Towards an Algebraic Theory of KD45-Like Logics

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Abstract

Algebraic semantics are introduced for a family of 'KD45-like' modal substructural logics as a generalization of Bezhanishivili's pseudomonadic algebras for the modal logic KD45. It is shown that these structures correspond to ordered pairs consisting of an $\mathsf{FL_e}$-algebra (or commutative pointed residuated lattice) and a subalgebra with a suitable lattice filter, extending a similar result for 'S5-like' logics. It is then shown that if the $\mathsf{FL_e}$-algebra reduct belongs to a variety that has the superamalgamation property, then the structure equipped with an additional constant is representable as an algebra of functions from a set of worlds to an $\mathsf{FL_e}$-algebra of the same variety.

Keywords: Modal Logic, Substructural Logic, $\mathsf{FL_e}$-algebra, Residuated Lattice.

1 Introduction

Non-classical modal logics — extensions of intermediate, substructural, and many-valued logics with various modal operators — have been motivated and studied in a wide range of contexts (see, e.g., [4, 7–9, 13, 14, 20, 22]). These logics are introduced either semantically via a relational semantics, or syntactically, via an axiomatization or class of algebraic structures, but general systematic accounts relating these two perspectives are mostly lacking in the literature. Such an account has been given in [12] for 'S5-like' modal logics corresponding to one-variable fragments of first-order logics, and the main aim of the current

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paper is to explore how far this approach extends to 'KD45-like' modal logics. In particular, we introduce, and provide representation theorems for, algebraic structures that generalize both Bezhanishivili's pseudomonadic algebras for the modal logic KD45 [3] and algebras for the 'KD45-like' many-valued modal logics with a possibilistic semantics considered in [5, 6, 10, 21].

In Section 2, we consider, as a general starting point, expansions with unary operations $\Box$ and $\Diamond$ of $\mathcal{L}$-*lattices*: algebraic structures for a signature $\mathcal{L}$ with a lattice reduct that provide semantics for a broad range of non-classical logics. As shown in [12], if the expansion satisfies equations corresponding to the modal axioms for S5, then its image under $\Box$ (equivalently, $\Diamond$) forms a subalgebra of its $\mathcal{L}$-lattice reduct. We show here that this is also the case for expansions that fail the 'reflexivity' equations $\Box x \leq x$ and $x \leq \Diamond x$, corresponding to the 'T' axiom distinguishing S5 from KD45. However, whereas the expansions satisfying the reflexivity equations are in one-to-one correspondence with $\mathcal{L}$-lattices equipped with a 'relatively complete' subalgebra [12], this is not the case in general.

In Section 3, we address this issue in the setting of $\mathsf{FL_e}$-algebras (also known as commutative pointed residuated lattices): algebraic structures that serve as semantics for a broad family of substructural logics (see [16, 18]). We introduce *pm-*$\mathsf{FL_e}$*-algebras* as expansions of $\mathsf{FL_e}$-algebras with unary operations $\Box$ and $\Diamond$ that include the aforementioned pseudomonadic algebras [3] and algebras for many-valued modal logics [5, 6, 10, 21]. We then prove that these structures are in one-to-one correspondence with ordered pairs consisting of an $\mathsf{FL_e}$-algebra and a subalgebra equipped with a suitable lattice filter (Theorem 3.9).

In Section 4, we establish a functional representation theorem for 'pointed' pm-$\mathsf{FL_e}$-algebras expanded with a constant that, as prescribed by two additional equations, generates the corresponding lattice filter. More precisely, following similar proofs in [2, 12] for one-variable fragments of first-order logics, we prove that if the $\mathsf{FL_e}$-reduct of a pointed pm-$\mathsf{FL_e}$-algebra belongs to a variety $\mathcal{V}$ that has the superamalgamation property, then the latter can be represented as an algebra of functions from a set of worlds to a member of $\mathcal{V}$ (Theorem 4.3). In the case where $\mathcal{V}$ is also closed under regular completions, this yields completeness results for a corresponding 'KD45-like' logic with respect to complex algebras based on frames consisting of a set of worlds W and a map $\pi\colon W \to A$ for some $\mathbf{A} \in \mathcal{V}$, generalizing the 'simplified semantics' for KD45 described in [19] and 'possibilistic semantics' for many-valued modal logics considered in [5, 21].

2 Modal Operators via Subalgebras

In this section, we consider a general class of lattice-ordered algebraic structures with unary operations $\Box$ and $\Diamond$ that satisfy equations corresponding to axioms common to 'KD45-like' logics. In particular, we show that this class includes exactly the algebraic structures in the general framework whose images under $\Box$ and $\Diamond$ coincide and form a common subalgebra.

Let us call an algebraic structure $\langle L, \wedge, \vee, \Box, \Diamond\rangle$ with a lattice reduct $\langle L, \wedge, \vee\rangle$ and unary operations $\Box$ and $\Diamond$ an *e-lattice* if it satisfies the following

equations satisfied by Bezhanishivili's pseudomonadic algebras [2] for KD45 [3]:

$$\begin{array}{ll} (\mathrm{L1}_\Box)\ \Box(x \wedge y) \approx \Box x \wedge \Box y & (\mathrm{L1}_\Diamond)\ \Diamond(x \vee y) \approx \Diamond x \vee \Diamond y \\ (\mathrm{L2}_\Box)\ \Box\Box x \approx \Box x & (\mathrm{L2}_\Diamond)\ \Diamond\Diamond x \approx \Diamond x \\ (\mathrm{L3}_\Box)\ \Box\Diamond x \approx \Diamond x & (\mathrm{L3}_\Diamond)\ \Diamond\Box x \approx \Box x. \end{array}$$

It follows directly, using $(\mathrm{L1}_\Box)$ and $(\mathrm{L1}_\Diamond)$, that the unary operations $\Box$ and $\Diamond$ of any e-lattice are order-preserving; that is, letting $\alpha \leq \beta$ stand for $\alpha \wedge \beta \approx \alpha$, every e-lattice satisfies the quasi-equations

$$(\mathrm{L4}_\Box)\ x \leq y \implies \Box x \leq \Box y \qquad (\mathrm{L4}_\Diamond)\ x \leq y \implies \Diamond x \leq \Diamond y.$$

Also $\Box A := \{\Box a \mid a \in A\} = \{\Diamond a \mid a \in A\}$, by $(\mathrm{L3}_\Box)$ and $(\mathrm{L3}_\Diamond)$, and $\Box$ and $\Diamond$ restrict to identity maps on this set, by $(\mathrm{L2}_\Box)$ and $(\mathrm{L2}_\Diamond)$.

Note that the equations $(\mathrm{L1}_\Box)$-$(\mathrm{L3}_\Box)$ and $(\mathrm{L1}_\Diamond)$-$(\mathrm{L3}_\Diamond)$ are associated with the 'K', '4', and '5' axioms of KD45, but the following equation corresponding to the 'D' axiom $\Box p \to \Diamond p$ is *not* satisfied by all e-lattices:

$$(\mathrm{LD})\ \Box x \leq \Diamond x.$$

Now let $\mathcal{L}$ be any algebraic signature containing binary operation symbols $\wedge$ and $\vee$, and denote by $\mathcal{L}_\Box$ the extension of $\mathcal{L}$ with unary operation symbols $\Box$ and $\Diamond$. We call an $\mathcal{L}$-structure $\mathbf{A}$ with a lattice reduct $\langle A, \wedge, \vee\rangle$ an *$\mathcal{L}$-lattice*, and call an $\mathcal{L}_\Box$-structure $\langle \mathbf{A}, \Box, \Diamond\rangle$ with an e-lattice reduct $\langle A, \wedge, \vee, \Box, \Diamond\rangle$ an *e-$\mathcal{L}$-lattice* if it satisfies for each n-ary operation symbol $\star$ of $\mathcal{L}$ the equation

$$(\star_\Box)\ \Box(\star(\Box x_1, \ldots, \Box x_n)) \approx \star(\Box x_1, \ldots, \Box x_n).$$

It follows easily, using $(\star_\Box)$, $(\mathrm{L3}_\Box)$, and $(\mathrm{L3}_\Diamond)$, that every e-$\mathcal{L}$-lattice $\langle \mathbf{A}, \Box, \Diamond\rangle$ also satisfies for each n-ary operation symbol $\star$ of $\mathcal{L}$ the equation

$$(\star_\Diamond)\ \Diamond(\star(\Diamond x_1, \ldots, \Diamond x_n)) \approx \star(\Diamond x_1, \ldots, \Diamond x_n).$$

Moreover, as the next result demonstrates, these additional equations express precisely what is required of an $\mathcal{L}_\Box$-structure $\langle \mathbf{A}, \Box, \Diamond\rangle$ with an e-lattice reduct $\langle A, \wedge, \vee, \Box, \Diamond\rangle$ to ensure that $\Box A$ forms a subalgebra of the $\mathcal{L}$-structure $\mathbf{A}$.

