# Toward the van Benthem Characterization Theorem for Non-Distributive Modal Logic

Yiwen Ding[a] Krishna Manoorkar[a] Mattia Panettiere[a]
Ruoding Wang[a,b]

[a] *Vrije Universiteit Amsterdam, The Netherlands*

[b] *Xiamen University, China*

**Abstract**

In this paper, we introduce the simulations and bisimulations on polarity-based semantics for non-distributive modal logic, which are natural generalizations of those notions on Kripke semantics for modal logic. We also generalize other important model-theoretic notions about Kripke semantics such as image-finite models, modally-saturated models, ultrafilter extension and ultrapower extension to the non-distributive setting. By using these generalizations, we prove the Hennessy-Milner theorem and the van Benthem characterization theorem for non-distributive modal logic based on polarity-based semantics.

*Keywords:* Polarity-based semantics, Non-distributive modal logics, Simulations, Bisimulations, Hennessy-Milner theorem, van Benthem characterization theorem.

## 1 Introduction

This paper focuses on advancing the model theory of polarity-based semantics for *normal non-distributive modal logics* [19,6,9]. These logics, also known as *LE-logics* (lattice expansion logics), are characterized by algebraic semantics provided by arbitrary lattices expanded with normal operators. The research on polarity-based semantics for normal non-distributive modal logics dates back to Gehrke's work on canonical extension of bounded lattices [22,21,17]. Leveraging the interpretation of polarities as formal contexts in Formal Concept Analysis [20], non-distributive modal logic is conceptualized as a logic of formal concepts or categories enriched with modal operators, such as approximation or knowledge operators [6,7]. Several model theoretic notions for polarity-based semantics like *disjoint unions*, *p-morphisms*, and *ultrafilter extensions* were introduced in prior work [10]. These concepts were used in [10] to prove

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*Goldblatt-Thomason theorem* for LE-logics. Simulations on polarity-based models were defined and employed to demonstrate the preservation of lattice-based modal $\mu$-calculus (an extension of basic LE-logic with the least and the greatest fixed point operators) under simulations [16, Section 6]. In this paper, we delve more into the problem of generalizing simulations and bisimulations from Kripke semantics to polarity-based semantics, and also prove fundamental results like Hennessy-Milner theorem and van Benthem characterization theorem for polarity-based semantics.

Bisimulations were initially defined by van Benthem [34] as p-relations on Kripke frames for classical modal logic. They aim to characterize points in Kripke models with equivalent behavior concerning modal logic formulas. Since their inception, bisimulations have played an important role in modal logic theory and its applications in computer science [34,32,25,23]. Hennessy and Milner [26] showed that on image-finite Kripke models, the semantic equivalence relation for modal logic forms a bisimulation relation, a result later extended to modally-saturated Kripke models [27]. Van Benthem [34] established the characterization theorem for modal logic, showing that classical modal logic is the bisimulation-invariant fragment of first-order logic.

Bisimulations have been extensively defined and studied across various semantic models for different logics. For instance, in topological semantics for (S4) modal logic, they are referred to as topo-bisimulations [1], where the Hennessy-Milner property for a class of topological models was established [11]. Similarly, in Kripke semantics for intuitionistic modal logic, they are known as asimulations, and results like the Hennessy-Milner theorem and the van Benthem theorem have been demonstrated [30]. Bezhanishvilli and Henke [3] proved the van Benthem theorem for models based on general descriptive frames using *Vietoris bisimulations* [2]. Similar studies have been carried out for different semantics of graded modal logics, dynamic modal logics, fuzzy modal logics, monotone modal logics, etc. [15,33,13,14,24]. De Groot [12] established a van Benthem theorem for the meet-semilattice semantics of non-distributive (positive) logic using the notion of meet simulations. In this work, we adopt a similar approach to prove the van Benthem theorem for the polarity-based semantics of normal non-distributive modal logic by defining appropriate notion of simulations. Along the way, we introduce essential model-theoretic notions like image-finite models, modally-saturated models, and $\omega$-saturated models for this semantics, and prove significant results like the Hennessy-Milner theorem and the Hennessy-Milner property for the class of modally-saturated models.

*Structure of the paper.* In Section 2, we collect the required preliminaries. In Section 3, we define the simulations and bisimulations on the polarity-based semantics and prove the Hennessy-Milner theorem. In Section 4, we define the modally-saturated polarity-based models, filter-ideal extensions of polarity-based models and show that two pointed polarity-based models are modally equivalent if and only if their filter-ideal extensions are bisimilar. In Section 5, we define the standard translation and prove the van Benthem characterization theorem for polarity-based semantics of non-distributive modal logic. In Section 6, we summarize the main findings of the paper and suggest some possible future works.

## 2 Preliminaries

In this section, we gather useful preliminaries about normal non-distributive modal logic and its polarity-based semantics based on [9] and [7]. We assume readers are familiar with the basic concepts from the model theory of classical modal logic such as simulations, bisimulations, ultrafilter extensions, standard translations, Hennessy-Milner theorem, and van Benthem characterization theorem. For a detailed discussion on the model theory of modal logic, we refer to [4, Section 2].

### 2.1 Non-Distributive Modal Logic

Let $\mathcal{V}$ be a countable set of propositional variables. The language $\mathcal{L}$ (i.e. set of formulas) is defined as follows:

$$\varphi ::= p \mid \bot \mid \top \mid \varphi \wedge \varphi \mid \varphi \vee \varphi \mid \Box\varphi \mid \Diamond\varphi,$$

where $p \in \mathcal{V}$. Given any $\varphi, \psi \in \mathcal{L}$, $\varphi \vdash \psi$ is an *$\mathcal{L}$-sequent*. A *normal non-distributive modal logic* is a set of $\mathcal{L}$-sequents containing the following axioms:

$$p \vdash p, \quad p \vdash \top \quad \bot \vdash p, \quad p \vdash p \vee q, \quad q \vdash p \vee q, \quad p \wedge q \vdash p, \quad p \wedge q \vdash q,$$
$$\top \vdash \Box\top, \quad \Diamond\bot \vdash \bot, \quad \Box p \wedge \Box q \vdash \Box(p \wedge q), \quad \Diamond(p \vee q) \vdash \Diamond p \vee \Diamond q$$

and closed under the following inference rules:

$$\frac{\varphi\vdash\chi \quad \chi\vdash\psi}{\varphi\vdash\psi} \quad \frac{\varphi\vdash\psi}{\varphi(\chi/p)\vdash\psi(\chi/p)} \quad \frac{\chi\vdash\varphi \quad \chi\vdash\psi}{\chi\vdash\varphi\wedge\psi} \quad \frac{\varphi\vdash\chi \quad \psi\vdash\chi}{\varphi\vee\psi\vdash\chi} \quad \frac{\varphi\vdash\psi}{\Box\varphi\vdash\Box\psi} \quad \frac{\varphi\vdash\psi}{\Diamond\varphi\vdash\Diamond\psi}$$

The smallest such set $\mathbb{L}$ is called the *basic* normal non-distributive modal logic. For any sequent $\varphi \vdash \psi$ in $\mathbb{L}$ we denote it as $\varphi \vdash_{\mathbb{L}} \psi$.

### 2.2 Polarity-Based Semantics

Given any relation $R \subseteq A \times X$ and any $B \subseteq A$, $Y \subseteq X$, the maps $R^{(1)} : \mathcal{P}(A) \to \mathcal{P}(X)$ and $R^{(0)} : \mathcal{P}(X) \to \mathcal{P}(A)$ are defined as $R^{(1)}[B] := \{x \in X \mid \forall b(b \in B \Rightarrow bRx)\}$ and $R^{(0)}[Y] := \{a \in A \mid \forall y(y \in Y \Rightarrow aRy)\}$. For any relation $R \subseteq A \times X$, we $R^c \subseteq A \times X$ be the relation defined as follows: For any $a \in A$, $x \in X$, $(a, x) \in R^c$ iff $(a, x) \notin R$.

**Definition 2.1** A *polarity* is a tuple $\mathfrak{P} = (A, X, I)$ such that $A$ and $X$ are sets and $I \subseteq A \times X$ is a binary relation. Given a polarity $\mathfrak{P} = (A, X, I)$, the maps $(\cdot)^\uparrow : \mathcal{P}(A) \to \mathcal{P}(X)$ and $(\cdot)^\downarrow : \mathcal{P}(X) \to \mathcal{P}(A)$ are defined as follows:

$$B^\uparrow := I^{(1)}[B] \text{ and } Y^\downarrow := I^{(0)}[Y], \text{ where } B \subseteq A \text{ and } Y \subseteq X.$$

The maps $(\cdot)^\uparrow$ and $(\cdot)^\downarrow$ form a *Galois connection* between posets $(\mathcal{P}(A), \subseteq)$ and $(\mathcal{P}(X), \subseteq)$, that is $Y \subseteq B^\uparrow$ iff $B \subseteq Y^\downarrow$ for all $B \in \mathcal{P}(A)$ and $Y \in \mathcal{P}(X)$.

**Definition 2.2** A *formal concept* of the polarity $\mathfrak{P} = (A, X, I)$ is a pair $([\![c]\!], (\![c]\!))$ such that $[\![c]\!] \subseteq A$, $(\![c]\!) \subseteq X$, and $[\![c]\!]^\uparrow = (\![c]\!)$, $(\![c]\!)^\downarrow = [\![c]\!]$. It follows that $[\![c]\!]$ and $(\![c]\!)$ are *Galois-stable*, i.e. $[\![c]\!]^{\uparrow\downarrow} = [\![c]\!]$ and $(\![c]\!)^{\downarrow\uparrow} = (\![c]\!)$. The set $\mathcal{C}(\mathfrak{P})$ of all the formal concepts of $\mathfrak{P}$ can be partially ordered as follows: for any $c, d \in \mathcal{C}(\mathfrak{P})$, $c \leq d$ iff $[\![c]\!] \subseteq [\![d]\!]$ iff $(\![d]\!) \subseteq (\![c]\!)$. The poset $\mathfrak{P}^+ = (\mathcal{C}(\mathfrak{P}), \leq)$ is a complete lattice – called the *concept lattice* of $\mathfrak{P}$ – such that for any $\mathrm{K} \subseteq \mathcal{C}(\mathfrak{P})$,

$$\bigwedge \mathrm{K} = (\bigcap\{[\![c]\!] \mid c \in \mathrm{K}\}, (\bigcap\{[\![c]\!] \mid c \in \mathrm{K}\})^\uparrow),$$

$$\bigvee \mathrm{K} = ((\bigcap\{(\![c]\!) \mid c \in \mathrm{K}\})^{\downarrow}, \bigcap\{(\![c]\!) \mid c \in \mathrm{K}\}).$$

**Proposition 2.3** *For any polarity $\mathfrak{P} = (A, X, I)$, the concept lattice $\mathfrak{P}^+$ is completely join-generated by the set $\{\boldsymbol{a} := (a^{\uparrow\downarrow}, a^{\uparrow}) \mid a \in A\}$, and is completely meet-generated by the set $\{\boldsymbol{x} := (x^{\downarrow}, x^{\downarrow\uparrow}) \mid x \in X\}$.*

**Theorem 2.4 (Birkhoff's representation theorem)** *Any complete lattice* $\mathbf{L}$ *is isomorphic to the concept lattice $\mathfrak{P}^+$ of some polarity $\mathfrak{P}$.*

**Definition 2.5** An LE-*frame* is a tuple $\mathfrak{F} = (\mathfrak{P}, R_\Diamond, R_\Box)$, where $\mathfrak{P} = (A, X, I)$ is a polarity, and $R_\Diamond \subseteq X \times A$ and $R_\Box \subseteq A \times X$ are *I-compatible* relations, i.e., for all $a \in A$ and $x \in X$, $R_\Diamond^{(0)}[a]$ (resp. $R_\Box^{(0)}[x]$) and $R_\Box^{(1)}[a]$ (resp. $R_\Diamond^{(1)}[x]$) are Galois-stable.

