On the Proof Theory of Apodictic Syllogistic

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Abstract

The success of syllogistic has been such that many of the features originally characterizing this logic have changed over decades. The present study aims at offering a faithful, possibly comprehensive, account of Aristotle's deductive logic, by extending the inherently proof-theoretical approach introduced by von Plato in 2009/16. Concretely, the main novelty consists in treating syllogistic as a natural deduction system of rules and 'perfecting' arguments (for imperfect moods) as derivability proofs in tree form. Nothing is added to the original source but, in this way, assertoric and apodictic syllogistic are transparently reconstructed and Aristotle's perfecting proofs are proved to be always correct. This would also be a first step to make the whole modal fragment fully comprehensible and to rehabilitate syllogistic as a fertile theory in the context of natural language reasoning.

Keywords: Proof Theory, Apodictic Syllogistic, Derivation Trees, Aristotle.

1 Introduction

It is *communis opinio* that, in contrast to the assertoric theory, Aristotle's modal syllogistic is obscure or even inconsistent. Following the proof-theoretical approach introduced in 2009/16 by the second author [47], which has already led to a transparent study of the assertoric fragment and to the remarkable normal form theorem, we provide a plain and perfectly intelligible reconstruction of apodictic syllogistic.

One of the primary goals of the *Analytica* was to present a well-defined deductive system and to study its properties. [2] Coherently, our approach is

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[2] Aristotle's syllogistic may be seen as a system of deductive logic (or as a prototypical logical theory [11]). In this respect, *An.Pr.* represents the historically-first attempt to develop a

essentially proof-theoretical. To meticulously inspect the original source, we treat syllogistic as a natural deduction system of rules and present perfecting (i.e. derivability) proofs in tree form. Indeed, in *An.Pr.* I, Aristotle himself systematically 'perfects' all imperfect syllogisms. Yet, the linearity of the text partially hides the deductive structure of these proofs, which is rather made explicit by means of our two-dimensional trees. Advantages offered by the arboreal form are evident, as it transparently shows the skeleton of the argument by keeping "track of what depends on what within the proof" [47, p. 16]. This brings great benefits for both exegesis and logical comprehension.

Our study aims at extending the approach by [47] and at offering a *comprehensive* and *faithful* account on Aristotle's deductive logic, in order to show that his modal syllogistic is not only comprehensible but also potentially relevant for contemporary logic. Methodologically, our analysis is loyal to the original source, to which nothing is added. Nonetheless, by presenting perfecting proofs in tree form, we have obtained a clear reconstruction not only for the plain assertoric system, but also for (part of) the controversial modal one. The readers themselves can compare our reconstructive proposal with the corresponding quotation and verify that Aristotle's proofs are always correct. In this way, purely- and mixed-apodictic syllogistic are fully vindicated.

1.1 The Structure of the Paper

Before effectively reconstructing assertoric and apodictic syllogistic, we try to clarify the *original* meaning of the term 'syllogism' in Aristotle's *corpus.*[3] This is done in Section 2. Then, in Section 3, we cursorily, but systematically, deal with the presentation of assertoric syllogistic. The reconstruction of this fragment is not problematic, but is relevant as its language and rules are at the basis of the mixed-apodictic system subsequently considered. In addition, already this investigation shows how the proof-theoretical approach offers a straightforward way to analyze the correctness of perfecting proofs. In Section 4, such an approach is extended to the apodictic fragment. In particular, we consider purely-apodictic syllogistic first (Section 4.2), then moving to the mixed-apodictic system, the language of which includes assertoric and apodictic propositions (Section 4.3). We conclude by comparing our work with related studies, in Section 5, and by pointing at possible directions for future research, in Section 6.

2 Rehabilitating Syllogistic

Aristotle's *Analytica Priora* marks the beginning of deductive logic. Later on, syllogistic gains an enormous success, becoming the hegemonic paradigm of Western logic until its crisis at the end of the XIX century. After the advent of mathematical logic, syllogistic has become more and more marginal, being

comprehensive and self-contained deductive system and to study its properties.

[3] Exegetical problems stem both from the 'double-nature' of the term in Aristotelian texts, where the author uses it with connected but different meanings, and from theoretical changes it has undertaken over decades. For further details, see Section 2.

considered as nothing but a monadic fragment of first-order logic, and nowadays it is commonly seen as a limited and quite sterile theory, the interest of which is merely historical. Actually, when looking at the paucity of its language, it appears striking that syllogistic has been the dominant deductive system for about two millennia.