Proposition 2.1 *Let $\langle \mathbf{A}, \Box, \Diamond\rangle$ be any $\mathcal{L}_\Box$-structure such that $\mathbf{A}$ is an $\mathcal{L}$-lattice and $\langle A, \wedge, \vee, \Box, \Diamond\rangle$ is an e-lattice. Then $\langle \mathbf{A}, \Box, \Diamond\rangle$ is an e-$\mathcal{L}$-lattice if and only if $\Box A$ forms a subalgebra $\Box\mathbf{A}$ of $\mathbf{A}$.*

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[2] A *pseudomonadic algebra* is an algebraic structure $\langle \mathbf{B}, \exists\rangle$ such that $\mathbf{B} = \langle B, \wedge, \vee, \neg, 0, 1\rangle$ is a Boolean algebra and $\exists$ is a unary operation satisfying the following equations:

$$\begin{array}{ll} (\mathrm{B1})\ \exists 0 \approx 0 & (\mathrm{B3})\ \exists(x \wedge \exists y) \approx \exists x \wedge \exists y \\ (\mathrm{B2})\ \exists(x \vee y) \approx \exists x \vee \exists y & (\mathrm{B4})\ \neg\exists x \leq \exists\neg x. \end{array}$$

Proof. If $\langle \mathbf{A}, \Box, \Diamond \rangle$ is an e-$\mathcal{L}$-lattice, then $\Box A$ is closed under each operation $\star^{\mathbf{A}}$ of $\mathbf{A}$, by $(\star_\Box)$, and hence forms a subalgebra. For the converse, suppose that $\Box\mathbf{A}$ is a subalgebra of $\mathbf{A}$ and consider any n-ary operation symbol $\star$ of $\mathcal{L}$ and $a_1, \ldots, a_n \in A$. By assumption, $\Box a_1, \ldots, \Box a_n \in \Box A$, so $\star(\Box a_1, \ldots, \Box a_n) \in \Box A$. Hence $\Box(\star(\Box a_1, \ldots, \Box a_n)) = \star(\Box a_1, \ldots, \Box a_n)$, by $(\mathrm{L2}_\Box)$. □

It is shown in [12] that every e-$\mathcal{L}$-lattice $\langle \mathbf{A}, \Box, \Diamond \rangle$ satisfying two additional 'reflexivity' axioms can be uniquely identified with the ordered pair $\langle \mathbf{A}, \Box\mathbf{A} \rangle$. Following [12], let us call an e-$\mathcal{L}$-lattice an *m-$\mathcal{L}$-lattice* if it satisfies the equations

$$(\mathrm{L5}_\Box)\ \ \Box x \leq x \qquad (\mathrm{L5}_\Diamond)\ \ x \leq \Diamond x.$$

It is easily checked that for any m-$\mathcal{L}$-lattice $\langle \mathbf{A}, \Box, \Diamond \rangle$ and $a \in A$,

$$\Box a = \max\{b \in \Box A \mid b \leq a\} \quad \text{and} \quad \Diamond a = \min\{b \in \Box A \mid a \leq b\}.$$

Conversely, call a subalgebra $\mathbf{A}_0$ of an $\mathcal{L}$-lattice $\mathbf{A}$ *relatively complete* if for any $a \in L$, the set $\{b \in L_0 \mid b \leq a\}$ has a maximum and the set $\{b \in L_0 \mid a \leq b\}$ has a minimum; in this case, defining $\Box_0 a := \max\{b \in A_0 \mid b \leq a\}$ and $\Diamond_0 a := \min\{b \in A_0 \mid a \leq b\}$ for each $a \in A$, yields an m-$\mathcal{L}$-lattice $\langle \mathbf{A}, \Box_0, \Diamond_0 \rangle$. Combining these two observations yields:

Theorem 2.2 ([12, Theorem 3.5]) *There exists a one-to-one correspondence between m-$\mathcal{L}$-lattices and ordered pairs $\langle \mathbf{A}, \mathbf{A}_0 \rangle$ such that $\mathbf{A}$ is an $\mathcal{L}$-lattice and $\mathbf{A}_0$ is a relatively complete subalgebra of $\mathbf{A}$, implemented by the maps $\langle \mathbf{A}, \Box, \Diamond \rangle \mapsto \langle \mathbf{A}, \Box\mathbf{A} \rangle$ and $\langle \mathbf{A}, \mathbf{A}_0 \rangle \mapsto \langle \mathbf{A}, \Box_0, \Diamond_0 \rangle$.*

A further 'functional' representation theorem is obtained in [12] for certain classes of m-$\mathcal{L}$-lattices. Let $\mathcal{V}$ be any variety of $\mathcal{L}$-lattices. A *V-formation* $\langle \mathbf{A}, \mathbf{B}_1, \mathbf{B}_2, f_1, f_2 \rangle$ in $\mathcal{V}$ consists of algebras $\mathbf{A}, \mathbf{B}_1, \mathbf{B}_2 \in \mathcal{V}$ and embeddings $f_1 \colon \mathbf{A} \to \mathbf{B}_1$, $f_2 \colon \mathbf{A} \to \mathbf{B}_2$. A *superamalgam* $\langle \mathbf{C}, g_1, g_2 \rangle$ in $\mathcal{V}$ of this V-formation consists of an algebra $\mathbf{C} \in \mathcal{V}$ and embeddings $g_1 \colon \mathbf{B}_1 \to \mathbf{C}$, $g_2 \colon \mathbf{B}_2 \to \mathbf{C}$ such that $g_1 \circ f_1 = g_2 \circ f_2$ and for any $b_1 \in B_1$, $b_2 \in B_2$ and distinct $i, j \in \{1, 2\}$,

$$g_i(b_i) \leq g_j(b_j) \implies g_i(b_i) \leq g_i \circ f_i(a) = g_j \circ f_j(a) \leq g_j(b_j) \text{ for some } a \in A.$$

$\mathcal{V}$ is said to have the *superamalgamation property* if every V-formation in $\mathcal{V}$ has a superamalgam in $\mathcal{V}$.

Theorem 2.3 ([12, Theorem 4.1]) *Let $\mathcal{V}$ be a variety of $\mathcal{L}$-lattices that has the superamalgamation property. Then every m-$\mathcal{L}$-lattice with an $\mathcal{L}$-lattice reduct in $\mathcal{V}$ is isomorphic to an m-$\mathcal{L}$-lattice $\langle \mathbf{B}, \Box, \Diamond \rangle$ such that $\mathbf{B}$ is a subalgebra of $\mathbf{A}^W$ for some $\mathbf{A} \in \mathcal{V}$ and set W, and contains for each $f \in B$ the constant maps*

$$\Box f \colon W \to A;\ u \mapsto \bigwedge_{v \in W} f(v) \quad \text{and} \quad \Diamond f \colon W \to A;\ u \mapsto \bigvee_{v \in W} f(v).$$

The main goal of this paper is to obtain versions of Theorems 2.2 and 2.3 for a broad family of e-$\mathcal{L}$-lattices serving as algebraic semantics for 'KD45-like' logics.

3 A Correspondence Theorem

The correspondence established in Theorem 2.2 does not apply to e-$\mathcal{L}$-lattices that fail to satisfy either (L5$_\Box$) or (L5$_\Diamond$). In this section, we establish a refined version of this correspondence in the setting of $\mathsf{FL_e}$-algebras.

Let us fix $\mathcal{L}$ to be the signature with binary operation symbols $\wedge$, $\vee$, $\cdot$, and $\to$ and constant symbols e and f. An $\mathsf{FL_e}$-*algebra* — also known as a *commutative pointed residuated lattice* — is an $\mathcal{L}$-lattice $\langle A, \wedge, \vee, \cdot, \to, \mathrm{e}, \mathrm{f}\rangle$ such that $\langle A, \cdot, \mathrm{e}\rangle$ is a commutative monoid and $\to$ is the residuum of $\cdot$, that is, $a \cdot b \leq c \iff a \leq b \to c$, for all $a, b, c \in A$.