**Definition 2.6** For any LE-frame $\mathfrak{F} = (\mathfrak{P}, R_\Diamond, R_\Box)$, the *complex algebra* of $\mathfrak{F}$ is $\mathfrak{F}^+ := (\mathfrak{P}^+, \Diamond^{\mathfrak{P}^+}, \Box^{\mathfrak{P}^+})$ where $\mathfrak{P}^+$ is the concept lattice of $\mathfrak{P}$, $\Diamond^{\mathfrak{P}^+}$ and $\Box^{\mathfrak{P}^+}$ are unary operators on $\mathfrak{P}^+$ defined as follows: for every $c \in \mathfrak{P}^+$,

$$\Diamond^{\mathfrak{P}^+}(c) := (R_\Diamond^{(0)}[[\![c]\!]]^{\downarrow}, R_\Diamond^{(0)}[[\![c]\!]]) \quad \text{and} \quad \Box^{\mathfrak{P}^+}(c) := (R_\Box^{(0)}[(\![c]\!)], R_\Box^{(0)}[(\![c]\!)]^{\uparrow}).$$

Note that $\Diamond^{\mathfrak{P}^+}$ (resp. $\Box^{\mathfrak{P}^+}$) is completely join (resp. meet) preserving.

**Definition 2.7** A *valuation* on an LE-frame $\mathfrak{F} = (\mathfrak{P}, R_\Diamond, R_\Box)$ is a map $V : \mathcal{V} \to \mathfrak{P}^+$. For each $p \in \mathcal{V}$, we let $[\![p]\!] := [\![V(p)]\!]$ (resp. $(\![p]\!) := (\![V(p)]\!)$) denote the extension (resp. intension) of the interpretation of $p$ under $V$. A valuation can be homomorphically extended to an unique valuation $\overline{V} : \mathcal{L} \to \mathfrak{P}^+$ on all the $\mathcal{L}$-formulas. An LE-*model* is a pair $(\mathfrak{F}, V)$, where $\mathfrak{F}$ is an LE-frame, and $V$ is a valuation on it.

**Definition 2.8** For any LE-model $\mathfrak{M} = (\mathfrak{F}, V)$, the *modal satisfaction* relations $\Vdash$ and $\succ$ are defined inductively as follows:

| | | | |
|---|---|---|---|
| $\mathfrak{M}, a \Vdash p$ | iff $a \in [\![p]\!]_{\mathfrak{M}}$ | $\mathfrak{M}, x \succ p$ | iff $x \in (\![p]\!)_{\mathfrak{M}}$ |
| $\mathfrak{M}, a \Vdash \top$ | always | $\mathfrak{M}, x \succ \top$ | iff $(\forall a \in A) aIx$ |
| $\mathfrak{M}, x \succ \bot$ | always | $\mathfrak{M}, a \Vdash \bot$ | iff $(\forall x \in X) aIx$ |
| $\mathfrak{M}, a \Vdash \varphi \wedge \psi$ | iff $\mathfrak{M}, a \Vdash \varphi$ and $\mathfrak{M}, a \Vdash \psi$ | $\mathfrak{M}, x \succ \varphi \wedge \psi$ | iff $(\forall a \in A)$ $(\mathfrak{M}, a \Vdash \varphi \wedge \psi \Rightarrow aIx)$ |
| $\mathfrak{M}, x \succ \varphi \vee \psi$ | iff $\mathfrak{M}, x \succ \varphi$ and $\mathfrak{M}, x \succ \psi$ | $\mathfrak{M}, a \Vdash \varphi \vee \psi$ | iff $(\forall x \in X)$ $(\mathfrak{M}, x \succ \varphi \vee \psi \Rightarrow aIx)$ |
| $\mathfrak{M}, a \Vdash \Box\varphi$ | iff $(\forall x \in X)(\mathfrak{M}, x \succ \varphi \Rightarrow aR_\Box x)$ | $\mathfrak{M}, x \succ \Box\varphi$ | iff $(\forall a \in A)(\mathfrak{M}, a \Vdash \Box\varphi \Rightarrow aIx)$ |
| $\mathfrak{M}, x \succ \Diamond\varphi$ | iff $(\forall a \in A)(\mathfrak{M}, a \Vdash \varphi \Rightarrow xR_\Diamond a)$ | $\mathfrak{M}, a \Vdash \Diamond\varphi$ | iff $(\forall x \in X)(\mathfrak{M}, x \succ \Diamond\varphi \Rightarrow aIx)$ |

Note that unlike the classical modal logic, the modal operators $\Box$ and $\Diamond$ are not inter-definable in LE-logic. For any $\mathcal{L}$-sequent $\varphi \vdash \psi$, and an LE-model $\mathfrak{M} = (\mathfrak{F}, V)$, $\mathfrak{M} \models \varphi \vdash \psi$ iff $[\![V(\varphi)]\!] \subseteq [\![V(\psi)]\!]$ iff $(\![V(\psi)]\!) \subseteq (\![V(\varphi)]\!)$.

The following theorem states that the basic normal non-distributive modal logic defined in Section 2.1 is sound and complete w.r.t. the class of LE-models.

**Theorem 2.9 (Proposition 3, [7])** *For any unprovable $\mathcal{L}$-sequent $\varphi \vdash \psi$ in the non-distributive modal logic, there is an* LE-*model* $\mathfrak{M}$ *such that* $\mathfrak{M} \not\models \varphi \vdash \psi$.

### 2.3 Two Sorted First-Order Logic

Given a countable set of propositional variables $\mathcal{V}$, we let $\mathcal{L}^1$ be the two sorted first-order language with equality built over two disjoint countable sets of variables $G$ and $M$, three binary predicates $I$, $R_\Box$, $R_\Diamond$, and two unary predicates

$P_A$ and $P_X$ for every $p \in \mathcal{V}$. A *domain* for $\mathcal{L}^1$ is a pair of disjoint sets $A$ and $X$. An *interpretation* I of $\mathcal{L}^1$ over domain $(A, X)$ assigns binary predicates $I$, $R_\Box$, and $R_\Diamond$ to relations $I^{\mathrm{I}} \subseteq A \times X$, $R^{\mathrm{I}}{}_\Box \subseteq A \times X$, and $R^{\mathrm{I}}{}_\Diamond \subseteq X \times A$, such that $R^{\mathrm{I}}{}_\Box$ and $R^{\mathrm{I}}{}_\Diamond$ are $I^{\mathrm{I}}$-compatible. For every propositional variable $p \in \mathcal{V}$, the unary predicates $P_A$ and $P_X$ are assigned to sets $P_A^{\mathrm{I}} \subseteq A$, $P_X^{\mathrm{I}} \subseteq X$, such that, $(I^{\mathrm{I}})^{(1)}[P_A^{\mathrm{I}}] = P_X^{\mathrm{I}}$, and $(I^{\mathrm{I}})^{(0)}[P_X^{\mathrm{I}}] = P_A^{\mathrm{I}}$, respectively. An $\mathcal{L}^1$-*structure* is a tuple $\mathfrak{M} = (A, X, \mathrm{I})$, where I is an interpretation of $\mathcal{L}^1$ over domain $A$ and $X$. A *valuation* on an $\mathcal{L}^1$- structure $\mathfrak{M}$ is a map $v$ which assigns every variable $g \in G$ (resp. $m \in M$) to an element $a \in A$ (resp. $x \in X$). The *satisfaction relation* for any $\mathcal{L}^1$-term w.r.t. any $\mathcal{L}^1$- structure $\mathfrak{M}$ and valuation $v$ on it is defined as follows:

1. $\mathfrak{M}, v \vDash g_1 = g_2$ iff $v(g_1) = v(g_2)$
2. $\mathfrak{M}, v \vDash m_1 = m_2$ iff $v(m_1) = v(m_2)$
3. $\mathfrak{M}, v \vDash P_A(g_1)$ iff $v(g_1) \in P_A^{\mathrm{I}}$
4. $\mathfrak{M}, v \vDash P_X(m_1)$ iff $v(m_1) \in P_X^{\mathrm{I}}$
5. $\mathfrak{M}, v \vDash g_1 I m_1$ iff $v(g_1) I^{\mathrm{I}} v(m_1)$
6. $\mathfrak{M}, v \vDash g_1 R_\Box m_1$ iff $v(g_1) R_\Box^{\mathrm{I}} v(m_1)$
7. $\mathfrak{M}, v \vDash m_1 R_\Diamond g_1$ iff $v(m_1) R_\Diamond^{\mathrm{I}} v(g_1)$.

Any LE-model $\mathfrak{M} = (A, X, I^{\mathfrak{M}}, R_\Box^{\mathfrak{M}}, R_\Diamond^{\mathfrak{M}}, V)$ can be seen as an $\mathcal{L}^1$-structure, where the domain of interpretation is formed by sets $A$ and $X$ from the polarity. The binary predicate symbols $I$, $R_\Box$, $R_\Diamond$ are interpreted by the corresponding relations on the polarity, and unary predicate $P_A$ (resp. $P_X$) is interpreted as the set $[\![V(p)]\!]$ (resp. $(\![V(p)]\!)$), for any $p \in \mathcal{V}$. For any LE-model $\mathfrak{M}$, we use the same name $\mathfrak{M}$ for the two sorted first-order structure when they are clearly distinguishable from the context.

## 3 Simulations and Bisimulations on LE-Models

In this section, we generalize simulations and bisimulations from Kripke models to LE-models and show invariance of $\mathcal{L}$-formulas under bisimulations. We give an example which shows that the Hennesy-Milner theorem does not hold with respect to this definition of bisimulation. Thus, this definition of bisimulation does not provide us notion of bisimilarity which has desired (or expected) properties. Hence, we give a more general definition of bisimilarity which is based on simulations, and prove Hennesy-Milner theorem for polarity-based semantics of non-distributive modal logic with respect to this generalized notion of bisimilarity.

**Definition 3.1** Let $\mathfrak{M}_1 = (\mathfrak{F}_1, V_1)$ and $\mathfrak{M}_2 = (\mathfrak{F}_2, V_2)$ be any LE-models, $a_1, x_1$ in $\mathfrak{M}_1$ and $a_2, x_2$ in $\mathfrak{M}_1$, then

1. $\mathfrak{M}_1, a_1 \rightsquigarrow_A \mathfrak{M}_2, a_2$ if for any $\varphi \in \mathcal{L}$, $\mathfrak{M}_1, a_1 \Vdash \varphi$ implies $\mathfrak{M}_2, a_2 \Vdash \varphi$.
2. $\mathfrak{M}_1, x_1 \rightsquigarrow_X \mathfrak{M}_2, x_2$ if for any $\varphi \in \mathcal{L}$, $\mathfrak{M}_1, x_1 \succ \varphi$ implies $\mathfrak{M}_2, x_2 \succ \varphi$.
3. $\mathfrak{M}_1, a_1 \leftsquigarrow_A \mathfrak{M}_2, a_2$ if $\mathfrak{M}_2, a_2 \rightsquigarrow_A \mathfrak{M}_1, a_1$.
4. $\mathfrak{M}_1, x_1 \leftsquigarrow_X \mathfrak{M}_2, x_2$ if $\mathfrak{M}_2, x_2 \rightsquigarrow_X \mathfrak{M}_1, x_1$.
5. $\mathfrak{M}_1, a_1 \leftrightsquigarrow_A \mathfrak{M}_2, a_2$ if $\mathfrak{M}_1, a_1 \rightsquigarrow_A \mathfrak{M}_2, a_2$ and $\mathfrak{M}_1, a_1 \leftsquigarrow_A \mathfrak{M}_2, a_2$.
6. $\mathfrak{M}_1, x_1 \leftrightsquigarrow_X \mathfrak{M}_2, x_2$ if $\mathfrak{M}_1, x_1 \rightsquigarrow_X \mathfrak{M}_2, x_2$ and $\mathfrak{M}_1, x_1 \leftsquigarrow_X \mathfrak{M}_2, x_2$.

For ease of notation, we will use $a_1 \rightsquigarrow_A a_2$ instead of $\mathfrak{M}_1, a_1 \rightsquigarrow_A \mathfrak{M}_2, a_2$ when $\mathfrak{M}_1$ and $\mathfrak{M}_2$ are clear from the context. We use similar shorthands for the other relations defined in Definition 3.1. The following definition generalizes

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$$\begin{tikzcd} x_1 \arrow[r, dashed, "T"] & x_2 \\ a_1 \arrow[u, dashed, "I_1^c"] \arrow[r, "S"'] & a_2 \arrow[u, "I_2^c"'] \end{tikzcd}$$

$$\begin{tikzcd} x_1 \arrow[r, "T"] & x_2 \\ a_1 \arrow[u, "I_1^c"] \arrow[r, dashed, "S"'] & a_2 \arrow[u, dashed, "I_2^c"'] \end{tikzcd}$$
4

$$\begin{tikzcd} x_1 \arrow[r, dashed, "T"] & x_2 \\ a_1 \arrow[u, dashed, "R^c_{\Box_1}"] \arrow[r, "S"'] & a_2 \arrow[u, "R^c_{\Box_2}"'] \end{tikzcd}$$
5

$$\begin{tikzcd} x_1 \arrow[r, "T"] & x_2 \\ a_1 \arrow[u, "R^c_{\Diamond_1}"] \arrow[r, dashed, "S"'] & a_2 \arrow[u, dashed, "R^c_{\Diamond_2}"'] \end{tikzcd}$$
6

Fig. 1. The 'back and forth' conditions of a simulation $(S, T)$

simulations and bisimulations on Kripke models to LE-models.

