Natural Deduction, Normalization and Subformula Property for Kreisel-Putnam Logic

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Abstract

We introduce a new natural deduction system for Kreisel-Putnam logic. Our system is based on Schroeder-Heister's *calculus of higher-level rules*, an extension of ordinary natural deduction in which not only *formulas*, but also *rules* can act as assumptions that may be discharged in the course of a derivation. We will establish a *normalization theorem*, i.e., we will show that every deduction in our system can be converted into a deduction without 'detours'. The method used for this is new and might also be applied to other non-classical logics. As a consequence of this result, we will obtain an unrestricted subformula property and a separation theorem for our system.

Keywords: Kreisel-Putnam logic, intermediate logics, natural deduction, proof theory, normalization, subformula property, calculus of higher-level rules.

1 Introduction

The intermediate logic KP, known as *Kreisel-Putnam logic*, can be defined as the result of adding the scheme $(\neg\varphi \to (\psi \vee \chi)) \to ((\neg\varphi \to \psi) \vee (\neg\varphi \to \chi))$ to intuitionistic propositional logic (IPC). It was introduced in 1957 by Kreisel and Putnam [9] in order to disprove a conjecture by Łukasiewicz [11] according to which IPC is the only intermediate logic having the disjunction property.

In this paper, we provide a new natural deduction system for KP. Our system is based on the so-called *calculus of higher-level rules*, a generalized natural deduction formalism introduced by Schroeder-Heister [18,19,20], in which not only *formulas*, but also *rules* can be used as assumptions that may be discharged in the course of constructing a proof tree. We present an innovative normalization procedure which allows to convert every deduction in our system into a *normal deduction*, i.e., a deduction containing no 'detours'. The underlying method might also be used to obtain normalization results for other non-classical logics. [2] We will see that our system is fully *analytic*, in the sense that normal deductions always satisfy an unrestricted subformula property.

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[2] We already succeeded in finding similar natural deduction systems for propositional inquisitive logic and for a small number of intermediate logics different from KP (cf. [13]).

Over the years, proof theorists have developed various methods which allow to construct analytic proof systems for non-classical logics in a more or less uniform way [1,3,6,10,15]. The system KP occupies a somewhat special position here, as most of these methods do not seem to work when applied to it. A labelled sequent calculus for KP is presented in [7, Sect. 11.2], but this calculus internalizes the semantic characterization of KP given in [2, p. 55], so it is not purely syntactic in a strict sense. Simple natural deduction systems for KP and related logics are provided in [4,16], along with a computational interpretation in the style of Curry [5] and Howard [8]. These calculi, however, do not seem to satisfy a full subformula property (for a closely related system, this was pointed out in [14, pp. 35–36]). As far as we know, it is still an open question whether KP admits an analytic hypersequent calculus or nested sequent calculus.

The paper is structured as follows. In Section 2, we first recall some basic notions from the literature. In Sections 3–4, we introduce our natural deduction system for KP and demonstrate its completeness in a syntactic way. The proof of our normalization result is presented in Sections 5–7. In Section 8, we establish the subformula property and a separation theorem for our system.

2 Basic Notions

We start by defining some basic notions from the theory of superintuitionistic logics. For further details, we refer to [2, Sect. 4.1]. Throughout this paper, we assume a fixed set P of propositional atoms, denoted by the meta-variables p, q, r, etc. The *language of intuitionistic logic*, notation $\mathcal{L}_\mathsf{I}$, is defined as follows:

$$\varphi ::= p \mid \bot \mid \varphi \wedge \varphi \mid \varphi \vee \varphi \mid \varphi \to \varphi \qquad (p \in \mathsf{P}).$$

As usual, we write $\neg\varphi$ as an abbreviation for $\varphi \to \bot$. A formula $\varphi \in \mathcal{L}_\mathsf{I}$ is said to be a *prime formula*, if it is either an atom from the set P, or it is of the form $\varphi = \bot$. And φ is called *disjunction-free*, if it does not contain any occurrences of the symbol $\vee$. We also write IPC for the set of all formulas valid in *intuitionistic logic* and CPC for the set of all formulas valid in *classical logic*. By a *substitution*, we will mean a function $s : \mathsf{P} \to \mathcal{L}_\mathsf{I}$ which assigns, to each atom $p \in \mathsf{P}$, a formula $s(p) \in \mathcal{L}_\mathsf{I}$. Such a function s is extended to arbitrary formulas by putting $s(\bot) := \bot$ as well as $s(\varphi \odot \psi) := s(\varphi) \odot s(\psi)$ for all $\odot \in \{\wedge, \vee, \to\}$.

Definition 2.1 [Superintuitionistic Logic] A set of formulas $L \subseteq \mathcal{L}_\mathsf{I}$ is said to be a *superintuitionistic logic* (or an *si-logic* for short), if the following holds:

- $\mathsf{IPC} \subseteq L$,
- L is closed under modus ponens: if $\varphi \in L$ and $(\varphi \to \psi) \in L$, then $\psi \in L$,
- L is closed under substitution: for any substitution s, if $\varphi \in L$, then $s(\varphi) \in L$.

An si-logic L is *consistent*, if it satisfies the condition $\bot \notin L$. It is not difficult to show that every consistent si-logic L is *intermediate* between intuitionistic logic and classical logic, in the sense that $\mathsf{IPC} \subseteq L \subseteq \mathsf{CPC}$. A consistent si-logic is therefore also referred to as an *intermediate logic* [2, p. 109].

Clearly, for any set of formulas $\Gamma \subseteq \mathcal{L}_\mathsf{I}$, one can construct an si-logic by adding the formulas from Γ to IPC and by closing the resulting set of formu-

(A1) $\varphi \to (\psi \to \varphi)$	(A2) $(\varphi \to (\psi \to \chi)) \to ((\varphi \to \psi) \to (\varphi \to \chi))$
(A3) $(\varphi \wedge \psi) \to \varphi$, $(\varphi \wedge \psi) \to \psi$	(A4) $\varphi \to (\psi \to (\varphi \wedge \psi))$
(A5) $\varphi \to (\varphi \vee \psi)$, $\psi \to (\varphi \vee \psi)$	(A6) $(\varphi \to \chi) \to ((\psi \to \chi) \to ((\varphi \vee \psi) \to \chi))$
(A7) $\bot \to \varphi$	(KP) $(\neg\varphi \to (\psi \vee \chi)) \to ((\neg\varphi \to \psi) \vee (\neg\varphi \to \chi))$

Modus ponens: from $\Gamma \vdash_{\mathrm{H}} \varphi$ and $\Delta \vdash_{\mathrm{H}} \varphi \to \psi$, infer $\Gamma, \Delta \vdash_{\mathrm{H}} \psi$.

Fig. 1. The Hilbert-style system HKP.

las under modus ponens and substitution. The si-logic thus obtained will be denoted by $\mathsf{IPC} + \Gamma$. If Γ is of the form $\Gamma = \{\varphi_1, \ldots, \varphi_n\}$, then we also write $\mathsf{IPC} + \varphi_1 + \ldots + \varphi_n$ instead of $\mathsf{IPC} + \Gamma$. The intermediate logic KP, known as *Kreisel-Putnam logic*, was first considered in [9] and can be defined as follows:

$$\mathsf{KP} := \mathsf{IPC} + (\neg p \to (q \vee r)) \to ((\neg p \to q) \vee (\neg p \to r)).$$

The system KP has some historical significance, because it was the first counterexample to a conjecture by Łukasiewicz [11] according to which IPC is the only intermediate logic having the disjunction property. This conjecture was refuted by Kreisel and Putnam [9] who proved that the disjunction property also holds for KP: if we have $(\varphi \vee \psi) \in \mathsf{KP}$, then also $\varphi \in \mathsf{KP}$ or $\psi \in \mathsf{KP}$.

A sound and complete Hilbert-style system for KP is presented in Figure 1. The provability relation of this system is denoted by $\vdash_{\mathrm{H}}$ and defined in the usual way. That is, we write $\Gamma \vdash_{\mathrm{H}} \varphi$ and say that φ is *provable* from Γ in HKP, if one of the following holds: (1) φ is an element of Γ, or (2) φ is an axiom of HKP, or (3) there are subsets $\Delta, \Sigma \subseteq \Gamma$ and a formula ψ such that $\Gamma = \Delta \cup \Sigma$ as well as $\Delta \vdash_{\mathrm{H}} \psi$ and $\Sigma \vdash_{\mathrm{H}} \psi \to \varphi$. Using essentially the same argument as in [14, Theorem 1.5.4], it is easy to prove that HKP satisfies the *deduction theorem*.

Theorem 2.2 (Deduction Theorem) *We have $\Gamma, \varphi \vdash_{\mathrm{H}} \psi$ iff $\Gamma \vdash_{\mathrm{H}} \varphi \to \psi$.*

For later purposes, we also need to show that every negated formula of our language is intuitionistically equivalent to a disjunction-free formula. We first assign, to each $\varphi \in \mathcal{L}_{\mathsf{I}}$, a disjunction-free formula φ^{G} in the following way:

- $\bot^{\mathrm{G}} := \bot$ and $p^{\mathrm{G}} := \neg\neg p$ for all atoms $p \in \mathsf{P}$,
- $(\varphi \odot \psi)^{\mathrm{G}} := \varphi^{\mathrm{G}} \odot \psi^{\mathrm{G}}$ for $\odot \in \{\wedge, \to\}$, and $(\varphi \vee \psi)^{\mathrm{G}} := \neg(\neg\varphi^{\mathrm{G}} \wedge \neg\psi^{\mathrm{G}})$.

This translation is also known as the *Gödel-Gentzen negative translation* [21, p. 49]. We will write $\varphi \equiv_{\mathsf{IPC}} \psi$, if φ and ψ are equivalent in IPC, i.e., if we have $(\varphi \to \psi) \in \mathsf{IPC}$ and $(\psi \to \varphi) \in \mathsf{IPC}$. One can now establish the desired result: every negated formula $\neg\varphi$ is IPC-equivalent to the disjunction-free formula $\neg\varphi^{\mathrm{G}}$.

Lemma 2.3 *For any $\varphi \in \mathcal{L}_{\mathsf{I}}$, we have $\neg\varphi \equiv_{\mathsf{IPC}} \neg\varphi^{\mathrm{G}}$.*

Proof. Using induction on φ, it is easy to prove that $\neg\neg\varphi \equiv_{\mathsf{IPC}} \varphi^{\mathrm{G}}$. Moreover, we clearly have $\neg\varphi \equiv_{\mathsf{IPC}} \neg\neg\neg\varphi$. Hence, it follows $\neg\varphi \equiv_{\mathsf{IPC}} \neg\neg\neg\varphi \equiv_{\mathsf{IPC}} \neg\varphi^{\mathrm{G}}$. □

3 Higher-Level Rules and Natural Deduction for KP

We will now introduce an analytic natural deduction system for Kreisel-Putnam logic. As explained above, our system is based on Schroeder-Heister's *calculus* *of higher-level rules*, a generalization of ordinary natural deduction in which not only *formulas*, but also *rules* can serve as assumptions that may be discharged in the course of a derivation [18,19,20]. The basic idea is as follows: each proof tree in our system will be built up from two types of rules, which we will call *primitive rules* and *assumption rules*. The primitive rules are used to formalize the 'meaning' of the logical symbols in our language, so an application of such a rule will never count towards the open assumptions of a derivation. An application of an assumption rule, on the other hand, is treated as a *supposition* (rather than as a valid inference step), and it is considered to be *open* as long as it has not been discharged by another rule application in the proof tree.