Remark 3.1 Varieties of $\mathsf{FL_e}$-algebras provide algebraic semantics for a broad family of non-classical logics (see, e.g., [16, 18]); in particular:

- $\mathsf{FL_e}$-algebras serve as algebraic semantics for the full Lambek calculus with exchange, and $\mathsf{FL_{ew}}$-algebras — $\mathsf{FL_e}$-algebras satisfying $\mathrm{f} \leq x \leq \mathrm{e}$ — play the same role for the extension of this calculus with weakening rules;
- Heyting algebras, serving as algebraic semantics for intuitionistic logic, are term-equivalent to $\mathsf{FL_{ew}}$-algebras satisfying $x \approx x \cdot x$ (just identify $\cdot$ and $\wedge$), and Boolean algebras, serving as algebraic semantics for classical logic, are term-equivalent to Heyting algebras satisfying $\neg\neg x \approx x$, where $\neg x := x \to \mathrm{f}$.
- BL-algebras, serving as algebraic semantics for Hájek's basic fuzzy logic, are term-equivalent to $\mathsf{FL_{ew}}$-algebras satisfying $(x \to y) \vee (y \to x) \approx \mathrm{e}$ and $x \wedge y \approx x \cdot (x \to y)$, while MV-algebras and Gödel algebras, serving as algebraic semantics for Łukasiewicz logic and Gödel-Dummett logic, are term-equivalent to BL-algebras satisfying $\neg\neg x \approx x$ and $x \approx x \cdot x$, respectively.

We call an e-$\mathcal{L}$-lattice $\langle \mathbf{A}, \Box, \Diamond\rangle$ with $\mathsf{FL_e}$-algebra reduct $\mathbf{A}$ a *pm-*$\mathsf{FL_e}$*-algebra* if it satisfies the additional equations

$$\text{(P1)}\ \ \Box(x \to \Box y) \approx \Diamond x \to \Box y \qquad \text{(P2)}\ \ \Box(\Box x \to y) \approx \Box x \to \Box y.$$

Every m-$\mathcal{L}$-lattice satisfies (P1) and (P2) (see [12, Example 3.1]) and will hence also be referred to as an *m-*$\mathsf{FL_e}$*-algebra*.

It follows easily using (L3$_\Box$), (P1), and (P2) that every pm-$\mathsf{FL_e}$-algebra $\langle \mathbf{A}, \Box, \Diamond\rangle$ also satisfies the equations

$$\text{(P3)}\ \ \Box(x \to \Diamond y) \approx \Diamond x \to \Diamond y \qquad \text{(P4)}\ \ \Box(\Diamond x \to y) \approx \Diamond x \to \Box y.$$

The following lemma establishes an additional property of pm-$\mathsf{FL_e}$-algebras that is useful in comparing them with algebras for other modal logics.

Lemma 3.2 *Every pm-*$\mathsf{FL_e}$*-algebra* $\langle \mathbf{A}, \Box, \Diamond\rangle$ *satisfies*

$$\text{(P5)}\ \ \Diamond(x \cdot \Diamond y) \approx \Diamond x \cdot \Diamond y.$$

Proof. Consider any $a, b \in A$. Using $(\cdot_\Diamond)$, $(\to_\Diamond)$, (P3), and standard properties

of $\mathsf{FL_e}$-algebras, we obtain

$$\begin{aligned}\Diamond(a \cdot \Diamond b) \to (\Diamond a \cdot \Diamond b) &= \Diamond(a \cdot \Diamond b) \to \Diamond(\Diamond a \cdot \Diamond b)\\ &= \Box((a \cdot \Diamond b) \to \Diamond(\Diamond a \cdot \Diamond b))\\ &= \Box(a \to (\Diamond b \to \Diamond(\Diamond a \cdot \Diamond b)))\\ &= \Box(a \to \Diamond(\Diamond b \to \Diamond(\Diamond a \cdot \Diamond b)))\\ &= \Diamond a \to \Diamond(\Diamond b \to \Diamond(\Diamond a \cdot \Diamond b))\\ &= \Diamond a \to (\Diamond b \to (\Diamond a \cdot \Diamond b))\\ &\geq \mathrm{e},\end{aligned}$$

and therefore, by residuation, $\Diamond(a \cdot \Diamond b) \leq \Diamond a \cdot \Diamond b$. Similarly,

$$\begin{aligned}(\Diamond a \cdot \Diamond b) \to \Diamond(a \cdot \Diamond b) &= \Diamond a \to (\Diamond b \to \Diamond(a \cdot \Diamond b))\\ &= \Diamond a \to \Diamond(\Diamond b \to \Diamond(a \cdot \Diamond b))\\ &= \Box(a \to \Diamond(\Diamond b \to \Diamond(a \cdot \Diamond b)))\\ &= \Box(a \to (\Diamond b \to \Diamond(a \cdot \Diamond b)))\\ &= \Box((a \cdot \Diamond b) \to \Diamond(a \cdot \Diamond b))\\ &= \Diamond(a \cdot \Diamond b) \to \Diamond(a \cdot \Diamond b)\\ &\geq \mathrm{e},\end{aligned}$$

and therefore, by residuation, $\Diamond a \cdot \Diamond b \leq \Diamond(a \cdot \Diamond b)$. □

Note that if $\langle \mathbf{A}, \Box, \Diamond\rangle$ is a pm-$\mathsf{FL_e}$-algebra such that $\mathbf{A}$ satisfies $x \leq \mathrm{e}$ and $x \wedge (x \to y) \leq y$ (in particular, if $\mathbf{A}$ is a Heyting algebra), then for any $a \in A$,

$$\Box a = \Box a \wedge \mathrm{e} = \Box a \wedge \Box(a \to \Diamond a) = \Box(a \wedge (a \to \Diamond a)) \leq \Box\Diamond a = \Diamond a,$$

that is, $\langle \mathbf{A}, \Box, \Diamond\rangle$ satisfies (LD).

Remark 3.3 Let us call a pm-$\mathsf{FL_e}$-algebra $\langle \mathbf{A}, \Box, \Diamond\rangle$ *Boolean* if $\mathbf{A}$ is a Boolean algebra according to the term-equivalence described in Remark 3.1. In this case, defining $\neg a := a \to \mathrm{f}$ produces a pseudomonadic algebra $\langle A, \wedge, \vee, \neg, \mathrm{f}, \mathrm{e}, \Diamond\rangle$. Conversely, given any pseudomonadic algebra $\langle \mathbf{B}, \exists\rangle$, defining $a \to b := \neg a \vee b$ and $\forall a := \neg\exists\neg a$ produces a Boolean pm-$\mathsf{FL_e}$-algebra $\langle A, \wedge, \vee, \wedge, \to, \mathrm{e}, \mathrm{f}, \forall, \exists\rangle$. So pseudomonadic algebras are term-equivalent to Boolean pm-$\mathsf{FL_e}$-algebras. Similarly, pm-$\mathsf{FL_e}$-algebras with a Gödel algebra reduct are term-equivalent to algebraic semantics for the modal Gödel-Dummett logic defined over a class of many-valued serial, transitive, and Euclidean frames in [10] and studied in [21] as a possibilistic logic.

Remark 3.4 It is straightforward to show that pm-$\mathsf{FL_e}$-algebras satisfying (LD) with a BL-algebra reduct are term-equivalent to the pseudomonadic BL-algebras proposed in [6] as algebraic semantics for '$\mathsf{KD45}$-like' extensions of Hájek's basic fuzzy logic. In particular, a class of pseudomonadic BL-algebras with an MV-algebra reduct is studied in [5] as algebraic semantics for a logic of possibilistic Kripke frames defined over finite totally ordered MV-algebras. To

see that not all pm-$\mathsf{FL_e}$-algebras with a BL-algebra (or even MV-algebra) reduct satisfy (LD), consider the MV-algebra $\mathbf{Ł}_3 = \langle \{0, \frac{1}{2}, 1\}, \min, \max, \cdot, \to, 0, 1\rangle$ (in the language of $\mathsf{FL_e}$-algebras), where $a \cdot b := \max(0, a + b - 1)$ and $a \to b := \min(1, 1 - a + b)$, and define

$$\Box a := \begin{cases} 1 & \text{if } a = 1 \text{ or } a = \frac{1}{2} \\ 0 & \text{if } a = 0, \end{cases} \qquad \Diamond a = \begin{cases} 1 & \text{if } a = 1 \\ 0 & \text{if } a = 0 \text{ or } a = \frac{1}{2}. \end{cases}$$

Then $\langle \mathbf{Ł}_3, \Box, \Diamond\rangle$ does not satisfy (LD), since $\Box\frac{1}{2} = 1 > 0 = \Diamond\frac{1}{2}$.