**Definition 3.2** Let $\mathfrak{M}_1 = (\mathfrak{F}_1, V_1)$ and $\mathfrak{M}_2 = (\mathfrak{F}_2, V_2)$ be any LE-models where $\mathfrak{F}_1 = (A_1, X_1, I_1, R_{\Box 1}, R_{\Diamond 1})$ and $\mathfrak{F}_2 = (A_2, X_2, I_2, R_{\Box 2}, R_{\Diamond 2})$. A *simulation* from $\mathfrak{M}_1$ to $\mathfrak{M}_2$ is a pair of relations $(S, T)$ such that $S \subseteq A_1 \times A_2$ and $T \subseteq X_1 \times X_2$ satisfying the following conditions: for any $a_1 \in A_1$, $x_1 \in X_1$, $a_2 \in A_2$ and $x_2 \in X_2$,

1. If $a_1 S a_2$, then $\forall p \in \mathcal{V}$, if $a_1 \in [\![V_1(p)]\!]$, then $a_2 \in [\![V_2(p)]\!]$.
2. If $x_1 T x_2$, then $\forall p \in \mathcal{V}$, if $x_2 \in (\![V_2(p)]\!)$, then $x_1 \in (\![V_1(p)]\!)$.
3. If $a_1 S a_2$ and $a_2 I_2^c x_2$, then $\exists x_1 \in X_1$ such that $a_1 I_1^c x_1$ and $x_1 T x_2$.
4. If $x_1 T x_2$ and $a_1 I_1^c x_1$, then $\exists a_2 \in A_2$ such that $a_2 I_2^c x_2$ and $a_1 S a_2$.
5. If $a_1 S a_2$ and $a_2 R^c_{\Box_2} x_2$, then $\exists x_1 \in X_1$ such that $a_1 R^c_{\Box_1} x_1$ and $x_1 T x_2$.
6. If $x_1 T x_2$ and $x_1 R^c_{\Diamond_1} a_1$, then $\exists a_2 \in A_2$ such that $x_2 R^c_{\Diamond_2} a_2$ and $a_1 S a_2$.

The conditions 3-6 are called 'back and forth' conditions of the simulation, which can be represented by using diagrams in Figure 1. We write $\mathfrak{M}_1, a_1 \rightrightarrows \mathfrak{M}_2, a_2$ (resp. $\mathfrak{M}_1, x_1 \rightrightarrows \mathfrak{M}_2, x_2$) if there exists a simulation $(S, T)$ from $\mathfrak{M}_1$ to $\mathfrak{M}_2$ such that $a_1 S a_2$ (resp. $x_1 T x_2$), and $\mathfrak{M}_1, a_1 \leftleftarrows \mathfrak{M}_2, a_2$ (resp. $\mathfrak{M}_1, x_1 \leftleftarrows \mathfrak{M}_2, x_2$) if $\mathfrak{M}_2, a_2 \rightrightarrows \mathfrak{M}_1, a_1$ (resp. $\mathfrak{M}_2, x_2 \rightrightarrows \mathfrak{M}_1, x_1$). A *bisimulation* between $\mathfrak{M}_1$ and $\mathfrak{M}_2$ is simulation $(S, T)$ from $\mathfrak{M}_1$ to $\mathfrak{M}_2$, such that $(S^{-1}, T^{-1})$ is a simulation from $\mathfrak{M}_2$ to $\mathfrak{M}_1$.

**Remark 3.3** The definition of bisimilarity implied by the above definition of bisimulation does not give us desired properties like the Hennessy-Milner theorem for polarity-based semantics. This will be illustrated in Example 3.8. Hence, a modified definition of bisimilarity is given in Definition 3.9 which allows us to prove Hennessy-Milner theorem.

This definition naturally generalizes sumulations and bisimulations on the Kripke models in the following sense. In [6, Section 3.3], it was showed that LE-models can be seen as a generalization of Kripke models by associating any Kripke model $\mathcal{M} = (W, R, V)$ with an LE-model $\mathrm{LE}(\mathcal{M}) = (W_A, W_X, \neq, R_\Box, R_\Diamond, V')$, where $W_A = W = W_B$ and $R_\Box = R_\Diamond \subseteq W \times W$ such that $w_1 R_\Box w_2$ iff $w_1 R^c w_2$ iff $w_1 R_\Diamond w_2$, and for any $p \in \mathcal{V}$, $V'(p) = (V(p), V(p)^c)$. This construction ensures that for any points in two Kripke models $\mathcal{M}$ and $\mathcal{M}'$ are modally equivalent, their liftings in the LE-models $\mathrm{LE}(\mathcal{M})$ and $\mathrm{LE}(\mathcal{M}')$ are modally equivalent w.r.t. the language $\mathcal{L}$. For more details, see [6]. The following lemma shows that any simulation (resp. bisimulation) on a Kripke model $\mathcal{M}$ naturally induces a simulation (resp. bisimulation) on LE($\mathcal{M}$).

**Lemma 3.4** *Let $\mathcal{M}_1 = (W_1, R_1, V_1)$ and $\mathcal{M}_2 = (W_2, R_2, V_2)$ be any Kripke models,* $\mathrm{LE}(\mathcal{M}_1) = (W_{A_1}, W_{X_1}, \neq, R_{\Box_1}, R_{\Diamond_1}, V_1')$ *and* $\mathrm{LE}(\mathcal{M}_2) = (W_{A_2}, W_{X_2}, \neq, R_{\Box_2}, R_{\Diamond_2}, V_2')$ *be their corresponding* LE*-models. A relation $Z \subseteq W_1 \times W_2$ between the Kripke models $\mathcal{M}_1$ and $\mathcal{M}_2$ is a simulation (resp. bisimulation) iff the tuple $(Z_A, Z_X)$, where $Z_A = Z = Z_X \subseteq W_{A_1} \times W_{A_2} = W_{X_1} \times W_{X_2}$ is a simulation (resp. bisimulation) between the* LE*-models* LE($\mathcal{M}_1$) *and* LE($\mathcal{M}_2$).

**Proof.** The proof follows straight-forwardly from the definitions of simulations and bisimulations on Kripke models and LE-models. $\Box$

The following theorem shows that the $\mathcal{L}$-formulas are preserved and reflected by simulations w.r.t. modal satisfaction relations $\Vdash$ and $\succ$, respectively.

**Theorem 3.5** *Let $\mathfrak{M}_1 = (\mathfrak{F}_1, V_1)$ and $\mathfrak{M}_2 = (\mathfrak{F}_2, V_2)$ be any* LE*-models, $a_1, x_1$ in $\mathfrak{M}_1$ and $a_2, x_2$ in $\mathfrak{M}_2$, then*

*1. If $\mathfrak{M}_1, a_1 \rightrightarrows \mathfrak{M}_2, a_2$, then $\mathfrak{M}_1, a_1 \rightsquigarrow_A \mathfrak{M}_2, a_2$.*

*2. If $\mathfrak{M}_1, x_1 \rightrightarrows \mathfrak{M}_2, x_2$, then $\mathfrak{M}_1, x_1 \leftsquigarrow_X \mathfrak{M}_2, x_2$.*

**Proof.** We give a proof by induction on the complexity of the formula $\varphi$. Without loss of generality, let $(S, T)$ be a simulation from $\mathfrak{M}_1$ to $\mathfrak{M}_2$. The initial step is when $\varphi \in \mathcal{V}$ is a propositional variable. For the first item, assume $a_1 S a_2$, then $a_1 \rightsquigarrow_A a_2$ follows directly from the item 1 of Definition 3.2. For the second item, assume $x_1 T a_2$, then $x_2 \rightsquigarrow_X x_1$ follows directly from the item 2 of Definition 3.2. Now, we consider induction step for all the connectives.

(1) Suppose $\varphi$ is $\psi_1 \wedge \psi_2$.

To prove item 1, suppose $a_1 S a_2$, and $\mathfrak{M}_1, a_1 \Vdash \psi_1 \wedge \psi_2$. Then, $\mathfrak{M}_1, a_1 \Vdash \psi_1$ and $\mathfrak{M}_1, a_1 \Vdash \psi_2$, which by induction hypothesis implies $\mathfrak{M}_2, a_2 \Vdash \psi_1$ and $\mathfrak{M}_2, a_2 \Vdash \psi_2$, which implies $\mathfrak{M}, a_2 \Vdash \psi_1 \wedge \psi_2$.

To prove item 2, suppose $x_1 T x_2$ and $\mathfrak{M}_1, x_1 \not\succ \psi_1 \wedge \psi_2$. Then, there exists $a_1 \in A_1$ such that $a_1 I_1^c x_1$ and $\mathfrak{M}_1, a_1 \Vdash \psi_1 \wedge \psi_2$. Then, by item 4 of Definition 3.2, there exists $a_2 \in A_2$ such that $a_2 I_2^c x_2$ and $a_1 S a_2$. Therefore, $\mathfrak{M}_2, a_2 \Vdash \psi_1 \wedge \psi_2$, which implies $\mathfrak{M}_2, x_2 \not\succ \psi_1 \wedge \psi_2$.

(2) Suppose $\varphi$ is $\psi_1 \vee \psi_2$. The proof is dual to the previous case.

(3) Suppose $\varphi = \Box\psi$.

To prove item 1, assume $a_1 S a_2$, and $\mathfrak{M}_2, a_2 \nVdash \Box\psi$. Then, there exists $x_2 \in X_2$ such that $\mathfrak{M}_2, x_2 \succ \psi$ and $a_2 R^c_{\Box_2} x_2$. By item 5 Definition 3.2, there exists $x_1 \in X_1$ such that $a_1 R^c_{\Box_1} x_1$, and $x_1 T x_2$. Therefore, by induction hypothesis $\mathfrak{M}_1, x_1 \succ \psi$. Hence, $\mathfrak{M}_1, a_1 \not\succ \Box\psi$.

To prove item 2, suppose $x_1 T x_2$, and $\mathfrak{M}_1, x_1 \not\succ \Box\psi$. Then, there exists $a_1 \in A_1$ such that $a_1 I_1^c x_1$ and $\mathfrak{M}_1, a_1 \Vdash \Box\psi$. By item 4 of Definition 3.2, there exists $a_2 \in A_2$ such that $a_2 I_2^c x_2$, and $a_1 S a_2$. Therefore, $\mathfrak{M}_2, a_2 \Vdash \Box\psi$, which implies $\mathfrak{M}_2, x_2 \not\succ \Box\psi$.

(4) Suppose $\varphi$ is $\Diamond\psi$. In this case, the proof is dual to the previous case.

This concludes the proof.

□

As a corollary, we get that the $\mathcal{L}$-formulas are bisimulation-invariant.

**Corollary 3.6** *Let $\mathfrak{M}_1 = (\mathfrak{F}_1, V_1)$ and $\mathfrak{M}_2 = (\mathfrak{F}_2, V_2)$ be any* LE*-models, where $\mathfrak{F}_1 = (A_1, X_1, I_1, R_{\Box 1}, R_{\Diamond 1})$ and $\mathfrak{F}_2 = (A_2, X_2, I_2, R_{\Box 2}, R_{\Diamond 2})$, and $(S, T)$ be a bisimulation between them. Then*

*1. For any $a_1 \in A_1$ and $a_2 \in A_2$, $a_1 S a_2$ implies $\mathfrak{M}_1, a_1 \leftrightsquigarrow_A \mathfrak{M}_2, a_2$.*

*2. For any $x_1 \in X_1$ and $x_2 \in X_2$, $x_1 T x_2$ implies $\mathfrak{M}_1, x_1 \leftrightsquigarrow_X \mathfrak{M}_2, x_2$.*

**Proof.** Note that by definition for any bisimulation $(S, T)$, both $(S, T)$ and $(S^{-1}, T^{-1})$ are simulations. The proof follows immediately by Theorem 3.5. □

We now try to prove the Hennessy-Milner theorem for the polarity-based semantics of non-distributive modal logic. We begin by generalizing the notion of image-finite models from Kripke semantics to polarity-based semantics.