First, we need to characterize the class of assumption rules that may occur in a proof tree of our system. An *assumption rule of level* 0 simply amounts to a formula $\varphi \in \mathcal{L}_\mathsf{I}$, so it behaves just like a *hypothesis* in ordinary natural deduction. By an *assumption rule of level* 1, we will mean a rule of the form

$$\frac{\varphi_1 \qquad \varphi_2 \qquad \cdots \qquad \varphi_n}{\psi}$$

where $n \geq 1$. That is, a rule of level 1 simply allows to infer a formula ψ from a given set of premises $\varphi_1, \ldots, \varphi_n$, without discharging any other rules in the proof tree. In what follows, we will also use the linear notation '$\varphi_1, \ldots, \varphi_n \Rightarrow \psi$' as a name for the level-1 rule with premises $\varphi_1, \ldots, \varphi_n$ and conclusion ψ. It will sometimes be convenient to treat a level-0 rule φ (i.e., an ordinary hypothesis) as a level-1 rule with an empty set of premises. For this reason, we also write '$\emptyset \Rightarrow \varphi$' instead of '$\varphi$'. An *assumption rule of level* $\ell > 1$ is a rule of the form

$$u\,\frac{\begin{array}{c}[\Gamma_1]^u\\ \mathcal{D}_1\\ \varphi_1\end{array} \qquad \begin{array}{c}[\Gamma_2]^u\\ \mathcal{D}_2\\ \varphi_2\end{array} \qquad \cdots \qquad \begin{array}{c}[\Gamma_n]^u\\ \mathcal{D}_n\\ \varphi_n\end{array}}{\psi}$$

where $\Delta = \{\Gamma_1 \Rightarrow \varphi_1, \ldots, \Gamma_n \Rightarrow \varphi_n\}$ is a non-empty set of rules whose maximal level is $\ell - 1$.[3] In other words, a rule of this kind allows to infer a formula ψ from a set of premises $\varphi_1, \ldots, \varphi_n$, while discharging all open applications of the rules from Γ_i in the subtree ending with the premise φ_i. A rule of this form will also be denoted by '$(\Gamma_1 \Rightarrow \varphi_1, \ldots, \Gamma_n \Rightarrow \varphi_n) \Rightarrow \psi$'. Throughout this paper, the discharging of rules will be indicated by square brackets and by writing a suitable label (such as u, v, w) to the left of the corresponding inference line.

Definition 3.1 The class of *assumption rules* is defined in the following way:

- Every formula $\varphi \in \mathcal{L}_\mathsf{I}$ is an assumption rule of level 0.
- If $\varphi \in \mathcal{L}_\mathsf{I}$ is a formula and $\Gamma = \{R_1, \ldots, R_n\}$ with $n \geq 1$ is a set of rules whose maximal level is $\ell \geq 0$, then $\Gamma \Rightarrow \varphi$ is an assumption rule of level $\ell + 1$.

Note that, according to this definition, assumption rules are always *non-schematic*, in the sense that they are built up from *specific formulas*, rather than from syntactic meta-variables ranging over arbitrary formulas. Furthermore, we do not assign any order to the expressions occurring on the left-hand side of

[3] That is, each $\Gamma_i \Rightarrow \varphi_i$ is of level $< \ell$ and at least one $\Gamma_i \Rightarrow \varphi_i$ is of level exactly $\ell - 1$.

$$\frac{\varphi \qquad \psi}{\varphi \wedge \psi}\wedge I \qquad \frac{\varphi_1 \wedge \varphi_2}{\varphi_k}\wedge E \quad (k=1,2) \qquad \frac{\varphi_k}{\varphi_1 \vee \varphi_2}\vee I \quad (k=1,2) \qquad \frac{\bot}{\varphi}\bot_{\mathsf{i}}$$

$$u\,\frac{\begin{array}{c}[\varphi]^u\\ \mathcal{D}\\ \psi\end{array}}{\varphi \to \psi}{\to}I \qquad \frac{\varphi \to \psi \qquad \varphi}{\psi}{\to}E \qquad u\,\frac{\begin{array}{c}[\Theta]^u\\ \mathcal{D}_1\\ \varphi \vee \psi\end{array} \qquad \begin{array}{c}[\Theta \Rightarrow \varphi]^u\\ \mathcal{D}_2\\ \chi\end{array} \qquad \begin{array}{c}[\Theta \Rightarrow \psi]^u\\ \mathcal{D}_3\\ \chi\end{array}}{\chi}\vee E$$

Side condition: In the rule $\vee E$, we require Θ to be a finite set of negative rules.

Fig. 2. Primitive rules of the natural deduction system NKP.

the symbol $\Rightarrow$ in an assumption rule. So, for example, $\varphi_1, (\varphi_2, \varphi_3 \Rightarrow \varphi_4) \Rightarrow \varphi_5$ and $(\varphi_3, \varphi_2 \Rightarrow \varphi_4), \varphi_1 \Rightarrow \varphi_5$ will be treated as *identical* assumption rules.

Definition 3.2 The *subrules* of an assumption rule are defined as follows:

- Every assumption rule is a subrule of itself.
- If $R = ((\Gamma_1 \Rightarrow \varphi_1, \ldots, \Gamma_n \Rightarrow \varphi_n) \Rightarrow \psi)$ is a rule of level $\ell \geq 2$, then every subrule of the rules contained in $\Gamma_1 \cup \ldots \cup \Gamma_n$ is also a subrule of R.

In other words, R is a subrule of R', just in case we either have $R = R'$, or R is one of the rules that are discharged by R' or by one of its other subrules.

Definition 3.3 [Prime Rule, Disjunction-Free Rule, Negative Rule] Consider an arbitrary assumption rule $R = (\Gamma \Rightarrow \varphi)$. We will say that:

- R is a *prime rule*, if every formula occurring in $\Gamma \cup \{\varphi\}$ is a prime formula.
- R is *disjunction-free*, if every formula occurring in $\Gamma \cup \{\varphi\}$ is disjunction-free.
- R is *negative*, if R is disjunction-free and of the form $R = (\Gamma \Rightarrow \bot)$ for $\Gamma \neq \emptyset$.

We are now ready to introduce a higher-level natural deduction system for Kreisel-Putnam logic.[4] Our system will be denoted by NKP and comprises the primitive rules depicted in Figure 2. As can be seen, the rules for conjunction, implication and falsum as well as the introduction rule for disjunction are simply the familiar ones from intuitionistic logic. In fact, the only new ingredient of our system is the elimination rule $\vee E$, which should be read as follows: let Θ be a finite set of *negative rules* and suppose that there exists a proof tree $\mathcal{D}_1$ for $\varphi \vee \psi$ such that each of the rules from Θ possibly has an undischarged application in $\mathcal{D}_1$. Furthermore, let $\mathcal{D}_2$ and $\mathcal{D}_3$ be two proof trees for χ such that $\Theta \Rightarrow \varphi$ possibly occurs among the open assumptions of $\mathcal{D}_2$, and $\Theta \Rightarrow \psi$ possibly occurs among the open assumptions of $\mathcal{D}_3$. Then, using the rule $\vee E$, we are allowed to infer χ and to discharge all open applications of the rules from Θ in $\mathcal{D}_1$, all open applications of $\Theta \Rightarrow \varphi$ in $\mathcal{D}_2$, and all open applications of $\Theta \Rightarrow \psi$ in $\mathcal{D}_3$. It is easy to see that the standard elimination rule for $\vee$ can be obtained as a special case of the rule $\vee E$, by taking Θ to be the *empty set*:

[4] A similar proof system for *inquisitive logic* was recently developed by the author in an unpublished manuscript [13]. An earlier version of this system was presented in [14, Ch. 2].

in this case, nothing is discharged in the subtree $\mathcal{D}_1$, and the additional rules $\Theta \Rightarrow \varphi$ and $\Theta \Rightarrow \psi$ simply amount to the level-0 rules φ and ψ, respectively.

Given an arbitrary application of $\vee E$, we will also use the following terminology: the negative rules from the set Θ are said to be the *auxiliary rules* of this application of $\vee E$. And the *side rules* are the two rules of the form $\Theta \Rightarrow \varphi$ and $\Theta \Rightarrow \psi$ which are discharged in the subtrees $\mathcal{D}_2$ and $\mathcal{D}_3$, respectively.

Definition 3.4 [The System NKP] We define NKP to be the higher-level natural deduction system having the primitive rules depicted in Figure 2.

Every proof tree built up from the primitive rules of our system and arbitrary (open or discharged) applications of assumption rules will be referred to as a *quasi-deduction*. By a *deduction* in NKP, we will mean a quasi-deduction in which all undischarged assumption rules are of level 0 (so these assumption rules are just ordinary *hypotheses*). The *derivability relation* of our system will be denoted by $\vdash_{\mathrm{N}}$. Thus, given an arbitrary set of assumption rules Γ, we will write $\Gamma \vdash_{\mathrm{N}} \varphi$ and say that φ is *derivable* from Γ, if there exists a quasi-deduction $\mathcal{D}$ with conclusion φ such that all open assumption rules in $\mathcal{D}$ are contained in Γ. In this case, $\mathcal{D}$ is also said to be a *quasi-deduction for* $\Gamma \vdash_{\mathrm{N}} \varphi$. If $\Gamma \vdash_{\mathrm{N}} \varphi$ holds for a set of *formulas* $\Gamma \subseteq \mathcal{L}_{\mathsf{I}}$, then φ is said to be *provable* from Γ.

Example 3.5 The following proof tree is a deduction in our system:

$$
\begin{prooftree}
\AxiomC{$\neg(p \wedge q) \to (\varphi \vee \psi)$}
\AxiomC{$[p \wedge q]^u$}
\RightLabel{$\wedge E$}
\UnaryInfC{p}
\RightLabel{$[p \Rightarrow \bot]^z$}
\UnaryInfC{$\bot$}
\LeftLabel{u}
\RightLabel{$\to I$}
\UnaryInfC{$\neg(p \wedge q)$}
\RightLabel{$\to E$}
\BinaryInfC{$\varphi \vee \psi$}
\AxiomC{$[\neg p]^w$}
\AxiomC{$[p]^v$}
\RightLabel{$\to E$}
\BinaryInfC{$\bot$}
\LeftLabel{v}
\RightLabel{$[(p \Rightarrow \bot) \Rightarrow \varphi]^z$}
\UnaryInfC{φ}
\LeftLabel{w}
\RightLabel{$\to I$}
\UnaryInfC{$\neg p \to \varphi$}
\RightLabel{$\vee I$}
\UnaryInfC{$(\neg p \to \varphi) \vee (\neg p \to \psi)$}
\AxiomC{$[\neg p]^y$}
\AxiomC{$[p]^x$}
\RightLabel{$\to E$}
\BinaryInfC{$\bot$}
\LeftLabel{x}
\RightLabel{$[(p \Rightarrow \bot) \Rightarrow \psi]^z$}
\UnaryInfC{ψ}
\LeftLabel{y}
\RightLabel{$\to I$}
\UnaryInfC{$\neg p \to \psi$}
\RightLabel{$\vee I$}
\UnaryInfC{$(\neg p \to \varphi) \vee (\neg p \to \psi)$}
\LeftLabel{z}
\RightLabel{$\vee E$}
\TrinaryInfC{$(\neg p \to \varphi) \vee (\neg p \to \psi)$}
\end{prooftree}
$$

In applications of primitive and non-primitive rules, we in general also permit *vacuous discharge of assumption rules*. However, one can show that, in applications of $\vee E$, vacuous discharge is always redundant. More precisely, every quasi-deduction $\mathcal{D}$ in our system can be rewritten in such a way that (1) each auxiliary rule and each side rule of any application of $\vee E$ in $\mathcal{D}$ is in fact discharged by this application, and (2) every proper subrule of an auxiliary rule (resp., side rule) of $\vee E$ in $\mathcal{D}$ is in fact discharged by at least one application of this auxiliary rule (resp., of a side rule). In what follows, we will tacitly assume that quasi-deductions always satisfy the conditions (1) and (2). In most cases, this does not make a big difference and the reader can easily transfer our arguments to the more general setting in which such a convention is not adopted.