Next, we introduce a generalization of the concepts of focal elements and c-relatively complete subalgebras defined in [6] for pseudomonadic BL-algebras (see Remark 3.4). Given any pm-$\mathsf{FL_e}$-algebra $\langle \mathbf{A}, \Box, \Diamond\rangle$, let

$$\mathcal{U}_{\mathbf{A}} := \{a \in A \mid \mathrm{e} \leq \Box a\}.$$

Recall that a *lattice filter* of an $\mathsf{FL_e}$-algebra $\mathbf{A}$ is a subset $L \subseteq A$ such that $L = {\uparrow}L := \{b \in A \mid a \leq b \text{ for some } a \in L\}$ and $a, b \in L$ implies $a \wedge b \in L$; if also $\mathrm{e} \in L$, then we call L an e-*lattice filter* of $\mathbf{A}$.

Lemma 3.5 *Let $\langle \mathbf{A}, \Box, \Diamond\rangle$ be any pm-$\mathsf{FL_e}$-algebra. Then $\mathcal{U}_{\mathbf{A}}$ is an* e*-lattice filter of $\mathbf{A}$ and $\mathcal{U}_{\mathbf{A}} = {\uparrow}\{\Box a \to a \mid a \in A\} \supseteq \{a \to \Diamond a \mid a \in A\}$. In particular, if $\mathbf{A}$ satisfies $x \leq \mathrm{e}$, then $\mathcal{U}_{\mathbf{A}} = \{a \in A \mid \mathrm{e} = \Box a\} = \{\Box a \to a \mid a \in A\}$.*

Proof. Observe first that $\mathrm{e} \in \mathcal{U}_{\mathbf{A}}$, since $\Box\mathrm{e} = \mathrm{e}$ by ($\mathrm{e}_\Box$). Also, if $a \in \mathcal{U}_{\mathbf{A}}$ and $b \in A$ satisfies $a \leq b$, then $\mathrm{e} \leq \Box a \leq \Box b$, by ($\mathrm{L4}_\Box$), so $b \in \mathcal{U}_{\mathbf{A}}$. Finally, if $a, b \in \mathcal{U}_{\mathbf{A}}$, then $\mathrm{e} \leq \Box a \wedge \Box b = \Box(a \wedge b)$, by ($\mathrm{L1}_\Box$), so $a \wedge b \in \mathcal{U}_{\mathbf{A}}$. Hence $\mathcal{U}_{\mathbf{A}}$ is an e-lattice filter of $\mathbf{A}$.

Next, for convenience, let $T := {\uparrow}\{\Box a \to a \mid a \in A\}$. If $a \in \mathcal{U}_{\mathbf{A}}$, then $a = \mathrm{e} \to a \geq \Box a \to a$, so $a \in T$. That is, $\mathcal{U}_{\mathbf{A}} \subseteq T$. But also $\Box a \to a \in \mathcal{U}_{\mathbf{A}}$ for each $a \in A$, since $\Box(\Box a \to a) = \Box a \to \Box a \geq \mathrm{e}$, by (P2), and, since $\mathcal{U}_{\mathbf{A}}$ is a lattice filter, $T = \mathcal{U}_{\mathbf{A}}$. Finally, $a \to \Diamond a \in \mathcal{U}_{\mathbf{A}}$ for each $a \in A$, since $\Box(a \to \Diamond a) = \Diamond a \to \Diamond a \geq \mathrm{e}$, by (P3). □

The next lemma shows that the operations $\Box$ and $\Diamond$ of any pm-$\mathsf{FL_e}$-algebra $\langle \mathbf{A}, \Box, \Diamond\rangle$ can be recovered from the subalgebra $\Box\mathbf{A}$ and e-lattice filter $\mathcal{U}_{\mathbf{A}}$.

Lemma 3.6 *Let $\langle \mathbf{A}, \Box, \Diamond\rangle$ be any pm-$\mathsf{FL_e}$-algebra. Then for any $a \in A$,*

$$\Box a = \max\{b \in \Box A \mid b \leq c \to a \text{ for some } c \in \mathcal{U}_{\mathbf{A}}\};$$
$$\Diamond a = \min\{b \in \Box A \mid c \cdot a \leq b \text{ for some } c \in \mathcal{U}_{\mathbf{A}}\}.$$

Proof. Fix $a \in A$, noting that $\Box a \in \{b \in \Box A \mid b \leq c \to a \text{ for some } c \in \mathcal{U}_{\mathbf{A}}\}$, since $\Box a \to a \in \mathcal{U}_{\mathbf{A}}$ and $\Box a \leq (\Box a \to a) \to a$. Now suppose that $b \in \Box A$ is such that $b \leq c \to a$ for some $c \in \mathcal{U}_{\mathbf{A}}$. Then, using residuation twice, $c \leq b \to a$. But $\Box b = b$, by ($\mathrm{L2}_\Box$), so, using ($\mathrm{L4}_\Box$) and (P2),

$$\mathrm{e} \leq \Box c \leq \Box(b \to a) = \Box(\Box b \to a) = \Box b \to \Box a = b \to \Box a,$$

and, by residuation, $b \leq \Box a$.

Similarly, $\Diamond a \in \{b \in \Box A \mid c \cdot a \leq b \text{ for some } c \in \mathcal{U}_{\mathbf{A}}\}$, using the fact that $a \to \Diamond a \in \mathcal{U}_{\mathbf{A}}$. Now suppose that $b \in \Box A$ is such that $c \cdot a \leq b$ for some $c \in \mathcal{U}_{\mathbf{A}}$. Then $c \leq a \to b = a \to \Box b$, since $b = \Box b$, and, using ($\mathrm{L4}_\Box$) and (P1),

$$\mathrm{e} \leq \Box c \leq \Box(a \to \Box b) = \Diamond a \to \Box b = \Diamond a \to b,$$

and, by residuation, $\Diamond a \leq b$. □

We now provide a characterization of the subalgebra and e-lattice filter of an FL_e-algebra required to obtain a corresponding pm-FL_e-algebra. For any subalgebra $\mathbf{B}$ and e-lattice filter $\mathcal{U}$ of an FL_e-algebra $\mathbf{A}$, let us call the ordered pair $\langle \mathbf{B}, \mathcal{U} \rangle$ a *$\mathcal{U}$-relatively complete subalgebra of* $\mathbf{A}$ if

(S1) for each $a \in A$, the set $\{b \in B \mid b \leq c \to a \text{ for some } c \in \mathcal{U}\}$ has a maximum and the set $\{b \in B \mid c \cdot a \leq b \text{ for some } c \in \mathcal{U}\}$ has a minimum;

(S2) $B \cap \mathcal{U} \subseteq {\uparrow}\mathrm{e}$.

Lemma 3.7 *Let $\langle \mathbf{A}, \Box, \Diamond \rangle$ be any pm-FL_e-algebra. Then $\langle \Box\mathbf{A}, \mathcal{U}_{\mathbf{A}} \rangle$ is a $\mathcal{U}_{\mathbf{A}}$-relatively complete subalgebra of $\mathbf{A}$.*

Proof. Recall that $\Box\mathbf{A}$ is a subalgebra of $\mathbf{A}$, by Proposition 2.1, and $\mathcal{U}_{\mathbf{A}}$ is an e-lattice filter of $\mathbf{A}$, by Lemma 3.5. Moreover, (S1) follows immediately from Lemma 3.6, so it remains to observe for (S2) that if $b \in \Box A \cap \mathcal{U}_{\mathbf{A}}$, then $\mathrm{e} \leq \Box b = b$, and therefore $\Box A \cap \mathcal{U}_{\mathbf{A}} \subseteq {\uparrow}\mathrm{e}$. □

Lemma 3.8 *Let $\langle \mathbf{B}, \mathcal{U} \rangle$ be any $\mathcal{U}$-relatively complete subalgebra of an FL_e-algebra $\mathbf{A}$, and define*

$$\Box_{\langle \mathbf{B}, \mathcal{U} \rangle} a := \max\{b \in B \mid b \leq c \to a \text{ for some } c \in \mathcal{U}\};$$
$$\Diamond_{\langle \mathbf{B}, \mathcal{U} \rangle} a := \min\{b \in B \mid c \cdot a \leq b \text{ for some } c \in \mathcal{U}\}.$$

Then $\langle \mathbf{A}, \Box_{\langle \mathbf{B}, \mathcal{U} \rangle}, \Diamond_{\langle \mathbf{B}, \mathcal{U} \rangle} \rangle$ is a pm-FL_e-algebra, $B = \Box_{\langle \mathbf{B}, \mathcal{U} \rangle} A$, and $\mathcal{U} = \mathcal{U}_{\mathbf{A}}$.

