# Positive Modal Logic Over Finite MV-Chains

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**Abstract**

In this paper, we introduce and study the positive (*i.e.*, the negation-free and implication-free) fragment of many-valued modal logic based on a finite MV-chain as algebra of truth-degrees. Besides the usual Kripke semantics, we introduce richer relational semantics based on partially ordered sets with local constraints on admissible valuations. We study this logic algebraically via the varieties of modal n-valued positive MV-algebras. Utilizing prior work on natural dualities for n-valued positive MV-algebras and the close relationship between these algebras and their distributive skeletons, we prove an algebraic completeness theorem. Furthermore, by means of an example regarding canonicity, we illustrate how the richer relational semantics presented in this paper are 'better-behaved' with respect to the positive many-valued modal logic than Kripke (or other intermediate) semantics are.

*Keywords:* Positive modal logic, many-valued modal logic, finite MV-chains, positive MV-algebras, algebraic completeness, distributive skeleton.

## 1 Introduction

*Positive modal logic*, introduced by Dunn [10], is the negation-free (and implication-free) fragment of classical modal logic. In this paper, we introduce and study the positive fragment of the *many-valued* modal logics which use some *finite* $\mathsf{MV}$-*chain* $\mathbf{Ł}_n$ as algebra of truth-degrees (see, *e.g.*, [4,13]).

Algebraically speaking, moving from classical modal logic to positive modal logic amounts to replacing Boolean algebras with operators by *modal distributive lattices* (Definition 2.1). Similarly, on the propositional level we shift from $\mathsf{MV}_n$-algebras [12] to certain *positive* $\mathsf{MV}$-*algebras* [2]. More specifically, we work over the variety of $\mathsf{PMV}_n$-*algebras* generated by the *finite positive* $\mathsf{MV}$-*chain* $\mathbf{PŁ}_n$ (Definition 2.8) for this purpose. While this shift clearly leads to *less structure* on the algebraic side, at the same time we consider relational semantics which carry *more structure* than usual Kripke frames (throughout this paper, we refer to Kripke frames as $\mathsf{Set}$-*frames*).

The semantics of positive modal logic on $\mathsf{Set}$-frames originally considered by Dunn [10] have the shortcoming that there are consequence pairs which, in

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the presence of negation, define the same class of $\mathsf{Set}$-frames and are canonical, but which are not inter-derivable in positive modal logic. To remedy this shortcoming, Celani and Jansana [5,6] introduced relational semantics on *ordered* frames (which we refer to as $\mathsf{Pos}$-*frames* in this paper, see Definition 2.2). Similarly, Hansoul and Teheux [13,23] studied richer semantics for $\mathbf{Ł}_n$-valued modal logic on *frames with local constraints* (which we refer to as $\mathsf{Set}_n$-*frames*, see Definition 2.5). Taking both of these into account, we introduce semantics for $\mathbf{PŁ}_n$-valued modal logic on $\mathsf{Pos}_n$-*frames*, that is, frames which carry a partial order *and* local constraints on valuations (Definition 3.2).

The methodology used in this paper is primarily algebraic. We introduce the varieties $\mathsf{mPMV}_n$ of *modal* $\mathsf{PMV}_n$-*algebras* (Definition 4.1), and prove a truth lemma (Lemma 4.5) which leads to an algebraic completeness theorem (Theorem 4.8). Furthermore, we show that the *canonical frames* arising from modal $\mathsf{PMV}_n$-algebras always are $\mathsf{Pos}_n$-frames (Proposition 4.4) and that $\mathsf{mPMV}_n$ is generated by the *complex algebras* of $\mathsf{Pos}_n$-frames (Theorem 4.7). The proofs of these main results rely on the author's prior work [21], where topological (natural) dualities for the varieties $\mathsf{PMV}_n$ were established. In particular, the relationship between $\mathsf{PMV}_n$-algebras and their *distributive skeletons* (Definition 2.9) studied therein [21, Subsection 4.2] provide a useful tool here.

While the distinction between $\mathsf{Set}$-frames and $\mathsf{Pos}_n$-frames does not play a role for the *minimal* modal logic, we show that it does make a difference when it comes to axiomatic extensions thereof. Indeed, towards the end of this paper we study an example of two consequence pairs which are canonical in $\mathbf{Ł}_n$-valued modal logic and define the same $\mathsf{Set}_n$-frames, but which are not inter-derivable in $\mathbf{PŁ}_n$-valued modal logic. In our semantics, this is reflected by the fact that these consequence pairs define different classes of $\mathsf{Pos}_n$-frames (Section 5). A more exhaustive study of this topic is one of many directions for future research we propose in the conclusion of this paper.

## 2 Preliminaries

In this section, we set up the scene by recalling some important definitions and prior results from positive modal logic (Subsection 2.1), modal logic over finite $\mathsf{MV}$-chains (Subsection 2.2) and positive $\mathsf{MV}_n$-algebras (Subsection 2.3).

### 2.1 Positive modal logic

Positive modal logic (with truth-constants), introduced by Dunn in [10], is the $\{\wedge, \vee, 0, 1, \Box, \Diamond\}$-fragment of classical modal logic. Algebraically speaking, removing the negation from the language amounts to replacing the underlying variety $\mathsf{BA}$ of Boolean algebras by the variety $\mathsf{DL}$ of bounded distributive lattices. Since $\Box$ and $\Diamond$ are then no longer inter-definable, their interplay is instead described by Dunn's *positivity-axioms*, that is, the axioms (P1)-(P2) in the following.

**Definition 2.1** [Modal distributive lattice] A *modal distributive lattice* is an algebra $\langle \mathbf{D}, \Box, \Diamond \rangle$, where $\mathbf{D} \in \mathsf{DL}$ is a bounded distributive lattice and $\Box, \Diamond \colon D \to D$ are unary operations satisfying the equations

(B1) $\Box 1 = 1$, (D1) $\Diamond 0 = 0$,
(B2) $\Box(x \wedge y) = \Box x \wedge \Box y$, (D2) $\Diamond(x \vee y) = \Diamond x \vee \Diamond y$,
(P1) $\Box(x \vee y) \leq \Box x \vee \Diamond y$, (P2) $\Box x \wedge \Diamond y \leq \Diamond(x \wedge y)$.

We denote the variety of modal distributive lattices by $\mathsf{mDL}$.

The exact relationship between the variety $\mathsf{mDL}$ and positive modal logic was described by Jansana in [14].

While Dunn [10] only considered the usual relational semantics on $\mathsf{Set}$-frames (throughout this paper, we use the term $\mathsf{Set}$-*frame* instead of *Kripke frame* to avoid confusion), Celani and Jansana [5] proposed some 'better-behaved' semantics based on the category of quasi-orders, which we now recall. For a slight increase in simplicity, in this paper we work with the category $\mathsf{Pos}$ of posets instead of quasi-orders.

Given a poset $(X, \leq)$, we use $\leq_{\mathrm{EM}}$ to denote the *Egli-Milner-lifting* of $\leq$ to the powerset $\mathcal{P}(X)$, defined for $A, B \subseteq X$ by

$$A \leq_{\mathrm{EM}} B \Leftrightarrow \begin{cases} \forall a \in A\colon \exists b \in B\colon a \leq b \text{ and} \\ \forall b \in B\colon \exists a \in A\colon a \leq b. \end{cases}$$

With this convention, we may paraphrase [5, Definition 4.1] as follows.

**Definition 2.2** [$\mathsf{Pos}$-frame & $\mathsf{Pos}$-model] A $\mathsf{Pos}$-*frame* is a structure $(X, \leq, R)$, where $(X, \leq)$ is a poset and $R \subseteq X^2$ is a binary relation which satisfies that $R[x] := \{x' \in X \mid xRx'\}$ is always convex and for all $x, y \in X$ it holds that

$$x \leq y \Rightarrow R[x] \leq_{\mathrm{EM}} R[y].$$

Fixing a countable set $\mathsf{Prop}$ of propositional variables, a $\mathsf{Pos}$-*model* is a structure $(X, \leq, R, \mathsf{Val})$ consisting of a $\mathsf{Pos}$-frame together with a $\mathsf{Pos}$-*valuation* $\mathsf{Val}\colon X \times \mathsf{Prop} \to \{0, 1\}$, which satisfies

$$x \leq y \Rightarrow \mathsf{Val}(x, p) \leq \mathsf{Val}(y, p)$$

for all $x, y \in X$ and $p \in \mathsf{Prop}$.

The condition on $\mathsf{Pos}$-frames (regarding the Egli-Milner order) ensures that the extension of $\mathsf{Val}$ to all formulas $\varphi$ built from $\mathsf{Prop}$ and connectives in $\{\wedge, \vee, 0, 1, \Box, \Diamond\}$ also has the property that $\mathsf{Val}(-, \varphi)$ is always order-preserving.

As pointed out in [5], one considerable advantage these $\mathsf{Pos}$-based semantics have over $\mathsf{Set}$-based semantics comes to light in the study of canonicity. For example, the two consequence pairs

$$\Box p \vdash p \text{ and } p \vdash \Diamond p$$

both define the class of reflexive $\mathsf{Set}$-frames but are not mutually inter-derivable in positive modal logic. In the richer semantics, however, both these formulas are canonical and correspond to the classes of $\mathsf{Pos}$-frames $(X, \leq, R)$ where

$$R_{\Box} := R \circ \leq \text{ and } R_{\Diamond} := R \circ \leq^{-1}$$

are reflexive, respectively [5, Sections 5 & 7].

The advantages of Pos-based semantics over Set-based ones also become apparent in the study of positive modal logic via duality theory [6] and coalgebras [20,3,9]. In particular, the latter fully embraces this view and proposes positive coalgebraic logic as the *logic of ordered transition systems*.