4 Soundness and Completeness

We will now prove that our system is sound and complete with respect to the Hilbert-style system HKP (see Figure 1). First, we establish the completeness of our system: everything which is provable in HKP is also provable in NKP.

Lemma 4.1 *Let $\Gamma \subseteq \mathcal{L}_{\mathsf{I}}$ be a set of formulas. If $\Gamma \vdash_{\mathrm{H}} \varphi$ holds in the Hilbert-style system* HKP*, then $\Gamma \vdash_{\mathrm{N}} \varphi$ holds in the natural deduction system* NKP*.*

Proof. By induction on the structure of a Hilbert-style proof for $\Gamma \vdash_{\mathrm{H}} \varphi$. It suffices to show that all axioms of HKP are provable in NKP. For the schemes `(A1)`–`(A7)`, this is straightforward. In order to show the provability of `(KP)`, let φ be an arbitrary formula. By Lemma 2.3, we have $\neg\varphi \equiv_{\mathsf{IPC}} \neg\varphi^{\mathrm{G}}$, so there must be a deduction $\mathcal{D}_1$ for $\neg\varphi \vdash_{\mathrm{N}} \neg\varphi^{\mathrm{G}}$ and a deduction $\mathcal{D}_2$ for $\neg\varphi^{\mathrm{G}} \vdash_{\mathrm{N}} \neg\varphi$ in our system. Using these proof trees, we now construct the following deduction:

$$
\begin{prooftree}
\AxiomC{$[\neg\varphi \to (\psi \vee \chi)]^a$}
\AxiomC{$[\varphi^{\mathrm{G}}]^u$}
\RightLabel{$[\varphi^{\mathrm{G}} \Rightarrow \bot]^z$}
\UnaryInfC{$\bot$}
\LeftLabel{u}
\RightLabel{$\to I$}
\UnaryInfC{$\neg\varphi^{\mathrm{G}}$}
\noLine
\UnaryInfC{$\mathcal{D}_2$}
\noLine
\UnaryInfC{$\neg\varphi$}
\RightLabel{$\to E$}
\BinaryInfC{$\psi \vee \chi$}
\AxiomC{$[\neg\varphi]^w$}
\noLine
\UnaryInfC{$\mathcal{D}_1$}
\noLine
\UnaryInfC{$\neg\varphi^{\mathrm{G}}$}
\AxiomC{$[\varphi^{\mathrm{G}}]^v$}
\RightLabel{$\to E$}
\BinaryInfC{$\bot$}
\LeftLabel{v}
\RightLabel{$[(\varphi^{\mathrm{G}} \Rightarrow \bot) \Rightarrow \psi]^z$}
\UnaryInfC{ψ}
\LeftLabel{w}
\RightLabel{$\to I$}
\UnaryInfC{$\neg\varphi \to \psi$}
\RightLabel{$\vee I$}
\UnaryInfC{$(\neg\varphi \to \psi) \vee (\neg\varphi \to \chi)$}
\AxiomC{$[\neg\varphi]^y$}
\noLine
\UnaryInfC{$\mathcal{D}_1$}
\noLine
\UnaryInfC{$\neg\varphi^{\mathrm{G}}$}
\AxiomC{$[\varphi^{\mathrm{G}}]^x$}
\RightLabel{$\to E$}
\BinaryInfC{$\bot$}
\LeftLabel{x}
\RightLabel{$[(\varphi^{\mathrm{G}} \Rightarrow \bot) \Rightarrow \chi]^z$}
\UnaryInfC{χ}
\LeftLabel{y}
\RightLabel{$\to I$}
\UnaryInfC{$\neg\varphi \to \chi$}
\RightLabel{$\vee I$}
\UnaryInfC{$(\neg\varphi \to \psi) \vee (\neg\varphi \to \chi)$}
\LeftLabel{z}
\RightLabel{$\vee E$}
\TrinaryInfC{$(\neg\varphi \to \psi) \vee (\neg\varphi \to \chi)$}
\LeftLabel{a}
\RightLabel{$\to I$}
\UnaryInfC{$(\neg\varphi \to (\psi \vee \chi)) \to ((\neg\varphi \to \psi) \vee (\neg\varphi \to \chi))$}
\end{prooftree}
$$

Note that, since φ^{G} is disjunction-free, $\varphi^{\mathrm{G}} \Rightarrow \bot$ is a *negative rule* (see Definition 3.3), so the application of $\vee E$ in this deduction is in fact correct. □

In order to establish the soundness of our system, we first assign, to each assumption rule R, a corresponding formula $F(R)$ in the following way:

- If $R = \varphi$ is of level 0, then we put $F(R) := \varphi$.
- If $R = (R_1, \ldots, R_n \Rightarrow \varphi)$ is of level $\ell \geq 1$, then $F(R) := (\bigwedge_{i=1}^{n} F(R_i)) \to \varphi$.

For any set of rules Γ, we also put $F(\Gamma) := \{F(R) \mid R \in \Gamma\}$. It is now easy to show that, if φ is derivable from a set of assumption rules Γ in NKP, then φ is provable from the set of formulas $F(\Gamma)$ in the Hilbert-style system HKP.

Lemma 4.2 *Let Γ be a set of assumption rules. If $\Gamma \vdash_{\mathrm{N}} \varphi$ holds in the natural deduction system* NKP*, then $F(\Gamma) \vdash_{\mathrm{H}} \varphi$ holds in the Hilbert-style system* HKP*.*

The proof proceeds by induction on the structure of a quasi-deduction for $\Gamma \vdash_{\mathrm{N}} \varphi$ in our system. A few representative cases are considered in Appendix A.

Theorem 4.3 (Soundness and Completeness) *Let Γ be a set of formulas. We have $\Gamma \vdash_{\mathrm{N}} \varphi$ in the system* NKP *iff $\Gamma \vdash_{\mathrm{H}} \varphi$ holds in the system* HKP*.*

Proof. The statement follows directly from Lemma 4.1 and Lemma 4.2. □

5 Detours in Deductions

In this and the next two sections, we will establish a *normalization theorem* for our system, so we will show that every deduction in NKP can be converted into a *normal deduction*. Intuitively, a deduction is said to be *normal*, if it does not contain any 'detours'. In order to make this precise, we first need to recall some standard terminology. Given an arbitrary application of a rule of inference, we will call the formula occurrences directly above the line the *premises* and the formula occurrence directly below the line the *conclusion* of the application. In applications of *elimination rules*, we will distinguish between two types of premises: the *major premise* is the premise which contains the 'eliminated' occurrence of the connective; all other premises (if any) are said to be *minor premises*. So, for example, the major premise of an application of $\to E$ is the premise of the form $\varphi \to \psi$, and the minor premise is the one of the form φ. The major premise of an application of $\vee E$, on the other hand, is the premise of the form $\varphi \vee \psi$, and the minor premises are the two premises of the form χ.

A detour in a deduction may now be conceived as a formula that is both the conclusion of an introduction rule and the major premise of an elimination rule. However, deductions may also contain *repetitions* of formulas, since the minor premises and the conclusion of an application of $\vee E$ are always of the same shape. Thus, rather than considering only single occurrences of formulas, we must consider sequences of consecutive occurrences of the same formula. Such a sequence is also referred to as a *segment*. The intuitive notion of a detour is now formalized by the concept of a *cut segment*, which is defined as follows.

Definition 5.1 [Cut Segment] A *cut segment of length* n in a proof tree $\mathcal{D}$ is a sequence $\xi_1, \ldots, \xi_n$ of n consecutive occurrences of a formula φ in $\mathcal{D}$ such that:

- ξ_1 is the conclusion of an introduction rule,
- each ξ_i with $1 \leq i < n$ is a minor premise of $\vee E$,
- ξ_n is the major premise of an elimination rule.

Example 5.2 The following deduction contains a cut segment of length 2:

$$\dfrac{\dfrac{\begin{array}{c}\vdots\\ \chi_1 \vee \chi_2\end{array} \qquad \dfrac{\begin{array}{cc}\mathcal{D}_1 & \mathcal{D}_2\\ \varphi & \psi\end{array}}{\varphi \wedge \psi}\wedge I \qquad \begin{array}{c}\vdots\\ \varphi \wedge \psi\end{array}}{\varphi \wedge \psi}\vee E}{\varphi}\wedge E$$

The unique formula occurring in a cut segment is said to be the *cut formula* of the segment. Our main goal is to show that cut segments can always be eliminated. However, this alone does not suffice to obtain a strong subformula property for our system, since there are also other types of detours that need to be considered. One problem arises from the intuitionistic absurdity rule $\bot_{\mathsf{i}}$, because this rule, too, can 'introduce' a formula which might then serve as the major premise of an elimination rule. This issue can easily be resolved by restricting $\bot_{\mathsf{i}}$ to instances in which the conclusion is atomic. Another, more severe, problem is related to the assumption rules involved in an application of $\vee E$. In order to understand this problem, consider the following deduction:

$$z\,\dfrac{\dfrac{\neg p \to (r_1 \vee r_2) \qquad u\,\dfrac{\dfrac{\dfrac{[p]^u \quad q}{p \wedge q}\wedge I}{\bot}[p \wedge q \Rightarrow \bot]^z}{\neg p}{\to}I}{r_1 \vee r_2}{\to}E \qquad \dfrac{w\,\dfrac{v\,\dfrac{\dfrac{[\neg p]^w \quad \dfrac{[p \wedge q]^v}{p}\wedge E}{\bot}{\to}E}{r_1}[(p \wedge q \Rightarrow \bot) \Rightarrow r_1]^z}{\neg p \to r_1}{\to}I}{(\neg p \to r_1) \vee (\neg p \to r_2)}\vee I \qquad \dfrac{y\,\dfrac{x\,\dfrac{\dfrac{[\neg p]^y \quad \dfrac{[p \wedge q]^x}{p}\wedge E}{\bot}{\to}E}{r_2}[(p \wedge q \Rightarrow \bot) \Rightarrow r_2]^z}{\neg p \to r_2}{\to}I}{(\neg p \to r_1) \vee (\neg p \to r_2)}\vee I}{(\neg p \to r_1) \vee (\neg p \to r_2)}\vee E$$

This deduction contains the formula $p \wedge q$ (acting as the premise of the auxiliary rule $p \wedge q \Rightarrow \bot$), but this formula is neither a subformula of an open hypothesis nor a subformula of the conclusion of the deduction. Note that, in a sense, $p \wedge q$ behaves just like a *cut formula* here: it is the conclusion of an introduction rule in the subtree on the left-hand side, and it is the major premise of an elimination rule in the two subtrees on the right-hand side. In order to avoid detours of this kind, we must not only eliminate cut segments, but we must also show that auxiliary rules can be restricted to *prime rules* (see Definition 3.3).