Proof. For convenience, let us abbreviate $\Box_{\langle \mathbf{B}, \mathcal{U} \rangle}$ and $\Diamond_{\langle \mathbf{B}, \mathcal{U} \rangle}$ by $\Box$ and $\Diamond$, respectively. We observe first that condition (S1) implies that the required maximum and minimum exist for each $a \in A$. Next we show that $\Box$ and $\Diamond$ restrict to identity maps on B.

Claim 1. $\Box a = a$, for each $a \in B$.

Proof. Note first that $a \leq \mathrm{e} \to a$, so, since $a \in B$ and $\mathrm{e} \in \mathcal{U}$, the definition of $\Box a$ yields $a \leq \Box a$. But also, by definition, $\Box a \leq c \to a$ for some $c \in \mathcal{U}$. Hence, by residuation twice, $c \leq \Box a \to a$, and, since $\mathbf{B}$ is a subalgebra of $\mathbf{A}$ and $\mathcal{U}$ is an e-lattice filter, $\Box a \to a \in B \cap \mathcal{U}$. So $\Box a \to a \in {\uparrow}\mathrm{e}$, by (S2), and, by residuation, $\Box a \leq a$. That is, $\Box a = a$. □

Claim 2. $\Diamond a = a$, for each $a \in B$.

Proof. Note first that $\mathrm{e} \cdot a \leq a$, so, since $a \in B$ and $\mathrm{e} \in \mathcal{U}$, the definition of $\Diamond a$ yields $\Diamond a \leq a$. But also, by definition, $c \cdot a \leq \Diamond a$ for some $c \in \mathcal{U}$. So, by residuation, $c \leq a \to \Diamond a$, and, since $\mathbf{B}$ is a subalgebra of $\mathbf{A}$ and $\mathcal{U}$ is an e-lattice

filter, $a \to \Diamond a \in B \cap \mathcal{U}$. Hence $a \to \Diamond a \in {\uparrow}\mathrm{e}$, by (S2), and, by residuation, $a \leq \Diamond a$. That is, $\Diamond a = a$. □

Claim 1 yields $B \subseteq \Box A$; hence, since the other inclusion is immediate, $B = \Box A$. Observe also that for any $c \in \mathcal{U}$, since $\mathrm{e} \leq c \to c$, it follows from the definition of $\Box c$ that $\mathrm{e} \leq \Box c$. So $\mathcal{U} \subseteq \mathcal{U}_{\mathbf{A}}$. For the converse, if $\mathrm{e} \leq \Box c$ for some $c \in A$, then, again by the definition of $\Box c$, there exists a $c' \in \mathcal{U}$ such that $\mathrm{e} \leq c' \to c$. So $c' \leq c$ and, since $\mathcal{U}$ is an e-lattice filter, also $c \in \mathcal{U}$. Hence $\mathcal{U}_{\mathbf{A}} = \mathcal{U}$.

We now check that $\langle \mathbf{A}, \Box, \Diamond \rangle$ satisfies the equations to be a pm-$\mathsf{FL_e}$-algebra. First consider (L1$_\Box$). Let $a, b \in A$. It is clear from the definition of $\Box$ that $\Box(a \wedge b) \leq \Box a \wedge \Box b$. For the converse inequality, let $c_a, c_b \in \mathcal{U}$ be such that $\Box a \leq c_a \to a$ and $\Box b \leq c_b \to b$. Since $c_a \wedge c_b \leq c_a$ and $c_a \wedge c_b \leq c_b$,

$$\Box a \wedge \Box b \leq \Box a \leq c_a \to a \leq (c_a \wedge c_b) \to a$$
$$\Box a \wedge \Box b \leq \Box b \leq c_b \to b \leq (c_a \wedge c_b) \to b,$$

and hence

$$\Box a \wedge \Box b \leq ((c_a \wedge c_b) \to a) \wedge ((c_a \wedge c_b) \to b) = (c_a \wedge c_b) \to (a \wedge b).$$

So, since $\mathcal{U}$ is a lattice filter, $c_a \wedge c_b \in \mathcal{U}$ and $\Box a \wedge \Box b \leq \Box(a \wedge b)$.

Now consider (L1$_\Diamond$). Let $a, b \in A$. It is clear from the definition of $\Diamond$ that $\Diamond(a \vee b) \geq \Diamond a \vee \Diamond b$. For the converse inequality, let $c_a, c_b \in \mathcal{U}$ be such that $\Diamond a \geq c_a \cdot a$ and $\Diamond b \geq c_b \cdot b$. Then

$$\Diamond a \vee \Diamond b \geq \Diamond a \geq c_a \cdot a \geq (c_a \wedge c_b) \cdot a$$
$$\Diamond a \vee \Diamond b \geq \Diamond b \geq c_b \cdot b \geq (c_a \wedge c_b) \cdot b,$$

and hence

$$\Diamond a \vee \Diamond b \geq ((c_a \wedge c_b) \cdot a) \vee ((c_a \wedge c_b) \cdot b) = (c_a \wedge c_b) \cdot (a \vee b).$$

So, since $\mathcal{U}$ is a lattice filter, $c_a \wedge c_b \in \mathcal{U}$ and $\Diamond a \vee \Diamond b \geq \Diamond(a \vee b)$.

The cases of (L2$_\Box$), (L2$_\Diamond$), (L3$_\Box$), (L3$_\Diamond$), and ($\star_\Box$) for $\star \in \{\cdot, \to, \wedge, \vee\}$ all follow easily from Claims 1 and 2 and the fact that $\mathbf{B}$ is a subalgebra of $\mathbf{A}$.

Next, consider (P1). For any $a, b \in A$,

$$\Diamond a \to \Box b \leq (a \cdot (a \to \Diamond a)) \to \Box b = (a \to \Diamond a) \to (a \to \Box b).$$

But $\Diamond a \to \Box b \in \Box A = B$ and $a \to \Diamond a \in \mathcal{U}_{\mathbf{A}} = \mathcal{U}$, by Lemma 3.5, so, using Claim 1 and the definition of $\Box$,

$$\Diamond a \to \Box b \leq \Box(a \to \Box b).$$

On the other hand, there is a $c \in \mathcal{U}$ such that $\Box(a \to \Box b) \leq c \to (a \to \Box b) = (c \cdot a) \to \Box b$. By residuation twice, $c \cdot a \leq \Box(a \to \Box b) \to \Box b$. From the definition of $\Diamond$ and using residuation once more, it follows that $\Box(a \to \Box b) \leq \Diamond a \to \Box b$.

Finally, consider (P2). Let $c \in \mathcal{U}$ be such that

$$\Box(\Box a \to b) \leq c \to (\Box a \to b) = \Box a \to (c \to b).$$

Then $\Box(\Box a \to b) \cdot \Box a \leq c \to b$. Recalling that $\mathbf{B} = \Box\mathbf{A}$ is a subalgebra of $\mathbf{A}$, $\Box(\Box a \to b) \cdot \Box a \in B$, so $\Box(\Box a \to b) \cdot \Box a \leq \Box b$ and $\Box(\Box a \to b) \leq \Box a \to \Box b$. For the converse inequality, if $c \in \mathcal{U}$ is such that $\Box b \leq c \to b$, then

$$\Box a \to \Box b \leq \Box a \to (c \to b) = c \to (\Box a \to b).$$

Using once more the fact that $\mathbf{B} = \Box\mathbf{A}$ is a subalgebra of $\mathbf{A}$, $\Box a \to \Box b \in B$ and $\Box a \to \Box b \leq \Box(\Box a \to b)$. □

Combining Lemmas 3.7 and 3.8, we obtain:

Theorem 3.9 *There is a one-to-one correspondence between pm-$\mathsf{FL_e}$-algebras and ordered pairs $\langle \mathbf{A}, \langle \mathbf{B}, \mathcal{U} \rangle \rangle$ such that $\mathbf{A}$ is an $\mathsf{FL_e}$-algebra and $\langle \mathbf{B}, \mathcal{U} \rangle$ is a $\mathcal{U}$-relatively complete subalgebra of $\mathbf{A}$, implemented by the maps $\langle \mathbf{A}, \Box, \Diamond \rangle \mapsto \langle \mathbf{A}, \langle \Box\mathbf{A}, \mathcal{U}_{\mathbf{A}} \rangle \rangle$ and $\langle \mathbf{A}, \langle \mathbf{B}, \mathcal{U} \rangle \rangle \mapsto \langle \mathbf{A}, \Box_{\langle \mathbf{B}, \mathcal{U} \rangle}, \Diamond_{\langle \mathbf{B}, \mathcal{U} \rangle} \rangle$.*

Remark 3.10 Theorem 3.9 is a proper generalization of Theorem 2.2 in the setting of $\mathsf{FL_e}$-algebras. Just observe that if $\langle \mathbf{A}, \Box, \Diamond \rangle$ is an m-$\mathsf{FL_e}$-algebra, then $\mathcal{U}_{\mathbf{A}} = {\uparrow}\mathrm{e}$ and $\langle \mathbf{A}, \langle \Box\mathbf{A}, \mathcal{U}_{\mathbf{A}} \rangle \rangle$ can be identified with $\langle \mathbf{A}, \Box\mathbf{A} \rangle$, where $\Box\mathbf{A}$ is a relatively complete subalgebra of $\mathbf{A}$.

