Vdash \beta$. So, for every $\psi \in \Delta$, $X^{\Delta}_{\varphi} \Vdash \psi$, which yields the desired result that $\Delta \nvDash_{(\mathsf{InqIL}^{\circ})} \varphi$.

Proof of Theorem 3.10: We can proceed as in the proof of Theorem 3.8, but using the fact that intuitionistic logic has the finite model property. Assume $\varphi \notin \mathsf{InqIL}^{\circ}$. Then $\alpha \notin \mathsf{InqIL}^{\circ}$, for every $\alpha \in \mathcal{R}(\varphi)$. By the finite model property of intuitionistic logic, for each $\alpha \in \mathcal{R}(\varphi)$ there is a finite Kripke model refuting α. We can take the disjoint union $\mathcal{M}$ of these models. Then, for each $\alpha \in \mathcal{R}(\varphi)$, there is a state in $inq(\mathcal{M})$ that does not support α. The union of these states does not support φ.

Proof of Theorem 4.1: The rule L1 coincides with $\circ$-*intro* and it is clear that all the rules L2, $\circ\bot/\bot$, *split** and *da* can be simulated in $S(\mathsf{InqIL}^{\circ})$. For instance, *da* is obtained as follows:

$$\dfrac{\circ p \qquad \overset{(1)}{[p]}}{p}\ (1),\ \circ\text{-}elim$$

Hence, if $\Delta \vdash_{S^*(\mathsf{InqIL}^{\circ})} \varphi$ then $\Delta \vdash_{S(\mathsf{InqIL}^{\circ})} \varphi$. In order to show also the converse implication we will simulate the rule $\circ$-*elim** in $S^*(\mathsf{InqIL}^{\circ})$. First, we will show by induction on declarative α that $\circ\alpha \vdash_{S^*(\mathsf{InqIL}^{\circ})} \alpha$. The cases of $\alpha = \bot$ and $\alpha = \circ\varphi$ are by DPLL. The case of atomic formulas is by *da*. As an inductive assumption (IndA) we assume that we already have for some declarative formulas β, γ that $\circ\beta \vdash_{S^*(\mathsf{InqIL}^{\circ})} \beta$ and $\circ\gamma \vdash_{S^*(\mathsf{InqIL}^{\circ})} \gamma$. We just sketch the derivations for $\alpha = \beta \wedge \gamma$ and $\alpha = \beta \to \gamma$:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\circ(\beta\wedge\gamma)}{\circ\beta\wedge\circ\gamma}\ \mathsf{PLL}}{\circ\beta}\ \wedge\text{-}elim}{\beta}\ \text{IndA} \qquad \dfrac{\dfrac{\dfrac{\circ(\beta\wedge\gamma)}{\circ\beta\wedge\circ\gamma}\ \mathsf{PLL}}{\circ\gamma}\ \wedge\text{-}elim}{\gamma}\ \text{IndA}}{\beta\wedge\gamma}\ \wedge\text{-}intro \qquad \dfrac{\dfrac{\dfrac{\circ(\beta\to\gamma)}{\circ\beta\to\circ\gamma}\ \mathsf{PLL}}{\circ\beta\to\gamma}\ \mathsf{IL},\ \text{IndA}}{\beta\to\gamma}\ \mathsf{PLL}$$

Now, $\varphi \to \alpha \vdash_{S^*(\mathsf{InqIL}^{\circ})} \circ\varphi \to \alpha$ can be derived as follows:

$$\dfrac{\dfrac{\dfrac{\overset{(1)}{[\circ\varphi]} \qquad \dfrac{\dfrac{\overset{(2)}{[\varphi]} \qquad \varphi\to\alpha}{\alpha}\ \to\text{-}elim}{\circ\alpha}\ \text{L1}}{\circ\alpha}\ (2),\ \text{L2}}{\alpha}}{\circ\varphi\to\alpha}\ (1),\ \to\text{-}intro$$

Proof of Theorem 4.2: The proof of soundness is straightforward. To prove completeness we can either directly construct the Lindenbaum-Tarski algebra of InqIL°, or use our completeness result with respect to inquisitive Kripke models and observe that the algebras of propositions in these models are inquisitive nuclear algebras.

Proofs of Theorems 5.2 and 5.3: We can proceed as in the proofs of Theorems 3.8 and 3.10. Let us focus on the logic $\mathsf{InqIL}^{\vee}$. The role of simple formulas in L° can be played by $\vee$-free $L^{\vee}$-formulas. We have to show (a)

that we can obtain a result analogous to Claim 3.6, and (b) that we obtain a suitable disjunctive normal form result. As regards (a) the introduction and elimination rules for $\Vee$-free $L^{\vee}$-formulas are obtained directly from $\vee$-*intro* and $\vee$-*elim*. In order to prove the disjunctive normal form we need to extend the definition of resolution for $\vee$. We define:

- $\mathcal{R}(\varphi \vee \psi) = \{\bigvee(\mathcal{R}(\varphi) \cup \mathcal{R}(\psi))\}$.