Definition 5.3 [Pure Quasi-Deduction] A quasi-deduction $\mathcal{D}$ is *pure*, if it satisfies the following two conditions: (1) all auxiliary rules of applications of $\vee E$ in $\mathcal{D}$ are prime rules, and (2) every conclusion of $\bot_{\mathsf{i}}$ in $\mathcal{D}$ is an atomic formula.

Definition 5.4 [Normal Quasi-Deduction] We will say that a quasi-deduction is *normal*, if it is a pure quasi-deduction and it also contains no cut segments.

Our normalization proof consists of two main steps: in Section 6, we will first see that every deduction can be turned into a *pure deduction*. In Section 7, we will then see that every pure deduction can be turned into a *normal deduction*.

6 Purification of Deductions

Let us start by showing that every deduction can be transformed into a pure deduction. We first note that, in every quasi-deduction of our system, $\bot_{\mathsf{i}}$ can be restricted to instances in which the conclusion is an atomic formula. The proof works in essentially the same way as in intuitionistic logic (see, e.g., [12, Proposition 4.2]). Thus, it only remains to show that auxiliary rules of $\vee E$ can be restricted to prime rules. To this end, we first assign, to each disjunction-free rule $R = (\Gamma \Rightarrow \varphi)$, a finite set of prime rules $\mathsf{pr}(R)$ in the following way:

- if φ is prime, then $\mathsf{pr}(\Gamma \Rightarrow \varphi) := \{\mathsf{pr}(\Gamma) \Rightarrow \varphi\}$ for $\mathsf{pr}(\Gamma) := \bigcup_{R' \in \Gamma} \mathsf{pr}(R')$, [5]
- $\mathsf{pr}(\Gamma \Rightarrow \psi \wedge \chi) := \mathsf{pr}(\Gamma \Rightarrow \psi) \cup \mathsf{pr}(\Gamma \Rightarrow \chi)$,
- $\mathsf{pr}(\Gamma \Rightarrow \psi \to \chi) := \mathsf{pr}(\Gamma, \psi \Rightarrow \chi)$.

Given any set of disjunction-free rules Γ, we also define $\mathsf{pr}(\Gamma) := \bigcup_{R \in \Gamma} \mathsf{pr}(R)$. Note that, if Γ is finite, then $\mathsf{pr}(\Gamma)$ is finite as well. The basic idea is to show that, if a formula φ is derivable using a disjunction-free rule R as an assumption, then it is also derivable using the prime rules from $\mathsf{pr}(R)$ as assumptions.

Definition 6.1 The *rank* of a formula φ, notation $rk(\varphi)$, is defined as follows:

- $rk(\bot) := 1$ and $rk(p) := 1$ for $p \in \mathsf{P}$,
- $rk(\psi \odot \chi) := rk(\psi) + rk(\chi) + 1$ for $\odot \in \{\wedge, \vee, \to\}$.

In other words, $rk(\varphi)$ simply amounts to the *length* of the string φ, provided that parentheses are ignored. By the *degree of a rule* $R = (\Gamma \Rightarrow \varphi)$, notation $dg(R)$, we will mean the sum of the ranks of all formulas occurring in $\Gamma \cup \{\varphi\}$. For any set of rules Γ, we also write $\Gamma \vdash_{\mathrm{P}} \varphi$, if φ is derivable from Γ by a *pure* quasi-deduction. We are now ready to prove the desired result: if a disjunction-

[5] If $\Gamma = \emptyset$, then we put $\mathsf{pr}(\Gamma) := \emptyset$. So, in this case, we have $\mathsf{pr}(\Gamma \Rightarrow \varphi) = \{\emptyset \Rightarrow \varphi\} = \{\varphi\}$.

free rule R occurs as (a component of) an open assumption in a quasi-deduction, then we can always replace R be the corresponding set of prime rules $\mathsf{pr}(R)$.

Lemma 6.2 *Let Γ and Δ be arbitrary sets of assumption rules and suppose that Δ is finite. Furthermore, let R be a disjunction-free rule. It holds:*

(i) *if $\Gamma, R \vdash_{\mathrm{P}} \varphi$, then $\Gamma, \mathsf{pr}(R) \vdash_{\mathrm{P}} \varphi$,*

(ii) *if $\Gamma, (\Delta, R \Rightarrow \psi) \vdash_{\mathrm{P}} \varphi$, then $\Gamma, (\Delta, \mathsf{pr}(R) \Rightarrow \psi) \vdash_{\mathrm{P}} \varphi$.*

Proof. Both statements are proved simultaneously, by induction on the degree of R. The base case is trivial. For the inductive step, let $dg(R) \geq 2$ and suppose that both of the claims hold for all disjunction-free rules of degree smaller than $dg(R)$. Without loss of generality, we may assume that $R = (\Pi \Rightarrow \chi)$, where χ is a disjunction-free formula and $\Pi = \{\Sigma_1 \Rightarrow \varphi_1, \ldots, \Sigma_n \Rightarrow \varphi_n\}$ is a possibly empty set of disjunction-free rules. There are the following possibilities.

Case 1: χ is a prime formula. In this case, one can easily derive (i) from the induction hypothesis for (ii), and (ii) from the induction hypothesis for (i).

Case 2: χ is not prime. In order to prove part (i), let $\mathcal{D}$ be a pure quasi-deduction for $\Gamma, R \vdash_{\mathrm{P}} \varphi$. We select a topmost open application of $R = (\Pi \Rightarrow \chi)$ in $\mathcal{D}$ and rewrite this application using one of the following two conversions:

$$\chi = \alpha \wedge \beta : \quad \begin{array}{c} [\Sigma_1]^u \qquad [\Sigma_n]^u \\ \mathcal{D}_1 \qquad \mathcal{D}_n \\ u\,\dfrac{\varphi_1 \quad \cdots \quad \varphi_n}{\alpha \wedge \beta}\,\Pi \Rightarrow \alpha \wedge \beta \\ \mathcal{D}_0 \end{array} \quad \rightsquigarrow \quad \begin{array}{c} \dfrac{\begin{array}{cc} \begin{array}{c} [\Sigma_1]^u \qquad [\Sigma_n]^u \\ \mathcal{D}_1 \qquad \mathcal{D}_n \\ u\,\dfrac{\varphi_1 \quad \cdots \quad \varphi_n}{\alpha}\,\Pi \Rightarrow \alpha \end{array} & \begin{array}{c} [\Sigma_1]^v \qquad [\Sigma_n]^v \\ \mathcal{D}_1 \qquad \mathcal{D}_n \\ v\,\dfrac{\varphi_1 \quad \cdots \quad \varphi_n}{\beta}\,\Pi \Rightarrow \beta \end{array} \end{array}}{\alpha \wedge \beta}\,\wedge I \\ \mathcal{D}_0 \end{array}$$

$$\chi = \alpha \to \beta : \quad \begin{array}{c} [\Sigma_1]^u \qquad [\Sigma_n]^u \\ \mathcal{D}_1 \qquad \mathcal{D}_n \\ u\,\dfrac{\varphi_1 \quad \cdots \quad \varphi_n}{\alpha \to \beta}\,\Pi \Rightarrow \alpha \to \beta \\ \mathcal{D}_0 \end{array} \quad \rightsquigarrow \quad \begin{array}{c} [\Sigma_1]^u \qquad [\Sigma_n]^u \\ \mathcal{D}_1 \qquad \mathcal{D}_n \\ u\,\dfrac{\varphi_1 \quad \cdots \quad \varphi_n \quad [\alpha]^v}{\beta}\,\Pi, \alpha \Rightarrow \beta \\ v\,\dfrac{}{\alpha \to \beta}\,{\to} I \\ \mathcal{D}_0 \end{array}$$

We only consider the case in which χ is of the form $\chi = \alpha \to \beta$, so we assume that the second of these conversions has been performed. As can be seen, the selected application of R is now replaced by an open application of $\Pi, \alpha \Rightarrow \beta$. Thus, by repeating the procedure for all other open applications of R, we finally obtain a pure quasi-deduction for $\Gamma, (\Pi, \alpha \Rightarrow \beta) \vdash_{\mathrm{P}} \varphi$. Now, observe that we have $dg(\Pi, \alpha \Rightarrow \beta) < dg(R)$. Hence, by the induction hypothesis for (i), we may conclude $\Gamma, \mathsf{pr}(\Pi, \alpha \Rightarrow \beta) \vdash_{\mathrm{P}} \varphi$ and therefore $\Gamma, \mathsf{pr}(R) \vdash_{\mathrm{P}} \varphi$, as desired.

In order to establish part (ii) of the lemma, let now $\mathcal{D}$ be a pure quasi-deduction for $\Gamma, (\Delta, R \Rightarrow \psi) \vdash_{\mathrm{P}} \varphi$. We select a topmost open application of $\Delta, R \Rightarrow \psi$ in $\mathcal{D}$ and rewrite this application using one of the following conversions (the vertical dots represent a collection of subtrees corresponding to the rules in Δ):

$$\chi = \alpha \wedge \beta : \qquad u\,\dfrac{\vdots\,\Delta \qquad \begin{array}{c}[\Pi]^u\\ \mathcal{D}_1\\ \alpha \wedge \beta\end{array}}{\begin{array}{c}\psi\\ \mathcal{D}_2\end{array}}\,\Delta, R \Rightarrow \psi \quad \rightsquigarrow \quad u\,\dfrac{\vdots\,\Delta \qquad \dfrac{\begin{array}{c}[\Pi]^u\\ \mathcal{D}_1\\ \alpha \wedge \beta\end{array}}{\alpha}\wedge E \quad \dfrac{\begin{array}{c}[\Pi]^u\\ \mathcal{D}_1\\ \alpha \wedge \beta\end{array}}{\beta}\wedge E}{\begin{array}{c}\psi\\ \mathcal{D}_2\end{array}}\,\Delta, (\Pi \Rightarrow \alpha), (\Pi \Rightarrow \beta) \Rightarrow \psi$$

$$\chi = \alpha \to \beta : \qquad u\,\dfrac{\vdots\,\Delta \qquad \begin{array}{c}[\Pi]^u\\ \mathcal{D}_1\\ \alpha \to \beta\end{array}}{\begin{array}{c}\psi\\ \mathcal{D}_2\end{array}}\,\Delta, R \Rightarrow \psi \quad \rightsquigarrow \quad u\,\dfrac{\vdots\,\Delta \qquad \dfrac{\begin{array}{c}[\Pi]^u\\ \mathcal{D}_1\\ \alpha \to \beta\end{array} \quad [\alpha]^u}{\beta}\to E}{\begin{array}{c}\psi\\ \mathcal{D}_2\end{array}}\,\Delta, (\Pi, \alpha \Rightarrow \beta) \Rightarrow \psi$$

Again, let us suppose that $\chi = \alpha \to \beta$, so the selected application of $\Delta, R \Rightarrow \psi$ is now replaced by an open application of $\Delta, R' \Rightarrow \psi$, where $R' := (\Pi, \alpha \Rightarrow \beta)$. By repeating the procedure, we thus obtain $\Gamma, (\Delta, R' \Rightarrow \psi) \vdash_{\mathrm{P}} \varphi$. Since we have $dg(R') < dg(R)$ and $\mathsf{pr}(R') = \mathsf{pr}(R)$, this implies $\Gamma, (\Delta, \mathsf{pr}(R') \Rightarrow \psi) \vdash_{\mathrm{P}} \varphi$ and therefore $\Gamma, (\Delta, \mathsf{pr}(R) \Rightarrow \psi) \vdash_{\mathrm{P}} \varphi$ by the induction hypothesis for part (ii). $\Box$