### 2.2 Modal logic over finite MV-chains

Recall that the *standard* MV*-algebra* is the algebra

$$\mathbf{Ł} = \langle [0,1], \odot, \oplus, \wedge, \vee, \neg, 0, 1 \rangle$$

based on the real unit interval with its usual bounded lattice structure $\wedge, \vee, 0, 1$ and additional operations defined by

$$x \odot y = \mathsf{max}\{0, x + y - 1\}, \qquad x \oplus y = \mathsf{min}\{1, x + y\}, \qquad \neg x = 1 - x.$$

A detailed overview of MV-algebras and their relationship to Łukasiewicz logic may be found in the books [7,18]. In this paper, we focus on (positive subreducts of) *finite* MV*-chains*, that is, the following finite subalgebras of the standard MV-algebra.

**Definition 2.3** [Finite MV-chain] Let $n \geq 1$ be a natural number. The $(n+1)$*-element* MV*-chain* is given by

$$\mathbf{Ł}_n = \langle \{0, \tfrac{1}{n}, \ldots \tfrac{n-1}{n}, 1\}, \odot, \oplus, \wedge, \vee, \neg, 0, 1 \rangle,$$

considered as a subalgebra of the standard MV-algebra. As usual, we use $\mathsf{MV}_n$ to denote the variety $\mathbb{HSP}(\mathbf{Ł}_n)$.

The varieties $\mathsf{MV}_n$, axiomatized by Grigolia [12], provide algebraic counterparts to Łukasiewicz *finitely-valued logics*. Many-valued modal logic over finite MV-chains was investigated by Hansoul and Teheux [13] (also see Bou *et al.* [4]) as follows.

The *modal* MV*-language* $\mathcal{L}^{\mathrm{m}}_{\mathsf{MV}} = \{\odot, \oplus, \wedge, \vee, \neg, 0, 1, \Box\}$ consists of the signature of MV together with a unary operation symbol $\Box$. The collection of *modal* MV *formulas* $\mathsf{Form}^{\mathrm{m}}_{\mathsf{MV}}$ is defined inductively from a countable collection of propositional variables Prop and the connectives of $\mathcal{L}^{\mathrm{m}}_{\mathsf{MV}}$ as usual.

Similarly to how Pos-frames provide richer semantics for positive modal logic, relational frames based on the following category were introduced by Hansoul and Teheux [13] as richer semantics for $\mathbf{Ł}_n$-valued modal logic.

**Definition 2.4** [The category $\mathsf{Set}_n$] The category $\mathsf{Set}_n$ has objects $(X, \mathsf{sub})$, where $X$ is a set and $\mathsf{sub}\colon X \to \mathbb{S}(\mathbf{Ł}_n)$ associates with every point of $X$ a subalgebra of $\mathbf{Ł}_n$.

A $\mathsf{Set}_n$*-morphism* $(X, \mathsf{sub}) \to (X', \mathsf{sub}')$ is a map $f\colon X \to X'$ which satisfies $\mathsf{sub}'(f(x)) \subseteq \mathsf{sub}(x)$ for all $x \in X$.

The duality-theoretical reason behind this choice of base category is that $\mathsf{Set}_n$ is the 'discrete counterpart' of the category $\mathsf{Stone}_n$ of structured Stone

spaces dually equivalent to $\mathsf{MV}_n$ [15,8,17]. From the logical perspective, we think of the additional structure on $\mathsf{Set}_n$ as *putting constraints on the admissible valuations*. This becomes apparent in the following definition of $\mathsf{Set}_n$-based relational frames and models [13, Definition 7.2].

**Definition 2.5** [$\mathsf{Set}_n$-frame & $\mathsf{Set}_n$-model] A $\mathsf{Set}_n$*-frame* is a structure of the form $(X, \mathsf{sub}, R)$, where $(X, \mathsf{sub})$ is a member of $\mathsf{Set}_n$ and $R \subseteq X^2$ is a binary relation such that the condition

$$xRy \Rightarrow \mathsf{sub}(y) \subseteq \mathsf{sub}(x)$$

holds for all $x, y \in X$.

A $\mathsf{Set}_n$*-model* is given by $(X, \mathsf{sub}, R, \mathsf{Val})$, consisting of a $\mathsf{Set}_n$-frame and a $\mathsf{Set}_n$*-valuation* $\mathsf{Val}\colon X \times \mathsf{Prop} \to \mathbf{Ł}_n$, which needs to satisfy $\mathsf{Val}(x, p) \in \mathsf{sub}(x)$ for all $x \in X$ and $p \in \mathsf{Prop}$.

The condition on the accessibility relation of $\mathsf{Set}_n$-frames is precisely what is needed to ensure that the extension of valuations to the set of all formulas in $\mathsf{Form}^{\mathrm{m}}_{\mathsf{MV}}$ still satisfies the constraints on valuations. More specifically, if $(X, \mathsf{sub}, R, \mathsf{Val})$ is a $\mathsf{Set}_n$-model, we inductively extend $\mathsf{Val}$ to a map $\mathsf{Val}\colon X \times \mathsf{Form}^{\mathrm{m}}_{\mathsf{MV}} \to \mathbf{Ł}_n$, which is defined in the obvious way for the $\mathsf{MV}$-connectives and for formulas of the form $\Box\varphi$ by

$$\mathsf{Val}(x, \Box\varphi) = \bigwedge\{\mathsf{Val}(x', \varphi) \mid xRx'\}.$$

The fact that $\mathsf{sub}(x)$ is a subalgebra of $\mathbf{Ł}_n$ and the condition on $\mathsf{Set}_n$-frames ensure that $\mathsf{Val}(x, \varphi) \in \mathsf{sub}(x)$ holds for all $x \in X$ and $\varphi \in \mathsf{Form}^{\mathrm{m}}_{\mathsf{MV}}$.

We say that a formula $\varphi \in \mathsf{Form}^{\mathrm{m}}_{\mathsf{MV}}$ is *satisfied* at $x \in X$ in a $\mathsf{Set}_n$-model $(X, \mathsf{sub}, R, \mathsf{Val})$ if $\mathsf{Val}(x, \varphi) = 1$ and $\mathsf{Set}_n$-frame *validity* is defined analogous to classical modal logic. We denote by $\Lambda^{\mathrm{m}}_{\mathsf{MV}_n}$ the set of all modal $\mathsf{MV}$-formulas which are valid in every $\mathsf{Set}_n$-frame.

Every $\mathsf{Set}$-frame $(X, R)$ can be identified with a $\mathsf{Set}_n$-frame $(X, \mathsf{sub}^{\mathrm{tr}}, R)$ which has the 'trivial constraint' $\mathsf{sub}^{\mathrm{tr}}(x) = \mathbf{Ł}_n$ at all $x \in X$ and it is easy to see that $\Lambda^{\mathrm{m}}_{\mathsf{MV}_n}$ coincides with the set of modal $\mathsf{MV}$-formulas valid on all $\mathsf{Set}$-frames in this sense. The additional 'richness' of semantics over $\mathsf{Set}_n$-frames again becomes visible when one considers canonicity [13, Sections 7-9] and frame-definability [23]. For example, the formulas

$$\Box(x \oplus x) \to \Box x \text{ and } \Diamond(x \oplus x) \to \Diamond x$$

both define the class of $\mathsf{Set}_n$-frames $(X, \mathsf{sub}, R)$ which satisfy

$$\forall x\colon \forall x' : xRx' \to \mathsf{sub}(x') = \mathbf{Ł}_1,$$

that is, frames where all states which are successors are crisp states (identifying $\mathbf{Ł}_1$ with the two-element Boolean algebra). Since the former formula is canonical [13, Example 8.28], the latter one is derivable from the axiomatic extension it yields. In Section 5, we study a similar situation in the positive fragment of the logic described here.

The algebraic counterpart to many-valued modal logic over $\mathbf{Ł}_n$ is given by the following variety of modal $\mathsf{MV}_n$-algebras.

**Definition 2.6** [Modal $\mathsf{MV}_n$-algebra] A *modal* $\mathsf{MV}_n$*-algebra* or $\mathsf{mMV}_n$-algebra is an algebra $\langle \mathbf{A}, \Box \rangle$, where $\mathbf{A} \in \mathsf{MV}_n$ is a $\mathsf{MV}_n$-algebra and $\Box\colon A \to A$ is a unary operation satisfying the equations

$$\text{(B1)}\ \ \Box 1 = 1, \qquad \text{(B2)}\ \ \Box(x \wedge y) = \Box x \wedge \Box y,$$
$$\text{(B3)}\ \ \Box(x \oplus x) = \Box x \oplus \Box x, \qquad \text{(B4)}\ \ \Box(x \odot x) = \Box x \odot \Box x.$$

We denote the variety of modal $\mathsf{MV}_n$-algebras by $\mathsf{mMV}_n$.

The following is a combined result of [13] and [4, Subsection 5.2].

**Theorem 2.7 ([13,4])** *Let $\varphi$ be a modal* MV*-formula. Then*

$$\varphi \in \Lambda^{\mathrm{m}}_{\mathsf{MV}_n} \text{ if and only if } \mathsf{mMV}_n \models \varphi \approx 1.$$

One of the main results of this paper is a similar theorem for the corresponding *positive* modal logic (see Theorem 4.8). On the underlying propositional level we consider positive $\mathsf{MV}_n$-algebras, introduced in the next subsection.

### 2.3 Finite positive MV-algebras

The quasi-variety PMV of *positive* MV*-algebras*, that is, negation-free (and implication-free) subreducts of MV-algebras, was recently introduced and finitely axiomatized in [2]. Here, we focus on finite positive MV-chains defined as follows.

**Definition 2.8** [Finite positive MV-chain] Let $n \geq 1$ be a natural number. The $(n+1)$*-element positive* MV*-chain* is given by

$$\mathbf{PŁ}_n = \langle \{0, \tfrac{1}{n}, \dots \tfrac{n-1}{n}, 1\}, \odot, \oplus, \wedge, \vee, 0, 1 \rangle,$$

understood as a reduct of $\mathbf{Ł}_n$. We write $\mathsf{PMV}_n$ for the (quasi-)variety $\mathbb{ISP}(\mathbf{PŁ}_n) = \mathbb{HSP}(\mathbf{PŁ}_n)$ generated by $\mathbf{PŁ}_n$, and we refer to members of $\mathsf{PMV}_n$ as *positive* $\mathsf{MV}_n$*-algebras* or $\mathsf{PMV}_n$*-algebras*.