**Definition 3.7** An LE-model $\mathfrak{M} = (A, X, I, R_\Box, R_\Diamond, V)$ is *image-finite* if for any $a \in A$ and $x \in X$, the sets $\{a' \mid a' I^c x\}, \{x' \mid a I^c x'\}, \{a' \mid a' R_\Box^c x\}$ and $\{x' \mid x' R_\Diamond^c a\}$ are all finite.

Note that a Kripke model $\mathcal{M}$ is image-finite iff LE($\mathcal{M}$) is image-finite. Thus, this definition can be seen as a generalization of image-finite models to polarity-based setting. However, as the language of non-distributive modal logic ($\mathcal{L}$) does not contain negation, the proof strategy for the classical Hennessy-Milner theorem cannot be adopted in the non-distributive setting. In fact, as the following example shows, Hennessy-Milner theorem does not hold for the polarity-based semantics of non-distributive modal logic for the notion of bisimulation given by Definition 3.2.

**Example 3.8** Let $\mathfrak{M}_i = (A_i, X_i, I_i, R_{\Box i}, R_{\Diamond i}, V_i)$ for $i \in \{1, 2\}$ be two image-finite models such that $A_1 = \{a_1, b_1\}$, $X_1 = \{x_1, y_1\}$, $I_1 = \{(b_1, x_1)\}$, $A_2 = \{a_2\}$, $X_2 = \{x_2\}$ and $I_2 = R_{\Box i} = R_{\Diamond i} = \emptyset$ for $i \in \{1, 2\}$. Let $p$, $q$ be any two fixed propositions. Suppose $V_1$ and $V_2$ are such that, $V_1(q) = (\{b_1\}, \{x_1\})$, $p \in \mathcal{V}$, $V_1(p) = (\{a_1, b_1\}, \emptyset)$, $V_2(q) = (\emptyset, \{x_2\})$ and $V_2(p) = (\{a_2\}, \emptyset)$. These models are depicted in Figure 2. It is easy to check that $a_1 \leftrightsquigarrow_A a_2$.

Suppose there exists a bisimulation $(S, T)$ between $\mathfrak{M}_1$ and $\mathfrak{M}_2$, such that, $a_1 S a_2$. Then, by item 3 of Definition 3.2 for simulation $(S^{-1}, T^{-1})$ and $a_1 I_1^c y_1$, we must have $y_1 T x_2$. However, $\mathfrak{M}_1, y_1 \not\succ q$ and $\mathfrak{M}_2, x_2 \succ q$, which contradicts item 2 of Definition 3.2 for simulation $(S, T)$. Hence, there can not be a bisimulation between $a_1$ and $a_2$.

Therefore, to obtain the Hennessy-Milner theorem for polarity-based semantics, we introduce the notion of *bisimilarity*, which is founded on the definition of simulations between LE-models. This notion naturally generalizes the definition of bisimulation outlined in Definition 3.2.

**Definition 3.9** Let $\mathfrak{M}_1 = (\mathfrak{F}_1, V_1)$ and $\mathfrak{M}_2 = (\mathfrak{F}_2, V_2)$ be any LE-models, where $\mathfrak{F}_1 = (A_1, X_1, I_1, R_{\Box 1}, R_{\Diamond 1})$, $\mathfrak{F}_2 = (A_2, X_2, I_2, R_{\Box 2}, R_{\Diamond 2})$. For any $a_1, x_1$ in $\mathfrak{M}_1$ and $a_2, x_2$ in $\mathfrak{M}_2$, we say that $a_1$ and $a_2$ (resp. $x_1$ and $x_2$) are *bisimilar*,

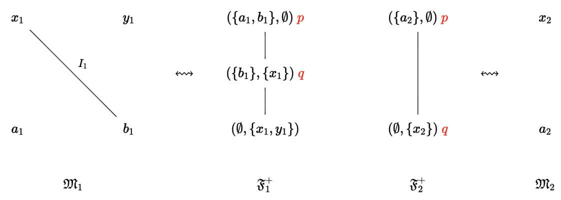


Fig. 2. Models $\mathfrak{M}_1$ and $\mathfrak{M}_2$ and their complex algebras

and denote it by $\mathfrak{M}_1, a_1 \rightleftarrows \mathfrak{M}_2, a_2$ (resp. $\mathfrak{M}_1, x_1 \rightleftarrows \mathfrak{M}_2, x_2$), if both $\mathfrak{M}_1, a_1 \rightrightarrows \mathfrak{M}_2, a_2$ and $\mathfrak{M}_1, a_1 \leftleftarrows \mathfrak{M}_2, a_2$ (resp. $\mathfrak{M}_1, x_1 \rightrightarrows \mathfrak{M}_2, x_2$ and $\mathfrak{M}_1, x_1 \leftleftarrows \mathfrak{M}_2, x_2$).

Henceforth, when we say that two elements are bisimilar, we refer to Definition 3.9, unless stated otherwise. We now show that even though Hennessy-Milner theorem for bisimulations does not generalize to the polarity-based setting, its counterpart for simulations does so.

**Theorem 3.10 (Hennesy-Milner Theorem)** *Let* $\mathfrak{M}_1 = (\mathfrak{F}_1, V_1)$ *and* $\mathfrak{M}_2 = (\mathfrak{F}_2, V_2)$ *be any image-finite* LE*-models,* $a_1, x_1$ *in* $\mathfrak{M}_1$ *and* $a_2, x_2$ *in* $\mathfrak{M}_2$*, then*

1. $\mathfrak{M}_1, a_1 \rightsquigarrow_A \mathfrak{M}_2, a_2$ *if and only if* $\mathfrak{M}_1, a_1 \rightrightarrows \mathfrak{M}_2, a_2$.
2. $\mathfrak{M}_1, x_1 \rightsquigarrow_X \mathfrak{M}_2, x_2$ *if and only if* $\mathfrak{M}_1, x_1 \leftleftarrows \mathfrak{M}_2, x_2$.

**Proof.** The right to left implication for both items follows from Theorem 3.5. We prove the left to right implication only for the first item, as the proof for the second is analogous.

For the left to right implication of the first item, we claim that $(\rightsquigarrow_A, \leftsquigarrow_X)$ is a simulation from $\mathfrak{M}_1$ to $\mathfrak{M}_2$, where $(a, a') \in \rightsquigarrow_A$ (denoted as $a \rightsquigarrow_A a'$) iff $\mathfrak{M}_1, a \rightsquigarrow_A \mathfrak{M}_2, a'$ and $(x, x') \in \leftsquigarrow_X$ (denoted as $x \leftsquigarrow_X x'$) iff $\mathfrak{M}_1, x \leftsquigarrow_X \mathfrak{M}_2, x'$. Assume $a_1 \rightsquigarrow_A a_2$ and $x_1 \leftsquigarrow_X x_2$, the items 1 and 2 of Definition 3.2 are satisfied immediately.

For item 3, assume $a_1 \rightsquigarrow_A a_2$ and $a_2 I_2^c x_2$. Therefore, $\mathfrak{M}_2, a_2 \not\Vdash \bot$. By definition of $\rightsquigarrow_A$, we have $\mathfrak{M}_1, a_1 \not\Vdash \bot$, which means that there exists $x_1$ such that $a_1 I_1^c x_1$. Therefore, the set $Y' := \{x \mid a_1 I_1^c x\}$ is a non-empty finite (by image-finiteness) set . Let $Y'$ be the set $\{x'_1, \cdots, x'_n\}$. Assume for any $x \in Y'$, it is not the case that $x \leftsquigarrow_X x_2$. So, there exist finitely many formulas $\psi_1, \cdots, \psi_n$ such that for each $i$, $\mathfrak{M}_2, x_2 \succ \psi_i$ and $\mathfrak{M}_1, x'_i \not\succ \psi_i$. As $a_2 I_2^c x_2$, there is $\mathfrak{M}_2, a_2 \not\Vdash \psi_1 \vee \cdots \vee \psi_n$. On the other hand, let $x'$ be any element in $X_1$. Thus, $\mathfrak{M}_1, x' \succ \psi_1 \vee \cdots \vee \psi_n$ implies $M_1, x' \succ \psi_i$ for each $i$, which means that $x' \notin Y'$, so $a_1 I_1 x'$. Hence, there is $\mathfrak{M}_1, a_1 \Vdash \psi_1 \vee ... \vee \psi_n$, which contradicts $a_1 \rightsquigarrow_A a_2$. So, there must exist $x_1$ such that $a_1 I_1^c x_1$ and $x_1 \leftsquigarrow_X x_2$.

Item 4 can be proved similarly to item 3.

For item 5, assume $a_1 \rightsquigarrow_A a_2$ and $a_2 R^c_{\Box_2} x_2$. Therefore, $\mathfrak{M}_2, a_2 \not\Vdash \Box\bot$, which implies $\mathfrak{M}_1, a_1 \not\Vdash \Box\bot$. Therefore, there exists $x_1 \in X_1$ such that $a_1 R^c_{\Box_1} x_1$. Thus, the set $Y' := \{x \mid a_1 R^c_{\Box_1} x\}$ is a non-empty finite (by image-finiteness) set. Let $Y'$ be the set $\{x'_1, \cdots, x'_n\}$. Assume for any $x' \in Y'$, it is not the case that $x' \leftsquigarrow_X x_2$. Therefore, there exist finitely many formulas $\psi_1, \cdots, \psi_n$ such

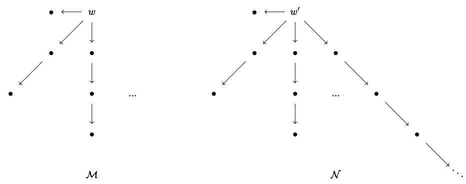

Fig. 3. Kripke models $\mathcal{M}$ and $\mathcal{N}$

that for each $i$, $\mathfrak{M}_2, x_2 \succ \psi_i$ and $\mathfrak{M}_1, x_i' \not\succ \psi_i$. Therefore, $\mathfrak{M}_2, x_2 \succ \psi_1 \vee \cdots \vee \psi_n$, which implies $\mathfrak{M}_2, a_2 \nVdash \Box(\psi_1 \vee \cdots \vee \psi_n)$. On the other hand, let $y'$ be any element in $X_1$. $\mathfrak{M}_1, y' \succ \psi_1 \vee \cdots \vee \psi_n$ implies $\mathfrak{M}_1, y' \Vdash \psi_i$ for each $i$, which means $y' \notin Y'$, so $a_1 R_{\Box_1} y'$. Therefore, $\mathfrak{M}_1, a_1 \Vdash \Box(\psi_1 \vee \cdots \vee \psi_n)$, which contradicts $a_1 \rightsquigarrow_A a_2$. So, there must exist $x_1$ such that $a_1 R^c_{\Box_1} x_1$ and $x_1 \leftrightsquigarrow_X x_2$.

Item 6 can be proved similarly to item 5.

□

The above theorem can be seen as a generalization of Henessey-Milner theorem to non-distributive setting as it relates invariance under formulas to simulations. The following corollary is immediate from the above theorem.

**Corollary 3.11** *Let* $\mathfrak{M}_1 = (\mathfrak{F}_1, V_1)$ *and* $\mathfrak{M}_2 = (\mathfrak{F}_2, V_2)$ *be any image-finite* LE-*models,* $a_1, x_1$ *in* $\mathfrak{M}_1$ *and* $a_2, x_2$ *in* $\mathfrak{M}_2$*, then*

*1.* $\mathfrak{M}_1, a_1 \leftrightsquigarrow_A \mathfrak{M}_2, a_2$ *if and only if* $\mathfrak{M}_1, a_1 \rightleftarrows \mathfrak{M}_2, a_2$*.*

*2.* $\mathfrak{M}_1, x_1 \leftrightsquigarrow_X \mathfrak{M}_2, x_2$ *if and only if* $\mathfrak{M}_1, x_1 \rightleftarrows \mathfrak{M}_2, x_2$*.*

As the simulations on LE-models can be seen as a generalizations of simulations on Kripke models, the converse of Theorem 3.5 does not always hold. That is, image-finiteness condition in Theorem 3.10 can not be dropped.