Proposition 6.3 *Let Γ be a set of formulas. If there exists a deduction for $\Gamma \vdash_{\mathrm{N}} \varphi$ in our system, then there also exists a pure deduction for $\Gamma \vdash_{\mathrm{N}} \varphi$.*

Proof. Suppose that there exists a deduction $\mathcal{D}$ for $\Gamma \vdash_{\mathrm{N}} \varphi$. Without loss of generality, we may assume that every conclusion of $\bot_{\mathsf{i}}$ in $\mathcal{D}$ is atomic. We now select an uppermost application of $\vee E$ in $\mathcal{D}$ such that not every auxiliary rule of this application is prime, and we then rewrite this application as follows:

$$u\,\dfrac{\begin{array}{c}[\Theta]^u\\ \mathcal{D}_3\\ \psi_1 \vee \psi_2\end{array} \quad \begin{array}{c}[\Theta \Rightarrow \psi_1]^u\\ \mathcal{D}_1\\ \chi\end{array} \quad \begin{array}{c}[\Theta \Rightarrow \psi_2]^u\\ \mathcal{D}_2\\ \chi\end{array}}{\begin{array}{c}\chi\\ \mathcal{D}_4\end{array}}\vee E \quad \rightsquigarrow \quad u\,\dfrac{\begin{array}{c}[\mathsf{pr}(\Theta)]^u\\ \mathcal{D}_3'\\ \psi_1 \vee \psi_2\end{array} \quad \begin{array}{c}[\mathsf{pr}(\Theta) \Rightarrow \psi_1]^u\\ \mathcal{D}_1'\\ \chi\end{array} \quad \begin{array}{c}[\mathsf{pr}(\Theta) \Rightarrow \psi_2]^u\\ \mathcal{D}_2'\\ \chi\end{array}}{\begin{array}{c}\chi\\ \mathcal{D}_4\end{array}}\vee E$$

where each of the proof trees $\mathcal{D}_i'$ with $i = 1, 2, 3$ is obtained from $\mathcal{D}_i$ by applying Lemma 6.2 to the disjunction-free rules in Θ. Observe that $\mathsf{pr}(\Theta)$ is still a finite set of *negative* rules, so the new application of $\vee E$ is in fact correct. Therefore, by repeating the procedure, we finally obtain a pure deduction for $\Gamma \vdash_{\mathrm{N}} \varphi$. $\Box$

7 Normalization

Next, we will show that every pure deduction can be transformed into a normal deduction. For technical reasons, we first note the following fact: if $\mathcal{D}_1$ is a quasi-deduction containing some open occurrences of a rule $\Theta \Rightarrow \varphi$, and if $\mathcal{D}_2$ is another quasi-deduction with conclusion φ, then we can always construct a new quasi-deduction $\mathcal{D}_1\langle \mathcal{D}_2 : \Theta \Rightarrow \varphi\rangle$ by 'substituting' $\mathcal{D}_2$ for all undischarged applications of $\Theta \Rightarrow \varphi$ in $\mathcal{D}_1$. Moreover, if Θ contains only *prime rules*, then φ is the only new cut formula that may arise in the course of this construction.

Lemma 7.1 *Let Θ be a finite set of prime rules, let $\mathcal{D}_1$ be a pure quasi-deduction for $\Gamma, (\Theta \Rightarrow \varphi) \vdash_{\mathrm{N}} \psi$ and let $\mathcal{D}_2$ be a pure quasi-deduction for $\Delta, \Theta \vdash_{\mathrm{N}} \varphi$.*

Then there exists a pure quasi-deduction $\mathcal{D}_3$ for $\Gamma, \Delta \vdash_{\mathrm{N}} \psi$ such that every cut formula occurring in $\mathcal{D}_3$, except possibly φ, already occurs as a cut formula in $\mathcal{D}_1$ or $\mathcal{D}_2$. Furthermore, if the conclusion of $\mathcal{D}_3$ belongs to a segment that starts with the conclusion of an introduction rule, then so does the conclusion of $\mathcal{D}_1$, provided that $\varphi \neq \psi$. This quasi-deduction $\mathcal{D}_3$ is denoted by $\mathcal{D}_1\langle \mathcal{D}_2 : \Theta \Rightarrow \varphi\rangle$.

A proof is provided in Appendix B. By the *rank* of a cut segment σ, notation $rk(\sigma)$, we will mean the rank of the cut formula occurring in σ. A cut segment σ in a deduction $\mathcal{D}$ is said to be *maximal*, if it is a cut segment of highest rank in this deduction. And by the *cut rank* of a deduction $\mathcal{D}$, we will mean the pair of natural numbers $\mathrm{CR}(\mathcal{D}) = (m, n)$ defined as follows: if $\mathcal{D}$ contains no cut segments, then we put $\mathrm{CR}(\mathcal{D}) := (0, 0)$. Otherwise, we put $\mathrm{CR}(\mathcal{D}) := (m, n)$, where m is the rank of a maximal cut segment in $\mathcal{D}$, and n is the number of maximal cut segments in $\mathcal{D}$. As usual, we will assume that cut ranks are ordered *lexicographically*. That is, given two deductions $\mathcal{D}$ and $\mathcal{D}'$ with $\mathrm{CR}(\mathcal{D}) = (m, n)$ and $\mathrm{CR}(\mathcal{D}') = (m', n')$, we will write $\mathrm{CR}(\mathcal{D}) < \mathrm{CR}(\mathcal{D}')$ and say that the cut rank of $\mathcal{D}$ is *smaller* than the cut rank of $\mathcal{D}'$, if we either have $m < m'$, or we have both $m = m'$ and $n < n'$. We first show that cut ranks of non-normal deductions can always be decreased: every pure deduction containing at least one cut segment can be transformed into a pure deduction of smaller cut rank.

Lemma 7.2 *Let $\mathcal{D}$ be a pure deduction for $\Gamma \vdash_{\mathrm{N}} \varphi$ such that $\mathrm{CR}(\mathcal{D}) > (0, 0)$. Then there exists a pure deduction $\mathcal{D}'$ for $\Gamma \vdash_{\mathrm{N}} \varphi$ with $\mathrm{CR}(\mathcal{D}') < \mathrm{CR}(\mathcal{D})$.*

Proof. Let $\mathcal{D}$ be a pure deduction for $\Gamma \vdash_{\mathrm{N}} \varphi$ such that $\mathrm{CR}(\mathcal{D}) > (0, 0)$. We select an *uppermost* maximal cut segment σ in $\mathcal{D}$, so σ is a cut segment of highest rank in $\mathcal{D}$, and it holds $rk(\pi) < rk(\sigma)$ for every cut segment π occurring above the conclusion of the elimination rule at the bottom of σ. Let ψ be the cut formula occurring in σ. While σ is of length greater than 1, we first permute the major premise at the end of σ over the minor premises of $\vee E$ in the 'middle part' of σ. This is achieved by applying the following *permutation conversion*, where (E) stands for an instance of an elimination rule with major premise ψ:

$$u\,\dfrac{\dfrac{\begin{matrix}[\Theta]^u\\ \mathcal{D}_3\\ \varphi_1 \vee \varphi_2\end{matrix} \quad \begin{matrix}[\Theta \Rightarrow \varphi_1]^u\\ \mathcal{D}_1\\ \psi\end{matrix} \quad \begin{matrix}[\Theta \Rightarrow \varphi_2]^u\\ \mathcal{D}_2\\ \psi\end{matrix}}{\psi}\,{\vee E} \qquad \mathcal{D}_4}{\begin{matrix}\chi\\ \mathcal{D}_5\end{matrix}}\,(\mathrm{E}) \quad \rightsquigarrow \quad u\,\dfrac{\begin{matrix}[\Theta]^u\\ \mathcal{D}_3\\ \varphi_1 \vee \varphi_2\end{matrix} \quad \dfrac{\begin{matrix}[\Theta \Rightarrow \varphi_1]^u\\ \mathcal{D}_1\\ \psi\end{matrix} \quad \mathcal{D}_4}{\chi}\,(\mathrm{E}) \quad \dfrac{\begin{matrix}[\Theta \Rightarrow \varphi_2]^u\\ \mathcal{D}_2\\ \psi\end{matrix} \quad \mathcal{D}_4}{\chi}\,(\mathrm{E})}{\begin{matrix}\chi\\ \mathcal{D}_5\end{matrix}}\,{\vee E}$$

A small difficulty arises from the fact that (E) itself could be an instance of $\vee E$ and, in this case, it might also discharge some open applications of auxiliary rules occurring in the subtree $\mathcal{D}_3$. Applying the permutation conversion in such a situation would turn these auxiliary rules into *open* assumptions. However, this problem has an easy solution, since it is always possible to 'shift' the corresponding auxiliary rules from the lower application of $\vee E$ to the upper application of $\vee E$. An explanation of the procedure is provided in Appendix C.

Clearly, after a finite number of permutations, we obtain a pure deduction in which σ has been turned into a cut segment of length 1. We now eliminate the cut formula ψ using one of the following *detour conversions* ($i = 1, 2$):

$$\psi = \varphi_1 \wedge \varphi_2 : \qquad \begin{matrix} \dfrac{\dfrac{\begin{matrix}\mathcal{D}_1\\ \varphi_1\end{matrix} \quad \begin{matrix}\mathcal{D}_2\\ \varphi_2\end{matrix}}{\varphi_1 \wedge \varphi_2}\wedge I}{\varphi_i}\wedge E \\ \mathcal{D}_3 \end{matrix} \qquad \rightsquigarrow \qquad \begin{matrix}\mathcal{D}_i\\ \varphi_i\\ \mathcal{D}_3\end{matrix}$$

$$\psi = \varphi_1 \to \varphi_2 : \qquad \begin{matrix} \dfrac{u\,\dfrac{\begin{matrix}[\varphi_1]^u\\ \mathcal{D}_1\\ \varphi_2\end{matrix}}{\varphi_1 \to \varphi_2}{\to}I \quad \begin{matrix}\mathcal{D}_2\\ \varphi_1\end{matrix}}{\varphi_2}{\to}E \\ \mathcal{D}_3 \end{matrix} \qquad \rightsquigarrow \qquad \begin{matrix}\mathcal{D}_2\\ \varphi_1\\ \mathcal{D}_1\\ \varphi_2\\ \mathcal{D}_3\end{matrix}$$

$$\psi = \varphi_1 \vee \varphi_2 : \qquad \begin{matrix} u\,\dfrac{\dfrac{\begin{matrix}[\Theta]^u\\ \mathcal{D}_3\\ \varphi_i\end{matrix}}{\varphi_1 \vee \varphi_2}\vee I \quad \begin{matrix}[\Theta \Rightarrow \varphi_1]^u\\ \mathcal{D}_1\\ \chi\end{matrix} \quad \begin{matrix}[\Theta \Rightarrow \varphi_2]^u\\ \mathcal{D}_2\\ \chi\end{matrix}}{\chi}\vee E \\ \mathcal{D}_4 \end{matrix} \qquad \rightsquigarrow \qquad \begin{matrix}\mathcal{D}_i\langle \mathcal{D}_3 : \Theta \Rightarrow \varphi_i\rangle\\ \chi\\ \mathcal{D}_4\end{matrix}$$

where $\mathcal{D}_i\langle \mathcal{D}_3 : \Theta \Rightarrow \varphi_i\rangle$ is the proof tree obtained from Lemma 7.1. It is easy to see that every new cut formula possibly arising from the conversion is of smaller rank than ψ. Hence, the resulting deduction $\mathcal{D}'$ satisfies $\textsc{cr}(\mathcal{D}') < \textsc{cr}(\mathcal{D})$. $\square$

Given an arbitrary deduction $\mathcal{D}$, we can now first apply Proposition 6.3 in order to convert $\mathcal{D}$ into a pure deduction $\mathcal{D}'$. Afterwards, we can repeatedly apply Lemma 7.2 in order to construct a sequence of pure deductions $\mathcal{D}_1$, $\mathcal{D}_2$, $\mathcal{D}_3$, etc., such that the associated cut ranks form a descending chain $\textsc{cr}(\mathcal{D}') > \textsc{cr}(\mathcal{D}_1) > \textsc{cr}(\mathcal{D}_2) > \ldots$ After a finite number of steps, this must yield a *normal deduction*, so we now obtain the desired normalization result for our system.