An axiomatization of $\mathsf{PMV}_n$ (involving a quasi-equation) can easily be obtained as combination of the axiomatization of PMV from [2] with the one of $\mathsf{MV}_n$ from [12], noting that the additional axioms which characterize $\mathsf{MV}_n$ inside MV do not involve negation or implication (for a purely equational axiomatization, also see the recent [1]).

Later on, we make frequent use of the fact that, for each $\ell \in \mathbf{PŁ}_n$, the unary map $\tau_\ell\colon \{0, \frac{1}{n}, \dots, \frac{n-1}{n}, 1\} \to \{0, \frac{1}{n}, \dots, \frac{n-1}{n}, 1\}$ given by

$$\tau_\ell(x) = \begin{cases} 1 & \text{if } \ell \leq x, \\ 0 & \text{otherwise} \end{cases}$$

is term-definable in $\mathbf{PŁ}_n$, since it can be term-defined in $\mathbf{Ł}_n$ as combination of expressions of the form $x \odot x$ and $x \oplus x$ only [19, pp. 344-345].

Another useful tool later on is a close relation between a $\mathsf{PMV}_n$-algebra and its distributive skeleton, defined similarly to the Boolean skeleton of a $\mathsf{MV}_n$-algebra.

**Definition 2.9** [Distributive skeleton] Let $\mathbf{A} \in \mathsf{PMV}_n$ be a positive $\mathsf{MV}_n$-algebra. The *distributive skeleton* of $\mathbf{A}$ is the bounded distributive lattice $\mathfrak{S}(\mathbf{A})$ with carrier set $\mathfrak{S}(A) = \{a \in \mathbf{A} \mid a \oplus a = a\}$ and the bounded lattice structure inherited from $\mathbf{A}$.

The following fact about distributive skeletons is crucial for the purposes of this paper. In practice, it means that prime filters of $\mathfrak{S}(\mathbf{A})$ can be uniquely extended to $\mathsf{PMV}_n$-homomorphisms $\mathbf{A} \to \mathbf{P}\Ł_n$.

**Theorem 2.10 ([21, Theorem 4.4])** *For every $\mathbf{A} \in \mathsf{PMV}_n$, there is a bijection between the sets of homomorphisms*

$$\mathsf{PMV}_n(\mathbf{A}, \mathbf{P}\Ł_n) \cong \mathsf{DL}(\mathfrak{S}(\mathbf{A}), \mathbf{2})$$

*given by restriction $u \mapsto u|_{\mathfrak{S}(\mathbf{A})}$.*

In Section 4, we consider modal extensions of $\mathsf{PMV}_n$-algebras as algebraic counterpart of the $\mathbf{P}\Ł_n$-valued modal logic introduced in the next section.

## 3 Positive modal logic over finite MV-chains

We work in the language $\mathcal{L}^{\mathrm{m}}_{\mathsf{PMV}} = \{\odot, \oplus, \wedge, \vee, 0, 1, \Box, \Diamond\}$, that is, the signature of $\mathsf{PMV}$ together with two unary operation symbols $\Box$ and $\Diamond$. We define the set $\mathsf{Form}^{\mathrm{m}}_{\mathsf{PMV}}$ of modal $\mathsf{PMV}$-formulas inductively from a countable collection $\mathsf{Prop}$ of propositional variables and the connectives in $\mathcal{L}^{\mathrm{m}}_{\mathsf{PMV}}$.

Similarly to what is discussed in Subsections 2.1 and 2.2, we consider relational semantics on frames with additional structure. We choose the following base-category, which entails both the categories $\mathsf{Pos}$ and $\mathsf{Set}_n$.

**Definition 3.1** [The category $\mathsf{Pos}_n$] We define the category $\mathsf{Pos}_n$ as follows. The objects of $\mathsf{Pos}_n$ are structures of the form $(X, \leq, \mathsf{sub})$, such that $(X, \leq)$ is a poset and $(X, \mathsf{sub}) \in \mathsf{Set}_n$, that is, $\mathsf{sub}\colon X \to \mathbb{S}(\mathbf{P}\Ł_n)$.

A morphism $f\colon (X, \leq, \mathsf{sub}) \to (X', \leq', \mathsf{sub}')$ in $\mathsf{Pos}_n$ is an order-preserving map $f\colon (X, \leq) \to (X', \leq')$ which is also a $\mathsf{Set}_n$-morphism, that is, $\mathsf{sub}'(f(x)) \subseteq \mathsf{sub}(x)$ holds for all $x \in X$.

Note that we can really consider the reduct $(X, \mathsf{sub})$ of an object of $\mathsf{Pos}_n$ as a member of $\mathsf{Set}_n$, since the subuniverses of $\mathbf{P}\Ł_n$ coincide with the ones of $\Ł_n$, as shown in [21, Proposition 3.2]. We also think of $\mathbf{P}\Ł_n$ itself as a member of $\mathsf{Pos}_n$ which we denote by

$$\underset{\sim}{\mathbf{P}\Ł}_n = \big(\{0, \tfrac{1}{n}, \ldots, \tfrac{n-1}{n}, 1\}, \leq, \mathsf{sub}^{\mathrm{gen}}\big).$$

Here, $\leq$ is the the usual chain-order and $\mathsf{sub}^{\mathrm{gen}}(\ell) = \langle \ell \rangle$ is the subalgebra of $\mathbf{P}\Ł_n$ generated by $\{\ell\}$.

Members of $\mathsf{Pos}_n$ endowed with a compatible accessibility relation constitute our semantics for positive modal logic over finite $\mathsf{MV}$-chains.

**Definition 3.2** [$\mathsf{Pos}_n$-frame] A $\mathsf{Pos}_n$-*frame* is a structure of the form

$$\mathfrak{F} = (X, \leq, \mathsf{sub}, R),$$

where $(X, \leq, \mathsf{sub}) \in \mathsf{Pos}_n$ and $R \subseteq X^2$ is a binary relation such that $R[x]$ is always convex and the following two *compatibility conditions* are satisfied:

(i) For all $x, y \in X$ it holds that

$$x \leq y \Rightarrow R[x] \leq_{\mathrm{EM}} R[y].$$

(ii) Whenever $x, y \in X$ satisfy $y \in R[x]$, there exist $y', y'' \in R[x]$ with
- $y' \leq y \leq y''$ and
- $\mathsf{sub}(y'), \mathsf{sub}(y'') \subseteq \mathsf{sub}(x)$.

Here, in condition (i) the subscript EM denotes the Egli-Milner lifting of $\leq$ to $\mathcal{P}(X)$ as defined in the paragraph before Definition 2.2. The significance of the compatibility conditions (i) and (ii) should become clear after Proposition 3.5.

**Example 3.3** The following are special examples of $\mathsf{Pos}_n$-frames.

(1) Every $\mathsf{Pos}$-frame $(X, \leq, R)$ as in Definition 2.2 can be identified with the $\mathsf{Pos}_n$-frame $(X, \leq, \mathsf{sub}^{\mathrm{tr}}, R)$, where $\mathsf{sub}^{\mathrm{tr}}$ are the 'trivial constraints' $\mathsf{sub}^{\mathrm{tr}}(x) = \mathbf{PL}_n$ for all $x \in X$.

(2) Every $\mathsf{Set}_n$-frame $(X, \mathsf{sub}, R)$ as in Definition 2.5 can be identified with the $\mathsf{Pos}_n$-frame $(X, \leq^{\mathrm{dis}}, \mathsf{sub}, R)$, where $\leq^{\mathrm{dis}}$ is the discrete order on $X$.

(3) In particular, combining (1) and (2), every $\mathsf{Set}$-frame $(X, R)$ can be identified with the $\mathsf{Pos}_n$-frame $(X, \leq^{\mathrm{dis}}, \mathsf{sub}^{\mathrm{tr}}, R)$.

Not surprisingly, the corresponding $\mathsf{Pos}_n$-models are also defined as a blend of $\mathsf{Pos}$-models and $\mathsf{Set}_n$-models.

**Definition 3.4** [$\mathsf{Pos}_n$-model] A $\mathsf{Pos}_n$-*model* is a structure of the form

$$\mathfrak{M} = (X, \leq, \mathsf{sub}, R, \mathsf{Val}),$$

where $(X, \leq, \mathsf{sub}, R)$ is a $\mathsf{Pos}_n$-frame and the $\mathsf{Pos}_n$-*valuation* $\mathsf{Val}\colon X \times \mathsf{Prop} \to \mathbf{PL}_n$ satisfies the following two conditions:

(i) If $x \leq y$, then $\mathsf{Val}(x, p) \leq \mathsf{Val}(y, p)$ for all $p \in \mathsf{Prop}$.

(ii) $\mathsf{Val}(x, p) \in \mathsf{sub}(x)$ for all $x \in X$ and $p \in \mathsf{Prop}$.

For example, every $\mathsf{Pos}$-model, $\mathsf{Set}_n$-model or $\mathsf{Set}$-model can be seen as a $\mathsf{Pos}_n$-model analogous to Example 3.3.

Given a $\mathsf{Pos}_n$-model $(X, \leq, \mathsf{sub}, R, \mathsf{Val})$, we inductively extend its $\mathsf{Pos}_n$-valuation to a map

$$\mathsf{Val}\colon X \times \mathsf{Form}^{\mathrm{m}}_{\mathsf{PMV}} \to \mathbf{PL}_n,$$

which is defined in the obvious way for the $\mathsf{PMV}$-connectives and for formulas of shape $\Box\varphi$ or $\Diamond\varphi$ via

$$\mathsf{Val}(x, \Box\varphi) = \bigwedge\{\mathsf{Val}(x', \varphi) \mid xRx'\} \text{ and } \mathsf{Val}(x, \Diamond\varphi) = \bigvee\{\mathsf{Val}(x', \varphi) \mid xRx'\}.$$

Conditions (i) and (ii) of Definition 3.4 are equivalent to the fact that $\mathsf{Val}(-,p)\colon (X,\leq,\mathsf{sub}) \to \underset{\sim}{\mathbf{PL}}_n$ is always a morphism in $\mathsf{Pos}_n$ (where we identify $\mathbf{PL}_n$ with a member of $\mathsf{Pos}_n$ as described after Definition 3.1). In the following, we show that the compatibility conditions required of $\mathsf{Pos}_n$-frames (Definition 3.2) assure that this property extends to $\mathsf{Val}(-,\varphi)$ for every $\varphi \in \mathsf{Form}^{\mathrm{m}}_{\mathsf{PMV}}$.