This result can also be specialized to pm-$\mathsf{FL_e}$-algebras that satisfy (LD), including (up to term-equivalence) the pseudomonadic BL-algebras discussed in Remark 3.4.

Lemma 3.11 *A pm-$\mathsf{FL_e}$-algebra $\langle \mathbf{A}, \Box, \Diamond \rangle$ satisfies* (LD) *if and only if $\{b \in \Box A \mid b \geq c \cdot d$ for some $c, d \in \mathcal{U}_{\mathbf{A}}\} \subseteq {\uparrow}\mathrm{e}$.*

Proof. Assume first that $\langle \mathbf{A}, \Box, \Diamond \rangle$ satisfies (LD). If $b \geq c \cdot d$ for some $b \in \Box A$ and $c, d \in \mathcal{U}_{\mathbf{A}}$, then $b \geq \Diamond c \geq \Box c \geq \mathrm{e}$. Now suppose that $\{b \in \Box A \mid b \geq c \cdot d$ for some $c, d \in \mathcal{U}_{\mathbf{A}}\} \subseteq {\uparrow}\mathrm{e}$. For any $a \in A$, since $\Box a \to a, a \to \Diamond a \in \mathcal{U}_{\mathbf{A}}$, $\Box a \to \Diamond a \in \Box A$, and $(\Box a \to a) \cdot (a \to \Diamond a) \leq \Box a \to \Diamond a$, it follows by assumption that $\mathrm{e} \leq \Box a \to \Diamond a$ which implies $\Box a \leq \Diamond a$. □

Corollary 3.12 *There is a one-to-one correspondence between pm-$\mathsf{FL_e}$-algebras that satisfy* (LD) *and ordered pairs $\langle \mathbf{A}, \langle \mathbf{B}, \mathcal{U} \rangle \rangle$ such that $\mathbf{A}$ is an $\mathsf{FL_e}$-algebra, $\langle \mathbf{B}, \mathcal{U} \rangle$ is a $\mathcal{U}$-relatively complete subalgebra of $\mathbf{A}$, and $\{b \in B \mid b \geq c \cdot d$ for some $c, d \in \mathcal{U}\} \subseteq {\uparrow}\mathrm{e}$.*

We conclude this section by introducing an expansion of pm-$\mathsf{FL_e}$-algebra $\langle \mathbf{A}, \Box, \Diamond \rangle$ with a constant u that generates the e-lattice filter $\mathcal{U}_A$. To this end, let us call an algebraic structure $\langle \mathbf{A}, \Box, \Diamond, u \rangle$ with a pm-$\mathsf{FL_e}$-algebra reduct $\langle \mathbf{A}, \Box, \Diamond \rangle$ and constant u a *pointed pm-$\mathsf{FL_e}$-algebra* if it satisfies the equations

$$\text{(U1)}\ \ \Box u \approx \mathrm{e} \qquad \text{(U2)}\ \ u \leq \Box x \to x.$$

Observe that $\langle \mathbf{A}, \Box, \Diamond, u \rangle$ satisfies $\Box u \approx u$ if and only if it satisfies $\Box x \leq x$ and $x \leq \Diamond x$, i.e., if and only if $\langle \mathbf{A}, \Box, \Diamond \rangle$ is an m-$\mathsf{FL_e}$-algebra.

If $\langle \mathbf{A}, \Box, \Diamond, u\rangle$ is a pointed pm-$\mathsf{FL_e}$-algebra, then $\mathcal{U}_A = {\uparrow}u$. Just observe that if $a \in {\uparrow}u$, then $\mathrm{e} = \Box u \leq \Box a$, by (U1) and (L4$_\Box$), so $a \in \mathcal{U}_A$, and, conversely, if $a \in \mathcal{U}_A$, then $a \in {\uparrow}\{\Box b \to b \mid b \in A\} \subseteq {\uparrow}u$, by Lemma 3.5 and (U2). On the other hand, if $\langle \mathbf{B}, \mathcal{U}\rangle$ is a $\mathcal{U}$-relatively complete subalgebra of an $\mathsf{FL_e}$-algebra $\mathbf{A}$ such that $\mathcal{U} = {\uparrow}u$ for some $u \in \mathcal{U}$, then $\langle \mathbf{A}, \Box, \Diamond, u\rangle$ is a pointed pm-$\mathsf{FL_e}$-algebra, where for each $a \in A$,

$$\Box a := \max\{b \in B \mid b \leq u \to a\} = \Box_{\langle B,\mathcal{U}\rangle} a;$$
$$\Diamond a := \min\{b \in B \mid a \cdot u \leq b\} = \Diamond_{\langle B,\mathcal{U}\rangle} a.$$

A direct application of Theorem 3.9 therefore yields the following:

Corollary 3.13 *Pointed pm-$\mathsf{FL_e}$-algebras are in one-to-one correspondence with ordered pairs $\langle \mathbf{A}, \langle \mathbf{B}, u\rangle\rangle$ such that $\mathbf{B}$ is a subalgebra of $\mathbf{A}$, $B \cap {\uparrow}u \subseteq {\uparrow}\mathrm{e}$, and the sets $\{b \in B \mid b \leq u \to a\}$ and $\{b \in B \mid au \leq b\}$ have a maximum and minimum, respectively, for each $a \in A$.*

4 A Functional Representation Theorem

Let $\mathcal{V}$ be any variety of $\mathsf{FL_e}$-algebras and let $pm\mathcal{V}_u$ denote the variety of pointed pm-$\mathsf{FL_e}$-algebras with an $\mathcal{L}$-lattice reduct in $\mathcal{V}$. In this section, we prove — adapting a strategy introduced for Heyting algebras in [2] and generalized to a universal algebraic setting in [12] — that if $\mathcal{V}$ has the superamalgamation property, then every $\langle \mathbf{A}, \Box, \Diamond, u\rangle \in pm\mathcal{V}_u$ can be represented as an algebraic structure consisting of functions from a set of worlds W to some $\mathbf{L} \in \mathcal{V}$.

Let $\mathbf{A}$ be any $\mathsf{FL_e}$-algebra. A *possibilistic* $\mathbf{A}$*-frame* is an ordered pair $\langle W, \pi\rangle$ consisting of a non-empty set W and a map $\pi\colon W \to A$ satisfying $\bigvee \pi[W] = \mathrm{e}$. A *universal* $\mathbf{A}$*-frame* is a possibilistic $\mathbf{A}$-frame $\langle W, \pi\rangle$ satisfying $\pi[W] = \{\mathrm{e}\}$. The following lemma describes the construction of pointed pm-$\mathsf{FL_e}$-algebras for a possibilistic $\mathbf{A}$-frame $\langle W, \pi\rangle$ consisting of maps $f\colon W \to A$, where the operations of $\mathbf{A}$ are defined pointwise and $\Box f$ and $\Diamond f$ are constant functions mapping each $w \in W$ to some fixed $c \in A$.

Lemma 4.1 *Let $\mathbf{A}$ be any $\mathsf{FL_e}$-algebra and let $\langle W, \pi\rangle$ be a possibilistic $\mathbf{A}$-frame. Suppose that $\mathbf{B}$ is a subalgebra of $\mathbf{A}^W$ such that $\pi \in B$ and for each $f \in B$, the following constant functions exist and belong to B:*

$$\Box f := \bigwedge\{\pi(w) \to f(w) \mid w \in W\}, \qquad \Diamond f := \bigvee\{\pi(w) \cdot f(w) \mid w \in W\}.$$

Then $\langle \mathbf{B}, \Box, \Diamond, \pi\rangle$ is a pointed pm-$\mathsf{FL_e}$-algebra.