Theorem 7.3 (Normalization Theorem) *Let Γ be a set of formulas. Every deduction for $\Gamma \mathrel{|\!\!\frac{\;\;}{\scriptscriptstyle N}} \varphi$ can be transformed into a normal deduction for $\Gamma \mathrel{|\!\!\frac{\;\;}{\scriptscriptstyle N}} \varphi$.*

8 Properties of Normal Deductions

We will now establish an unrestricted *subformula property* for our system. First, we need to find a suitable notion of a *track* in a deduction. Intuitively, a track may be seen as a sequence of formula occurrences that are sufficiently 'related' to each other by their syntactic shape. In particular, any two consecutive formulas in a track (except possibly those involved in $\perp_{\mathrm{i}}$) should satisfy some kind of *local* subformula property, in the sense that one of these formulas is always a subformula of the other (cf. [21, Sect. 6.2]). Defining tracks for our system turns out to be somewhat complicated, because a deduction may now also contain premises of *assumption rules*, and it is not obvious what the successor of such a formula should be in a track. In order to address this difficulty, we first need to take a closer look at the relationship between the *auxiliary rules* and the *side rules* of an application of $\vee E$. We will say that φ is a P-*formula* (resp., a C-*formula*) of an assumption rule R, if φ is a premise (resp., the conclusion) of some subrule of R.[6] Given any instance of $\vee E$ with set of auxiliary rules Θ and major premise $\varphi \vee \psi$, it is now easy to verify the following facts:

- if a formula χ occurs as a P-formula in an auxiliary rule $R \in \Theta$, then χ also occurs as a C-formula in each of the two side rules $\Theta \Rightarrow \varphi$ and $\Theta \Rightarrow \psi$,
- if χ occurs as a P-formula in one of the side rules $\Theta \Rightarrow \varphi$ and $\Theta \Rightarrow \psi$, then χ also occurs as a C-formula in at least one auxiliary rule $R \in \Theta$.

In other words, for every P-formula of an auxiliary rule of $\vee E$, there exists a corresponding C-formula in each of the two side rules of this application of $\vee E$. And for every P-formula of a side rule, there exists a corresponding C-formula in one of the auxiliary rules. The notion of a track is now defined as follows.

Definition 8.1 [Track] A *track* in a deduction $\mathcal{D}$ is a sequence of pairwise distinct formula occurrences $\xi_1, \ldots, \xi_n$ in $\mathcal{D}$ such that the following holds:[7]

(i) ξ_1 is a hypothesis of $\mathcal{D}$, but not a C-formula of a side rule of $\vee E$.

(ii) Each ξ_i with $1 \leq i < n$ is not a minor premise of $\to E$ and it holds:
  (a) If ξ_i is the major premise $\varphi \vee \psi$ of an application of $\vee E$, then ξ_{i+1} is the conclusion of one of the side rules of this application.
  (b) If ξ_i is a P-formula of an auxiliary rule (resp., side rule) of $\vee E$, then ξ_{i+1} is an occurrence of the corresponding C-formula belonging to one of the side rules (resp., auxiliary rules) of this application of $\vee E$.
  (c) If ξ_i is neither a major premise of $\vee E$ nor a premise of an assumption rule, then ξ_{i+1} is the formula occurrence standing directly below ξ_i.

(iii) ξ_n is a minor premise of $\to E$ or the conclusion of $\mathcal{D}$.

A track in a deduction $\mathcal{D}$ is said to be a *main track*, if it ends with the conclusion of $\mathcal{D}$. For example, the normal deduction displayed below contains exactly six tracks: the main tracks of the deduction are the sequences $\tau_1 = (1, 5, 9, 10, 11, 18)$ and $\tau_2 = (1, 5, 15, 16, 17, 18)$. And the other four tracks are the sequences $\tau_3 = (6, 8, 3, 4)$, $\tau_4 = (12, 14, 3, 4)$, $\tau_5 = (2, 7)$ and $\tau_6 = (2, 13)$.

$$
z\,\dfrac{\dfrac{{\scriptstyle(1)}\ \neg p \to (\varphi \vee \psi) \qquad u\,\dfrac{{\scriptstyle(3)}\ \dfrac{{\scriptstyle(2)}\ [p]^u}{\bot}\,[p \Rightarrow \bot]^z}{{\scriptstyle(4)}\ \neg p}\,{\to}I}{{\scriptstyle(5)}\ \varphi \vee \psi}\,{\to}E \qquad \dfrac{w\,\dfrac{{\scriptstyle(9)}\ v\,\dfrac{{\scriptstyle(8)}\ \dfrac{{\scriptstyle(6)}\ [\neg p]^w \quad {\scriptstyle(7)}\ [p]^v}{\bot}\,{\to}E}{\varphi}\,[(p \Rightarrow \bot) \Rightarrow \varphi]^z}{{\scriptstyle(10)}\ \neg p \to \varphi}\,{\to}I}{{\scriptstyle(11)}\ (\neg p \to \varphi) \vee (\neg p \to \psi)}\,{\vee}I \qquad \dfrac{y\,\dfrac{{\scriptstyle(15)}\ x\,\dfrac{{\scriptstyle(14)}\ \dfrac{{\scriptstyle(12)}\ [\neg p]^y \quad {\scriptstyle(13)}\ [p]^x}{\bot}\,{\to}E}{\psi}\,[(p \Rightarrow \bot) \Rightarrow \psi]^z}{{\scriptstyle(16)}\ \neg p \to \psi}\,{\to}I}{{\scriptstyle(17)}\ (\neg p \to \varphi) \vee (\neg p \to \psi)}\,{\vee}I}{{\scriptstyle(18)}\ (\neg p \to \varphi) \vee (\neg p \to \psi)}\,{\vee}E
$$

Lemma 8.2 *Let $\mathcal{D}$ be a deduction. Every formula occurrence in $\mathcal{D}$ belongs to some track. Thus, in particular, $\mathcal{D}$ has at least one main track.*[8]

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[6] In this definition, a level-0 rule ψ is treated as a zero-premise rule with conclusion ψ.

[7] We say that two formula occurrences are *distinct*, if they are *distinct occurrences* of formulas (but not necessarily occurrences of *distinct formulas*). Requiring $\xi_1, \ldots, \xi_n$ to be pairwise distinct is not essential here, but it simplifies our treatment a little: without this restriction, a deduction might also contain *infinitely many* tracks. Further details are provided in [13].

[8] Here, we are assuming the discharge convention introduced at the end of Section 3.

The proof idea is explained in Appendix D. By a *strictly positive subformula* (or an *sp-subformula*) of φ, we will mean (an occurrence of) a subformula of φ which is not included in the antecedent of an implication. A formula φ is said to be a *Harrop formula*, if it does not contain an sp-subformula with main connective $\vee$. Obviously, every track in a deduction may also be divided into a sequence of *segments*, rather than into a sequence of single occurrences of formulas. Here, a segment is understood to be any (maximal) subsequence of a track that consists of repeated occurrences of the same formula. So, in particular, a segment may now also contain repetitions of formulas arising from the 'switching' between P-formulas and C-formulas. We will say that a segment σ is the *conclusion* (resp., a *premise*) of a rule application, if the first (resp., last) formula occurrence in σ is the conclusion (resp., a premise) of this application.

Proposition 8.3 (Structure of Tracks) *Let $\mathcal{D}$ be a normal deduction, let τ be a track in $\mathcal{D}$, and let $\sigma_1, \ldots, \sigma_n$ be the sequence of segments in τ. There exists a segment σ_k in τ, called the* minimum segment *of the track, such that:*

(i) *Each σ_i with $1 \leq i < k$ is a major premise of an elimination rule and the formula occurring in σ_i is an sp-subformula of the one occurring in σ_1.*

(ii) *If $k \neq n$, then σ_k is a premise of an introduction rule or a premise of $\bot_\mathsf{i}$.*

(iii) *Each σ_i with $k < i < n$ is a premise of an introduction rule and the formula occurring in σ_i is an sp-subformula of the one occurring in σ_n.*

Proof. Let $\mathcal{D}$ be a normal deduction, let τ be a track in $\mathcal{D}$, and let $\sigma_1, \ldots, \sigma_n$ be the sequence of segments in τ. We first show that every major premise of an E-rule in τ must precede all premises of I-rules and all premises of $\bot_\mathsf{i}$ in τ. Towards a contradiction, suppose that there exists a premise of an I-rule or a premise of $\bot_\mathsf{i}$ that precedes a major premise of an E-rule in τ. Then, clearly, there must be a segment σ_j which is both the conclusion of an I-rule or the conclusion of $\bot_\mathsf{i}$, and the major premise of an E-rule. By the pureness of $\mathcal{D}$, we know that σ_j cannot be a conclusion of $\bot_\mathsf{i}$, and it also cannot contain any premises of assumption rules: otherwise, the formula occurring in σ_j would be prime, and such a formula can never be the major premise of an E-rule. Hence, σ_j is the conclusion of an I-rule and every formula occurrence in σ_j, except the last one, is a minor premise of $\vee E$. But this means that σ_j is a *cut segment*, which is a contradiction to the assumption that $\mathcal{D}$ is normal. Thus, every major premise of an E-rule precedes all premise of I-rules and all premises of $\bot_\mathsf{i}$ in τ. Using a similar argument, it is also easy to show that there can be at most one premise of $\bot_\mathsf{i}$ in τ, and this premise must precede all premises of I-rules in τ.

The desired minimum segment can now be determined as follows: if τ contains a premise of an I-rule or a premise of $\bot_\mathsf{i}$, then let σ_k be the segment that ends with the first such formula in τ. Otherwise, let σ_k be the end segment of τ. By what was said above, every segment preceding σ_k in τ is a major premise of an E-rule, and every segment succeeding σ_k in τ, except the last one, is a premise of an I-rule. Hence, σ_k does in fact have the desired properties. $\Box$

We are now ready to establish an unrestricted version of the subformula property for our system: every formula occurring in a normal deduction $\mathcal{D}$ is a subformula of an open hypothesis or a subformula of the conclusion of $\mathcal{D}$.