**Proposition 3.5** *Let $(X,\leq,\mathsf{sub},R,\mathsf{Val})$ be a $\mathsf{Pos}_n$-model. Then, for every modal $\mathsf{PMV}$-formula $\varphi \in \mathsf{Form}^{\mathrm{m}}_{\mathsf{PMV}}$, the following two conditions hold.*

(i) *If $x \leq y$ then $\mathsf{Val}(x,\varphi) \leq \mathsf{Val}(y,\varphi)$ for all $x,y \in X$.*

(ii) *$\mathsf{Val}(x,\varphi) \in \mathsf{sub}(x)$ for all $x \in X$.*

**Proof.** We proceed by induction on the formula $\varphi$. The case $\varphi = p \in \mathsf{Prop}$ is covered by Definition 3.4. If $\varphi = \psi_1 * \psi_2$ with $* \in \{\odot,\oplus,\wedge,\vee\}$ being a $\mathsf{PMV}$-connective, condition (i) holds because all of these connectives are order-preserving and (ii) holds because $\mathsf{sub}(x)$ always is a subalgebra of $\mathbf{PL}_n$.

This leaves us with the case $\varphi = \Box\psi$ (the case $\varphi = \Diamond\psi$ being analogous). To prove condition (i), suppose that $\mathsf{Val}(y,\varphi) < 1$. Then there exists some $y' \in R[y]$ with $\mathsf{Val}(y,\varphi) = \mathsf{Val}(y',\psi)$. Since $R[x] \leq_{\mathrm{EM}} R[y]$, there exists some $x' \in R[x]$ with $x' \leq y'$ and therefore, by the inductive hypothesis, $\mathsf{Val}(x',\psi) \leq \mathsf{Val}(y',\psi)$. This yields

$$\mathsf{Val}(x,\Box\psi) = \bigwedge\{\mathsf{Val}(\tilde{x},\psi) \mid xR\tilde{x}\} \leq \mathsf{Val}(x',\psi) \leq \mathsf{Val}(y',\psi) = \mathsf{Val}(y,\Box\psi)$$

as desired.

To prove condition (ii), suppose towards contradiction that $\mathsf{Val}(x,\Box\psi) \notin \mathsf{sub}(x)$. Since $1 \in \mathsf{sub}(x)$, there needs to exist some $y \in R[x]$ with $\mathsf{Val}(x,\varphi) = \mathsf{Val}(y,\psi)$. By Definition 3.2(ii), there is some $y' \in R[x]$ with $y' \leq y$ and $\mathsf{sub}(y') \subseteq \mathsf{sub}(x)$. By the inductive hypothesis we have $\mathsf{Val}(y',\psi) \neq \mathsf{Val}(y,\psi)$ (since otherwise $\mathsf{Val}(y',\psi) \notin \mathsf{sub}(y') \subseteq \mathsf{sub}(x)$). This yields

$$\mathsf{Val}(y',\psi) < \mathsf{Val}(y,\psi) = \mathsf{Val}(x,\Box\psi) = \bigwedge\{\mathsf{Val}(\tilde{y},\psi) \mid xR\tilde{y}\},$$

a contradiction to $xRy'$. □

We say that a modal $\mathsf{PMV}$-formula $\varphi \in \mathsf{Form}^{\mathrm{m}}_{\mathsf{PMV}}$ is *satisfied* at $x \in X$ in a $\mathsf{Pos}_n$-model $(X,\leq,\mathsf{sub},R,\mathsf{Val})$ if $\mathsf{Val}(x,\varphi) = 1$. As usual, we say that $\varphi$ is *valid* in a $\mathsf{Pos}_n$-frame if it is satisfied at every state in every model based on that frame. We denote by $\Lambda^{\mathrm{m}}_{\mathsf{PMV}_n}$ the set of all modal $\mathsf{PMV}$-formulas which are valid in every $\mathsf{Pos}_n$-frame. More generally, similar to [10], we consider *consequence pairs* $\psi \vdash \varphi$ and say that such a consequence pair is *valid in a $\mathsf{Pos}_n$-frame* if $\mathsf{Val}(x,\psi) = 1$ implies $\mathsf{Val}(x,\varphi) = 1$ in every $\mathsf{Pos}_n$-model based on that frame. We say the consequence pair is *valid* if it is valid in all $\mathsf{Pos}_n$-frames. In this case, we write $\psi \models_n \varphi$. The valid formulas $\varphi \in \Lambda^{\mathrm{m}}_{\mathsf{PMV}_n}$ are thus precisely the ones for which $1 \models_n \varphi$ holds.

As we show in the following, regarding the *minimal* modal logic over $\mathbf{PL}_n$, the distinction between $\mathsf{Pos}_n$-based semantics and $\mathsf{Set}$-based semantics is 'redundant'.

**Proposition 3.6** *A consequence pair $\psi \vdash \varphi$ is valid (in all $\mathsf{Pos}_n$-frames) if and only if it is valid in all $\mathsf{Set}$-frames.*

**Proof.** If the consequence pair is valid in all $\mathsf{Pos}_n$-frames, then it is valid in all $\mathsf{Set}$-frames, since the latter can be identified with $\mathsf{Pos}_n$-frames as in Example 3.3(3).

Conversely, if the consequence pair is not valid in all $\mathsf{Pos}_n$-frames, then there is a $\mathsf{Pos}_n$-model $(X, \leq, \mathsf{sub}, R, \mathsf{Val})$ witnessing this. The underlying $\mathsf{Set}$-model $(X, R, \mathsf{Val})$ then witnesses that the consequence pair is not valid in the $\mathsf{Set}$-frame $(X, R)$. □

Nevertheless, as in the case of classical positive modal logic (Subsection 2.1) and modal logic over $\mathbf{Ł}_n$ (Subsection 2.2), the additional structure of $\mathsf{Pos}_n$-frames becomes interesting when considering questions about *modal definability* or *canonicity*. We begin to touch upon these questions (with a specific example) in Section 5. Before that, we study the logic introduced here algebraically in the following section.

## 4 Algebraic framework

In this section, we study the logic introduced in the previous section by algebraic means. The corresponding variety of modal algebras arises as a combination of the axioms of modal distributive lattices (see Definition 2.1) and modal $\mathsf{MV}_n$-algebras (see Definition 2.6) as follows.

**Definition 4.1** [Modal $\mathsf{PMV}_n$-algebra] A *modal $\mathsf{PMV}_n$-algebra* or $\mathsf{mPMV}_n$-algebra is an algebra $\langle \mathbf{A}, \Box, \Diamond \rangle$, where $\mathbf{A} \in \mathsf{PMV}_n$ is a positive $\mathsf{MV}_n$-algebra and $\Box, \Diamond \colon A \to A$ are unary operations satisfying the equations

$$\begin{array}{ll}
\text{(B1)}\ \Box 1 = 1, & \text{(D1)}\ \Diamond 0 = 0, \\
\text{(B2)}\ \Box(x \wedge y) = \Box x \wedge \Box y, & \text{(D2)}\ \Diamond(x \vee y) = \Diamond x \vee \Diamond y, \\
\text{(B3)}\ \Box(x \oplus x) = \Box x \oplus \Box x, & \text{(D3)}\ \Diamond(x \oplus x) = \Diamond x \oplus \Diamond x, \\
\text{(B4)}\ \Box(x \odot x) = \Box x \odot \Box x, & \text{(D4)}\ \Diamond(x \odot x) = \Diamond x \odot \Diamond x, \\
\text{(P1)}\ \Box(x \vee y) \leq \Box x \vee \Diamond y, & \text{(P2)}\ \Box x \wedge \Diamond y \leq \Diamond(x \wedge y).
\end{array}$$

We denote the variety of modal $\mathsf{PMV}_n$-algebras by $\mathsf{mPMV}_n$.

In practice, the following equivalent axiomatization of $\mathsf{mPMV}_n$, using the unary terms $\tau_\ell$ defined in Subsection 2.1, will usually be more convenient to work with.

**Lemma 4.2** *An algebra $\langle \mathbf{A}, \Box, \Diamond \rangle$ with $\mathbf{A} \in \mathsf{PMV}_n$ is a modal $\mathsf{PMV}_n$-algebra if and only if it satisfies* (B1)-(B2)*,* (D1)-(D2)*,* (P1)-(P2) *and*

$$\begin{array}{ll}
\text{(B}\tau\text{)} & \Box\tau_\ell(x) = \tau_\ell(\Box x) \text{ for all } \ell \in \mathbf{PŁ}_n \backslash \{0\}, \\
\text{(D}\tau\text{)} & \Diamond\tau_\ell(x) = \tau_\ell(\Diamond x) \text{ for all } \ell \in \mathbf{PŁ}_n \backslash \{0\}.
\end{array}$$

**Proof.** It is shown in [19, pp. 344-345] that every $\tau_\ell(x)$ can be obtained exclusively as a combination of terms $x \oplus x$ and $x \odot x$. Therefore, the axioms

(B3)-(B4) imply (B$\tau$) and (D3)-(D4) imply (D$\tau$). Conversely, note that the equations

$$\tau_\ell(x \oplus x) = \tau_{\lceil \frac{\ell}{2} \rceil}(x) \text{ and } \tau_\ell(x \odot x) = \tau_{\lceil \frac{\ell+1}{2} \rceil}(x)$$

are satisfied in $\mathbf{P\L}_n$, where for $q \in \mathbb{Q}$ we define $\lceil q \rceil$ to be the smallest element of $\mathbf{P\L}_n$ which is above $q$. Therefore, using (B$\tau$), for every $\ell \in \mathbf{P\L}_n \backslash \{0\}$ we can compute

$$\tau_\ell(\Box x \oplus \Box x) = \tau_{\lceil \frac{\ell}{2} \rceil}(\Box x) = \Box \tau_{\lceil \frac{\ell}{2} \rceil}(x) = \Box \tau_\ell(x \oplus x) = \tau_\ell(\Box(x \oplus x)).$$

But this implies $\Box x \oplus \Box x = \Box(x \oplus x)$ since $\mathbf{P\L}_n$ satisfies the quasi-equation

$$\bigwedge_{\ell \neq 0} (\tau_\ell(x) \approx \tau_\ell(y)) \to x \approx y.$$

The case for $\Box(x \odot x)$, $\Diamond(x \oplus x)$ and $\Diamond(x \odot x)$ is analogous. □

Next, we explain how to obtain $\mathsf{Pos}_n$-frames from $\mathsf{mPMV}_n$-algebras.

