Logics of Knowability

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Abstract

Knowability has a significant history in the field of epistemic logic dating back to the introduction of Fitch's paradox in 1963. More recently it has received attention from the perspectives of dynamic epistemic logic and epistemic temporal logic. However, existing treatments of knowability either address a knowability-like concept that deviates from an intuitive understanding of knowability in some important way or take knowability to be a complex notion merely definable in the language. In this paper, we take 'it is knowable that' as a primitive modality and offer axiomatizations of two knowability logics—one of knowledge and knowability and one of pure knowability. We prove that these logics are sound and complete with respect to a distinguished subclass of birelational frames. These proofs are of technical interest, as well; because knowability is, we argue, best understood as a non-normal modality, the completeness proofs require novel constructions.

Keywords: epistemic logic, knowability, non-normal modal logic, completeness

1 Introduction

In the setting of dynamic epistemic logic, knowability has recently received treatment in a number of papers (see, e.g., [1,3,4,10,14,15]). From this perspective, a proposition φ is knowable for an agent if the agent would come to know φ upon receiving some information or after the successful execution of some action, like a true public announcement. One might be interested in an approach that is, on the face of it, simpler, however. Namely, much in the way that one project of epistemic logic has been to work out the logic of the knowledge operator K (see, e.g., [9,11,12]), it seems natural to introduce a knowability operator $\Box$ and propose some candidate logic for it.

Something like this is done in [5] where a logic of knowability, knowledge, and belief is introduced in order to refine the picture of knowledge and belief that Stalnaker presents in [12]. The notion captured in [5] is not quite knowability as one might naïvely conceive of it, however. Bjorndahl and Özgün propose the following gloss for $\Box\varphi$: "one could come to know what φ used to express (before you came to know it)" [5, p. 753]. Their knowability operator in a sense deindexicalizes the proposition on which it acts. Though this notion is

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interesting in its own right, it seems to exhibit significant logical deviation from knowability simpliciter.

For example, after accepting positive introspection of knowledge, it seems as though one ought to be committed to the inference that if φ is knowable, then φ is knowably known. But $\Box\varphi \to \Box K\varphi$ is not a theorem of logic found in [5]. (This fact is intuitive given their gloss and can be easily shown by finding a countermodel and appealing to their soundness result.) Perhaps even more glaring is that this notion of knowability seems to commit us to the principle $\Box\varphi \wedge \Box\psi \to \Box(\varphi \wedge \psi)$. On the simpler reading of *knowable* this sounds wrong: presumably we commonly find ourselves in situations where we must choose to direct our attention in one direction or the other. Upon attending to one matter we may come to know φ, but in doing so preclude ourselves from knowing ψ, which we would have known had we otherwise directed our attention.

The approach that we take in this paper to capture the simpler notion of knowability is heavily inspired by the considerable literature on Fitch's paradox of knowability, introduced in [8]. The paradox is typically rendered in a bimodal logic where the *knowable* is decomposed as *possibly known*. Without significant logical assumptions imposed, such a treatment of knowability will avoid validating principles like $\Box\varphi \wedge \Box\psi \to \Box(\varphi \wedge \psi)$, but will, with rather minor logical assumptions, validate principles like $\Box\varphi \to \Box K\varphi$. We also take inspiration from Burgess's approach to knowability in [7] where the possibility part of the decomposition has a temporal flavor, though unlike Burgess, we do not assume linearity of time.

The goal of this paper is to propose two logics—one of knowledge and knowability and one of pure knowability—and establish their completeness with respect to a class of models that have an intuitive appeal. In order to achieve this, a third auxiliary logic in a more expressive language will be introduced. Completeness of that logic with respect to the restricted model class will be established and the result transferred over to the two knowability logics by way of establishing full and faithful embeddings of them into the auxiliary one. While the paper is motivated by epistemic considerations, the results and proofs are also of technical interest, as the completeness theorems are proved with novel canonical model constructions; more interesting techniques are required given that we are producing relational models for logics with a non-normal modal operator.

The outline of the paper is as follows. In Section 2 we introduce the three languages that we are going to work with, as well as a few syntactic notions, including the translations. Birelational semantics are then presented in Section 3. In Section 4 we establish the auxiliary logic and prove that it is complete with respect to the smaller model class defined in Section 3. Finally, Sections 5 and 6 propose the logics of knowledge and knowability and of pure knowability, proving the expected soundness and completeness results.

2 Languages

2.1 The Languages of Knowability

Fix a countably infinite set $\mathbb{P} = \{p, q, \ldots\}$ of propositional variables. The language $\mathcal{L}_{K\Box}$ is given by the formal grammar

$$\varphi ::= p \,|\, \neg\varphi \,|\, (\varphi \wedge \varphi) \,|\, K\varphi \,|\, \Box\varphi$$

where $p \in \mathbb{P}$. We will use the following standard bits of shorthand: $\bot$ for $p \wedge \neg p$, $\top$ for $\neg(p \wedge \neg p)$, $\varphi \vee \psi$ for $\neg(\neg\varphi \wedge \neg\psi)$, $\varphi \to \psi$ for $\neg(\varphi \wedge \neg\psi)$, $\hat{K}\varphi$ for $\neg K \neg\varphi$, and $\Diamond\varphi$ for $\neg\Box\neg\varphi$. Additionally, we write $\mathcal{L}_{\Box}$ for the K-free fragment of $\mathcal{L}_{K\Box}$. The proposed readings of the modalities are *the agent knows that* φ for $K\varphi$ and *it is knowable for the agent that* φ for $\Box\varphi$.

Definition 2.1 We define *complexity* $c : \mathcal{L}_{K\Box} \to \mathbb{N}$ recursively as follows:

- $c(p) = 0$
- $c(\neg\varphi) = c(\varphi) + 1$
- $c(\varphi \wedge \psi) = \max(c(\varphi), c(\psi)) + 1$
- $c(K\varphi) = c(\varphi) + 1$
- $c(\Box\varphi) = c(\varphi) + 2$

Note the $+2$ in the last clause. This definition allows us to apply the inductive hypothesis to the formula $K\varphi$ when checking the $\Box\varphi$ case. We also distinguish a special class formulas that play a central roll in our axiomatizations.

Definition 2.2 We define the set of *sharp formulas* inductively as follows:

- $\top$ and $\bot$ are sharp;
- For any φ, $\Diamond\varphi$ is sharp;
- If φ and ψ are sharp, then $\varphi \wedge \psi$ and $\varphi \vee \psi$ are sharp.
- If φ is sharp, then $K\varphi$ is sharp;

We have the dual notion of a flat formula, which is not strictly speaking necessary to introduce but will make some definitions and arguments later on a bit cleaner.

Definition 2.3 We define the set of *flat formulas* inductively as follows:

- $\top$ and $\bot$ are flat;
- For any φ, $\Box\varphi$ is flat;
- If φ and ψ are flat, then $\varphi \wedge \psi$ and $\varphi \vee \psi$ are flat.
- If φ is flat, then $\hat{K}\varphi$ is flat;

With just classical reasoning, one can see that every sharp formula is equivalent to the negation of a flat one and vice versa. (This holds even in the restricted language $\mathcal{L}_{\Box}$.) For a set of formulas Γ, we write $\Gamma^{\sharp}$ for $\{\varphi \in \Gamma : \varphi \text{ is sharp}\}$ and $\Gamma^{\flat}$ for $\{\varphi \in \Gamma : \varphi \text{ is flat}\}$.

2.2 The Language of Knowledge and the Future

Our third language of interest $\mathcal{L}_{KG}$ is given by the formal grammar

$$\varphi ::= p \,|\, \neg\varphi \,|\, (\varphi \wedge \varphi) \,|\, K\varphi \,|\, G\varphi$$

where $p \in \mathbb{P}$. We will use the same bits of shorthand as before in addition to writing $F\varphi$ for $\neg G \neg \varphi$. $G\varphi$ is to be read as *from now on,* φ. As a warning, this reading differs slightly from the usual one found in tense logic. We want to hear our reading of G as naturally satisfying the **T** axiom.