**Example 3.12** Consider two Kripke models $\mathcal{M}$ and $\mathcal{N}$ in Figure 3, where any $p \in \mathcal{V}$ is false at all the points. It is easy to check that $w$ and $w'$ are modally equivalent in $\mathcal{M}$ and $\mathcal{N}$. Therefore $w'_A \rightsquigarrow_A w_A$ on their corresponding LE-models LE($\mathcal{N}$) and LE($\mathcal{M}$). Assume that there exists a simulation $(S, T)$ from LE($\mathcal{N}$) to LE($\mathcal{M}$) such that $w'_A S w_A$. Then, $S$ must be a simulation from $\mathcal{N}$ to $\mathcal{M}$. However, it is not possible to have simulations from $\mathcal{N}$ to $\mathcal{M}$, so there is a contradiction. Therefore, there is no simulation from LE($\mathcal{N}$) to LE($\mathcal{M}$) which can link $w'_A$ and $w_A$. This shows that converse of item 1 in Theorem 3.10 is not true in general. We can similarly show that converse of item 2 in Theorem 3.10 need not hold in general.

## 4 Modal Saturation via Filter-Ideal Extensions

In this section, we continue to investigate the relation between modal equivalence and bisimilarity. We first introduce the notion of modally-saturated (M-saturated) LE-models and the Hennessy-Milner property. Then, we show that the class of M-saturated LE-models has the Hennessy-Milner property. After that, we introduce the notion of filter-ideal extension of LE-models and show that any points in LE-models are modally equivalent if and only if they are bisimilar on their filter-ideal extensions.

### 4.1 Modally-saturated LE-models and Hennessy-Milner Property

**Definition 4.1** Let K be a class of LE-models. K is a *Hennessy-Milner class* or has the *Hennessy-Milner property* if for any LE-models $\mathfrak{M}_1, \mathfrak{M}_2 \in \mathrm{K}$, and $a_1$, $x_1$ in $\mathfrak{M}_1$, and $a_2$, $x_2$ in $\mathfrak{M}_2$,

1. $\mathfrak{M}_1, a_1 \rightsquigarrow_X \mathfrak{M}_2, a_2$ implies $\mathfrak{M}_1, a_1 \rightrightarrows \mathfrak{M}_2, a_2$, and
2. $\mathfrak{M}_1, x_1 \rightsquigarrow_A \mathfrak{M}_2, x_2$ implies $\mathfrak{M}_1, x_1 \leftleftarrows \mathfrak{M}_2, x_2$.

Theorem 3.10 shows that image-finite LE-models have the Hennessy-Milner property; we now introduce a larger class of models which retains the property.

**Definition 4.2** Let $\mathfrak{M} = (A, X, I, R_\Box, R_\Diamond, V)$ be an LE-model, $A' \subseteq A$ and $X' \subseteq X$. Let $\Sigma$ be a set of $\mathcal{L}$-formulas. $\Sigma$ is *satisfiable* in $A'$ (resp. $X'$) if for any $\varphi \in \Sigma$, there exists a point $a$ (resp. $x$) such that, $\mathfrak{M}, a \Vdash \varphi$ (resp. $\mathfrak{M}, x \succ \varphi$). $\Sigma$ is *finitely satisfiable* in $A'$ (resp. $X'$) if any finite subset of $\Sigma$ is satisfiable in $A'$ (resp. $X'$). $\mathfrak{M}$ is *modally-saturated* (M-saturated for short) if for any set of formulas $\Sigma$, $a \in A$ and $x \in X$ the following conditions hold:

1. If $\Sigma$ is finitely satisfiable in $\{x' \mid aI^c x'\}$, then $\Sigma$ is satisfiable in $\{x' \mid aI^c x'\}$.
2. If $\Sigma$ is finitely satisfiable in $\{a' \mid a'I^c x\}$, then $\Sigma$ is satisfiable in $\{a' \mid a'I^c x\}$.
3. If $\Sigma$ is finitely satisfiable in $\{x' \mid aR^c_\Box x'\}$, then $\Sigma$ is satisfiable in $\{x' \mid aR^c_\Box x'\}$.
4. If $\Sigma$ is finitely satisfiable in $\{a' \mid xR^c_\Diamond a'\}$, then $\Sigma$ is satisfiable in $\{a' \mid xR^c_\Diamond a'\}$.

**Theorem 4.3** *The class of* M*-saturated models has Hennessy-Milner property.*

**Proof.** Let $\mathfrak{M}_1$ and $\mathfrak{M}_2$ be two M-saturated models, and let $a_1, x_1 \in \mathfrak{M}_1$ and $a_2, x_2 \in \mathfrak{M}_2$. We only prove the first item of Definition 4.1, and the proof of the second item is similar. Assume that $a_1 \rightsquigarrow_A a_2$, we claim that $(\rightsquigarrow_A, \leftsquigarrow_X)$ is a simulation from $\mathfrak{M}_1$ to $\mathfrak{M}_2$, where $(a, a') \in \rightsquigarrow_A$ (denoted as $a \rightsquigarrow_A a'$) iff $\mathfrak{M}_1, a \rightsquigarrow_A \mathfrak{M}_2, a'$ and $(x, x') \in \leftsquigarrow_X$ (denoted as $x \leftsquigarrow_X x'$) iff $\mathfrak{M}_1, x \leftsquigarrow_X \mathfrak{M}_2, x'$. Hence, it is sufficient to show that $(\rightsquigarrow_A, \leftsquigarrow_X)$ satisfies item 1-6 of Definition 3.2.

Items 1 and 2 are easy to check. To prove item 3, assume that $a'_1 \rightsquigarrow_A a'_2$ and $a'_2 I^c_2 x'_2$. Let $\Sigma$ be the set of all the formulas satisfied at $x'_2$. Therefore, for any finite subset $\Delta$ of $\Sigma$, $\mathfrak{M}_2, x'_2 \succ \bigvee\Delta$. Therefore, $\mathfrak{M}_2, a'_2 \nVdash \bigvee\Delta$, which by $a'_1 \rightsquigarrow_A a'_2$ implies $\mathfrak{M}_1, a'_1 \nVdash \bigvee\Delta$. So, there exists $x'_1$, such that $\mathfrak{M}_1, x'_1 \succ \bigvee\Delta$ and $a'_1 I^c_1 x'_1$. Hence, $\Sigma$ is finitely satisfiable in $\{x'_1 \mid a'_1 I^c_1 x'_1\}$. Therefore, by M-saturation of $\mathfrak{M}_1$, $\Sigma$ is satisfiable in $\{x'_1 \mid a'_1 I^c_1 x'_1\}$. Therefore, there exists $x_\Sigma$ such that $a'_1 I^c_1 x_\Sigma$ and all formulas in $\Sigma$ are satisfied at $x_\Sigma$ which means $x_\Sigma \leftsquigarrow_X x'_2$. The item 4 is proved similarly to item 3.

For item 5, assume that $a'_1 \rightsquigarrow_A a'_2$ and $a'_2 R^c_{\Box_2} x'_2$. Let $\Sigma$ be the set of all the formulas satisfied at $x'_2$. Therefore, for any finite subset $\Delta$ of $\Sigma$, $\mathfrak{M}_2, x'_2 \succ \bigvee\Delta$. Therefore $\mathfrak{M}_2, a'_2 \nVdash \Box(\bigvee\Delta)$, which by $a'_1 \rightsquigarrow_A a'_2$ implies $\mathfrak{M}_1, a'_1 \nVdash \Box(\bigvee\Delta)$. So, there exists $x'_1$, such that $\mathfrak{M}_1, x'_1 \succ \bigvee\Delta$ and $a'_1 R^c_{\Box_1} x'_1$. Hence, $\Sigma$ is finitely satisfiable in $\{x'_1 \mid a'_1 R^c_{\Box_1} x'_1\}$. Therefore, by M-saturation of $\mathfrak{M}_1$, $\Sigma$ is satisfiable in $\{x'_1 \mid a'_1 R^c_{\Box_1} x'_1\}$. Therefore, there exists $x_\Sigma$ such that $a'_1 R^c_{\Box_1} x_\Sigma$ and all formulas in $\Sigma$ are satisfied at $x_\Sigma$, which means $x_\Sigma \leftsquigarrow_X x'_2$. The item 6 is proved similarly to item 5. This concludes the proof. $\Box$

### 4.2 Filter-Ideal Extension of LE-Models

Let $\mathbf{A} = (\mathbf{L}, \Box, \Diamond)$ be any LE-algebra (see [10, Definition 1]). $\mathbf{A}_+ = (\mathsf{Fi}_\mathbf{A}, \mathsf{Id}_\mathbf{A}, I, R_\Box, R_\Diamond)$ is the *filter-ideal frame* of $\mathbf{A}$ (c.f. [10, Definition 29]), where $\mathsf{Fi}_\mathbf{A}$ (resp. $\mathsf{Id}_\mathbf{A}$) is the set of all filters (resp. ideals) of $\mathbf{L}$ such that for any $F \in \mathsf{Fi}_\mathbf{A}$ and $J \in \mathsf{Id}_\mathbf{A}$,

1. $FIJ$ if and only if $F \cap J \neq \emptyset$.
2. $FR_\Box J$ if and only if there exists $a \in J$ such that $\Box a \in F$.
3. $JR_\Diamond F$ if and only if there exists $a \in F$ such that $\Diamond a \in J$.

**Definition 4.4** Let $\mathfrak{M} = (\mathfrak{F}, V)$ be any LE-model with $\mathfrak{F} = (A, X, I, R_\Box, R_\Diamond)$. The *filter-ideal extension* of $\mathfrak{M}$ is the tuple $\mathfrak{M}^{\mathsf{FI}} = (\mathfrak{F}^{\mathsf{FI}}, V^{\mathsf{FI}})$, where $\mathfrak{F}^{\mathsf{FI}} = (\mathfrak{F}^+)_+$ and for any $p \in \mathcal{V}$, $V^{\mathsf{FI}}(p) = ([\![V^{\mathsf{FI}}(p)]\!], (\![V^{\mathsf{FI}}(p)]\!)) = (\{F \mid ([\![p]\!], (\![p]\!)) \in F\}, \{J \mid ([\![p]\!], (\![p]\!)) \in J\})$.

**Remark 4.5** In order to check that $\mathfrak{M}^{\mathsf{FI}}$ is well defined we need to check that for any $p \in \mathcal{V}$, $([\![V^{\mathsf{FI}}(p)]\!], (\![V^{\mathsf{FI}}(p)]\!))$ indeed forms a concept of $\mathfrak{F}$. We only prove $[\![V^{\mathsf{FI}}(p)]\!]^\uparrow = (\![V^{\mathsf{FI}}(p)]\!)$ here. The proof for $(\![V^{\mathsf{FI}}(p)]\!)^\downarrow = [\![V^{\mathsf{FI}}(p)]\!]$ is similar. The right to left inclusion is trivial. As for inclusion in other direction, suppose there exists $J \in [\![V^{\mathsf{FI}}(p)]\!]^\uparrow$ and $J \notin (\![V^{\mathsf{FI}}(p)]\!)$. Thus, $([\![p]\!], (\![p]\!)) \notin J$ and for any filter $F$ which contains $([\![p]\!], (\![p]\!))$, $F \cap J \neq \emptyset$. As $([\![p]\!], (\![p]\!)) \notin J$, by the filter-ideal theorem, there exists a filter $F'$ such that $([\![p]\!], (\![p]\!)) \in F'$ and $F' \cap J = \emptyset$. This contradicts our assumption. So, the left to right inclusion holds.

The following lemma shows that the filter-ideal extensions of LE-models preserve modal satisfaction.