**Definition 4.3** [Canonical frame of a $\mathsf{mPMV}_n$-algebra] Let $\langle \mathbf{A}, \Box, \Diamond \rangle$ be a modal $\mathsf{PMV}_n$-algebra. The *canonical* $\mathsf{Pos}_n$*-frame of* $\langle \mathbf{A}, \Box, \Diamond \rangle$ is the $\mathsf{Pos}_n$-frame

$$\langle \mathbf{A}, \Box, \Diamond \rangle^+ := \big(\mathsf{PMV}_n(\mathbf{A}, \mathbf{P\L}_n), \leq^{\mathrm{pw}}, \mathsf{im}, R^{\mathrm{m}}\big),$$

where $\leq^{\mathrm{pw}}$ is the point-wise order, $\mathsf{im}$ takes a homomorphism $u$ to its image $\mathsf{im}(u) = u(\mathbf{A}) \subseteq \mathbf{P\L}_n$ and the binary relation $R^{\mathrm{m}}$ is defined via

$$uR^{\mathrm{m}}u' \iff \forall a \in \mathbf{A} : u(\Box a) \leq u'(a) \leq u(\Diamond a).$$

This definition is justified, since in the following we show that canonical frames of $\mathsf{mPMV}_n$-algebras really are $\mathsf{Pos}_n$-frames.

**Proposition 4.4** *Let* $\langle \mathbf{A}, \Box, \Diamond \rangle$ *be a modal* $\mathsf{PMV}_n$*-algebra. Then its canonical frame* $\langle \mathbf{A}, \Box, \Diamond \rangle^+$ *is a* $\mathsf{Pos}_n$*-frame.*

**Proof.** It is clear by definition that $(\mathsf{PMV}_n(\mathbf{A}, \mathbf{P\L}_n), \leq^{\mathrm{pw}}, \mathsf{im})$ is an object of the category $\mathsf{Pos}_n$. We need to show that $R^{\mathrm{m}}$ satisfies the compatibility conditions (i) and (ii) from Definition 3.2.

To verify (i), first suppose that $u, w \in \mathsf{PMV}_n(\mathbf{A}, \mathbf{P\L}_n)$ are homomorphisms for which $u \leq^{\mathrm{pw}} w$ and $wR^{\mathrm{m}}w'$ hold. Then we need to find $u'$ which satisfies $uR^{\mathrm{m}}u' \leq w'$. We define a filter $F \subseteq \mathfrak{S}(\mathbf{A})$ on the distributive skeleton of $\mathbf{A}$ (recall Definition 2.9) by

$$F = \{b \in \mathfrak{S}(\mathbf{A}) \mid u(\Box b) = 1\}$$

and an ideal $J \subseteq \mathfrak{S}(\mathbf{A})$ generated by the set

$$\{b \in \mathfrak{S}(\mathbf{A}) \mid w'(b) = 0 \text{ or } u(\Diamond b) = 0\}.$$

If $b \in J$, there are $b', b'' \in \mathfrak{S}(\mathbf{A})$ with $b \leq b' \vee b''$, $w'(b') = 0$ and $u(\Diamond b'') = 0$. We then compute

$$\begin{aligned} u(\Box b) &\leq u(\Box(b' \vee b'')) \\ &\leq u(\Box b') \vee u(\Diamond b'') && \text{(P1)} \\ &\leq w(\Box b') \vee u(\Diamond b'') && (u \leq^{\mathrm{pw}} w) \\ &\leq w'(b') \vee u(\Diamond b'') = 0. && (wR^{\mathrm{m}}w') \end{aligned}$$

Therefore, $b \notin F$ and we showed that $F$ and $J$ are disjoint. There is a prime filter $U'$ extending $F$ with $U' \cap J = \varnothing$, and by Theorem 2.10 there is a homomorphism $u' \colon \mathbf{A} \to \mathbf{P}\text{Ł}_n$ which extends (the characteristic function of) $U'$.

We show that $u'$ has the desired properties. To see that $u'(a) \leq w'(a)$, in case $w'(a) \neq 1$ we can use $\ell = w'(a) + \frac{1}{n}$ and find $\tau_\ell(a) \in J$, which yields

$$\tau_\ell(u'(a)) = u'(\tau_\ell(a)) = 0,$$

that is, $u'(a) < \ell$ as desired. The argument for $u'(a) \leq u(\Diamond a)$ is similar, using (D$\tau$), and for $u(\Box a) \leq u'(a)$ using the construction of $F$ and (B$\tau$).

For the second half of condition (i) we proceed similarly. Assuming that $u \leq^{\mathrm{pw}} w$ and $uR^{\mathrm{m}}u'$ hold, we want to find $w'$ with $wR^{\mathrm{m}}w'$ and $u' \leq^{\mathrm{pw}} w'$. This time, let $F \subseteq \mathfrak{S}(\mathbf{A})$ be the filter generated by

$$\{b \in \mathfrak{S}(\mathbf{A}) \mid u'(b) = 1 \text{ or } w(\Box b) = 1\},$$

and let $J \subseteq \mathfrak{S}(\mathbf{A})$ be the ideal

$$J = \{b \in \mathfrak{S}(\mathbf{A}) \mid w(\Diamond b) = 0\}.$$

Given $b \in F$, we have $b \geq b' \wedge b''$ for some $b', b'' \in \mathfrak{S}(\mathbf{A})$ which satisfy $u'(b') = w(\Box b'') = 1$. We compute

$$\begin{aligned} w(\Diamond b) &\geq w(\Diamond(b' \wedge b'')) \\ &\geq w(\Diamond b') \wedge w(\Box b'') && \text{(P2)} \\ &\geq u(\Diamond b') \wedge w(\Box b'') && (u \leq^{\mathrm{pw}} w) \\ &\geq u'(b') \wedge w(\Box b'') = 1. && (uR^{\mathrm{m}}u') \end{aligned}$$

Therefore, $b \notin J$ and we showed that $J$ and $F$ are disjoint. The filter $F$ can be extended to a prime filter $W' \subseteq \mathfrak{S}(\mathbf{A})$ which is disjoint from $J$, and by Theorem 2.10, there is a homomorphism $w' \colon \mathbf{A} \to \mathbf{P}\text{Ł}_n$ which restricts to (the characteristic function of) $W'$ on $\mathfrak{S}(\mathbf{A})$.

To see that $w'$ has the desired properties,

$$u'(a) \leq w'(a)$$

follows from $\tau_\ell(a) \in F$ with $\ell = u'(a)$ and similarly we get $w(\Box a) \leq w'(a)$. To see that $w'(a) \leq w(\Diamond a)$ holds, suppose $w(\Diamond a) \neq 1$ and set $\ell = w(\Diamond a) + \frac{1}{n}$.

Then $\tau_\ell(a) \in J$ implies $0 = w'(\tau_\ell(a)) = \tau_\ell(w'(a))$, which means $w'(a) \leq w(\Diamond a)$ by our choice of $\ell$.

We now verify condition (ii) of Definiton 3.2, using an argument similar to [13, Lemma 7.4]. For the first part of condition (ii), let $u \in \mathsf{PMV}_n(\mathbf{A}, \mathbf{P\L}_n)$ with $\mathsf{im}(u) = \mathbf{P\L}_k$ and let $uR^{\mathrm{m}}w$. Suppose towards contradiction that there is no $w' \leq^{\mathrm{pw}} w$ with $uR^{\mathrm{m}}w'$ and $\mathsf{im}(w') \leq \mathbf{P\L}_k$. Choose $\tilde{w} \in R^{\mathrm{m}}[u]$ and $a \in \mathbf{A}$ such that $\tilde{w} \leq^{\mathrm{pw}} w$ and $\tilde{w}(a) = \frac{1}{m} \in \mathbf{P\L}_n \backslash \mathbf{P\L}_k$ is the minimal value obtained by a member of $R^{\mathrm{m}}[u]$ below $w$. By the duality for $\mathsf{PMV}_n$ established in [21], we can think of $(\mathsf{PMV}_n(\mathbf{A}, \mathbf{P\L}_n), \leq^{\mathrm{pw}})$ as Priestley space in which the subsets $\mathsf{PMV}_n(\mathbf{A}, \mathbf{P\L}_k)$ and $R^{\mathrm{m}}[u]$ are closed. The algebra $\mathbf{A}$ can be identified with the collection of structure-preserving continuous maps $\mathsf{PMV}_n(\mathbf{A}, \mathbf{P\L}_n) \to \underset{\sim}{\mathbf{P\L}}_n$ via the isomorphism $a \mapsto \mathsf{ev}_a$. Now consider the closed sets

$$F = R^{\mathrm{m}}[u] \cap \mathsf{PMV}_n(\mathbf{A}, \mathbf{P\L}_k) \text{ and } G = R^{\mathrm{m}}[u] \cap \tilde{w}{\downarrow}.$$

We have $F{\uparrow} \cap G{\downarrow} = \varnothing$, since otherwise there would be $w' \in R^{\mathrm{m}}[u] \cap \mathsf{PMV}_n(\mathbf{A}, \mathbf{P\L}_k)$ with $w' \leq^{\mathrm{pw}} \tilde{w} \leq^{\mathrm{pw}} w$, contradicting our initial assumption. Therefore, there exists a clopen down-set $\Omega$ which contains $G$ and is disjoint from $F$. Now we define the element

$$a' = a|_\Omega \cup 1|_{\mathsf{PMV}_n(\mathbf{A}, \mathbf{P\L}_n) \backslash \Omega},$$

which is a well-defined member of $\mathbf{A}$ since $\Omega$ is downwards closed. Now by Lemma 4.5 below we find

$$u(\Box a') = \bigwedge \{v(a') \mid v \in R^{\mathrm{m}}[u]\} = \bigwedge \{v(a) \mid v \in R^{\mathrm{m}}[u] \cap \Omega\} = \tilde{w}(a),$$

which yields a contradiction to $\mathsf{im}(u) = \mathbf{P\L}_k$, since $\tilde{w}(a) \notin \mathbf{P\L}_k$.