For the rest of this paper, we fix the translation $t : \mathcal{L}_{K\Box} \to \mathcal{L}_{KG}$, which is defined as follows:

- $t(p) = p$;
- $t(\neg\varphi) = \neg t(\varphi)$;
- $t(\varphi \wedge \psi) = t(\varphi) \wedge t(\psi)$;
- $t(K\varphi) = Kt(\varphi)$;
- $t(\Box\varphi) = FKt(\varphi)$.

We will write $t_\Box$ for $t|_{\mathcal{L}_\Box}$.

2.3 Logics

First we remark that all of the logics considered this paper will be classical in the sense that this will contain every substitution instance of every theorem of the classical propositional calculus (CPC). Given some logic L in one of the above languages, we will write $\vdash_{\mathsf{L}} \varphi$ to mean that φ is a theorem of L. Further, we write $\Gamma \vdash_{\mathsf{L}} \varphi$ to mean that for some finite set $\{\gamma_i\}_i \subseteq \Gamma$ we have $\vdash_{\mathsf{L}} \bigwedge_i \gamma_i \to \varphi$. In either case, we might suppress the subscript when it is clear from context which logic is being discussed.

Definition 2.4 A set of formulas Γ is an L-*theory* just in case whenever $\Gamma \vdash_{\mathsf{L}} \varphi$, we have $\varphi \in \Gamma$.

Again, the L- prefix might be omitted when there is only one logic in play. As one final piece of notation, we write $\mathrm{cl}(\Gamma)$ for the smallest theory containing Γ. This is sure to exist as the property of being a theory is evidently closed under intersection. We also have the notion of maximal consistency.

Definition 2.5 An L-theory Γ is called L-*maximally consistent* if $\Gamma \nvdash_{\mathsf{L}} \bot$, but for any $\varphi \notin \Gamma$, $\Gamma \cup \{\varphi\} \vdash_{\mathsf{L}} \bot$.

Accordingly, we will be able to make free use of Lindenbaum's lemma.

Lemma 2.6 (Lindenbaum's Lemma) *Every consistent* L*-theory can be extended to maximally consistent* L*-theory.*

Proof. See, e.g., [6]. $\Box$

3 Semantics

3.1 Binary Relations

As the languages introduced in the previous section naturally have both an epistemic and a temporal flavor, this paper will be primarily concerned with birelational structures.

Definition 3.1 A *frame* is a triple $(W, R, \leqslant)$ such that W is a set and R and $\leqslant$ are binary relations on W.

We will typically refer to the first relation of a frame (R in the above definition) as the *epistemic relation* and the second relation of a frame ($\leqslant$ in the above definition) as the *temporal relation.* Of course, we will be interested in frames where the relations satisfy some additional properties. Recall the following standard definition.

Definition 3.2 A binary relation $\lhd$ on a set A is a *quasi-order* if it satisfies both of the following conditions.

- (Reflexivity) For all $x \in A$, $x \lhd x$.
- (Transitivity) For all $x, y, z \in A$, if $x \lhd y$ and $y \lhd z$, then $x \lhd z$.

A set equipped with a quasi-order is called a *quasi-ordered set*, which we might shorten to *qoset.* We also have distinguished maps of qosets.

Definition 3.3 Where $Q = (A, \lhd_A)$ and $P = (B, \lhd_B)$ an *order-preserving map* from Q to P is a function $f : A \to B$ such that for all $a, a' \in A$, $a \lhd_A a'$ implies $f(a) \lhd_B f(a')$.

We will also make use of linear orders as index sets for times.

Definition 3.4 A relation $\lhd$ is a *linear order* on a set A if it is quasi-order that moreover satisfies the following:

- (Antisymmtery) For all $x, y \in A$, if $x \lhd y$ and $y \lhd x$, then $x = y$.
- (Totality) For all $x, y \in A$, $x \lhd y$ or $y \lhd x$.

3.2 ET Frames and ET Diagrams

In our pursuit of frames suitable for understanding knowability, we begin with a frame-like notion that hopefully has some intuitive pull.

Definition 3.5 An *ET diagram* is a triple $\mathscr{D} = (\mathcal{I}, Q_-, P_{-,-})$ such that the following hold:

- $\mathcal{I} = (I, \preccurlyeq)$ is a linearly ordered set;
- Q_- assigns to each $\tau \in I$ a quasi-order Q_τ;
- $P_{-,-}$ assigns to each pair (τ, σ) with $\tau \preccurlyeq \sigma$ an order-preserving map $P_{\tau,\sigma} : Q_\sigma \to Q_\tau$ where $P_{\tau,\tau} = \mathrm{id}_{Q_\tau}$ for all $\tau \in I$ and $P_{\tau,\sigma} \circ P_{\sigma,\rho} = P_{\tau,\rho}$.

In such a diagram, I is a collection of times with $\preccurlyeq$ serving as the natural *at least as late as* order, Q assigns to each time τ a qoset meant to capture the agent's epistemic state at the different possible worlds at τ, and $P_{\tau,\sigma}$ maps each possible world at time σ to the possible world in its past at time τ. The order-preserving condition is a perfect-recall-like condition, enforcing that as the agent's epistemic state evolves in time, it only rules out possibilities, never ruling in a possibility already ruled out. To shed light on the definition of an ET diagram, we will present an example.

Example 3.6 Consider an urn containing two marbles, and assume that Betty knows that each marble is exactly black or white. She then retrieves the marbles from the urn, one at a time. The relevant times for us will be before she has retrieved any marbles (t_0), after she has retrieved one (t_1), and after she has retrieved both (t_2). Accordingly, the ET diagram will contain three qosets [2]. At t_0, Betty knows nothing about the colors of the marbles in the urn, so the the qoset will be one with three states—there are either zero, one, or two white marbles—that are all in the same equivalence class. At t_1, Betty has drawn either a white marble or a black marble, so there will be two equivalence classes with two states each; if she drew a black marble, then she is either in the case where there are no white marbles or the case where there is one (and similarly if she drew white). Finally at t_2, Betty has reached the end of inquiring about the urn, so the equivalence relation will just be identity. Note that we will still have four states, since in the case where there was a marble of each color in the urn, there were two possible histories.

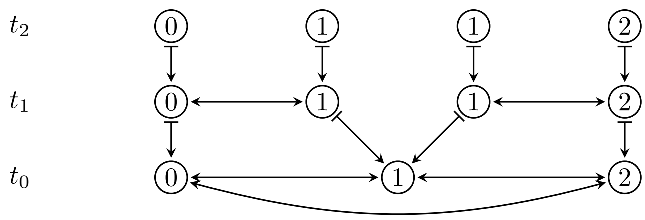

Fig. 1. The ET diagram for Example 3.6, where each state is marked with the number of white marbles in the urn. The identity maps have been omitted, as has the map from t_2 to t_0, which can be computed by composition. The reflexive arrows have also been omitted for each quasi-order.

Though it would be possible to work with ET diagrams directly, it will be easier to extract birelational structures so that we can use standard relation semantics.

Definition 3.7 Given an ET diagram $\mathscr{D} = (I, Q_-, P_{-,-})$, its *induced frame* is $F(\mathscr{D}) = (W_{\mathscr{D}}, R_{\mathscr{D}}, \leqslant_{\mathscr{D}})$, where $(W_{\mathscr{D}}, R_{\mathscr{D}})$ is the qoset obtained by taking the disjoint union of the Q_τs and $w \leqslant_{\mathscr{D}} v$ just in case $P_{\tau,\sigma}(v) = w$ where $w \in Q_\tau$ and $v \in Q_\sigma$.

[2] In this case, Betty's information is completely transparent to her, so the quasi-orders will in fact be equivalence relations.

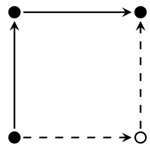

Fig. 2. A diagram representing the progressivity condition from Definition 3.9

Definition 3.8 A birelational frame $\mathscr{F}$ is an *ET frame* if $\mathscr{F} = F(\mathscr{D})$ for some ET diagram $\mathscr{D}$.