Proof. To see that $\langle \mathbf{B}, \Box, \Diamond\rangle$ satisfies (L1$_\Box$), observe that for any $f, g \in B$,

$$\begin{aligned}
\Box(f \wedge g) &= \bigwedge\{\pi(w) \to (f \wedge g)(w) \mid w \in W\}\\
&= \bigwedge\{(\pi(w) \to f(w)) \wedge (\pi(w) \to g(w)) \mid w \in W\}\\
&= \bigwedge\{\pi(w) \to f(w) \mid w \in W\} \wedge \bigwedge\{\pi(w) \to g(w) \mid w \in W\}\\
&= \Box f \wedge \Box g.
\end{aligned}$$

The condition (L1$_\Diamond$) is analogous. To check the remaining conditions for an e-$\mathcal{L}$-lattice, it suffices to observe that $\Box f$ and $\Diamond f$ are constant functions for any $f \in B$, and for any constant function $c\colon W \to A$ (writing c also for the constant value taken by the function c),

$$\Box c = \bigwedge\{\pi(w) \to c \mid w \in W\} = \bigvee \pi[W] \to c = \mathrm{e} \to c = c$$
$$\Diamond c = \bigvee\{\pi(w) \cdot c \mid w \in W\} = \bigvee \pi[W] \cdot c = \mathrm{e} \cdot c = c.$$

In particular, for $\star \in \{\wedge, \vee, \cdot, \to\}$ and any $f, g \in B$, the functions $\Box f$, $\Box g$, and $\Box f \star \Box g$ are all constant and hence $\Box(\Box f \star \Box g) = \Box f \star \Box g$.

Observe next for (P1) that for any $f, g \in B$,

$$\begin{aligned}\Box(f \to \Box g) &= \bigwedge\{\pi(w) \to (f \to \Box g)(w) \mid w \in W\}\\ &= \bigwedge\{(\pi(w) \cdot f(w)) \to \Box g \mid w \in W\}\\ &= \bigvee\{\pi(w) \cdot f(w) \mid w \in W\} \to \Box g\\ &= \Diamond f \to \Box g,\end{aligned}$$

and, similarly, $\Box(\Box f \to g) = \Box f \to \Box g$, for (P2).

For (U1), observe that $\pi(w) \leq \mathrm{e}$ for each $w \in W$, since $\bigvee \pi[W] = \mathrm{e}$, so

$$\mathrm{e} \leq \bigwedge\{\pi(w) \to \pi(w) \mid w \in W\} \leq \bigwedge\{\pi(w) \to \mathrm{e} \mid w \in W\} = \bigvee \pi[W] \to \mathrm{e} = \mathrm{e},$$

that is, $\Box\pi = \mathrm{e}$. Finally, for (U2), given any $f \in B$, clearly $\pi(w) \leq (\pi(w) \to f(w)) \to f(w)$ for each $w \in W$, so $\pi \leq \Box f \to f$. □

Observe that if $\mathbf{A}$ is a *complete* $\mathsf{FL_e}$-algebra (i.e., $\bigwedge X$ and $\bigvee X$ exist in A for all $X \subseteq A$), then every possibilistic $\mathbf{A}$-frame $\langle W, \pi\rangle$ is associated with the *complex* $\mathbf{A}$*-algebra* $\langle \mathbf{A}^W, \Box, \Diamond, \pi\rangle$, where for each $f \in A^W$,

$$\Box f := \bigwedge\{\pi(w) \to f(w) \mid w \in W\}, \qquad \Diamond f := \bigvee\{\pi(w) \cdot f(w) \mid w \in W\}.$$

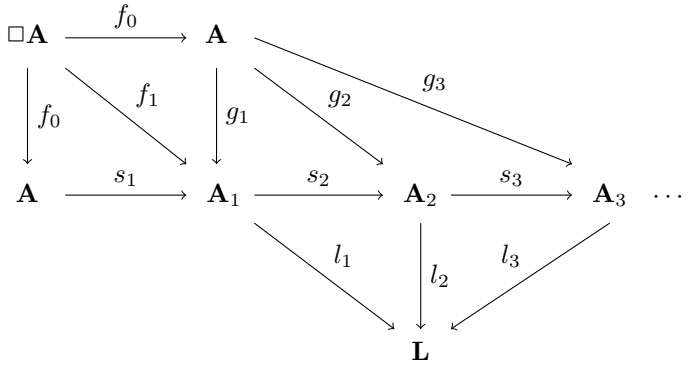
For any class $\mathcal{K}$ of $\mathsf{FL_e}$-algebras, we call a (pointed) pm-$\mathsf{FL_e}$-algebra $\mathcal{K}$*-functional* if it is isomorphic to one constructed as in Lemma 4.1 with $\mathbf{A} \in \mathcal{K}$.

Remark 4.2 If $\langle W, \pi\rangle$ is a universal $\mathbf{A}$-frame, then $\langle \mathbf{B}, \Box, \Diamond\rangle$, as constructed in Lemma 4.1, satisfies $\Box f = \bigwedge\{f(w) \mid w \in W\}$ and $\Diamond f = \bigvee\{f(w) \mid w \in W\}$ for all $f \in B$, and is an m-$\mathsf{FL_e}$-algebra. Conversely, if $\langle \mathbf{B}, \Box, \Diamond\rangle$, as constructed in Lemma 4.1, is an m-$\mathsf{FL_e}$-algebra, then $\mathrm{e} \leq \Box\pi \to \pi = \mathrm{e} \to \pi = \pi \leq \mathrm{e}$, so $\pi = \mathrm{e}$ and $\langle W, \pi\rangle$ is a universal $\mathbf{A}$-frame. That is, an m-$\mathsf{FL_e}$-algebra is $\mathcal{V}$-functional if and only if it is isomorphic to an algebraic structure constructed as described in Lemma 4.1 with $\mathbf{A} \in \mathcal{V}$, where $\langle W, \pi\rangle$ is a universal $\mathbf{A}$-frame.

Theorem 4.3 *Let $\mathcal{V}$ be a variety of $\mathsf{FL_e}$-algebras that has the superamalgamation property. Then every member of $pm\mathcal{V}_u$ is $\mathcal{V}$-functional.*

Proof. Consider any $\langle \mathbf{A}, \Box, \Diamond, u\rangle \in pm\mathcal{V}_u$. We define for $W := \mathbb{N}^{>0}$ a sequence of algebras $\langle \mathbf{A}_i\rangle_{i\in W}$ in $\mathcal{V}$ and sequences of embeddings $\langle f_i\colon \Box\mathbf{A} \to \mathbf{A}_i\rangle_{i\in W}$, $\langle g_i\colon \mathbf{A} \to \mathbf{A}_i\rangle_{i\in W}$, and $\langle s_i\colon \mathbf{A}_{i-1} \to \mathbf{A}_i\rangle_{i\in W}$ by induction as follows. Let $\mathbf{A}_0 := \mathbf{A}$ and let $f_0\colon \Box\mathbf{A} \to \mathbf{A}$ be the inclusion embedding. Inductively, for each $i \in W$, there exists a superamalgam $\langle \mathbf{A}_i, s_i, g_i\rangle$ of the V-formation $\langle \Box\mathbf{A}, \mathbf{A}_{i-1}, \mathbf{A}, f_{i-1}, f_0\rangle$, and we define also $f_i := s_i \circ f_{i-1} = g_i \circ f_0 = g_i|_{\Box A}$.

Now let $\mathbf{L}$ be the direct limit of the system $\langle\langle \mathbf{A}_i, s_i\rangle\rangle_{i\in W}$ with associated sequence of embeddings $\langle l_i\colon \mathbf{A}_i \to \mathbf{L}\rangle_{i\in W}$. The first two superamalgamation steps of this construction are depicted in the following diagram:



The operations of $\mathbf{L}^W$ are defined pointwise, so $B := \{\langle l_i \circ g_i(a)\rangle_{i\in W} \mid a \in A\}$ is the universe of a subalgebra $\mathbf{B}$ of $\mathbf{L}^W$ that contains $\pi := \langle l_i \circ g_i(u)\rangle_{i\in W}$. Now consider any $a \in A$ and let

$$S_a := \{\pi(i) \to (l_i \circ g_i(a)) \mid i \in W\} \quad \text{and} \quad T_a := \{\pi(i) \cdot (l_i \circ g_i(a)) \mid i \in W\}.$$

Fix $i \in W$. We show that $l_i \circ g_i(\Box a) = \bigwedge S_a$, the proof that $l_i \circ g_i(\Diamond a) = \bigvee T_a$ being very similar. In particular, it then follows that for any $i \in W$,

$$\mathrm{e} = \Diamond(\mathrm{e}) = l_i \circ g_i(\Diamond \mathrm{e}) = \bigvee T_{\mathrm{e}} = \bigvee \pi[W].$$