Then we need to prove interchangeability of equivalent formulas for $\vee$ which follows from the fact that we can derive $(\varphi \vee \chi) \to (\psi \vee \chi)$ from $\varphi \to \psi$ in the following way (using the fact that $\psi \vee \chi$ is declarative):

$$\dfrac{[\varphi \vee \chi]^{(1)} \qquad \dfrac{\dfrac{[\varphi]^{(2)} \qquad \varphi \to \psi}{\psi}\,{\to}\text{-}elim}{\psi \vee \chi}\,\vee\text{-}intro \qquad \dfrac{[\chi]^{(3)}}{\psi \vee \chi}\,\vee\text{-}intro}{\psi \vee \chi}\,(2),\ (3),\ \vee\text{-}elim$$
$$\dfrac{\psi \vee \chi}{(\varphi \vee \chi) \to (\psi \vee \chi)}\,(1),\ {\to}\text{-}intro$$

In the inductive step for $\vee$ in the proof of disjunctive normal form we have to show that

$$\vdash_{S(\mathsf{InqIL}^{\vee})} (\alpha_1 \Vee \ldots \Vee \alpha_n) \vee (\beta_1 \Vee \ldots \Vee \beta_m) \leftrightarrow (\alpha_1 \vee \ldots \vee \alpha_n \vee \beta_1 \vee \ldots \vee \beta_m)$$

which is obtained by application of the introduction and elimination rules for $\vee$ and $\Vee$ using the fact that the formulas on both sides of the equivalence are declarative. In the case of $\mathsf{InqIL}^{\otimes}$ we need to employ a different resolution condition:

- $\mathcal{R}(\varphi \otimes \psi) = \{\alpha \otimes \beta \mid \alpha \in \mathcal{R}(\varphi), \beta \in \mathcal{R}(\psi)\}$.

The rule T5 is used in the inductive step for $\otimes$ in the proof of the disjunctive normal form result. In the case of $\mathsf{InqIL}^{\vee\otimes}$ we also use the fact that $(\alpha \vee \beta) \leftrightarrow (\alpha \otimes \beta)$ is provable in $S(\mathsf{InqIL}^{\vee\otimes})$ (using $\vee$-*intro*, $\vee$-*elim* and T1–T4).

Proof of Theorem 5.4: (a)–(d) follow from the mutual definability of $\vee$ and $\circ$. (c) is based on the fact that $\circ\varphi$ is equivalent to $\circ(\varphi \Vee \varphi)$ in InqIL°, and (d) is based on the fact that $\varphi \vee \psi$ is equivalent to $(\varphi \Vee \psi) \vee (\varphi \Vee \psi)$ in $\mathsf{InqIL}^{\vee}$.

Proof of Theorem 6.3: We can first observe that $\vee$-*intro* and $\vee$-*elim* are sound with respect to Beth semantics and thus can be derived in $S(\mathsf{HInqIL})$ because of its completeness. The soundness of $\vee$-*intro* is immediate. In order to show the soundness of $\vee$-*elim* we can take any state s of any Beth model and assume that $\varphi \vee \psi$, $\varphi \to \alpha$ and $\psi \to \alpha$ all hold at s. It follows from these assumptions that every path through s intersects $|\alpha|$. Since α is declarative we obtain $s \in |\alpha|$, i.e. $s \Vdash \alpha$, as desired.

In the other direction, we can observe that all instances of H1–H5 are derivable from $\vee$-*intro* and $\vee$-*elim*. For an illustration, we show how to derive both directions of H5. The left-to-right direction of H5 is obtained as follows:

$$\dfrac{(\varphi \mathbin{\mathbb{V}} \psi) \vee (\varphi \mathbin{\mathbb{V}} \psi) \qquad \dfrac{[\varphi \mathbin{\mathbb{V}} \psi]^{(1)} \qquad \dfrac{[\varphi]^{(2)}}{\varphi \vee \psi}\vee\text{-}intro \qquad \dfrac{[\psi]^{(3)}}{\varphi \vee \psi}\vee\text{-}intro}{\varphi \vee \psi}\ (2),\ (3),\ \mathbb{V}\text{-}elim}{\varphi \vee \psi}\ (1),\ \vee\text{-}elim$$

The right-to-left direction of H5 is derived in the following way:

$$\dfrac{\varphi \vee \psi \qquad \dfrac{\dfrac{[\varphi]^{(1)}}{\varphi \mathbin{\mathbb{V}} \psi}\mathbb{V}\text{-}intro}{(\varphi \mathbin{\mathbb{V}} \psi) \vee (\varphi \mathbin{\mathbb{V}} \psi)}\vee\text{-}intro \qquad \dfrac{\dfrac{[\psi]^{(2)}}{\varphi \mathbin{\mathbb{V}} \psi}\mathbb{V}\text{-}intro}{(\varphi \mathbin{\mathbb{V}} \psi) \vee (\varphi \mathbin{\mathbb{V}} \psi)}\vee\text{-}intro}{(\varphi \mathbin{\mathbb{V}} \psi) \vee (\varphi \mathbin{\mathbb{V}} \psi)}\ (1),\ (2),\ \vee\text{-}elim$$

Proof of Claim 6.4: Take any state s of any Beth model based on a Beth frame satisfying the condition (C) and assume that in this model s does not force $(\alpha \to \varphi) \mathbb{V} (\alpha \to \psi)$. Then there are $t, u \geq s$ such that $t \Vdash \alpha$ but not $t \Vdash \varphi$ and $u \Vdash \alpha$ but not $u \Vdash \psi$. Since the model satisfies (C) there is $v \geq u$ such that $v \leq t$ and $v \leq u$ and every path through v joins a path that goes through t or through u. It follows that every path through v intersects $|\alpha|$. Since α is declarative, $|\alpha|$ is Beth closed and thus $v \Vdash \alpha$. But, because of persistence, v does not force $\varphi \mathbb{V} \psi$. It follows that s does not force $\alpha \to (\varphi \mathbb{V} \psi)$.

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