Theorem 8.4 (Subformula Property) *Every formula occurring in a normal deduction for $\Gamma \vdash_{\mathrm{N}} \varphi$ is a subformula of some formula in $\Gamma \cup \{\varphi\}$.*

Proof. Let Γ be a set of formulas and let $\mathcal{D}$ be a normal deduction for $\Gamma \vdash_{\mathrm{N}} \varphi$. Towards a contradiction, suppose that $\mathcal{D}$ contains a formula ψ which is *not* a subformula of some element of $\Gamma \cup \{\varphi\}$. Without loss of generality, we may assume that ψ is also a *largest* such formula in $\mathcal{D}$, i.e., every formula of rank greater than $rk(\psi)$ in $\mathcal{D}$ is a subformula of some element of $\Gamma \cup \{\varphi\}$. Let τ be a track to which ψ belongs and let $\xi_1, \ldots, \xi_n$ be the sequence of formulas in τ.

First, assume that ψ occurs after the minimum segment in τ. Then, by Proposition 8.3, ψ is a subformula of ξ_n, which is either the conclusion of $\mathcal{D}$ or a minor premise of $\rightarrow E$. If ξ_n would be the conclusion of $\mathcal{D}$, then ψ would be a subformula of φ, which is a contradiction to our assumption. Thus, ξ_n must be a minor premise of $\rightarrow E$. But then, clearly, ψ is a proper subformula of the associated major premise $\xi_n \rightarrow \chi$, which is a formula of rank greater than $rk(\psi)$ and therefore a subformula of some element of $\Gamma \cup \{\varphi\}$. As a consequence, ψ must also be such a subformula, so we arrive again at a contradiction.

Next, assume that ψ occurs in the minimum segment or before the minimum segment in τ. Then, by Proposition 8.3, ψ is a subformula of the hypothesis ξ_1 occurring at the top of τ. Clearly, ξ_1 cannot be discharged by an assumption rule: in this case, it would be a C-formula of an auxiliary rule of $\vee E$, so there would be a corresponding P-formula (belonging to a side rule of this application of $\vee E$) which would precede ξ_1 in τ. Hence, ξ_1 is either an open hypothesis, or it is discharged by $\rightarrow I$. If ξ_1 is an open hypothesis, then ψ is a subformula of some element of Γ, which is a contradiction to our assumption about ψ. And if ξ_1 is discharged by an application of $\rightarrow I$, then ψ must be a proper subformula of the conclusion $\xi_1 \rightarrow \chi$ of this application, which is a formula of rank greater than $rk(\psi)$ and therefore a subformula of some element of $\Gamma \cup \{\varphi\}$. Hence, ψ must also be such a subformula, so we obtain again a contradiction. $\Box$

As an immediate consequence of this result, we now also obtain a *separation theorem* for our system. Furthermore, using Proposition 8.3, it is easy to establish a strong form of the *disjunction property* for KP in a proof-theoretical manner. The argument is essentially the same as in [17, pp. 55–56].

Theorem 8.5 (Separation) *The only primitive rules applied in a normal deduction for $\Gamma \vdash_{\mathrm{N}} \varphi$ are the rules for the logical symbols occurring in $\Gamma \cup \{\varphi\}$.*

Theorem 8.6 (Disjunction Property) *Let Γ be a set of Harrop formulas. If we have $\Gamma \vdash_{\mathrm{N}} \varphi \vee \psi$ in our system, then also $\Gamma \vdash_{\mathrm{N}} \varphi$ or $\Gamma \vdash_{\mathrm{N}} \psi$.*

9 Conclusion

We introduced a higher-level natural deduction system for KP, presented a new normalization procedure and established an unrestricted subformula property for our system. Our results might be of interest not only with regard to KP itself, but also with regard to the proof theory of non-classical logics in general. In this area of research, one focuses primarily on *sequent-style* proof systems (such as, e.g., hypersequents, nested sequent calculi or labelled sequent calculi), but natural deduction systems often play only a minor role. Using the expressive power of higher-level rules, one might be able to overcome this limitation and to find analytic natural deduction systems for logics that do not admit a full subformula property within the framework of ordinary natural deduction.

For future work, we plan to adapt our normalization strategy to other non-classical logics, which might include not only intermediate logics different from KP, but also various modal logics. This is not entirely trivial, because some of our arguments are tailored to cases in which certain assumption rules are required to be *disjunction-free* (this applies primarily to the proof of Lemma 6.2). Apart from considering specific examples of non-classical logics, we also intend to develop a more general procedure which allows to construct higher-level natural deduction systems for modal and intermediate logics in a systematic way. This might be done by converting axiom schemes directly into 'equivalent' natural deduction rules and by establishing a generic normalization result for the proof systems thus obtained. First attempts in this direction have already led to various promising ideas that we plan to address in a subsequent study.

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Appendix

A Proof of Lemma 4.2

Let Γ be a set of assumption rules and suppose that there exists a quasi-deduction $\mathcal{D}$ for $\Gamma \vdash_{\mathrm{N}} \varphi$ in the system NKP. Using induction on the structure of $\mathcal{D}$, we show that there also exists a Hilbert-style proof for $F(\Gamma) \vdash_{\mathrm{H}} \varphi$ in HKP.

For the base case, assume that $\mathcal{D} = \varphi$ is a one-element quasi-deduction. Then, since $\varphi \in \Gamma$ and $F(\varphi) = \varphi$, we trivially also have $F(\Gamma) \vdash_{\mathrm{H}} \varphi$ in HKP.

For the inductive step, suppose that $\mathcal{D}$ ends with a primitive rule of NKP or with an assumption rule of level $\ell \geq 1$. We only consider the following cases.

Case 1: $\mathcal{D}$ ends with an application of $\vee E$. In this case, $\mathcal{D}$ is of the form

$$\dfrac{\begin{matrix}[\Theta]^u \\ \mathcal{D}_3 \\ \psi_1 \vee \psi_2\end{matrix} \qquad \begin{matrix}[\Theta \Rightarrow \psi_1]^u \\ \mathcal{D}_1 \\ \varphi\end{matrix} \qquad \begin{matrix}[\Theta \Rightarrow \psi_2]^u \\ \mathcal{D}_2 \\ \varphi\end{matrix}}{\varphi}\, u\ \vee E$$

where $\Theta = \{\Pi_1 \Rightarrow \bot, \ldots, \Pi_n \Rightarrow \bot\}$ is a finite set of negative rules, each $\mathcal{D}_i$ with $i = 1, 2$ is a quasi-deduction for $\Gamma, (\Theta \Rightarrow \psi_i) \vdash_{\mathrm{N}} \varphi$, and $\mathcal{D}_3$ is a quasi-deduction for $\Gamma, \Theta \vdash_{\mathrm{N}} \psi_1 \vee \psi_2$. Using the induction hypothesis, we first obtain

$$F(\Gamma), F(\Theta \Rightarrow \psi_i) \vdash_{\mathrm{H}} \varphi \quad \text{for } i = 1, 2, \tag{A.1}$$

$$F(\Gamma), F(\Theta) \vdash_{\mathrm{H}} \psi_1 \vee \psi_2 \tag{A.2}$$

in the system HKP. For simplicity, let us assume that $\Theta \neq \emptyset$. In this case, we have $F(\Theta \Rightarrow \psi_i) = \bigwedge F(\Theta) \rightarrow \psi_i$ for each $i = 1, 2$. Thus, from (A.1), it follows

$$F(\Gamma) \vdash_{\mathrm{H}} (\bigwedge F(\Theta) \rightarrow \psi_i) \rightarrow \varphi \quad \text{for } i = 1, 2 \tag{A.3}$$

by the deduction theorem for HKP (Theorem 2.2). For every k with $1 \leq k \leq n$, let now π_k be the formula given by $\pi_k := \bigwedge F(\Pi_k)$. Then, clearly, we have $F(\Pi_k \Rightarrow \bot) = \neg\pi_k$ for all $1 \leq k \leq n$, so it follows $F(\Theta) = \{\neg\pi_1, \ldots, \neg\pi_n\}$. Consequently, from (A.2), we obtain $F(\Gamma), \neg\pi_1, \ldots, \neg\pi_n \vdash_{\mathrm{H}} \psi_1 \vee \psi_2$. Using the deduction theorem and a subsequent application of axiom scheme `(KP)`, this implies $F(\Gamma), \neg\pi_2, \ldots, \neg\pi_n \vdash_{\mathrm{H}} (\neg\pi_1 \rightarrow \psi_1) \vee (\neg\pi_1 \rightarrow \psi_2)$. By repeating the argument and by using intuitionistic reasoning, it is easy to derive

$$F(\Gamma) \vdash_{\mathrm{H}} (\bigwedge F(\Theta) \rightarrow \psi_1) \vee (\bigwedge F(\Theta) \rightarrow \psi_2). \tag{A.4}$$

The desired Hilbert-style proof for $F(\Gamma) \vdash_{\mathrm{H}} \varphi$ can now be obtained from (A.3) and (A.4) using axiom scheme `(A6)` and three applications of modus ponens.

Case 2: $\mathcal{D}$ ends with an application of an assumption rule R of level $\ell \geq 1$. Clearly, this application must be open, so $\mathcal{D}$ is a quasi-deduction for $\Delta, R \vdash_{\mathrm{H}} \varphi$, where Δ is a set of rules satisfying $\Gamma = \Delta \cup \{R\}$. By assumption, R is of the form $R = (\Pi \Rightarrow \varphi)$, where $\Pi = \{\Sigma_1 \Rightarrow \psi_1, \ldots, \Sigma_n \Rightarrow \psi_n\}$ is a set of rules whose maximal level is $\ell - 1$. As a consequence, $\mathcal{D}$ must be of the form

$$\begin{array}{c} \begin{array}{ccc} [\Sigma_1]^u & & [\Sigma_n]^u \\ \mathcal{D}_1 & & \mathcal{D}_n \\ \psi_1 & \cdots & \psi_n \end{array} \\ \hline \varphi \end{array} \; R \quad (u)$$

where each $\mathcal{D}_i$ is a quasi-deduction for $\Delta, \Sigma_i \vdash_{\mathrm{N}} \psi_i$. Let now i with $1 \leq i \leq n$ be arbitrary and let $R_i := (\Sigma_i \Rightarrow \psi_i)$. By induction hypothesis, we have $F(\Delta), F(\Sigma_i) \vdash_{\mathrm{H}} \psi_i$ in the system HKP. We first show that this yields $F(\Delta) \vdash_{\mathrm{H}} F(R_i)$. For simplicity, let us assume that $\Sigma_i \neq \emptyset$. From $F(\Delta), F(\Sigma_i) \vdash_{\mathrm{H}} \psi_i$, it follows $F(\Delta) \vdash_{\mathrm{H}} \bigwedge F(\Sigma_i) \rightarrow \psi_i$ by Theorem 2.2 and intuitionistic reasoning. But then, since $F(R_i) = \bigwedge F(\Sigma_i) \rightarrow \psi_i$, we also have $F(\Delta) \vdash_{\mathrm{H}} F(R_i)$, as desired. Now, as we have seen, it holds $F(\Delta) \vdash_{\mathrm{H}} F(R_i)$ for all $1 \leq i \leq n$. Hence, using axiom `(A4)` and modus ponens, we may conclude $F(\Delta) \vdash_{\mathrm{H}} \bigwedge F(\Pi)$. Moreover, since $F(R) = \bigwedge F(\Pi) \rightarrow \varphi$, it also holds $F(R) \vdash_{\mathrm{H}} \bigwedge F(\Pi) \rightarrow \varphi$. Together with $F(\Delta) \vdash_{\mathrm{H}} \bigwedge F(\Pi)$, this implies $F(\Delta), F(R) \vdash_{\mathrm{H}} \varphi$ by modus ponens. □

B Proof of Lemma 7.1

Let Θ be a finite set of prime rules, let $\mathcal{D}_1$ be a pure quasi-deduction for $\Gamma, (\Theta \Rightarrow \varphi) \vdash_{\mathrm{N}} \psi$ and let $\mathcal{D}_2$ be a pure quasi-deduction for $\Delta, \Theta \vdash_{\mathrm{N}} \varphi$. We establish the existence of $\mathcal{D}_1\langle\mathcal{D}_2 : \Theta \Rightarrow \varphi\rangle$ by induction on the level of $R := (\Theta \Rightarrow \varphi)$.