We call the class of ET frames $\mathfrak{E}$. If one is convinced by the naturality of ET diagrams, then $\mathfrak{E}$ seems to be the appropriate frame class for modeling knowability. One of the key moves that we want to make is recognizing that the class of ET frames sits inside a larger frame class that is more readily defined, easier to work with, and as it turns out, has the same logic. We now define these frames and verify that they constitute a superclass of $\mathfrak{E}$.

Definition 3.9 A *weak ET frame* is a triple $(W, R, \leqslant)$ where W is a set, R and $\leqslant$ are quasi-orders, and the following condition holds:

- (Progressivity) If $w \leqslant x \; R \; y$, then there exists $z \in W$ such that $w \; R \; z \leqslant y$.

We call the class of weak ET frames $\mathfrak{W}$.

Proposition 3.10 *If $\mathscr{F}$ is an ET frame, then it is a weak ET frame.*

Proof. By assumption, we have $\mathscr{F} = F(\mathscr{D}) = (W_{\mathscr{D}}, R_{\mathscr{D}}, \leqslant_{\mathscr{D}})$ for some ET diagram $\mathscr{D}$. $R_{\mathscr{D}}$ is a quasi-order since disjoint unions of quasi-orders are quasi-orders. $\leqslant_{\mathscr{D}}$ is also a quasi-order by the assumptions on $P_{-,-}$. Namely, $w \leqslant w$ since $P_{\tau,\tau}(w) = \mathrm{id}_{Q_\tau}(w) = w$, and if $w \leqslant_{\mathscr{D}} v \leqslant_{\mathscr{D}} u$, we have $P_{\tau,\rho}(u) = P_{\tau,\sigma}(P_{\sigma,\rho}(u)) = P_{\tau,\sigma}(v) = w$, so $w \leqslant_{\mathscr{D}} u$. Progression holds due to each $P_{\tau,\sigma}$ being order-preserving. If $w \leqslant_{\mathscr{D}} x \; R_{\mathscr{D}} \; y$, then the desired z is just $P_{\tau,\sigma}(y)$ where $w \in Q_\tau$ and $x, y \in Q_\sigma$. $\square$

Definition 3.11 A *weak ET model* is a 4-tuple $(W, R, \leqslant, V)$ where $(W, R, \leqslant)$ is a weak ET frame and $V : \mathbb{P} \to \mathcal{P}(W)$ is a valuation.

Definition 3.12 Truth of a formula (in any of the aforementioned languages) at a world w in a weak ET model $\mathscr{M} = (W, R, \leqslant, V)$ is defined as recursively as follows:

$$\begin{array}{lll}
\mathscr{M}, w \vDash p & \text{iff} & w \in V(p) \\
\mathscr{M}, w \vDash \neg\varphi & \text{iff} & \mathscr{M}, w \nvDash \varphi \\
\mathscr{M}, w \vDash \varphi \wedge \psi & \text{iff} & \mathscr{M}, w \vDash \varphi \text{ and } \mathscr{M}, w \vDash \psi \\
\mathscr{M}, w \vDash K\varphi & \text{iff} & \text{for all } v \text{ such that } w \; R \; v,\ \mathscr{M}, v \vDash \varphi \\
\mathscr{M}, w \vDash G\varphi & \text{iff} & \text{for all } v \text{ such that } w \leqslant v,\ \mathscr{M}, v \vDash \varphi \\
\mathscr{M}, w \vDash \Box\varphi & \text{iff} & \text{there exists } v \text{ such that } w \leqslant v \\
 & & \text{and for all } u \text{ such that } v \; R \; u,\ \mathscr{M}, u \vDash \varphi
\end{array}$$

Just by inspecting the truth clauses, we can see that our translation t preserves truth.

Lemma 3.13 *Let $\mathscr{M} = (W, R, \leqslant, V)$ be a weak ET model, $w \in W$, and $\varphi \in \mathcal{L}_{K\Box}$. Then $\mathscr{M}, w \vDash \varphi$ if and only if $\mathscr{M}, w \vDash t(\varphi)$.*

Proof. Induction on formula complexity. The only step that is not completely trivial simply requires recognizing that $FK\varphi$ and $\Box\varphi$ have the same truth conditions. $\Box$

We conclude the section by proving that sharp (respectively, flat) formulas are persistent (respectively, conversely persistent) along the temporal relation. This should make the axiomatizations of the knowability logics in the later sections more perspicuous.

Proposition 3.14 *Let $\mathscr{M}$ be a weak ET model and suppose that $w \leqslant x$. For a sharp formula χ, if $w \vDash \chi$, then $x \vDash \chi$. Dually, for a flat formula ε, if $x \vDash \varepsilon$ then $w \vDash \varepsilon$.*

Proof. We proceed by induction on rank. If $\chi = \Diamond\varphi$ for some φ, then $w \vDash \chi$ means that whenever $y \geq w$, $y\ R\ z$ for some z such that $z \vDash \varphi$. Clearly x will satisfy this condition, as well, since $y \geq x$ implies $y \geq w$ by transitivity. The case where χ is the conjunction or disjunction of two sharp formulas follows from the fact that the intersection and union of two upwardly closed sets is again upwardly closed. Finally suppose that $\chi = K\chi'$ where χ' is sharp. We want to show that if $x\ R\ y$, then $y \vDash \chi'$. For any such y, however, by excision we have a z such that $w\ R\ z \leqslant y$. Since $w \vDash K\chi'$, $z \vDash \chi'$. By induction, $y \vDash \chi'$, so indeed $x \vDash K\chi'$.

The converse persistence of flat formulas can be easily recovered from the persistence of sharp formulas together with the observation that the negation of a flat formula is equivalent to a sharp formula. $\Box$

4 Knowledge and the Future

The first logic that we will consider is LKF (presented in Figure 3), a logic of knowledge and the future, which we will show is sound and complete with respect to both of the model classes introduced in the previous section. This logic is fairly straightforward, only consisting in the fusion of S4 with itself plus one additional axiom to capture the progressivity condition. We begin with the following easy result, which is almost certainly known.

Theorem 4.1 LKF *is sound with respect to $\mathfrak{W}$.*

Proof. A bulk of the work is just checking the soundness of S4 with respect to quasi-orders, which is known (see, e.g., [6]). To verify the soundness of **P**, let $\mathscr{M} = (W, R, \leqslant, V)$ be a weak ET model, and suppose that some $w \in W$ satisfies $w \vDash KG\varphi$. Let x be arbitrary such that $w \leqslant x$, and let y be arbitrary such that $x\ R\ y$. By progressivity, there is some z such that $w\ R\ z \leqslant y$. Since $w \vDash KG\varphi$, $z \vDash G\varphi$, so $y \vDash \varphi$. As y was an arbitrary R-successor of x, $x \vDash K\varphi$, and since x was an arbitrary $\leqslant$-successor of w, $w \vDash GK\varphi$. $\Box$

Corollary 4.2 LKF *is sound with respect to $\mathfrak{E}$*

The Logic LKF	
C	any substitution instance of a theorem of CPC
$\mathbf{K}_K$	$K(\varphi \to \psi) \to (K\varphi \to K\psi)$
$\mathbf{T}_K$	$K\varphi \to \varphi$
$\mathbf{4}_K$	$K\varphi \to KK\varphi$
$\mathbf{K}_G$	$G(\varphi \to \psi) \to (G\varphi \to G\psi)$
$\mathbf{T}_G$	$G\varphi \to \varphi$
$\mathbf{4}_G$	$G\varphi \to GG\varphi$
P	$KG\varphi \to GK\varphi$
MP	from $\varphi \to \psi$ and φ infer ψ
$\mathtt{Nec}_K$	from φ infer $K\varphi$
$\mathtt{Nec}_G$	from φ infer $G\varphi$

Fig. 3.

Now we verify completeness with respect to $\mathfrak{E}$. We will in fact be able to get a slightly stronger result, which is that LKF is complete with respect to $\mathfrak{E}_\omega$, the class of frames induced by an ET diagram where the linear order is ω. Our approach will be to define a canonical ET diagram, from which a model can be easily recovered.

Definition 4.3 Let C and D be sets of $\mathcal{L}_{KG}$-formulas. We say that the pair (C, D) is *coherent* just in case D is LKF-consistent and for all formulas φ, if $C \vdash_{\mathsf{LKF}} G\varphi$, then $D \vdash_{\mathsf{LKF}} \varphi$.