For the proof of the second part of condition (ii) we proceed similarly. Take $u \in \mathsf{PMV}_n(\mathbf{A}, \mathbf{P\L}_n)$ with $\mathsf{im}(u) = \mathbf{P\L}_k$ and $uR^{\mathrm{m}}w$ and suppose towards contradiction that there is no $w \leq^{\mathrm{pw}} w''$ with $uR^{\mathrm{m}}w''$ and $\mathsf{im}(w'') \subseteq \mathbf{P\L}_k$. Choose $\tilde{w} \in R^{\mathrm{m}}[u] \cap w{\uparrow}$ and $a \in \mathbf{A}$ such that $\tilde{w}(a) = \frac{m-1}{m}$ is the maximal value obtained by a member of $R[u]$ above $w$. Separate the closed sets

$$F = R^{\mathrm{m}}[u] \cap \tilde{w}{\uparrow} \text{ and } G = R^{\mathrm{m}}[u] \cap \mathsf{PMV}_n(\mathbf{A}, \mathbf{P\L}_k),$$

by a clopen upset $\Omega \supseteq F$ and take

$$a' = a|_\Omega \cup 0|_{\mathsf{PMV}_n(\mathbf{A}, \mathbf{P\L}_n) \backslash \Omega},$$

which is a well-defined member of $\mathbf{A}$ since $\Omega$ is upwards closed. Then, using Lemma 4.5, we compute

$$u(\Diamond a') = \bigvee \{v(a') \mid v \in R^{\mathrm{m}}[u]\} = \bigvee \{v(a) \mid v \in R^{\mathrm{m}}[u] \cap \Omega\} = \tilde{w}(a),$$

which yields $u(\Diamond a') = \frac{m-1}{m} \notin \mathbf{P\L}_k$, which is a contradiction to our assumption $\mathsf{im}(u) = \mathbf{P\L}_k$. □

As usual, the following truth lemma is the most significant ingredient to obtain an algebraic completeness result.

**Lemma 4.5 (Truth Lemma)** *Let $\langle \mathbf{A}, \Box, \Diamond \rangle$ be a $\mathsf{mPMV}_n$-algebra. Then, in the canonical frame $\langle \mathbf{A}, \Box, \Diamond \rangle^+$, we have*

$$u(\Box a) = \bigwedge \{v(a) \mid uR^{\mathrm{m}}v\} \text{ and } u(\Diamond a) = \bigvee \{v(a) \mid uR^{\mathrm{m}}v\}$$

*for all $u \in \mathsf{PMV}_n(\mathbf{A}, \mathbf{P}\Ł_n)$ and $a \in \mathbf{A}$.*

**Proof.** We start with the first equation. It is clear by definition that $u(\Box a) \leq \bigwedge \{v(a) \mid uR^{\mathrm{m}}v\}$ holds. Suppose towards contradiction that this inequality is strict, say

$$u(\Box a) < \bigwedge \{v(a) \mid uR^{\mathrm{m}}v\} =: d$$

holds for some $a \in \mathbf{A}$. We define a filter $F$ and an ideal $J$ on the distributive skeleton $\mathfrak{S}(\mathbf{A})$ as follows. Set

$$F = \{b \in \mathfrak{S}(\mathbf{A}) \mid u(\Box b) = 1\},$$

which is a filter due to axioms (B1) and (B2) and let $J \subseteq \mathfrak{S}(\mathbf{A})$ be the ideal generated by the set

$$\{\tau_d(a) \vee b' \in \mathfrak{S}(\mathbf{A}) \mid u(\Diamond b') = 0\}.$$

If $b \in J$, then $b \leq \tau_d(a) \vee b'_1 \vee \cdots \vee b'_n$ for some $b'_1, \ldots b'_n$ with $u(\Diamond b'_i) = 0$ for all $i = 1, \ldots, n$. Then we use the fact that $u$ is a homomorphism and the properties of modal $\mathsf{PMV}_n$-algebras to find

$$\begin{aligned}
u(\Box b) &\leq u\big(\Box(\tau_d(a) \vee (b'_1 \vee \cdots \vee b'_n))\big) && \text{(B2)}\\
&\leq u(\Box\tau_d(a) \vee \Diamond(b'_1 \vee \cdots \vee b'_n)) && \text{(P1)}\\
&= u(\Box\tau_d(a)) \vee u(\Diamond b'_1 \vee \cdots \vee \Diamond b'_n) && \text{(D2)}\\
&= \tau_d(u(\Box a)) \vee u(\Diamond b'_1) \vee \cdots \vee u(\Diamond b'_n) = 0. && \text{(B}\tau\text{)}
\end{aligned}$$

Therefore, $u(\Box b) = 0$, which implies $b \notin F$. So we showed that $F$ and $J$ are disjoint. By the prime ideal theorem for distributive lattices, we can find a prime filter $W \subseteq \mathfrak{S}(\mathbf{A})$ with $F \subseteq W$ and $W \cap J = \varnothing$. By Theorem 2.10, there is a homomorphism $w\colon \mathbf{A} \to \mathbf{P}\Ł_n$ which restricts to (the characteristic function of) $W$ on $\mathfrak{S}(\mathbf{A})$.

We now show that $uR^{\mathrm{m}}w$ holds. Indeed, take $a' \in \mathbf{A}$ and set $u(\Box a') = \ell$. Then

$$\tau_\ell(w(a')) = w(\tau_\ell(a')) = 1$$

because $u(\Box\tau_\ell(a')) = \tau_\ell(u(\Box a')) = 1$ implies $\tau_\ell(a') \in F$. Therefore, we have

$$u(\Box a') \leq w(a') \text{ for all } a' \in \mathbf{A}.$$

Similarly, if $u(\Diamond a') \neq 1$, set $\ell = u(\Diamond a') + \frac{1}{n}$ and find

$$\tau_\ell(w(a')) = w(\tau_\ell(a')) = 0$$

because $\tau_\ell(a') \leq \tau_d(a) \vee \tau_\ell(a') \in J$ implies $\tau_\ell(a') \in J$. Therefore, we have $w(a') < u(\Diamond a') + \frac{1}{n}$, which implies

$$w(a') \leq u(\Diamond a') \text{ for all } a' \in \mathbf{A}$$

as desired.

So we showed that $uR^{\mathrm{m}}w$ holds. However, note that $\tau_d(a) \in J$ implies that

$$\tau_d(w(a)) = w(\tau_d(a)) = 0,$$

which by our choice of $d$ means $w(a) < \bigwedge\{v(a) \mid uR^{\mathrm{m}}v\}$, a contradiction to $uR^{\mathrm{m}}w$. This finishes the proof of the first equation.

To prove the second equation, suppose towards contradiction that

$$\bigvee\{v(a) \mid uR^{\mathrm{m}}v\} < u(\Diamond a)$$

holds for some $a \in \mathbf{A}$. Set $d := \bigvee\{v(a) \mid uR^{\mathrm{m}}v\} + \frac{1}{n}$. Define a filter $F$ and an ideal $J$ of $\mathfrak{S}(\mathbf{A})$ as follows. Let

$$J = \{b \in \mathfrak{S}(\mathbf{A}) \mid u(\Diamond b) = 0\},$$

which is an ideal due to axioms (D1)-(D2) and let $F$ be the filter generated by the set

$$\{\tau_d(a) \wedge b' \mid u(\Box b') = 1\}.$$

If $b \in F$, then $\tau_d(a) \wedge b'_1 \wedge \cdots \wedge b'_n \leq b$ for some $b'_1, \ldots b'_n$ with $u(\Box b'_i) = 1$ for all $i = 1, \ldots, n$. We compute

$$\begin{aligned}
u(\Diamond b) &\geq u\big(\Diamond(\tau_d(a) \wedge (b'_1 \wedge \cdots \wedge b'_n))\big) && \text{(D2)}\\
&\geq u(\Diamond\tau_d(a) \wedge \Box(b'_1 \wedge \cdots \wedge b'_n)) && \text{(P2)}\\
&= u(\Diamond\tau_d(a)) \wedge u(\Box b'_1 \wedge \cdots \wedge \Box b'_n) && \text{(B2)}\\
&= \tau_d(u(\Diamond a)) \wedge u(\Box b'_1) \wedge \cdots \wedge u(\Box b'_n) = 1. && \text{(D}\tau\text{)}
\end{aligned}$$

Therefore, $u(\Diamond b) = 1$, which implies $b \notin J$ and we showed that $F$ and $J$ are disjoint. As before, we can find a homomorphism $w\colon \mathbf{A} \to \mathbf{P}\text{Ł}_n$ with $w(F) = \{1\}$ and $w(J) = \{0\}$.