For the base case, assume that R is of level $\ell = 0$, so Θ is empty and $R = \varphi$. In this case, we define $\mathcal{D}_1\langle\mathcal{D}_2 : \Theta \Rightarrow \varphi\rangle$ to be the pure quasi-deduction obtained by writing the tree $\mathcal{D}_2$ above each open occurrence of the hypothesis $R = \varphi$ in $\mathcal{D}_1$. Clearly, φ is the only new cut formula that may arise in the course of this construction, so $\mathcal{D}_1\langle\mathcal{D}_2 : \Theta \Rightarrow \varphi\rangle$ does in fact have the desired properties.

For the inductive step, suppose that R is of level $\ell \geq 1$, so $\Theta = \{\Pi_1 \Rightarrow \chi_1, \ldots, \Pi_n \Rightarrow \chi_n\} \neq \emptyset$, where each $\Pi_i \Rightarrow \chi_i$ is a prime rule of level $< \ell$. We select a topmost open application of $\Theta \Rightarrow \varphi$ in $\mathcal{D}_1$, i.e., we write $\mathcal{D}_1$ in the form

$$\begin{array}{c} u\,\dfrac{\begin{array}{ccc}[\Pi_1]^u & & [\Pi_n]^u \\ \mathcal{D}_1^* & & \mathcal{D}_n^* \\ \chi_1 & \cdots & \chi_n\end{array}}{\varphi}\,\Theta \Rightarrow \varphi \\ \mathcal{D}_0 \\ \psi \end{array}$$

where none of $\mathcal{D}_1^*, \ldots, \mathcal{D}_n^*$ contains another open application of $\Theta \Rightarrow \varphi$. For each i with $1 \leq i \leq n$, let now Σ_i be the smallest set of rules such that $\mathcal{D}_i^*$ is a quasi-deduction for $\Sigma_i, \Pi_i \mid\!\overline{}_{\mathrm{N}}\, \chi_i$. By assumption, $\mathcal{D}_2$ is a quasi-deduction for $\Delta, (\Pi_1 \Rightarrow \chi_1), \ldots, (\Pi_n \Rightarrow \chi_n) \mid\!\overline{}_{\mathrm{N}}\, \varphi$, and each $\Pi_i \Rightarrow \chi_i$ is of level $< \ell$. Hence, using the induction hypothesis n times, we obtain the pure quasi-deduction

$$\mathcal{D}^* := \mathcal{D}_2 \langle \mathcal{D}_1^* : \Pi_1 \Rightarrow \chi_1 \rangle \cdots \langle \mathcal{D}_n^* : \Pi_n \Rightarrow \chi_n \rangle$$

for $\Delta, \Sigma_1, \ldots, \Sigma_n \mid\!\overline{}_{\mathrm{N}}\, \varphi$ such that all cut formulas occurring in $\mathcal{D}^*$, except possibly $\chi_1, \ldots, \chi_n$, already occur in $\mathcal{D}_2$ or in one of the trees $\mathcal{D}_1^*, \ldots, \mathcal{D}_n^*$ (and therefore in $\mathcal{D}_1$). But note that, by assumption, each χ_i is prime, and such a formula can never be a cut formula. Thus, every cut formula in $\mathcal{D}^*$ already occurs as a cut formula in $\mathcal{D}_1$ or $\mathcal{D}_2$. We now construct the pure quasi-deduction

$$\begin{array}{c} \mathcal{D}^* \\ \varphi \\ \mathcal{D}_0 \\ \psi \end{array}$$

Clearly, φ is the only new cut formula possibly arising from this construction. Hence, by repeating the procedure for the other open applications of $\Theta \Rightarrow \varphi$, we finally obtain the desired quasi-deduction $\mathcal{D}_1 \langle \mathcal{D}_2 : \Theta \Rightarrow \varphi \rangle$ for $\Gamma, \Delta \mid\!\overline{}_{\mathrm{N}}\, \psi$. □

C Supplement to the Proof of Lemma 7.2

We need to show that the permutation conversion can also be performed, if the elimination rule (E) at the bottom of a cut segment is itself an instance of $\vee E$. In order to understand the problem, let us consider a deduction $\mathcal{D}$ of the form

$$\begin{array}{c} v\,\dfrac{u\,\dfrac{\begin{array}{c}[\Theta]^u\ [\Omega]^v \\ \mathcal{D}_3 \\ \varphi_1 \vee \varphi_2\end{array} \quad \begin{array}{c}[\Theta \Rightarrow \varphi_1]^u\ [\Omega]^v \\ \mathcal{D}_1 \\ \psi_1 \vee \psi_2\end{array} \quad \begin{array}{c}[\Theta \Rightarrow \varphi_2]^u\ [\Omega]^v \\ \mathcal{D}_2 \\ \psi_1 \vee \psi_2\end{array}}{\psi_1 \vee \psi_2}\,\vee E \quad \begin{array}{c}[\Omega \Rightarrow \psi_1]^v \\ \mathcal{D}_4 \\ \chi\end{array} \quad \begin{array}{c}[\Omega \Rightarrow \psi_2]^v \\ \mathcal{D}_5 \\ \chi\end{array}}{\chi}\,\vee E \\ \mathcal{D}_6 \end{array}$$

The upper application of $\vee E$ in this proof tree has the set of auxiliary rules Θ, and the lower application has the set of auxiliary rules Ω. As indicated by the label v, the lower application of $\vee E$ also discharges some of the rules from Ω in the subtree $\mathcal{D}_3$, so we cannot apply the desired permutation conversion directly, as this would turn the respective rules into *open* assumptions.

In order to resolve this issue, we will now demonstrate that one can 'shift' each of the auxiliary rules from Ω in $\mathcal{D}_3$ to the upper application of $\vee E$. For this purpose, let us say that a quasi-deduction is *structural*, if it contains only applications of assumption rules, but no applications of primitive rules. The following fact can be verified by an easy induction on the level of a rule $\Gamma \Rightarrow \varphi$.

Fact C.1 (Modus Ponens for Assumption Rules) *For every assumption rule $\Gamma \Rightarrow \varphi$, there exists a structural quasi-deduction for $(\Gamma \Rightarrow \varphi), \Gamma \vdash_{\mathrm{N}} \varphi$.*

Suppose now that, in the deduction above, Θ and Ω are the sets of negative rules given by $\Theta = \{\Pi_1 \Rightarrow \bot, \ldots, \Pi_n \Rightarrow \bot\}$ and $\Omega = \{\Sigma_1 \Rightarrow \bot, \ldots, \Sigma_m \Rightarrow \bot\}$. By applying Fact C.1 to the rules in Ω, we may obtain, for each $1 \leq i \leq m$, a structural quasi-deduction $\mathcal{D}_i^*$ for $(\Sigma_i \Rightarrow \bot), \Sigma_i \vdash_{\mathrm{N}} \bot$ and thus for $\Omega, \Sigma_i \vdash_{\mathrm{N}} \bot$. Let now $k = 1, 2$ be arbitrary and let $\mathcal{D}_k^+$ be the quasi-deduction obtained from the subtree $\mathcal{D}_k$ by rewriting every open application of $\Theta \Rightarrow \varphi_k$ as follows:

$$
\dfrac{\begin{matrix}[\Pi_1]^w & & [\Pi_n]^w \\ \mathcal{D}_1' & & \mathcal{D}_n' \\ \bot & \cdots & \bot\end{matrix}}{\begin{matrix}\varphi_k \\ \mathcal{D}_0\end{matrix}}{\scriptstyle w}\ \Theta \Rightarrow \varphi_k
\quad \rightsquigarrow \quad
\dfrac{\begin{matrix}[\Pi_1]^w & & [\Pi_n]^w & \Omega\ [\Sigma_1]^w & & \Omega\ [\Sigma_m]^w \\ \mathcal{D}_1' & & \mathcal{D}_n' & \mathcal{D}_1^* & & \mathcal{D}_m^* \\ \bot & \cdots & \bot & \bot & \cdots & \bot\end{matrix}}{\begin{matrix}\varphi_k \\ \mathcal{D}_0\end{matrix}}{\scriptstyle w}\ \Theta, \Omega \Rightarrow \varphi_k
$$

In the course of this conversion, additional open occurrences of the rules from Ω are added to the proof tree, and each open application of $\Theta \Rightarrow \varphi_k$ is replaced by an open application of $\Theta, \Omega \Rightarrow \varphi_k$. Note that, since every $\mathcal{D}_i^*$ is structural, the conversion does not introduce any new cut formulas. Using the proof trees $\mathcal{D}_1^+$ and $\mathcal{D}_2^+$ thus obtained, we now transform the whole deduction $\mathcal{D}$ into

$$
\dfrac{\dfrac{\begin{matrix}[\Theta, \Omega]^u \\ \mathcal{D}_3 \\ \varphi_1 \vee \varphi_2\end{matrix} \quad \begin{matrix}[\Theta, \Omega \Rightarrow \varphi_1]^u\ [\Omega]^v \\ \mathcal{D}_1^+ \\ \psi_1 \vee \psi_2\end{matrix} \quad \begin{matrix}[\Theta, \Omega \Rightarrow \varphi_2]^u\ [\Omega]^v \\ \mathcal{D}_2^+ \\ \psi_1 \vee \psi_2\end{matrix}}{\psi_1 \vee \psi_2}{\scriptstyle u}\ \vee E \quad \begin{matrix}[\Omega \Rightarrow \psi_1]^v \\ \mathcal{D}_4 \\ \chi\end{matrix} \quad \begin{matrix}[\Omega \Rightarrow \psi_2]^v \\ \mathcal{D}_5 \\ \chi\end{matrix}}{\begin{matrix}\chi \\ \mathcal{D}_6\end{matrix}}{\scriptstyle v}\ \vee E
$$

Clearly, this deduction has the same cut rank as the original one. However, the lower application of $\vee E$ does not discharge any assumption rules in the subtree $\mathcal{D}_3$ anymore, so one can now safely apply the desired permutation conversion.

D Proof of Lemma 8.2

We first define the notion of a *pre-track* in a quasi-deduction in exactly the same way as the notion of a track, except that (1) a pre-track may also start with an arbitrary C-formula belonging to an open application of an assumption rule, and (2) a pre-track always stops as soon as a P-formula of an open application of an assumption rule is encountered. Using induction on the structure of an arbitrary quasi-deduction $\mathcal{D}$, it is now possible to show that every formula occurrence in $\mathcal{D}$ belongs to some pre-track. The desired statement now follows from the fact that, if $\mathcal{D}$ is a *deduction* (i.e., if each assumption rule of level greater than 0 is discharged in $\mathcal{D}$), then every pre-track in $\mathcal{D}$ is also a track. $\square$

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