Lemma 4.4 *Suppose that (C, Δ) is coherent, where Δ is a* LKF*-maximally consistent theory. Then there exists a* LKF*-maximally consistent theory $\Gamma \supseteq C$ such that (Γ, Δ) is coherent.*

Proof. Set $\Gamma_0 = C \cup \{\neg G\varphi \in \mathsf{LKF} : \neg\varphi \in \Delta\}$. We now check the consistency of Γ_0. If it were not consistent, then $C \vdash \neg(\bigwedge_{i=1}^n \neg G\varphi_i)$ where each $\neg\varphi_i \in \Delta$ for each i. By propositional logic, $C \vdash \bigvee_{i=1}^n G\varphi_i$. Since $\bigvee_{i=1}^n G\varphi_i \to G(\bigvee_{i=1}^n G\varphi_i)$ is a theorem of S4, we in fact have $C \vdash G(\bigvee_{i=1}^n G\varphi_i)$. By the coherence of (C, Δ), $\Delta \vdash \bigvee_{i=1}^n G\varphi_i$. The maximality of Δ ensure that $\Delta \vdash G\varphi_k$ for some k. By $\mathbf{T}_G$, $\Delta \vdash \varphi_k$, which contradicts the consistency of Δ. Let Γ be some maximally consistent theory extending Γ_0. It is clear that (Γ, Δ) is coherent, since whenever $\Delta \nvdash \varphi$, $\Gamma \nvdash G\varphi$ by construction. □

Definition 4.5 A *history* $h = \langle \Gamma_1, \ldots, \Gamma_n \rangle$ is a finite sequence of complete theories such that for any $1 \leq i < n$ the pair (Γ_i, Γ_{i+1}) is coherent.

We introduce some pieces of notation to facilitate talking about histories.

- We write $\ell(h)$ for the length of a history.
- For $n \leq \ell(h)$, we write $h|_n$ for the history obtained by keeping only the first n entries h.
- For $i \leq \ell(h)$ we write $h(i)$ for the ith entry of h.
- We write $h_\bullet$ for $h(\ell(h))$.

- The set of all history of length n is denoted H_n.
- For two histories h and h' with $\ell(h) = \ell(h') = n$, we write $h\ R_n\ h'$ if for all $1 \leq i \leq n$, whenever $K\varphi \in h(i)$, $\varphi \in h'(i)$.

Lemma 4.6 *For each n, (H_n, R_n) is a qoset.*

Proof. Where Γ and Δ are maximally consistent theories, define a relation r by $\Gamma\ r\ \Delta$ just in case whenever $K\varphi \in \Gamma$, $\varphi \in \Delta$. It follows from the logic of the modality K being $\mathsf{S4}$ that r is a quasi-order. (This is checked when one verifies that the canonical frame for $\mathsf{S4}$ is in the desired frame class, for example. For a reference, see [6].) It is easy to see that R_n is just the product r^n restricted to the set of histories of length n. Quasi-orders are closed under products and restriction, so we are done. □

Definition 4.7 The *canonical diagram for* LKF is $D^{\mathsf{LKF}} = (\omega, Q_{-}^{\mathsf{LKF}}, P_{-,-}^{\mathsf{LKF}})$ where

- Q_n^{LKF} is the set of all histories of length n ordered by R_n, and
- $P_{n,m}^{\mathsf{LKF}} : H_m \to H_n$ is given by $h \mapsto h|_n$.

Proposition 4.8 *The canonical diagram for* LKF *is an ET diagram.*

Proof. This follows from the previous lemma and the facts that $h|_{\ell(h)} = h$ and that $(h|_n)_m = h_m$ where $m \leq n \leq \ell(h)$. □

Lemma 4.9 *Suppose $h \in H_n$ and $K\varphi \notin h(n)$. Then there exists $h' \in H_n$ such that $\varphi \notin h'(n)$ and $h\ R_n\ h'$.*

Proof. Write $h = \langle \Gamma_1, \ldots, \Gamma_n \rangle$. We first construct a sequence

$$\langle \Gamma_1^K, \ldots, \Gamma_{n-1}^K, \Gamma_n^K \cup \{\neg\varphi\} \rangle$$

where Γ_i^K is just notation for $\{\psi \in \mathcal{L}_{KG} : K\psi \in \Gamma_i\}$. Now we check that each consecutive pair is coherent. By $\mathbf{T}_K$, for each $1 \leq i \leq n-1$, we have $\Gamma_i^K \subseteq \Gamma_i$, so Γ_i^K is consistent. To see that the last entry is consistent, we assume for the sake of contradiction that $\Gamma_n^K \vdash \varphi$. Then for some $\psi \in \Gamma_n^K$, we have $\vdash \psi \to \varphi$. Applying $\mathtt{Nec}_K$, $\vdash K(\psi \to \varphi)$. Then by $\mathbf{K}_K$, $\vdash K\psi \to K\varphi$. Since $K\psi \in \Gamma_n$, we have $K\varphi \in \Gamma_n$. Contradiction. Now let $1 \leq i \leq n-1$ and suppose $\Gamma_i^K \vdash G\psi$. Then $\Gamma_i \vdash KG\psi$. By $\mathbf{P}$, $\Gamma_i \vdash GK\psi$. Because (Γ_i, Γ_{i+1}) is coherent, $\Gamma_{i+1} \vdash K\psi$ so $\Gamma_{i+1}^K \vdash \psi$.

Let Δ_n be some maximally consistent theory extending $\Gamma_n^K \cup \{\neg\varphi\}$. We apply Lemma 4.4 to $(\Gamma_{n-1}^K, \Delta_n)$ to obtain Δ_{n-1} extending Γ_{n-1}^K such that (Δ_{n-1}, Δ_n) is coherent. Next, apply the lemma to $(\Gamma_{n-2}^K, \Delta_{n-1})$ to obtain Δ_{n-2}. Continue in this way until Δ_1 has been defined, and then set $h' = \langle \Delta_1, \ldots, \Delta_n \rangle$. By construction $h\ R_n\ h'$. □

Definition 4.10 The *canonical ET model for* LKF is $\mathscr{M}^{\mathsf{LKF}} = (W^{\mathsf{LKF}}, R^{\mathsf{LKF}}, \leqslant^{\mathsf{LKF}}, V^{\mathsf{LKF}})$ where $W^{\mathsf{LKF}} = W_{D^{\mathsf{LKF}}}$, $R^{\mathsf{LKF}} = R_{D^{\mathsf{LKF}}}$, $\leqslant^{\mathsf{LKF}} = \leqslant_{D^{\mathsf{LKF}}}$, and $V(p) = \{h \in W^{\mathsf{LKF}} : p \in h_\bullet\}$.

Lemma 4.11 *For any $\varphi \in \mathcal{L}_{KG}$ and $h \in W^{\mathsf{LKF}}$, $\mathscr{M}^{\mathsf{LKF}}, h \models \varphi$ if and only if $\varphi \in h_\bullet$.*

Proof. The proof is by induction on formula complexity. The atomic and boolean connective cases are trivial, so we restrict our attention to the two modal cases.

First we consider when $\varphi = K\psi$ for some ψ. Suppose that $K\psi \in h_\bullet$. Let h' be arbitrary such that $h \; R^{\mathsf{LKF}} \; h'$. This means that $h \; R_{\ell(h)} \; h'$, so by the definition of this relation, $\psi \in h'(\ell(h))$. By induction, $h' \vDash \psi$, so $h \vDash K\psi$. In the other direction, suppose that $K\psi \notin h_\bullet$. By Lemma 4.9, there is another history $h' \in H_{\ell(h)}$ such that $h \; R_{\ell(h)} \; h'$ and $\psi \notin h'(\ell(h))$. By induction $h' \nvDash \psi$, so $h \nvDash K\psi$.