**Lemma 4.6** *Let $\mathfrak{M} = (\mathfrak{F}, V)$ be any* LE*-model where $\mathfrak{F} = (A, X, I, R_\Box, R_\Diamond)$, and $\mathfrak{M}^{\mathsf{FI}} = (\mathfrak{F}^{\mathsf{FI}}, V^{\mathsf{FI}})$ is the filter-ideal extension of $\mathfrak{M}$. Let $F$ be a filter of $\mathfrak{F}^+$ and $J$ be an ideal of $\mathfrak{F}^+$. Then*

$$\mathfrak{M}^{\mathsf{FI}}, F \Vdash \varphi \textit{ iff } ([\![\varphi]\!], (\![\varphi]\!)) \in F \quad \textit{and} \quad \mathfrak{M}^{\mathsf{FI}}, J \succ \varphi \textit{ iff } ([\![\varphi]\!], (\![\varphi]\!)) \in J.$$

**Proof.** It is straight-forward by induction on the complexity of formulas. $\Box$

For any LE-frame $\mathfrak{F}$ and $\mathrm{K} \subseteq \mathfrak{F}^+$, let $\mathsf{Fi}(\mathrm{K})$ (resp. $\mathsf{Id}(\mathrm{K})$) denote the filter (resp. the ideal) generated by K. When $\mathrm{K} = \{k\} \subseteq \mathfrak{F}^+$, let $\mathsf{Fi}(k)$ (resp. $\mathsf{Id}(k)$) denote the principal filter (resp. principal ideal) $\mathsf{Fi}(\{k\})$ (resp. $\mathsf{Id}(\{k\})$).

**Corollary 4.7 (truth lemma)** *Let $\mathfrak{M} = (\mathfrak{F}, V)$ be any* LE*-model where $\mathfrak{F} = (A, X, I, R_\Box, R_\Diamond)$, and $\mathfrak{M}^{\mathsf{FI}}$ be the filter-ideal extension of $\mathfrak{M}$. For all $\varphi \in \mathcal{L}$,*

*1. For any $a \in A$, $\mathfrak{M}, a \Vdash \varphi$ if and only if $\mathfrak{M}^{\mathsf{FI}}, \mathsf{Fi}(\mathbf{a}) \Vdash \varphi$, where $\mathbf{a} = (a^{\uparrow\downarrow}, a^\uparrow)$.*

*2. For any $x \in X$, $\mathfrak{M}, x \succ \varphi$ if and only if $\mathfrak{M}^{\mathsf{FI}}, \mathsf{Id}(\mathbf{x}) \succ \varphi$, where $\mathbf{x} = (x^\downarrow, x^{\downarrow\uparrow})$.*

**Proof.**

$$\begin{array}{rclcrcl}
\mathfrak{M}, a \Vdash \varphi & \Leftrightarrow & \mathbf{a} \leq ([\![\varphi]\!], (\![\varphi]\!)) & \quad & \mathfrak{M}, x \succ \varphi & \Leftrightarrow & ([\![\varphi]\!], (\![\varphi]\!)) \leq \mathbf{x} \\
& \Leftrightarrow & ([\![\varphi]\!], (\![\varphi]\!)) \in \mathsf{Fi}(\mathbf{a}) & & & \Leftrightarrow & ([\![\varphi]\!], (\![\varphi]\!)) \in \mathsf{Id}(\mathbf{x}) \\
& \Leftrightarrow & \mathfrak{M}^{\mathsf{FI}}, \mathsf{Fi}(\mathbf{a}) \Vdash \varphi & & & \Leftrightarrow & \mathfrak{M}^{\mathsf{FI}}, \mathsf{Id}(\mathbf{x}) \succ \varphi
\end{array}$$

The first equivalences follow from the relationship between $\Vdash$ and $\succ$ on a polarity-based model and order $\leq$ on its complex algebra. The second equivalence follows from the definitions of $\mathsf{Fi}(\mathbf{a})$ and $\mathsf{Id}(\mathbf{x})$, and the third equivalence follows from Lemma 4.6. $\square$

The above lemma implies that for any LE-model $\mathfrak{M}$ with filter-ideal extension $\mathfrak{M}^{\mathsf{FI}}$, for any $a \in A$, and $x \in X$, $a \leftrightsquigarrow_A \mathsf{Fi}(\mathbf{a})$ and $x \leftrightsquigarrow_X \mathsf{Id}(\mathbf{x})$.

**Lemma 4.8** *The filter-ideal extension of an* LE*-model is* M*-saturated.*

**Proof.** Let $\mathfrak{M}$ be an LE-model and $\mathfrak{M}^{\mathsf{FI}}$ be its filter-ideal extension. We only prove items 1 and 3 of Definition 4.2 hold for $\mathfrak{M}^{\mathsf{FI}}$. The proofs for items 2 and 4 can be obtained dually.

1. Let $F$ be any filter of $\mathfrak{F}^+$. Let $\Sigma$ be a set of $\mathcal{L}$-formulas which is finitely satisfiable in $\{J \mid FI^cJ\}$. By definition of filter-ideal extension this set is same as $\{J \mid F \cap J = \emptyset\}$. By Lemma 4.6, for any formula $\varphi$, $\mathfrak{M}^{\mathsf{FI}}, J \succ \varphi$ iff $([\![\varphi]\!], (\![\varphi]\!)) \in J$. Thus, $\Sigma$ is finitely satisfiable in $\{J \mid FI^cJ\}$ iff for any finite $\Delta \subseteq \Sigma$, the ideal generated by the set $\{([\![\varphi]\!], (\![\varphi]\!)) \mid \varphi \in \Delta\}$ (denoted by $Id(\Delta)$) is disjoint with $F$. Let

$$Y = \bigcup\{Id(\Delta) \mid \Delta \subseteq \Sigma,\, \Delta \text{ is finite}\}.$$

It is clear that $F \cap Y = \emptyset$ and $([\![\varphi]\!], (\![\varphi]\!)) \in Y$ for all $\varphi \in \Sigma$. We claim that $Y$ is an ideal of $\mathfrak{F}^+$. Suppose $c \in Y$, and $d \leq c$. By definition of $Y$, $c$ must be in $Id(\Delta)$ for some finite $\Delta \subseteq \Sigma$. Then, we must have $d \in Id(\Delta) \subseteq Y$. Finally, suppose $c_1, c_2 \in Y$. Then, we must have finite $\Delta_1, \Delta_2 \subseteq \Sigma$ such that, $c_1 \in Id(\Delta_1)$ and $c_2 \in Id(\Delta_2)$. Therefore, $c_1, c_2 \in Id(\Delta_1 \cup \Delta_2)$ implying $c_1 \vee c_2 \in Id(\Delta_1 \cup \Delta_2)$. As $\Delta_1 \cup \Delta_2 \subseteq \Sigma$ is finite, we must have $Id(\Delta_1 \cup \Delta_2) \subseteq Y$. Thus, we have $c_1 \vee c_2 \in Y$. So, $Y$ is an ideal such that $FI^cY$ and $\mathfrak{M}^{\mathsf{FI}}, Y \succ \Sigma$. Hence proved.

3. Let $\mathfrak{M} = (\mathfrak{F}, V)$ and $F$ be any filter of $\mathfrak{F}^+$. Let $\Sigma$ be a set of formulas which is finitely satisfiable in $\{J \mid FR^c_\Box J\}$. By definition of filter-ideal extension, this set is same as $\{J \mid F \cap \Box J = \emptyset\}$, where for any $C \subseteq \mathfrak{F}^+$, $\Box C = \{\Box c \mid c \in C\}$. By Lemma 4.6, for any formula $\varphi$, $\mathfrak{M}^{\mathsf{FI}}, J \succ \varphi$ iff $([\![\varphi]\!], (\![\varphi]\!)) \in J$. Thus, $\Sigma$ is finitely satisfiable in $\{J \mid FR^c_\Box J\}$ iff for any finite $\Delta \subseteq \Sigma$, the ideal generated by the set $\{([\![\varphi]\!], (\![\varphi]\!)) \mid \varphi \in \Delta\}$ (denoted by $Id(\Delta)$) satisfies the condition $\Box Id(\Delta) \cap F = \emptyset$. Let

$$Y = \bigcup\{Id(\Delta) \mid \Delta \subseteq \Sigma,\, \Delta \text{ is finite}\}.$$

It is clear that $F \cap \Box Y = \emptyset$ and $([\![\varphi]\!], (\![\varphi]\!)) \in Y$ for all $\varphi \in \Sigma$. By same proof as in item 1, we have that $Y$ is an ideal. Therefore, $Y$ is an ideal such that $FR^c_\Box Y$ and $\mathfrak{M}^{\mathsf{FI}}, Y \succ \Sigma$. Hence proved. $\square$

This lemma gives us the following theorem and corollary immediately, saying filter-ideal extensions LE-models have the Hennessy-Milner property.

**Theorem 4.9** *For all* LE*-models* $\mathfrak{M}_1$ *and* $\mathfrak{M}_2$, $a_1, x_1$ *in* $\mathfrak{M}_1$, *and* $a_2, x_2$ *in* $\mathfrak{M}_2$,

1. $\mathfrak{M}_1, a_1 \rightsquigarrow_A \mathfrak{M}_2, a_2$ *if and only if* $\mathfrak{M}_1^{\mathsf{FI}}, \mathsf{Fi}(\mathbf{a_1}) \rightrightarrows \mathfrak{M}_2^{\mathsf{FI}}, \mathsf{Fi}(\mathbf{a_2})$.
2. $\mathfrak{M}_1, x_1 \rightsquigarrow_X \mathfrak{M}_2, x_2$ *if and only if* $\mathfrak{M}_1^{\mathsf{FI}}, \mathsf{Id}(\mathbf{x_1}) \leftleftarrows \mathfrak{M}_2^{\mathsf{FI}}, \mathsf{Id}(\mathbf{x_2})$.

**Proof.** Immediate by Corollary 4.7, Lemma 4.8, and Theorem 4.3. $\square$

**Corollary 4.10** *For all* LE-*models* $\mathfrak{M}_1$ *and* $\mathfrak{M}_2$*, any* $a_1, x_1$ *in* $\mathfrak{M}_1$ *and any* $a_2, x_2$ *in* $\mathfrak{M}_2$*,*

1. $\mathfrak{M}_1, a_1 \leftrightsquigarrow_A \mathfrak{M}_2, a_2$ *if and only if* $\mathfrak{M}_1^{\mathsf{Fl}}, \mathsf{Fi}(\mathbf{a_1}) \rightleftarrows \mathfrak{M}_2^{\mathsf{Fl}}, \mathsf{Fi}(\mathbf{a_2})$.
2. $\mathfrak{M}_1, x_1 \leftrightsquigarrow_X \mathfrak{M}_2, x_2$ *if and only if* $\mathfrak{M}_1^{\mathsf{Fl}}, \mathsf{Id}(\mathbf{x_1}) \rightleftarrows \mathfrak{M}_2^{\mathsf{Fl}}, \mathsf{Id}(\mathbf{x_2})$.

## 5 Toward van Benthem Characterization Theorem

In this section, we firstly introduce the standard translation for non-distributive modal logic based on polarity-based semantics, and then define the ultrapower extension for LE-models. Following that, we introduce $\omega$-saturated LE-models and show that $\omega$-saturated LE-models are M-saturated. Finally, we show that for any LE-model, there exists an ultrapower extension of it which is $\omega$-saturated. We use this result to prove the van Benthem characterization theorem of non-distributive modal logic.

### 5.1 Standard Translation

The following definition of standard translation for non-distributive modal logic on polarity-based semantics is inspired from [8, Section 2.1].

**Definition 5.1** Let $g \in G$ and $m \in M$ be two $\mathcal{L}_1$-variables and $\varphi \in \mathcal{L}$ (see Section 2.3). The *standard translations* $ST_g : \mathcal{L} \to \mathcal{L}^1$ and $ST_m : \mathcal{L} \to \mathcal{L}^1$ are defined as follows:

| | |
|---|---|
| $ST_g(\bot) := \forall m(gIm)$ | $ST_m(\bot) := m = m$ |
| $ST_g(\top) := g = g$ | $ST_m(\bot) := \forall g(gIm)$ |
| $ST_g(p) := P_A(g)$ | $ST_m(p) := P_X(m)$ |
| $ST_g(\varphi \vee \psi) := \forall m(ST_m(\varphi) \wedge ST_m(\psi) \to gIm)$ | $ST_m(\varphi \vee \psi) := ST_m(\varphi) \wedge ST_m(\psi)$ |
| $ST_g(\varphi \wedge \psi) := ST_g(\varphi) \wedge ST_g(\psi)$ | $ST_m(\varphi \wedge \psi) := \forall g(ST_g(\varphi) \wedge ST_g(\psi) \to gIm)$ |
| $ST_g(\Diamond\varphi) := \forall m(ST_m(\Diamond\varphi) \to gIm)$ | $ST_m(\Diamond\varphi) := \forall g(ST_g(\varphi) \to mR_\Diamond g)$ |
| $ST_g(\Box\varphi) := \forall m(ST_m(\varphi) \to gR_\Box m)$ | $ST_m(\Box\varphi) := \forall g(ST_g(\Box\varphi) \to gIm)$ |

The following lemma is immediate from the definition above.