Note first that for any $k \in W$, using the definition of f_k and the fact that $\mathbf{L}$ is a direct limit,

$$l_k \circ g_k(\Box a) = l_k \circ f_k(\Box a) = l_{k+1} \circ s_{k+1} \circ f_k(\Box a) = l_{k+1} \circ g_{k+1}(\Box a).$$

Note also that $\Box a \leq u \to a$. Hence for each $j \in W$,

$$l_i \circ g_i(\Box a) = l_j \circ g_j(\Box a) \leq l_j \circ g_j(u \to a) = \pi(j) \to (l_j \circ g_j(a)).$$

So $l_i \circ g_i(\Box a)$ is a lower bound of S_a. Now suppose that $c \in L$ is another lower bound of S_a. Then there exist $k \in W$ and $d \in A_k$ such that $c = l_k(d)$ and

$$l_{k+1} \circ s_{k+1}(d) = l_k(d) = c \leq \pi(k+1) \to (l_{k+1} \circ g_{k+1}(a)) = l_{k+1} \circ g_{k+1}(u \to a).$$

But l_{k+1} is an embedding, so $s_{k+1}(d) \leq g_{k+1}(u \to a)$. Hence there exists, since $\langle \mathbf{A}_{k+1}, s_{k+1}, g_{k+1}\rangle$ is a superamalgam of $\langle \Box\mathbf{A}, \mathbf{A}_k, \mathbf{A}, f_k, f_0\rangle$, an element $b \in \Box A$ such that

$$s_{k+1}(d) \leq s_{k+1} \circ f_k(b) = g_{k+1} \circ f_0(b) \leq g_{k+1}(u \to a).$$

But s_{k+1} and g_{k+1} are embeddings and f_0 is the inclusion map, so $d \leq f_k(b)$ and $b \leq u \to a$. Hence $b \leq \Box a$, since $b \in \Box A$ and $\Box a = \max\{b' \in \Box A \mid b' \leq u \to a\}$. So $f_k(b) \leq f_k(\Box a) = g_k(\Box a)$, and, using the first inequality,

$$c = l_k(d) \leq l_k \circ f_k(b) \leq l_k \circ g_k(\Box a) = l_i \circ g_i(\Box a).$$

So $l_i \circ g_i(\Box a)$ is the greatest lower bound of S_a.

It remains to prove that the following map is an isomorphism:

$$f\colon \langle \mathbf{A}, \Box, \Diamond, u\rangle \to \langle \mathbf{B}, \Box, \Diamond, \pi\rangle; \quad a \mapsto \langle l_i \circ g_i(a)\rangle_{i\in W}.$$

Since the operations of $\mathbf{L}^W$ are defined pointwise, it is easily checked that f is an isomorphism of $\mathsf{FL_e}$-algebras. Moreover, for any $a \in L$ and $i \in W$,

$$\begin{aligned} f(\Box a)(i) &= l_i \circ g_i(\Box a) \\ &= \bigwedge\{\pi(j) \to l_j \circ g_j(a) \mid j \in W\} \\ &= \Box(l_i \circ g_i(a)) \\ &= \Box f(a)(i), \end{aligned}$$

and, similarly, $f(\Diamond a)(i) = \Diamond f(a)(i)$. □

Theorem 4.3 can also be understood as a general completeness theorem for a family of 'KD45-like' logics. Let $\mathbf{Fm}^u_\Box$ denote the formula algebra of the language of pointed pm-$\mathsf{FL_e}$-algebras over a countably infinite set of variables. For any class $\mathcal{K}$ of pointed pm-$\mathsf{FL_e}$-algebras, we obtain a consequence relation (logic) over $\mathrm{Fm}^u_\Box$ by defining for $\Gamma \cup \{\varphi\} \subseteq \mathrm{Fm}^u_\Box$,

$$\begin{aligned} \Gamma \vDash_{\mathcal{K}} \varphi :\Longleftrightarrow\ & \text{for every } \mathbf{A} \in \mathcal{K} \text{ and homomorphism } h\colon \mathbf{Fm}^u_\Box \to \mathbf{A}, \\ & \mathrm{e} \leq h(\psi) \text{ for all } \psi \in \Gamma \implies \mathrm{e} \leq h(\varphi). \end{aligned}$$

Then the theorem states that for any variety $\mathcal{V}$ of $\mathsf{FL_e}$-algebras that has the superamalgamation property, the class of $\mathcal{V}$-functional pointed pm-$\mathsf{FL_e}$-algebras provides a semantics for the logic $\vDash_{pm\mathcal{V}_u}$.

Remark 4.4 It is well-known that exactly eight varieties of Heyting algebras have the superamalgamation property [17]. Also, it has been proved recently that uncountably many varieties of $\mathsf{FL_e}$-algebras — including the varieties of $\mathsf{FL_e}$-algebras and $\mathsf{FL_{ew}}$-algebras — have the superamalgamation property [15], equivalent in this setting to the Craig interpolation property for the associated substructural logic (see, e.g., [18]). Hence, for such a variety $\mathcal{V}$, every member of $pm\mathcal{V}_u$ is $\mathcal{V}$-functional, and $\mathcal{V}$-functional pointed pm-$\mathsf{FL_e}$-algebras provide a semantics for a 'KD45-like' modal intermediate or substructural logic.

In certain cases, we obtain a more refined result. Let us say that a variety $\mathcal{V}$ of $\mathsf{FL_e}$-algebras *admits regular completions* if, for any $\mathbf{A} \in \mathcal{V}$, there exists an embedding of $\mathbf{A}$ into a complete member $\mathbf{B}$ of $\mathcal{V}$ that preserves all existing meets and joins. For such a variety, every member of $pm\mathcal{V}_u$ embeds into a complex $\mathbf{A}$-algebra. Just observe that for any $\langle \mathbf{A}, \Box, \Diamond, u\rangle \in pm\mathcal{V}_u$, the direct limit $\mathbf{L} \in \mathcal{V}$ constructed in the proof of Theorem 4.3 embeds into a complete member $\bar{\mathbf{L}}$ of $\mathcal{V}$ and hence $\langle \mathbf{A}, \Box, \Diamond, u\rangle$ is isomorphic to a subalgebra of the complex algebra $\langle \bar{\mathbf{L}}^W, \Box, \Diamond, \pi\rangle$.

Corollary 4.5 *Let $\mathcal{V}$ be a variety of $\mathsf{FL_e}$-algebras that is closed under regular completions and has the superamalgamation property, and let $\mathcal{P}_\mathcal{V}$ denote the class of complex algebras $\langle \mathbf{A}^W, \Box, \Diamond, \pi\rangle$ where $\mathbf{A}$ is a complete member of $\mathcal{V}$ and $\langle W, \pi\rangle$ is a possibilistic $\mathbf{A}$-frame. Then for any $\Gamma \cup \{\varphi\} \subseteq \mathrm{Fm}^u_\Box$,*

$$\Gamma \vDash_{pm\mathcal{V}_u} \varphi \iff \Gamma \vDash_{\mathcal{P}_\mathcal{V}} \varphi.$$

Remark 4.6 A sufficient condition for a class of $\mathsf{FL_e}$-algebras to admit regular completions is closure under MacNeille completions. In particular, this is the case for $\mathcal{BA}$ and $\mathcal{HA}$; indeed, they are the only non-trivial varieties of Heyting algebras that have this property [1]. Infinitely many varieties of $\mathsf{FL_e}$-algebras — in particular, the varieties of $\mathsf{FL_e}$-algebras and $\mathsf{FL_{ew}}$-algebras — are closed under MacNeille completions, which is equivalent in this setting to the existence of a certain form of analytic sequent calculus [11].

In this paper, we have taken the first steps towards an algebraic theory of a family of '$\mathsf{KD45}$-logics' that are complete with respect to semantics defined by possibilistic frames over a variety $\mathcal{V}$ of $\mathsf{FL_e}$-algebras, obtaining a representation theorem for $pm\mathcal{V}_u$ (Theorem 4.3) when $\mathcal{V}$ has the superamalgamation property and a completeness theorem with respect to complex algebras when $\mathcal{V}$ also admits regular completions (Corollary 4.5). However, there are several issues to address in future work. First, it would be preferable to obtain analogous results for $pm\mathcal{V}$, i.e., without assuming the constant u in the language. Second, we aim to extend our approach to varieties that lack the superamalgamation property such as the varieties of MV-algebras and BL-algebras that serve as algebraic semantics for Łukasiewicz logic and Hájek's basic logic, respectively.

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