We show that $uR^{\mathrm{m}}w$ holds. Given $a' \in \mathbf{A}$, let $\ell = u(\Box a')$. Then $\tau_\ell(a') \geq \tau_d(a) \wedge \tau_\ell(a') \in F$ implies $w(\tau_\ell(a')) = 1$ and thus we showed

$$u(\Box a') \leq w(a') \text{ for all } a \in \mathbf{A}.$$

Conversely, if $u(\Diamond a') \neq 1$, take $\ell = u(\Diamond a') + \frac{1}{n}$. Then $u(\Diamond\tau_\ell(a')) = 0$ yields $\tau_\ell(a') \in J$ and thus $w(\tau_\ell(a')) = 0$. This means $w(a') < u(\Diamond a) + \frac{1}{n}$. Thus we showed

$$w(a') \leq u(\Diamond a') \text{ for all } a' \in \mathbf{A},$$

and altogether that $uR^{\mathrm{m}}w$ holds.

However, since $\tau_d(a) \in F$ holds, we have $\tau_d(w(a)) = w(\tau_d(a)) = 1$, which by our choice means $\bigvee\{v(a) \mid uR^{\mathrm{m}}v\} < w(a)$, a contradiction. □

In the converse direction, we now explain how to obtain $\mathsf{mPMV}_n$-algebras from $\mathsf{Pos}_n$-frames.

**Definition 4.6** [Complex algebra of a $\mathsf{Pos}_n$-frame] Let $\mathfrak{F} = (X, \leq, \mathsf{sub}, R)$ be a $\mathsf{Pos}_n$-frame. The *complex* $\mathsf{mPMV}_n$*-algebra* of $\mathfrak{F}$ is the $\mathsf{mPMV}_n$-algebra

$$\mathfrak{F}_+ := \langle \underset{\sim}{\mathbf{PL}}_n^{\mathfrak{F}}, \Box_R, \Diamond_R \rangle,$$

where $\underset{\sim}{\mathbf{PL}}_n^{\mathfrak{F}}$ consists of all $\mathsf{Pos}_n$-morphisms $(X, \leq, \mathsf{sub}) \to \underset{\sim}{\mathbf{PL}}_n$ with point-wise $\mathsf{PMV}_n$-operations and

$$(\Box_R \alpha)(x) = \bigwedge_{xRx'} \alpha(x') \text{ and } (\Diamond_R \alpha)(x) = \bigvee_{xRx'} \alpha(x').$$

It is easy to check that this really is a $\mathsf{mPMV}_n$-algebra.

Every $\mathsf{mPMV}_n$-algebra can be embedded in the complex algebra of its canonical frame and vice versa. The proof of the following theorem is mostly straightforward.

**Theorem 4.7 (Representation Theorem)** *Let* $\mathbf{A}$ *be a* $\mathsf{mPMV}_n$*-algebra and let* $\mathfrak{F}$ *be a* $\mathsf{Pos}_n$*-frame.*

(i) *There is an embedding* $\mathbf{A} \hookrightarrow (\mathbf{A}^+)_+$ *via evaluations* $a \mapsto \mathsf{ev}_a$.

(ii) *There is an embedding* $\mathfrak{F} \hookrightarrow (\mathfrak{F}_+)^+$ *via evaluations* $x \mapsto \mathsf{ev}_x$.

**Proof.** The 'non-modal parts' of (i) and (ii) are both consequences of the dualities established in [21]. For the 'modal part' of (i), we need to show that $\Box_{R^{\mathrm{m}}} \mathsf{ev}_a = \mathsf{ev}_{\Box a}$ and $\Diamond_{R^{\mathrm{m}}} \mathsf{ev}_a = \mathsf{ev}_{\Diamond a}$. This follows from a direct computation

$$\Box_{R^{\mathrm{m}}} \mathsf{ev}_a(u) = \bigwedge_{uR^{\mathrm{m}}v} \mathsf{ev}_a(v) = \bigwedge_{uR^{\mathrm{m}}v} v(a) = u(\Box a) = \mathsf{ev}_{\Box a},$$

where we used Lemma 4.5. The other equation is shown completely analogous.

For the 'modal part' of (ii), we need to show

$$xRx' \Leftrightarrow \mathsf{ev}_x R^{\mathrm{m}} \mathsf{ev}_{x'}.$$

The direction '$\Rightarrow$' is immediate by definition since

$$\mathsf{ev}_x R^{\mathrm{m}} \mathsf{ev}_{x'} \Leftrightarrow \mathsf{ev}_x(\Box_R \alpha) \leq \mathsf{ev}_{x'}(\alpha) \leq \mathsf{ev}_x(\Diamond_R \alpha) \Leftrightarrow \bigwedge_{xRy} \alpha(y) \leq \alpha(x') \leq \bigvee_{xRy} \alpha(y)$$

and the latter clearly holds for all $\alpha$ if $xRx'$.

For the direction '$\Leftarrow$', suppose $x' \notin R[x]$. Then $R[x] \cap x'{\uparrow} = \varnothing$ or $R[x] \cap x'{\downarrow} = \varnothing$ needs to hold (since otherwise $xRx'$ holds by convexity of $R[x]$). In the former case, define $\alpha \colon \mathfrak{F} \to \underset{\sim}{\mathbf{PL}}_n$ by $\alpha(y) = 0$ if $y \in R[x]{\downarrow}$ and $\alpha(y) = 1$ otherwise. Then

$$\mathsf{ev}_{x'}(\alpha) = \alpha(x') = 1 \text{ but } \mathsf{ev}_x(\Diamond_R \alpha) = \bigvee_{xRy} \alpha(y) = 0$$

shows $\mathsf{ev}_{x'} \notin R^{\mathrm{m}}[\mathsf{ev}_x]$. In the other case a similar argument works with $\alpha(y) = 1$ if $y \in R[x]{\uparrow}$ and $\alpha(y) = 0$ otherwise. □

We are now ready to state the main result of this section. In the presence of the prior results of this section (in particular Lemma 4.5), the proof is fairly routine.

**Theorem 4.8 (Algebraic Completeness)** *Let* $\psi, \varphi \in \mathsf{Form}^{\mathrm{m}}_{\mathsf{PMV}}$.

(i) $\psi \models_n \varphi$ *if and only if* $\mathsf{mPMV}_n \models \psi \leq \varphi$.

(ii) *In particular,* $\varphi \in \Lambda^{\mathrm{m}}_{\mathsf{PMV}_n}$ *if and only if* $\mathsf{mPMV}_n \models \varphi \approx 1$.

**Proof.** Let $\langle \mathbf{F}, \Box, \Diamond \rangle = \mathsf{Free}_{\mathsf{mPMV}_n}(\mathsf{Prop})$ be the free $\mathsf{mPMV}_n$-algebra generated by the countable set of propositional variables $\mathsf{Prop}$. The *canonical* $\mathsf{Pos}_n$-*model* is based on its canonical frame $\langle \mathbf{F}, \Box, \Diamond \rangle^+$ together with the *canonical* $\mathsf{Pos}_n$-*valuation* defined by

$$\mathsf{Val}^{\mathrm{can}}(u, p) = u([p]),$$

where $[p]$ denotes the equivalence class of $p$. By Proposition 4.4, we know that $\langle \mathbf{F}, \Box, \Diamond \rangle^+$ really is a $\mathsf{Pos}_n$-frame. Furthermore, we can easily check that the canonical $\mathsf{Pos}_n$-model really is a $\mathsf{Pos}_n$-model (Definition 3.4) since

(i) If $u \leq^{\mathrm{pw}} v$, then $\mathsf{Val}^{\mathrm{can}}(u, p) = u([p]) \leq v([p]) = \mathsf{Val}^{\mathrm{can}}(v, p)$.

(ii) $\mathsf{Val}^{\mathrm{can}}(u, p) = u([p]) \in \mathsf{im}(u)$.

Using the Truth Lemma (Lemma 4.5), it is easy to verify that the property

$$\mathsf{Val}^{\mathrm{can}}(u, \varphi) = u([\varphi])$$

extends to all modal $\mathsf{PMV}$-formulas $\varphi \in \mathsf{Form}^{\mathrm{m}}_{\mathsf{PMV}}$.

Now suppose that $\mathsf{mPMV}_n \not\models \psi \leq \varphi$. Then in particular $[\psi] \leq [\varphi]$ does not hold in $\mathbf{F}$. Thus, there exists a homomorphism $u \colon \mathbf{F} \to \mathbf{P\text{Ł}}_n$ with $u([\psi]) = 1$ and $u([\varphi]) = 0$. This means that $\mathsf{Val}^{\mathrm{can}}(u, [\psi]) = 1$ and $\mathsf{Val}^{\mathrm{can}}(u, [\varphi]) = 0$, witnessing that $\psi \not\models_n \varphi$ in the canonical $\mathsf{Pos}_n$-model. □

For the minimal $\mathbf{P\text{Ł}}_n$-valued modal logic, the additional structure of $\mathsf{Pos}_n$-frames is 'redundant' (recall Proposition 3.6). However, in the next section we study a consequence pair which is only canonical with respect to the semantics over $\mathsf{Pos}_n$.

## 5 A case study in canonicity

In this section, we give an example of a consequence pair which is canonical with respect to the semantics over $\mathsf{Pos}_n$-frames but isn't with respect to either of the semantics over $\mathsf{Pos}$-frames or $\mathsf{Set}_n$-frames.

From now on, for simplicity we assume that $n$ is a prime number, which implies $\mathbb{S}(\mathbf{P\text{Ł}}_n) = \{\mathbf{P\text{Ł}}_n, \mathbf{P\text{Ł}}_1\}$. We consider the consequence pairs

$$\Box(x \oplus x) \vdash \Box x \text{ and } \Diamond(x \oplus x) \vdash \Diamond x.$$

As noted in Subsection 2.2, over $\mathsf{MV}_n$ (*i.e.*, with negation) the formula $\Box(x \oplus x) \to \Box x$ is canonical [13, Example 8.28] and defines the class of $\mathsf{Set}_n$-frames in which 'all successors are crisp'. In particular, this means that the formula $\Diamond(x \oplus x) \to \Diamond x$ which defines the same class of $\mathsf{Set}_n$-frames is derivable from the corresponding axiomatic extension of modal logic over $\mathsf{MV}_n$. In this section, we show that in modal logic over $\mathsf{PMV}_n$ this is not the case anymore. In

the following, we identify the classes of $\mathsf{Pos}_n$-frames which are defined by the above consequence pairs. As it turns out, the former consequence pair defines $\mathsf{Pos}_n$-frames in which 'every successor has a crisp successor of the same element below' while the latter one defines the ones in which 'every successor has a crisp successor of the same element above'. Note that a $\mathsf{Pos}_n$-frame $(X, \leq, \mathsf{sub}, R)$ can be identified with a first-order structure where $\mathsf{sub}$ is described by unary relations for all $\mathbf{P\L}_k \in \mathbb{S}(\mathbf{P\L}_n)$.