**Lemma 5.2** *Let* $\mathfrak{M} = (A, X, I, R_\Box, R_\Diamond, V)$ *be an* LE-*model, then for any* $a \in A$ *and* $x \in X$*, and any* $\mathcal{L}_1$*-variables* $g \in G$ *and* $m \in M$*:*

1. $\mathfrak{M}, a \Vdash \varphi$ *iff* $\mathfrak{M} \vDash ST_g(\varphi)[a]$ *and* $\mathfrak{M}, x \succ \varphi$ *iff* $\mathfrak{M} \vDash ST_m(\varphi)[x]$.
2. $\mathfrak{M}, a \Vdash \varphi$ *iff* $\mathfrak{M} \vDash \forall g ST_g(\varphi)$ *and* $\mathfrak{M}, x \succ \varphi$ *iff* $\mathfrak{M} \vDash \forall m ST_m(\varphi)$.

**Proof.** It is straight-forward by induction on the complexity of formulas. $\Box$

### 5.2 Ultrapower Extension and $\omega$-Saturated Model

**Definition 5.3** Let K be a non-empty index set, and $\mathfrak{M} = (\mathfrak{F}, V)$ be an LE-model, where $\mathfrak{F} = (A, X, I, R_\Box, R_\Diamond)$ and for any $p \in \mathcal{V}$, $V(p) = ([\![p]\!], (\![p]\!))$. The K-*power* of $\mathfrak{M}$ is the LE-model $\mathfrak{M}^{\mathrm{K}} = (A^{\mathrm{K}}, X^{\mathrm{K}}, I^{\mathrm{K}}, R_\Box^{\mathrm{K}}, R_\Diamond^{\mathrm{K}}, V^{\mathrm{K}})$, where

1. $A^{\mathrm{K}}$ is the set of functions $s : \mathrm{K} \to A$.
2. $X^{\mathrm{K}}$ is the set of functions from $t : \mathrm{K} \to X$.
3. For any $s \in A^{\mathrm{K}}$ and $t \in X^{\mathrm{K}}$, $sI^{\mathrm{K}}t$ iff for any $k \in \mathrm{K}$, $s(k)It(k)$.
4. For any $s \in A^{\mathrm{K}}$ and $t \in X^{\mathrm{K}}$, $sR_\Box^{\mathrm{K}}t$ iff for any $k \in \mathrm{K}$, $s(k)R_\Box t(k)$.
5. For any $s \in A^{\mathrm{K}}$ and $t \in X^{\mathrm{K}}$, $tR_\Diamond^{\mathrm{K}}s$ iff for any $k \in \mathrm{K}$, $t(k)R_\Diamond s(k)$.

6. For any $p \in \mathcal{V}, V^{\mathrm{K}}(p) = ([\![p]\!]^{\mathrm{K}}, (\![p]\!)^{\mathrm{K}})$ such that $s \in [\![p]\!]^{\mathrm{K}}$ iff for any $k \in \mathrm{K}$, $s(k) \in [\![p]\!]$, and $t \in (\![p]\!)^{\mathrm{K}}$ iff for any $k \in \mathrm{K}$, $t(k) \in (\![p]\!)$.

Let $\mathcal{U} \subseteq \mathcal{P}(\mathrm{K})$ be an ultrafilter over K. We define equivalence relations $\sim_A$ and $\sim_X$ on $A^{\mathrm{K}}$ and $X^{\mathrm{K}}$, respectively, as follows:

1. $s_1 \sim_A s_2$ if and only if $\{k \in \mathrm{K} \mid s_1(k) = s_2(k)\} \in \mathcal{U}$.
2. $t_1 \sim_X t_2$ if and only if $\{k \in \mathrm{K} \mid t_1(k) = t_2(k)\} \in \mathcal{U}$.

The *ultrapower* of $\mathfrak{M}$ modulo K and $\mathcal{U}$ is a structure $\mathfrak{M}^{\mathrm{K}}_{\mathcal{U}} = (A^{\mathrm{K}}_{\mathcal{U}}, X^{\mathrm{K}}_{\mathcal{U}}, I^{\mathrm{K}}_{\mathcal{U}}, R^{\mathrm{K}}_{\Box\mathcal{U}}, R^{\mathrm{K}}_{\Diamond\mathcal{U}}, V^{\mathrm{K}}_{\mathcal{U}})$, where $A^{\mathrm{K}}_{\mathcal{U}}$ (resp. $X^{\mathrm{K}}_{\mathcal{U}}$) is quotient of $A^{\mathrm{K}}$ (resp. $X^{\mathrm{K}}$) over $\sim_A$ (resp. $\sim_X$), and for any $[s] \in A^{\mathrm{K}}_{\mathcal{U}}$, $[t] \in X^{\mathrm{K}}_{\mathcal{U}}$,

1. $[s]I^{\mathrm{K}}_{\mathcal{U}}[t]$ if and only if $\{k \in \mathrm{K} \mid s(k)It(k)\} \in \mathcal{U}$.
2. $[s]R^{\mathrm{K}}_{\Box\mathcal{U}}[t]$ if and only if $\{k \in \mathrm{K} \mid s(k)R_\Box t(k)\} \in \mathcal{U}$.
3. $[t]R^{\mathrm{K}}_{\Diamond\mathcal{U}}[s]$ if and only if $\{k \in \mathrm{K} \mid t(k)R_\Diamond s(k)\} \in \mathcal{U}$.
4. For any $p \in \mathcal{V}$, $V^{\mathrm{K}}_{\mathcal{U}}(p) = ([\![p]\!]^{\mathrm{K}}_{\mathcal{U}}, (\![p]\!)^{\mathrm{K}}_{\mathcal{U}})$, where $[s] \in [\![p]\!]^{\mathrm{K}}_{\mathcal{U}}$ if and only if $\{k \in \mathrm{K} \mid s(k) \in [\![p]\!]\} \in \mathcal{U}$, and $[t] \in (\![p]\!)^{\mathrm{K}}_{\mathcal{U}}$ if and only if $\{k \in \mathrm{K} \mid t(k) \in (\![p]\!) \in \mathcal{U}\}$.

The following theorem is an immediate consequence of Łós's Theorem for first-order logic.

**Theorem 5.4 (Łós's Theorem)** *Let* K *be a non-empty index set and* $\mathcal{U}$ *be an ultrafilter over* K. *Let* $\mathfrak{M}$ *be an* LE*-model and* $\mathfrak{M}^{\mathrm{K}}_{\mathcal{U}}$ *be its ultrapower modulo* K *and* $\mathcal{U}$. *For any two-sorted first-order formula* $\varphi \in \mathcal{L}^1$ *(see Section 2.3),*

$$\mathfrak{M}^{\mathrm{K}}_{\mathcal{U}} \vDash \varphi[s,t] \quad \textit{if and only if} \quad \{k \in \mathrm{K} \mid \mathfrak{M} \vDash \varphi[s(k), t(k)]\} \in \mathcal{U}.$$

**Remark 5.5** Any ultrapower $\mathfrak{M}^{\mathrm{K}}_{\mathcal{U}}$ of an LE-model $\mathfrak{M}$ is indeed an LE-model. In order to see this, we just need to check that relations $R^{\mathrm{K}}_{\Box\mathcal{U}}$ and $R^{\mathrm{K}}_{\Diamond\mathcal{U}}$ are $I$-compatible, and for any $p \in \mathcal{V}$, $[\![p]\!]^{\mathrm{K}}_{\mathcal{U}}$ and $(\![p]\!)^{\mathrm{K}}_{\mathcal{U}}$ are Galois stable. This is true because all these conditions are first-order definable, and so they will be preserved in the ultrapower extension by Łós's Theorem.

We now define $\kappa$-saturated first-order models for any infinite cardinal $\kappa$.

**Definition 5.6** Let $\kappa$ be an infinite cardinal, and $\mathfrak{M}$ be an LE-model. For any set $S$ of elements appearing in $\mathfrak{M}$, let $\mathcal{L}^1_S$ be the extension of $\mathcal{L}^1$ with constant symbols for every element of $S$. $\mathfrak{M}$ is *$\kappa$-saturated* if for any set of elements $S$ appearing in $\mathfrak{M}$ with $|S| < \kappa$, and any set $\Sigma$ of $\mathcal{L}^1_S$ formulas with finite many free variables, if $\Sigma$ is finitely satisfiable in $\mathfrak{M}$, then $\Sigma$ is satisfiable in $\mathfrak{M}$.

The following theorem follows from [5, Theorem 6.1.8].

**Theorem 5.7** *For any* LE*-model* $\mathfrak{M}$*, there exist an non-empty index set* K *and an ultrafilter* $\mathcal{U}$ *over* K *such that* $\mathfrak{M}^{\mathrm{K}}_{\mathcal{U}}$ *is* $\omega$*-saturated.*

**Proposition 5.8** *Any* $\omega$*-saturated* LE*-model is* M*-saturated.*

**Proof.** Let $\mathfrak{M} = (\mathfrak{F}, V)$ be an $\omega$-saturated LE-model, where $\mathfrak{F} = (A, X, I, R_\Box, R_\Diamond)$. We only prove the fist item of Definition 4.2, and other conditions can be proved similarly. Let $\Sigma$ be a set of $\mathcal{L}$-formulas which are finitely satisfiable in set $\{x' \mid aI^c x'\}$. Define $\Sigma'$ to be the set

$$\Sigma' = \{aI^c x\} \cup ST_x(\Sigma).$$

Clearly $\Sigma'$ contains only finite many free variables (only $x$) and every finite subset of $\Sigma'$ is satisfiable in $(\mathfrak{M}, x)_{x \in X}$. Hence, by $\omega$-saturation $\Sigma'$ is satisfiable

$$
\begin{tikzcd}
\mathfrak{M}_1^{\mathsf{Fl}}, \mathsf{Fi}(\mathbf{a}_1) \arrow[r, "\rightrightarrows"] & \mathfrak{M}_2^{\mathsf{Fl}}, \mathsf{Fi}(\mathbf{a}_2) \\
\mathfrak{M}_1, a_1 \arrow[r, "\rightsquigarrow_A"] \arrow[u, "\mathsf{Fl}"] \arrow[d, "\omega_1"'] & \mathfrak{M}_2, a_2 \arrow[u, "\mathsf{Fl}"'] \arrow[d, "\omega_2"] \\
\mathfrak{M}_1^{\omega}, \omega_1(a_1) \arrow[r, "\rightrightarrows"'] & \mathfrak{M}_2^{\omega}, \omega_2(a_2)
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathfrak{M}_1^{\mathsf{Fl}}, \mathsf{Id}(\mathbf{x}_1) & \mathfrak{M}_2^{\mathsf{Fl}}, \mathsf{Id}(\mathbf{x}_2) \arrow[l, "\leftleftarrows"'] \\
\mathfrak{M}_1, x_1 \arrow[r, "\rightsquigarrow_X"] \arrow[u, "\mathsf{Fl}"] \arrow[d, "\omega_1"'] & \mathfrak{M}_2, x_2 \arrow[u, "\mathsf{Fl}"'] \arrow[d, "\omega_2"] \\
\mathfrak{M}_1^{\omega}, \omega_1(x_1) & \mathfrak{M}_2^{\omega}, \omega_2(x_2) \arrow[l, "\leftleftarrows"]
\end{tikzcd}
$$

Fig. 4. Detour Lemma

in $(\mathfrak{M}, x)_{x \in X}$ at some $x_0$ with $aI^c x_0$ . Then, by Lemma 5.2 we have $\mathfrak{M}, x_0 \succ \Sigma$. Therefore, $\Sigma$ is satisfiable in $\{x' \mid aI^c x'\}$. □

It follows immediately from the above proposition and Theorem 4.3 that the class of $\omega$-saturated LE-models satisfy Hennessy-Milner property.

### 5.3 The van Benthem Characterization Theorem

We are now ready to prove the van Benthem characterization theorem for non-distributive modal logic based on the polarity-based semantics, which follows from the detour lemma below.