Now we consider the case where $\varphi = G\psi$. If $h \leqslant^{\mathsf{LKF}} h'$, then $G\psi \in h_\bullet = h'(\ell(h))$. By $\mathbf{4}_G$ and $\mathbf{T}_G$ [3], $G^k\psi \in h'(\ell(h))$ where $k = \ell(h') - \ell(h)$. Since h' is a history, we will have $G^{k-l}\psi \in h'(\ell(h)+l)$ for all $0 \leq l \leq k$. In particular, $\psi \in h'(\ell(h)+k) = h'_\bullet$. By induction, $h' \vDash \psi$, so $h \vDash G\psi$. If $G\psi \notin h_\bullet$, a standard argument shows that $h_\bullet^G \cup \{\neg\psi\}$ is consistent. Let Γ be some maximally consistent theory extending this set. $(h_\bullet, \Gamma)$ is coherent by construction, so we set $h' = h^\frown\Gamma$. Since $\psi \notin h'_\bullet$, by induction $h' \nvDash \psi$. But $h \leqslant^{\mathsf{LKF}} h'$, so $h \nvDash G\psi$. □

Theorem 4.12 LKF *is strongly complete with respect to* $\mathfrak{E}$*.*

Proof. Suppose that $\Gamma \nvdash_{\mathsf{LKF}} \varphi$. Then $\Gamma \cup \{\neg\varphi\}$ is consistent and can be extended to a maximally consistent theory Γ'. Rather trivially $\langle\Gamma'\rangle$ is a history, so we have $\mathscr{M}^{\mathsf{LKF}}, \langle\Gamma'\rangle \nvDash \varphi$ by the truth lemma. □

Corollary 4.13 LKF *is complete with respect to* $\mathfrak{W}$*.*

5 The Logic of Knowledge and Knowability

We now present our first knowability logic LKKa, which is more specifically a logic of knowledge and knowability. The axioms are presented in Figure 4.

The Logic LKKa		
C	any substitution instance of a theorem of CPC	
$\mathbf{K}_K$	$K(\varphi \to \psi) \to (K\varphi \to K\psi)$	
$\mathbf{T}_K$	$K\varphi \to \varphi$	
$\mathbf{4}_K$	$K\varphi \to KK\varphi$	
A	$K\varphi \to \Box\varphi$	
`MP`	from $\varphi \to \psi$ and φ infer ψ	
`Nec`$_K$	from φ infer $K\varphi$	
`Sh`	from $\chi \to \hat{K}\psi$ infer $\chi \to \Diamond\psi$	χ is sharp

Fig. 4.

Theorem 5.1 LKKa *is sound with respect to* $\mathfrak{W}$*.*

Proof. The only interesting case to check is the validity of `Sh`. Suppose that $\chi \to \hat{K}\psi$ is globally true in some model $\mathscr{M} = (W, R, \leqslant, V)$. Then if $w \vDash \chi$,

[3] You need to use $\mathbf{T}_G$ only in the case where $h = h'$.

we have $v \vDash \chi$ for all $v \geqslant w$ by Proposition 3.14. This means that every $\leqslant$-successor of w must make true $\hat{K}\psi$, implying $w \vDash \Diamond\psi$, as desired. $\square$

Definition 5.2 The *canonical model for* LKKa is defined as $(W^{\mathsf{LKKa}}, R^{\mathsf{LKKa}}, \leqslant^{\mathsf{LKKa}}, V^{\mathsf{LKKa}})$ where

(i) W^{LKKa} is the set of all LKKa-maximally consistent theories;

(ii) $\Gamma\ R^{\mathsf{LKKa}}\ \Delta$ just in case for all φ, if $K\varphi \in \Gamma$ then $\varphi \in \Delta$;

(iii) $\Gamma \leqslant^{\mathsf{LKKa}} \Delta$ just in case for all flat formulas φ, if $\varphi \in \Delta$, then $\varphi \in \Gamma$;

(iv) $V^{\mathsf{LKKa}}(p) = \{\Gamma \in W^{\mathsf{LKKa}} : p \in \Gamma\}$.

Note that item (iii) in the above definition could be replaced by declaring $\Gamma \leqslant^{\mathsf{LKKa}} \Delta$ just in case for all sharp formulas φ, if $\varphi \in \Gamma$, then $\varphi \in \Delta$.

Lemma 5.3 *Suppose that* $\Box\varphi \in \Gamma$. *Then there exists a* Δ *such that* $\Gamma \leqslant^{\mathsf{LKKa}} \Delta$ *and* $K\varphi \in \Delta$.

Proof. We want to show that $\Gamma^{\sharp} \cup \{K\varphi\}$ is consistent. If not, we have some formula $\psi \in \Gamma^{\sharp}$ such that $\vdash \psi \to \hat{K}\neg\varphi$. By `Sh`, $\vdash \psi \to \Diamond\neg\varphi$. Therefore $\Gamma \vdash \Diamond\neg\varphi$, which is a contradiction. $\square$

Lemma 5.4 *The frame* $(W^{\mathsf{LKKa}}, R^{\mathsf{LKKa}}, \leqslant^{\mathsf{LKKa}})$ *is progressive.*

Proof. Suppose that $\Gamma \leqslant^{\mathsf{LKKa}} \Delta$ and $\Delta\ R^{\mathsf{LKKa}}\ \Theta$. We wish to construct a Λ such that $\Gamma\ R^{\mathsf{LKKa}}\ \Lambda$ and $\Lambda \leqslant^{\mathsf{LKKa}} \Theta$. To this end, we will show that $\Gamma^K \cup \Theta^{\flat}$ is consistent. If not, then for some $\varphi \in \Gamma^K$ and some $\psi \in \Theta^{\flat}$, we have $\vdash \varphi \to \neg\psi$. By $\mathtt{Nec}_K$ and $\mathbf{K}_K$, $\vdash K\varphi \to K\neg\psi$, so $\Gamma \vdash K\neg\psi$. On the other hand, since $\Delta \leqslant^{\mathsf{LKKa}} \Theta$, we have $\Delta \vdash \hat{K}\psi$. Since ψ is flat, $\hat{K}\psi$ is also flat, so $\Gamma \vdash \hat{K}\psi$. Contradiction. We take Λ to be a maximally consistent extension of $\Gamma^K \cup \Theta^{\flat}$. By construction, $\Gamma\ R^{\mathsf{LKKa}}\ \Lambda$ and $\Lambda \leqslant^{\mathsf{LKKa}} \Theta$. $\square$

Lemma 5.5 $\varphi \in \Gamma$ *if and only if* $\Gamma \vDash \varphi$.

Proof. We proceed by induction on complexity. The atomic and boolean connective cases are trivial. The K case follows from a standard argument. Now we suppose that $\varphi = \Box\psi$. If $\Box\psi \in \Gamma$, then by Lemma 5.3 there is some Δ such that $K\psi \in \Delta$ and $\Gamma \leqslant^{\mathsf{LKKa}} \Delta$. By induction, $\Delta \vDash K\psi$, so $\Gamma \vDash \Box\psi$. On the other hand, if $\Box\psi \notin \Gamma$, which means that $\Diamond\neg\psi \in \Gamma$. Let Δ be arbitrary such that $\Gamma \leqslant^{\mathsf{LKKa}} \Delta$. Since $\Diamond\neg\psi$ is sharp, we have $\Diamond\neg\psi \in \Delta$. This implies that $\hat{K}\neg\psi \in \Delta$. $\square$

Theorem 5.6 LKKa *is strongly complete with respect to* $\mathfrak{W}$.

Corollary 5.7 *The translation* t *fully and faithfully embeds* LKKa *into* LKF.

Proof. Suppose that φ is a theorem of LKKa. By soundness of LKKa, φ is true in all weak ET models. Applying Lemma 3.13, $t(\varphi)$ is also true in all weak ET models. By completeness of LKF, $\vdash_{\mathsf{LKF}} t(\varphi)$. In the other direction, if φ is not a theorem of LKKa, because LKKa is complete with respect to $\mathfrak{W}$, φ is refuted in some model in the class. Lemma 3.13 then tells us that $t(\varphi)$ is refuted in the same model. By the soundness of LKF, $\nvdash_{\mathsf{LKF}} t(\varphi)$. $\square$

Corollary 5.8 LKKa *is complete with respect to* $\mathfrak{E}$.