**Proposition 5.1** *Let $\mathfrak{F} = (X, \leq, \mathsf{sub}, R)$ be a $\mathsf{Pos}_n$-frame.*

(i) *The consequence pair $\Box(p \oplus p) \vdash \Box p$ is valid in $\mathfrak{F}$ if and only if $\mathfrak{F}$ satisfies*

$$\forall x \forall y \colon \big(xRy \to \exists y' : (xRy' \wedge y' \leq y \wedge \mathsf{sub}(y') = \mathbf{P\L}_1)\big).$$

(ii) *The consequence pair $\Diamond(p \oplus p) \vdash \Diamond p$ is valid in $\mathfrak{F}$ if and only if $\mathfrak{F}$ satisfies*

$$\forall x \forall y \colon \big(xRy \to \exists y'' : (xRy'' \wedge y \leq y'' \wedge \mathsf{sub}(y'') = \mathbf{P\L}_1)\big).$$

**Proof.** We first prove part (i) of the proposition. Assume that $\mathfrak{F}$ satisfies the first-order condition in the statement and let $(X, \leq, \mathsf{sub}, R, \mathsf{Val})$ be an arbitrary $\mathsf{Pos}_n$-model based on $\mathfrak{F}$ which satisfies $\mathsf{Val}(x, \Box(p \oplus p)) = 1$ and $R[x] \neq \varnothing$ (otherwise, the consequence pair is trivially satisfied at $x$). If $y \in R[x]$, there is some $y' \in R[x] \cap y{\downarrow}$ with $\mathsf{sub}(y') = \mathbf{P\L}_1$. Then it must hold that $\mathsf{Val}(y', p) = 1$ (since otherwise $\mathsf{Val}(y', p) = 0$ implies $\mathsf{Val}(x, \Box(p \oplus p)) = 0$). Since $y' \leq y$, this now implies $\mathsf{Val}(y, p) = 1$ as well. Since $y$ was an arbitrary successor of $x$, we get that $\mathsf{Val}(x, \Box p) = 1$ as desired.

For the converse, assume that $\mathfrak{F}$ does not satisfy the first-order formula of the statement. In other words, there are $xRy$ such that no $xRy'$ with $y' \leq y$ satisfies $\mathsf{sub}(y') = \mathbf{P\L}_1$. Then choose some $\ell \in \mathbf{P\L}_n$ with $\frac{1}{2} \leq \ell < 1$ and consider any $\mathsf{Pos}_n$-valuation which satisfies

$$\mathsf{Val}(w, p) = \begin{cases} \ell & \text{if } xRw \text{ and } w \leq y \\ 1 & \text{if } xRw \text{ and } w \not\leq y. \end{cases}$$

The $\mathsf{Pos}_n$-model thus arising witnesses $\mathsf{Val}(x, \Box(p \oplus p)) = 1$ but $\mathsf{Val}(x, \Box p) = \ell$, which shows that the consequence pair is not valid in $\mathfrak{F}$.

To similarly prove part (ii) of the proposition, first suppose that $\mathfrak{F}$ satisfies the first-order formula in the statement, and suppose $(X, \leq, \mathsf{sub}, R, \mathsf{Val})$ is a $\mathsf{Pos}_n$-model based on $\mathfrak{F}$ with $\mathsf{Val}(x, \Diamond(p \oplus p)) = 1$. Then there exists some successor $y \in R[x]$ with $\mathsf{Val}(y, p) \geq \frac{1}{2}$. By our assumption, there exists another successor $y'' \in R[x]$ with $y \leq y''$ and $\mathsf{sub}(y'') = \mathbf{P\L}_1$. Since $\mathsf{Val}(y, p) \neq 0$ and $y''$ is crisp, we necessarily have $\mathsf{Val}(y'', p) = 1$, which implies $\mathsf{Val}(x, \Diamond p) = 1$ as desired.

For the converse, assume that the $\mathfrak{F}$ does not satisfy the first-order condition in the statement. Then there exist some $xRy$ such that all $w \in R[x]$ with $y \leq w$ satisfy $\mathsf{sub}(w) = \mathbf{P\L}_n$. Now pick some $\ell \in \mathbf{P\L}_n$ with $\frac{1}{2} \leq \ell < 1$ and consider any valuation which satisfies

$$\mathsf{Val}(w, p) = \begin{cases} \ell & \text{if } xRw \text{ and } y \leq w \\ 0 & \text{if } xRw \text{ and } y \not\leq w. \end{cases}$$

The $\mathsf{Pos}_n$-model thus arising witnesses $\mathsf{Val}(x, \Diamond(p \oplus p)) = 1$ but $\mathsf{Val}(x, \Diamond p) = \ell$, which shows that the consequence pair is not valid in $\mathfrak{F}$. □

To conclude this section, we note that the consequence pair $\Box(p \oplus p) \vdash \Box p$ is canonical with respect to $\mathsf{Pos}_n$-frames. Together with Section 4, this is a consequence of the following.

**Proposition 5.2** *Let $\langle \mathbf{A}, \Box, \Diamond \rangle$ be a modal $\mathsf{PMV}_n$-algebra which satisfies the equation $\Box(x \oplus x) \leq \Box x$. Then the canonical frame $\langle \mathbf{A}, \Box, \Diamond \rangle^+$ satisfies the first-order condition from Proposition 5.1(i).*

**Proof.** The idea is similar to the proof of part (ii) of Proposition 4.4. Suppose towards contradiction that there are some $u, v \in \mathsf{PMV}_n(\mathbf{A}, \mathbf{P}\Ł_n)$ with $uR^{\mathrm{m}}v$ and $R^{\mathrm{m}}[u] \cap \mathsf{PMV}_n(\mathbf{A}, \mathbf{P}\Ł_1) \cap v{\downarrow} = \varnothing$. Consider the closed sets

$$F = R^{\mathrm{m}}[u] \cap \mathsf{PMV}_n(\mathbf{A}, \mathbf{P}\Ł_1) \text{ and } G = R^{\mathrm{m}}[u] \cap v{\downarrow}.$$

As in the proof of Proposition 4.4, we can separate these sets by a clopen downset $\Omega \supseteq G$ with $\Omega \cap F = \varnothing$. Thus, from $a \in \mathbf{A}$ with $v(a) = \frac{1}{n}$ (which exists because $\mathsf{im}(v) = \mathbf{P}\Ł_n$), we can construct a witness for $u(\Box a') = v(a) = \frac{1}{n}$, which yields $u(\Box(a' \oplus a')) > u(\Box a')$, contradicting the initial assumption that $\Box(a' \oplus a') \leq \Box a'$ holds in $\mathbf{A}$. □

Together with Proposition 5.1 and the results of Section 4, this implies that the consequence pair $\Diamond(p \oplus p) \vdash \Diamond p$ is not derivable in the corresponding axiomatic extension. While the semantics over $\mathsf{Pos}_n$ reflect this situation adequately, the semantics restricted to $\mathsf{Set}$, $\mathsf{Set}_n$ or $\mathsf{Pos}$ all fail to do so.

## 6 Conclusion and further research

We introduced the positive fragment of $\Ł_n$-valued modal logic and studied it via its algebraic counterpart, modal $\mathsf{PMV}_n$-algebras. We introduced relational semantics on $\mathsf{Pos}_n$-frames for this logic and showed that canonical frames of modal $\mathsf{PMV}_n$-algebras are $\mathsf{Pos}_n$-frames. In an example, we illustrated how these richer semantics are 'well-behaved' with respect to canonicity. In the following, we collect some potential directions for future research along similar lines.

(1) Establish a 'Jónsson-Tarski-style' *topological duality* for the variety $\mathsf{mPMV}_n$ similar to the topological duality for $\mathsf{mDL}$ from [6] (which should correspond to the special case $n = 1$). Similarly to how the duality for $\mathsf{mDL}$ may be seen as extending Priestley duality, a duality for the variety $\mathsf{mPMV}_n$ may be be obtained as extension of the natural duality for $\mathsf{PMV}_n$ from [21].

(2) Investigate *canonicity and frame-definability* over $\mathsf{Pos}_n$-frames more rigorously, for example by providing an analogue of the Sahlqvist Theorem or the Goldblatt-Thomason Theorem. This will likely be similar to what is found in [6] for classical positive modal logic and in [13,23] for $\Ł_n$-valued modal logic.

(3) Study positive modal logic over $\mathbf{P}\Ł_n$ *coalgebraically*, similarly to [3,9] for classical positive modal logic. Similarly to [16], one could also attempt

to study the relationship between classical positive modal logic and $\mathbf{P}\Ł_n$-valued positive modal logic coalgebraically.

(4) Find axiomatizations and 'richer' semantics for broader classes of positive many-valued modal logics. For example, consider positive versions of some of the common infinitely-valued modal *fuzzy* logics [13,22,24]. Furthermore, the author agrees with one of the anonymous referees, who deemed it interesting to which extent the results of this paper can be extended to arbitrary finite lattice-based algebras of truth degrees (for which natural dualities always exist by [8, Corollary 3.3.9]).

In general, we hope this paper inspires future discussions of relational semantics *beyond* Set, in particular in many-valued modal logic where it still is customary to exclusively consider Set-based semantics (see, *e.g.*, [11,4,22]).

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