**Theorem 5.9 (Detour Lemma)** *For all* LE-*models* $\mathfrak{M}_1$ *and* $\mathfrak{M}_2$*, any* $a_1, x_1$ *in* $\mathfrak{M}_1$ *and* $a_2, x_2$ *in* $\mathfrak{M}_2$*,*

*1.* $\mathfrak{M}_1, a_1 \rightsquigarrow_A \mathfrak{M}_2, a_2$ *if and only if there exist two* $\omega$*-saturated* LE-*models* $\mathfrak{M}_1^{\omega}$ *and* $\mathfrak{M}_2^{\omega}$*, and two elementary embeddings* $\omega_1 : \mathfrak{M}_1 \to \mathfrak{M}_1^{\omega}$ *and* $\omega_2 : \mathfrak{M}_2 \to \mathfrak{M}_2^{\omega}$*, such that* $\mathfrak{M}_1^{\omega}, \omega_1(a_1) \rightrightarrows \mathfrak{M}_2^{\omega}, \omega_2(a_2)$*.*

*2.* $\mathfrak{M}_1, x_1 \rightsquigarrow_X \mathfrak{M}_2, x_2$ *if and only if there exist two* $\omega$*-saturated* LE-*models* $\mathfrak{M}_1^{\omega}$ *and* $\mathfrak{M}_2^{\omega}$*, and two elementary embeddings* $\omega_1 : \mathfrak{M}_1 \to \mathfrak{M}_1^{\omega}$ *and* $\omega_2 : \mathfrak{M}_2 \to \mathfrak{M}_2^{\omega}$*, such that* $\mathfrak{M}_1^{\omega}, \omega_1(x_1) \leftleftarrows \mathfrak{M}_2^{\omega}, \omega_2(x_2)$*.*

**Proof.** We only prove the item 1 here, the proof of the item 2 is similar. The implication from right to left proved by contradiction. Suppose $\mathfrak{M}_1^{\omega}, \omega_1(a_1) \rightrightarrows \mathfrak{M}_2^{\omega}, \omega_2(a_2)$, and $\omega_1 : \mathfrak{M}_1 \to \mathfrak{M}_1^{\omega}$, $\omega_2 : \mathfrak{M}_2 \to \mathfrak{M}_2^{\omega}$ are two elementary embeddings. Suppose there exists $\varphi \in \mathcal{L}$, such that $\mathfrak{M}, a_1 \Vdash \varphi$, and $\mathfrak{M}, a_2 \nVdash \varphi$. Therefore, by Lemma 5.2, $\mathfrak{M}_1 \vDash ST_g(\varphi)[a_1]$ and $\mathfrak{M}_2 \nVdash ST_g(\varphi)[a_2]$, which implies that $\mathfrak{M}_1^{\omega} \vDash ST_g(\varphi)[\omega_1(a_1)]$ and $\mathfrak{M}_2^{\omega} \nvDash ST_g(\varphi)[\omega_2(a_2)]$. Therefore, by Lemma 5.2 again, $\mathfrak{M}_1^{\omega}, \omega_1(a_1) \Vdash \varphi$ and $\mathfrak{M}_2^{\omega}, \omega_2(a_2) \nVdash \varphi$. However, this contradicts $\mathfrak{M}_1^{\omega}, \omega_1(a_1) \rightrightarrows \mathfrak{M}_2^{\omega}, \omega_2(a_2)$ due to Theorem 3.10. Hence proved.

For the implication from left to right, assume that $\mathfrak{M}_1, a_1 \rightsquigarrow_A \mathfrak{M}_2, a_2$. We consider their respective ultrapower extensions ${\mathfrak{M}_1^{\mathrm{K}}}_{\mathcal{U}}$, ${\mathfrak{M}_2^{\mathrm{K}}}_{\mathcal{U}}$, and natural elementary embeddings $\omega_i : \mathfrak{M}_i \to {\mathfrak{M}_i^{\mathrm{K}}}_{\mathcal{U}}$ where $i \in \{1, 2\}$ such that for any $a_i \in A_i$, $\omega_i(a_i) = [s^i_{a_i}]$, where $s^i_{a_i} : \mathrm{K} \to A_i$ are the constant functions which send all elements in K to $a_i$. Therefore, ${\mathfrak{M}_1^{\mathrm{K}}}_{\mathcal{U}}, \omega_1(a_1) \rightsquigarrow_A {\mathfrak{M}_2^{\mathrm{K}}}_{\mathcal{U}}, \omega_2(a_2)$. By Theorem 5.7, ${\mathfrak{M}_1^{\mathrm{K}}}_{\mathcal{U}}$ and ${\mathfrak{M}_2^{\mathrm{K}}}_{\mathcal{U}}$ are $\omega$-saturated LE-models. Therefore, by Proposition 5.8 and Theorem 4.3, ${\mathfrak{M}_1^{\mathrm{K}}}_{\mathcal{U}}, \omega_1(a_1) \rightrightarrows {\mathfrak{M}_2^{\mathrm{K}}}_{\mathcal{U}}, \omega_2(a_2)$. □

**Remark 5.10** Notice that in both items of the above lemma, "there exist" can be substituted by "for any". It is because the only if direction follows from Proposition 5.8 and the definition of elementary embeddings, and the if direction follows trivially from the existence of the $\omega$-saturated elementary extensions.

**Theorem 5.11 (van Benthem Characterization Theorem)** *Let $\varphi(g)$ (resp. $\varphi(m)$) be any two sorted first-order formula in $\mathcal{L}^1$, then $\varphi(g)$ (resp. $\varphi(m)$) is preserved (resp. reflected) by simulation if and only if there exist finite many $\mathcal{L}$-formulas $\psi_1, \psi_2, \cdots, \psi_n$ such that $\varphi(g)$ (resp. $\varphi(m)$) is equivalent to $\bigvee_{1\leq i\leq n} ST_g(\psi_i)$ (resp. $\bigvee_{1\leq i\leq n} ST_m(\psi_i)$).*

**Proof.** We only provide the proof for $\varphi(g)$, the the proof for $\varphi(m)$ is similar. The implication from right to left follows from Theorem 3.5. For the implication from left to right, assume that $\varphi(g)$ is preserved by simulation. We define a set of two sorted first-order formulas $\mathrm{JMOC}(\varphi) := \{\bigvee_{1\leq i\leq n} ST_g(\psi_i) \mid \psi_i \in \mathcal{L}, n \in \omega, \varphi(g) \vDash \bigvee_{1\leq i\leq n} ST_g(\psi_i)\}$. We first check that the following is true: if $\mathrm{JMOC}(\varphi) \vDash \varphi(g)$, then $\varphi(g)$ is equivalent to $\bigvee_j ST_g(\psi_j)$ for some finite set of $\mathcal{L}$-formulas $\{\psi_j \mid j \in \mathcal{J}\}$. Assume that $\mathrm{JMOC}(\varphi) \vDash \varphi(g)$. By compactness, there exists an finite set $\{\bigvee_{1\leq i\leq n} ST_g(\psi_{j,i}) \mid j \in \mathcal{J}, n \in \omega\} \subseteq \mathrm{JMOC}(\varphi)$ such that $\vDash \bigwedge_{j\in\mathcal{J}} \bigvee_{1\leq i\leq n} ST_g(\psi_{j,i}) \to \varphi(g)$. Therefore, we have $\vDash \varphi(g) \leftrightarrow \bigwedge_{j\in\mathcal{J}} \bigvee_{1\leq i\leq n} ST_g(\psi_{j,i})$. Note that $\bigwedge_{j\in\mathcal{J}} \bigvee_{1\leq i\leq n} ST_g(\psi_{j,i})$ is equivalent to $\bigvee_{1\leq i\leq n} \bigwedge_{j\in\mathcal{J}} ST_g(\psi_{j,i})$, and the latter is equivalent to $\bigvee_{1\leq i\leq n} ST_g(\bigwedge_{j\in\mathcal{J}} \psi_{j,i})$.

Therefore, it suffices to show that $\mathrm{JMOC}(\varphi) \vDash \varphi(g)$. Let $\mathfrak{M}$ be an LE-model such that $\mathfrak{M} \vDash \mathrm{JMOC}(\varphi)[a]$. Let $\mathrm{NT}(g) := \{\neg ST_g(\psi) \mid \psi \in \mathcal{L}, \mathfrak{M} \nvDash ST_g(\psi)[a]\}$. We want to show that $\mathrm{NT}(g) \cup \{\varphi(g)\}$ is consistent. Suppose not. Then, by compactness there exists a finite subset $\mathrm{NT}_0(g) = \{\neg ST_g(\psi_k) \mid k \in \mathcal{K}\}$ of $\mathrm{NT}(g)$ such that $\mathrm{NT}_0(g) \cup \{\varphi(g)\}$ is inconsistent. Therefore, $\varphi(g) \to \neg \bigwedge_{k\in\mathcal{K}} \neg ST_g(\psi_i)$ is valid, which means that $\varphi(g) \to \bigvee_{k\in\mathcal{K}} ST_g(\psi_i)$ is valid. Therefore, $\bigvee_{k\in\mathcal{K}} ST_g(\psi_i) \in \mathrm{JMOC}(\varphi)$, which implies $\mathfrak{M} \vDash \bigvee_{k\in\mathcal{K}} ST_g(\psi_i)[a]$. However, $\mathfrak{M} \nvDash ST_g(\psi_i)[a]$ for any $k \in \mathcal{K}$, which implies $\mathfrak{M} \nvDash \bigvee_{k\in\mathcal{K}} ST_g(\psi_i)[a]$, so there is a contradiction. Hence, $\mathrm{NT}(g) \cup \{\varphi(g)\}$ must be consistent. Therefore, there exists an $\mathcal{L}^1$-structure $\mathfrak{N}$ and $a'$ in $\mathfrak{N}$ such that $\mathfrak{N} \vDash \mathrm{NT}(g) \cup \{\varphi(g)\}[a']$.

For any formula $\psi$, $\mathfrak{M}, a \not\Vdash \psi$ iff $\neg ST_g(\psi) \in NT(g)$, which implies $\mathfrak{N}, a' \not\Vdash \psi$. Therefore, $\mathfrak{N}, a' \rightsquigarrow_A \mathfrak{M}, a$. By Theorem 5.9 (Detour Lemma), there exist two $\omega$-saturated LE-models $\mathfrak{N}^\omega$ and $\mathfrak{M}^\omega$, and two elementary embeddings $\omega_n : \mathfrak{N} \to \mathfrak{N}^\omega$ and $\omega_m : \mathfrak{M} \to \mathfrak{M}^\omega$, such that $\mathfrak{N}^\omega, \omega_n(a') \rightrightarrows \mathfrak{M}^\omega, \omega_m(a)$. As $\omega_n$ is an elementary embedding and $\mathfrak{N} \vDash \varphi(g)[a']$, we have $\mathfrak{N}^\omega \vDash \varphi(g)[\omega_n(a')]$. Therefore, as $\varphi(g)$ is preserved by simulation, $\mathfrak{M}^\omega \vDash \varphi(g)[\omega_m(a)]$. As $\omega_m$ is an elementary embedding, we have $\mathfrak{M} \vDash \varphi(g)[a]$. This concludes the proof. □

## 6 Conclusion

In this paper, we introduced bisimulations on polarity-based semantics for non-distributive modal logic, demonstrating their application in proving the Hennessy-Milner and van Benthem theorems. These results show that the non-distributive modal logic is bisimulation invariant fragment (within a specific signature and axioms) of two-sorted first-order logic. This work suggests the following directions for future research.

**Characterizing theorem for lattice-based modal $\mu$-calculus:** Lattice based modal $\mu$-calculus, i.e. the extension of non-distributive modal logic with least and greatest fixed point operators, was defined and studied in [16]. In [16], it was showed that the formulas of modal $\mu$-calculus are bisimulation-invariant. The classical modal $\mu$-calculus is characterized as the bisimulation invariant fragment of monadic second-order logic. It would be interesting to see if a similar characterization holds in its lattice-based generalization.

**Computational properties of bisimulations:** In classical modal logic, computational complexity of different bisimulation related problems like checking if two models are bisimlar, computing the largest bisimulation on a model have been studied extensively [31]. It would be interesting to study similar complexity problems in a non-distributive setting and compare the results with the classical setting.

**Bisimulations and network analysis:** In Social Network Analysis (SNA), different notions of equivalences between nodes are defined to study the structural nodes in one or two social networks [28,18]. On of the most prominent structural equivalences used in SNA is the notion of *regular equivalence*, which is analogous to bisimulations on (multi-modal) Kripke frames [35,29]. Enriched formal contexts (polarities) can be seen as bi-partite networks; so, it would be interesting to study if simulations and bisimulations defined in this paper can be useful in bi-partite network analysis.

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