6 Pure Knowability

The final logic we introduce is the logic of pure knowability LPKa. An axiomatization is presented in Figure 5

The Logic of Knowability		
C	any substitution instance of a theorem of CPC	
$\mathbf{Ka}_\Box$	$\Box\top$	
$\mathbf{4}^*_\Box$	$\Box\varphi \to \Box(\varphi \wedge \Box\varphi)$	
$\mathbf{T}^\flat_\Box$	$\Box\varepsilon \to \varepsilon$	ε is flat
`MP`	from $\varphi \to \psi$ and φ infer ψ	
`Reg`	from $\varphi \to \psi$ infer $\Box\varphi \to \Box\psi$	
`PR`	from $\chi \to (\varphi \to \psi)$ infer $\Box(\theta \wedge \chi) \to \Box(\theta \wedge (\Box\varphi \to \Box\psi))$	χ is sharp

Fig. 5.

The only deductive fact beyond the axioms and rules that we will need is that $\vdash_{\mathsf{LPKa}} \neg\Box\bot$, which follows from taking $\varepsilon = \bot$ in $\mathbf{T}^\flat_\Box$.

Theorem 6.1 LPKa *is sound with respect to* $\mathfrak{E}$.

Proof. **C**, $\mathbf{Ka}_\Box$, and $\mathbf{4}^*_\Box$ are all straightforward to verify. $\mathbf{T}^\flat_\Box$ follows quickly from Lemma 3.14. `MP` and `Reg` are also easy. This leaves `PR`. Suppose that $\vDash \chi \to (\varphi \to \psi)$, and let x be an arbitrary world in a model $\mathscr{M} = (W, R, \leqslant)$. Suppose that $x \vDash \Box(\theta \wedge \chi)$. Then there exists a world $x' \geq x$ such that each R-successor y' of x' satisfies $y' \vDash \theta \wedge \chi$. Suppose that such a y' also has $y' \vDash \Box\varphi$. Then there is some $y'' \geq y'$ such that each R-successor z'' of y'' has $z'' \vDash \varphi$. By progressivity, there is a $z' \leqslant z''$ such that $y' \; R \; z'$. By the transitivity of R, $x' \; R \; z'$, so $z' \vDash \chi$. Applying Lemma 3.14, $z'' \vDash \chi$. Therefore $z'' \vDash \psi$. Since z'' was an arbitrary R-successor of y'' and $y' \leqslant y''$, we must have $y' \vDash \Box\psi$. As $y' \vDash \theta$ and y' is an arbitrary R-successor of x' satisfying $\Box\varphi$, so $x \leqslant x'$ implies that $x \vDash \Box(\theta \wedge \chi) \to \Box(\theta \wedge (\Box\varphi \to \Box\psi))$. □

In establishing completeness, one might seek a model construction similar to the one found for LKKa in the previous section. The obvious obstacle to doing this, however, is that we no longer have a K modality with which to define the epistemic relation. As a workaround, we take inspiration from a standard canonical model argument. In a canonical model, we settle on using maximally consistent theories as points because they are exactly the kinds of theories that can appear as the set of all true formulas at a world in a model. We will attempt to do the same thing for *known formulas*. That is, we will offer a characterization of theories which can appear as the set of all formulas that are true at all epistemically accessible worlds from some world in some model.

Definition 6.2 A theory $\mathcal{E} \subseteq \mathcal{L}_\Box$ is called *epistemic* if it satisfies the following conditions

- (Consistency) $\bot \notin \mathcal{E}$;

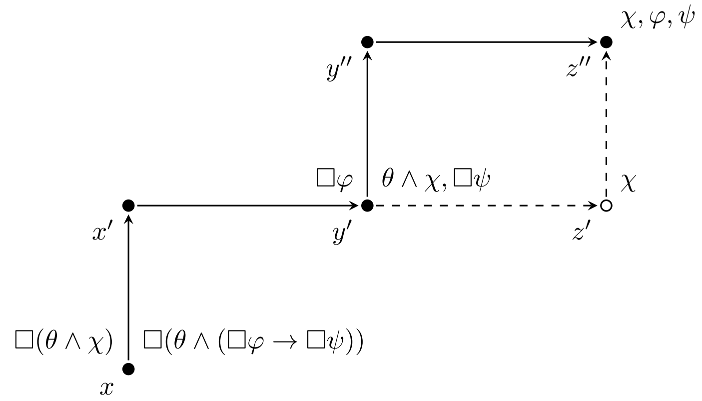

Fig. 6. A diagram illustrating the argument in the proof of Theorem 6.1.

- (Box closure) If $\alpha \in \mathcal{E}$, then $\Box\alpha \in \mathcal{E}$;
- (Sharp regularity) If $\chi \in \mathcal{E}^\sharp$ and $\vdash \chi \to (\beta \to \gamma)$, then $\Box\beta \to \Box\gamma \in \mathcal{E}$.

The points in the model we construct will then be pairs of maximally consistent theories (the truths) and epistemic theories (the known truths). But which such pairs will we use? As it turns out, the only compatibility condition we need is that the epistemic theory is contained in the maximally consistent theory, guaranteeing factivity of knowledge.

Definition 6.3 A pair of theories $(\Gamma, \mathcal{E})$ is called *concordant* if Γ is a maximally consistent theory, $\mathcal{E}$ is an epistemic theory, and $\mathcal{E} \subseteq \Gamma$.

We are now positioned to define the model. We use upward preservation of sharp formulas and of known sharp formulas to define the temporal relation, while the epistemic relation now has a simple definition that we extract from the epistemic theories.

Definition 6.4 We define the *canonical model for* LPKa $\mathscr{M}^{\mathsf{LPKa}} = (W^{\mathsf{LPKa}}, R^{\mathsf{LPKa}}, \leqslant^{\mathsf{LPKa}}, V^{\mathsf{LPKa}})$ as follows:

(i) W^{LPKa} is the set of concordant pairs.

(ii) $(\Gamma, \mathcal{E})\ R^{\mathsf{LPKa}}\ (\Delta, \mathcal{F})$ just in case $\mathcal{E} \subseteq \mathcal{F}$.

(iii) $(\Gamma, \mathcal{E}) \leqslant^{\mathsf{LPKa}} (\Delta, \mathcal{F})$ just in case $\Gamma^\sharp \subseteq \Delta^\sharp$ and $\mathcal{E}^\sharp \subseteq \mathcal{F}^\sharp$.

(iv) $V^{\mathsf{LPKa}}(p) = \{(\Gamma, \mathcal{E}) \in W : p \in \Gamma\}$.

The way that we have defined $\mathscr{M}^{\mathsf{LPKa}}$ makes it fairly easy to check that it is in the correct frame class.

Proposition 6.5 $(W^{\mathsf{LPKa}}, R^{\mathsf{LPKa}}, \leqslant^{\mathsf{LPKa}})$ *is in* $\mathfrak{E}$.

Proof. It is immediate that R^{LPKa} and $\leqslant^{\mathsf{LPKa}}$ are quasi-orders, so what remains is to check progressivity. Suppose that $(\Gamma, \mathcal{E}) \leqslant^{\mathsf{LPKa}} (\Delta, \mathcal{F})\ R^{\mathsf{LPKa}}\ (\Theta, \mathcal{G})$. We first show that $\mathcal{E} \cup \Theta^\flat$ is consistent. If not, then $\vdash \alpha \to \neg\varepsilon$ for some $\alpha \in \mathcal{E}$ and

$\varepsilon \in \Theta^\flat$. This implies that $\neg\varepsilon \in \mathcal{E}$, and because $\neg\varepsilon$ is equivalent to some sharp formula, we have $\neg\varepsilon \in \mathcal{F}$, which implies $\neg\varepsilon \in \mathcal{G}$, contradicting the consistency of Θ. Let Λ be some maximally consistent theory extending $\mathcal{E} \cup \Theta^\flat$. We now claim that $(\Gamma, \mathcal{E})\ R^{\mathsf{LPKa}}\ (\Lambda, \mathcal{E}) \leqslant^{\mathsf{LPKa}} (\Theta, \mathcal{G})$. It is trivial that $(\Gamma, \mathcal{E})\ R^{\mathsf{LPKa}}\ (\Lambda, \mathcal{E})$ since $\mathcal{E} = \mathcal{E}$. We also have $\Lambda^\sharp \subseteq \Theta^\sharp$ by construction, as this is equivalent to $\Theta^\flat \subseteq \Lambda^\flat$. What remains to show is that $\mathcal{E}^\sharp \subseteq \mathcal{G}^\sharp$. But this is immediate, since the assumptions imply $\mathcal{E}^\sharp \subseteq \mathcal{F}^\sharp \subseteq \mathcal{G}^\sharp$. $\square$

We now establish some lemmas that will facilitate the proof of the truth lemma. The first is rather quick, and intuitively just says that formulas omitted from the epistemic theory of a concordant are in fact not know there.

Lemma 6.6 *Suppose that* $(\Gamma, \mathcal{E}) \in W$ *has* $\varphi \notin \mathcal{E}$*. Then there exists a concordant pair* $(\Delta, \mathcal{F})$ *such that* $\varphi \notin \Delta$ *and* $(\Gamma, \mathcal{E})\ R^{\mathsf{LPKa}}\ (\Delta, \mathcal{F})$*.*

Proof. It is immediate that $\mathcal{E} \cup \{\neg\varphi\}$ is consistent. Let Δ be a maximally consistent theory extending $\mathcal{E} \cup \{\neg\varphi\}$, and let $\mathcal{F} = \mathcal{E}$. $\square$

It will be much more difficult to verify that if the maximally consistent theory of a concordant pair contains a knowability claim, then the purportedly knowable proposition actually appears in the epistemic theory of some temporal successor of the pair. This will require a recursive construction whose core machinery is established in the following three claims.

Claim 6.7 *Suppose that* $(\Gamma, \mathcal{E})$ *is concordant and that* $\Box\alpha \in \Gamma$*. Then* $\mathcal{E}^\sharp \cup \{\alpha \wedge \Box\alpha\}$ *is consistent.*

Proof. Suppose not. Then $\vdash \chi \to \neg(\alpha \wedge \Box\alpha)$ for some $\chi \in \mathcal{E}^\sharp$. This can be equivalently written as $\vdash \chi \to ((\alpha \wedge \Box\alpha) \to \bot)$. By sharp regularity, $\Box(\alpha \wedge \Box\alpha) \to \Box\bot \in \mathcal{E}$. Contraposing and using $\vdash \neg\Box\bot$, we see that $\neg\Box(\alpha \wedge \Box\alpha) \in \mathcal{E} \subseteq \Gamma$. But since $\Box\alpha \in \Gamma$, by $\mathbf{4}^*_\Box$, $\Box(\alpha \wedge \Box\alpha) \in \Gamma$, which is a contradiction. $\square$

Claim 6.8 *Suppose that* $(\Gamma, \mathcal{E})$ *is concordant and that* $\Box\alpha \in \Gamma$*. If* $\mathcal{E}^\sharp \vdash (\alpha \wedge \Box\alpha) \to \beta$*, then* $\Box(\alpha \wedge \beta) \in \Gamma$*.*

Proof. As $\mathcal{E}^\sharp$ is closed under conjunction, the hypothesis gives us $\vdash \chi \to ((\alpha \wedge \Box\alpha) \to \beta)$ for some $\chi \in \mathcal{E}^\sharp$. This is classically equivalent to $\vdash \chi \to ((\alpha \wedge \Box\alpha) \to (\alpha \wedge \beta))$. By sharp regularity, $\Box(\alpha \wedge \Box\alpha) \to \Box(\alpha \wedge \beta) \in \mathcal{E}$. Since $\vdash \Box\alpha \to \Box(\alpha \wedge \Box\alpha)$, we have $\Box\alpha \to \Box(\alpha \wedge \beta) \in \mathcal{E}$. Finally, since $\mathcal{E} \subseteq \Gamma$ and $\Box\alpha \in \Gamma$, we have $\Box(\alpha \wedge \beta) \in \Gamma$. $\square$

Claim 6.9 *Suppose that* $(\Gamma, \mathcal{E})$ *is concordant and that* $\Box\alpha \in \Gamma$*. If* $\mathcal{E}^\sharp \vdash (\alpha \wedge \Box\alpha) \to \chi$ *and* $\vdash \chi \to (\beta \to \gamma)$ *for some sharp formula* χ*, then* $\Box(\alpha \wedge (\Box\beta \to \Box\gamma)) \in \Gamma$*.*

Proof. First note that by Claim 6.8, $\Box(\alpha \wedge \chi) \in \Gamma$. Applying `PR` to the assumption $\vdash \chi \to (\beta \to \gamma)$, we obtain $\vdash \Box(\alpha \wedge \chi) \to \Box(\alpha \wedge (\Box\beta \to \Box\gamma))$. These facts entail $\Box(\alpha \wedge (\Box\beta \to \Box\gamma)) \in \Gamma$, as desired. $\square$

We can now prove the key lemma for the completeness argument.

Lemma 6.10 *Suppose that* $(\Gamma, \mathcal{E}) \in W$ *and* $\Box\varphi \in \Gamma$*. Then there exists a concordant pair* $(\Delta, \mathcal{F})$ *such that* $\varphi \in \mathcal{F}$ *and* $(\Gamma, \mathcal{E}) \leqslant^{\mathsf{LPKa}} (\Delta, \mathcal{F})$*.*

Proof. First we construct $\mathcal{F}$. Fix an enumeration $(\psi_i)_{i<\omega}$ of all formulas and an enumeration $(\theta_i)_{i<\omega}$ of all formulas whose outermost connective is a conditional. For convenience, write $\theta_i = \beta_i \to \gamma_i$. We will recursively define a sequence of formulas $(\alpha_i)_{i<\omega}$ as follows:

- $\alpha_0 = \varphi$.
- If $n > 0$ is odd, let k_n be the least natural such that $\mathcal{E}^\sharp \cup \{\alpha_{n-1} \wedge \Box\alpha_{n-1}\} \vdash \psi_{k_n}$ but $\mathcal{E}^\sharp \cup \{\alpha_{n-1} \wedge \Box\alpha_{n-1}\} \nvdash \Box\psi_{k_n}$. Set $\alpha_n = \alpha_{n-1} \wedge \psi_{k_n}$. If there is no such k_n, take $\alpha_n = \alpha_{n-1}$.
- If $n > 0$ is even, let k_n be the least natural such that $\mathcal{E}^\sharp \cup \{\alpha_{n-1} \wedge \Box\alpha_{n-1}\} \vdash \chi$ and $\vdash \chi \to \theta_{k_n}$ but $\mathcal{E}^\sharp \cup \{\alpha_{n-1} \wedge \Box\alpha_{n-1}\} \nvdash \Box\beta_i \to \Box\gamma_i$. Set $\alpha_n = \alpha_{n-1} \wedge (\Box\beta_i \to \Box\gamma_i)$. If there is no such k_n, take $\alpha_n = \alpha_{n-1}$.

First we claim that for each n, $\Box\alpha_n \in \Gamma$. The proof is by induction. The base case is by assumption. Now consider the case where $n > 0$ is odd. If $\alpha_n = \alpha_{n-1}$ we are done by induction. Otherwise, by Claim 6.8, $\Box(\alpha_{n-1} \wedge \psi_{k_n}) \in \Gamma$. Since $\alpha_n = \alpha_{n-1} \wedge \psi_{k_n}$, we are done. Last is the case where $n > 0$ is even. Again, if $\alpha_n = \alpha_{n-1}$, we are done by induction. Otherwise, Claim 6.8 tells us that $\Box(\alpha_{n-1} \wedge (\Box\beta_{k_n} \to \Box\gamma_{k_n})) \in \Gamma$.

For each n, let $\mathcal{F}_n = \mathrm{cl}(\mathcal{E}^\sharp \cup \{\alpha_n \wedge \Box\alpha_n\})$. Note that by Claim 6.7, each $\mathcal{F}_n$ is consistent. Now we claim that for any n, $\mathcal{F}_n \subseteq \mathcal{F}_{n+1}$. It suffices to show that $\alpha_{n+1} \wedge \Box\alpha_{n+1} \vdash \alpha_n \wedge \Box\alpha_n$. It is clear that $\alpha_{n+1} \vdash \alpha_n$, since α_{n+1} is either α_n or the conjunction of α_n with another formula. By $\mathtt{Reg}$, we then also have $\Box\alpha_{n+1} \vdash \Box\alpha_n$.

Now we set $\mathcal{F} = \bigcup \mathcal{F}_n$. We want to check that $\mathcal{F}$ is an epistemic theory.

- $\mathcal{F}$ is an increasing union of consistent theories, so it is a consistent theory.
- Suppose that $\alpha \in \mathcal{F}$. If $\Box\alpha \notin \mathcal{F}$, then at some odd stage $n > 0$, we would have set $\alpha_n = \alpha_{n-1} \wedge \alpha$. But then $\Box\alpha \in \mathrm{cl}(\mathcal{E}^\sharp \cup \{\alpha_n \wedge \Box\alpha_n\}) = \mathcal{F}_n$, which is a contradiction. Therefore $\Box\alpha \in \mathcal{F}$, so $\mathcal{F}$ is box closed.
- Suppose that $\chi \in \mathcal{F}^\sharp$ and $\vdash \chi \to (\beta \to \gamma)$. If $\Box\beta \to \Box\gamma \notin \mathcal{F}$, then at some even stage $n > 0$, we would have set $\alpha_n = \alpha_{n_1} \wedge (\Box\beta \to \Box\gamma)$. Again, this implies that $\Box\beta \to \Box\gamma \in \mathcal{F}_n$, which is a contradiction, so indeed $\Box\beta \to \Box\gamma \in \mathcal{F}$, and we can conclude that $\mathcal{F}$ is sharply regular.

To finish the argument, we need to find a maximally consistent theory Δ such that $(\Delta, \mathcal{F})$ is concordant and $(\Gamma, \mathcal{E}) \leqslant^{\mathsf{LPKa}} (\Delta, \mathcal{F})$. To this end, we verify the consistency of $\Gamma^\sharp \cup \mathcal{F}$. First, we claim that if $\alpha \in \mathcal{F}$ then $\Box\alpha \in \Gamma$. If $\alpha \in \mathcal{F}$, then $\alpha \in \mathcal{F}_n$ for some n. For such an n, $\vdash \chi \to ((\alpha_n \wedge \Box\alpha_n) \to \alpha)$ for some $\chi \in \mathcal{E}^\sharp$. By sharp regularity, $\Box(\alpha_n \wedge \Box\alpha_n) \to \Box\alpha \in \mathcal{E}$, so by $\mathbf{4}^*_\Box$, $\Box\alpha_n \to \Box\alpha \in \mathcal{E}$. We already established that $\Box\alpha_n \in \Gamma$ for each n, so it must be that $\Box\alpha \in \mathcal{E}$. Now suppose that $\Gamma^\sharp \cup \mathcal{F}$ is inconsistent. This implies that $\vdash \alpha \to \neg\chi$ for some $\alpha \in \mathcal{F}$ and $\chi \in \Gamma^\sharp$. By $\mathtt{Reg}$, $\vdash \Box\alpha \to \Box\neg\chi$, and since the claim tells us that $\Box\alpha \in \Gamma$, we have $\Box\neg\chi \in \Gamma$. But $\chi \in \Gamma^\sharp$, so $\Diamond\chi \in \Gamma$ by $\mathbf{T}^\sharp_\Diamond$, contradicting the consistency of Γ. Let Δ be some maximally consistent theory extending Γ. We have $\Gamma^\sharp \subseteq \Delta^\sharp$ and $\mathcal{E}^\sharp \subseteq \mathcal{F}^\sharp$, so $(\Gamma, \mathcal{E}) \leqslant^{\mathsf{LPKa}} (\Delta, \mathcal{F})$, as desired. $\Box$

Lemma 6.11 *For any concordant pair* $(\Gamma, \mathcal{E})$*,* $\varphi \in \Gamma$ *if and only if* $\mathscr{M}, (\Gamma, \mathcal{E}) \vDash \varphi$*.*

Proof. The proof is by induction on formula complexity. The only non-trivial case is when $\varphi = \Box\psi$ for some ψ.

Suppose $\Box\psi \in \Gamma$. By Lemma 6.10, there exists a concordant pair $(\Delta, \mathcal{F})$ such that $\psi \in \mathcal{F}$. Now let $(\Theta, \mathcal{G})$ be an arbitrary concordant pair such that $(\Delta, \mathcal{F})\ R^{\mathsf{LPKa}}\ (\Theta, \mathcal{G})$. Since $\mathcal{F} \subseteq \mathcal{G}$, $\psi \in \mathcal{G}$. But then $\psi \in \Theta$, so by the induction hypothesis, $\mathscr{M}, (\Theta, \mathcal{G}) \vDash \psi$. We conclude that $\mathscr{M}, (\Gamma, \mathcal{E}) \vDash \Box\psi$.

In the other direction, suppose that $\Box\psi \notin \Gamma$. Then $\Diamond\neg\psi \in \Gamma$. Let $(\Delta, \mathcal{F})$ be arbitrary such that $(\Gamma, \mathcal{E}) \leqslant^{\mathsf{LPKa}} (\Delta, \mathcal{F})$. Since $\Diamond\neg\psi$ is sharp, $\Diamond\neg\psi \in \Delta$. This implies $\psi \notin \mathcal{F}$, as otherwise we would have $\Box\psi \in \mathcal{F} \subseteq \Delta$, contradicting the consistency of Δ. By Lemma 6.6, there is a concordant pair $(\Theta, \mathcal{G})$ such that $\psi \notin \Theta$. By induction, $(\Theta, \mathcal{G}) \nvDash \psi$. Therefore, $(\Gamma, \mathcal{E}) \nvDash \Box\psi$. $\Box$

Theorem 6.12 LPKa *is strongly complete with respect to* $\mathfrak{W}$*.*

Proof. Suppose that $\Gamma \nvdash \varphi$. Then $\Gamma \cup \{\neg\varphi\}$ is consistent and can be extended to some maximally consistent Γ'. We first want to find some theory $\mathcal{E}$ such that $(\Gamma', \mathcal{E})$ is concordant. We propose $\mathcal{E} = \mathrm{cl}(\varnothing)$. Clearly, we have $\mathcal{E} \subseteq \Gamma'$, so we just need to check that $\mathcal{E}$ is epistemic.

- Consistency of $\mathcal{E}$ follows from soundness and the nonemptiness of $\mathfrak{W}$.
- Box closure follows from $\mathbf{Ka}_\Box$.
- Sharp regularity follows from `Reg`.

Since $(\Gamma', \mathcal{E})$ is concordant, $\mathscr{M}, (\Gamma', \mathcal{E}) \vDash \Gamma$ but $\mathscr{M}, (\Gamma', \mathcal{E}) \nvDash \varphi$ by the truth lemma, so $\Gamma \nvDash \varphi$. $\Box$

Corollary 6.13 *The translation* $t_\Box$ *fully and faithfully embeds* LPKa *into* LKF*.*

Proof. The proof is essentially identical to that of Corollary 5.7. $\Box$

Corollary 6.14 LPKa *is complete with respect to* $\mathfrak{E}$*.*

7 Future Work

The ideas in this paper suggests a systematic approach to thinking about knowability after one settles on a particular logic for knowledge. Namely, provided the logic of knowledge is Kripke complete, one could redefine an ET diagram so that each Q_τ is of the appropriate frame class. Then the logics of knowledge and knowability and of pure knowability coming from that notion of ET diagram could be axiomatized, providing an understanding how knowability behaves relative to that particular conception of knowledge. Interesting cases to treat include $\mathsf{S5}$, which is often used in applications of epistemic logic to game theory (see, e.g., [2] and [13]) and $\mathsf{S4.2}$, which Stalnaker claims to be the correct logic of knowledge in [12]. It might also be of interest to investigate the analogs of these knowability logics that come from removing or relaxing the progressivity condition on frames.

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