In fact, historical inspection, both of the original source and of its subsequent interpretations, proves a wide scope of this logic, which cannot be confined to the assertoric fragment. Many occurrences of the term in ancient and medieval sources draw an image of syllogistic which massively diverges from today's logic handbooks and, when delving deeper into the Aristotelian *corpus* and the development of the disciplines, perplexities arise on what exactly a 'syllogism' is. We will show that, against what is commonly thought, syllogistic holds valuable properties related to its affinity with natural language syntax and to ordinary reasoning, which have also been recently revived by studies in Natural Logic [45,36,26].

2.1 Aristotle's 'syllogism'

The term 'syllogism' is not a neologism introduced by Aristotle. In earlier Greek texts, this word is witnessed as signifying 'to reason', 'to calculate' or 'to conclude an argument', see [8,5,38]. These definitions are not far from the ones appearing in the *Organon*, where syllogisms are presented as deductions or valid arguments [4, 24b19-22], [33, 100a25-29], and not as *specific kinds* of inference, namely the assertoric ones [4, I, 1-6]. In addition, Aristotle's effective use of the term is quite liberal, not excluding 'syllogistic-style' (non-assertoric) arguments, e.g. modal syllogisms [4, I, 3,8ff.] or syllogisms with more than two premisses or with singular subjects [4, 70a27]. [4]

As some scholars noticed, the original meaning of the word 'syllogism' does not correspond to the one of today's handbooks, which roughly overlaps with the assertoric theory presented in [4, 1-6]. This latter seems actually connected to the "secondary sense" that Aristotle attributes to the word, the "primary" one being that of deduction in general [5, pp. 23-25]. So, strikingly, the best translation for the word 'syllogism' is not its English calque, but a term with different linguistic roots:

> Logicians normally use "syllogism" to *mean* one of the specific forms of valid argument Aristotle discussed in *An.Pr.* I 1-6 but Aristotle's definition of *syllogismos* comprehend a much wider class: pretty much any valid argument, or at least any argument with a conclusion different from any of its premisses. If we translate *sullogismos* as "syllogism", the broad scope of this definition is obscured and with it the nature of Aristotle's logical theorizing. [39, p. 30]

Also the reception of Aristotle's work in early commentators and in the late ancient tradition suggests a 'wide scope' interpretation of the notion. [5]

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[4] This polysemous, sometimes ambiguous use of the term is coherent with Aristotle's use of the language in general [31].

[5] In Late Antiquity, hypothetical and assertoric syllogistic coexist, and the term is equally used for both, see [40, p. 2]. For further details, see for instance [6,7,13,40].

On the other hand, the subsequent success of syllogistic has been such that many original features of this theory have been modified and misread several times and in a variety of ways, both by restricting and by broadening it. In fact, even assuming that syllogistic had to be originally identified with assertoric syllogistic, it clearly emerges that this core theory has been variously (re-)interpreted over centuries. For instance, during the Middle Ages, both Arabic and Latin traditions tended to extend Aristotelian definitions, by considering relational or three-premisses syllogisms or syllogisms with singular terms. [6]

For all these reasons, it seems reasonable to distinguish (at least) between syllogism *lato sensu*, the meaning of which corresponds to 'deduction' or 'valid argument', and *stricto sensu*, referring to specific assertoric inferences constituted by two categorical premisses and by three terms. As anticipated, it is likely that Aristotle's original use of the word is closer to that recently developed by Natural Logic, i.e. denoting a (correct) inference expressed in natural language, than to the (toy) definition attributed to it from the Nineteenth century on in contraposition to mathematical logic. This simple consideration would shed new light on the real scope of syllogistic and on its potential interest for contemporary logic. When syllogistic is seen for what it is – namely, a (first) logical system for human reasoning – the study of its fragments, including the (non-assertoric) modal and relational ones, acquires new importance.

3 On Assertoric Syllogistic

In this Section, we briefly, but systematically, reconstruct the fragment introduced by Aristotle in *An.Pr.* I, 1-7. [7] This is crucial as, although it is not controversial, [8] "assertoric syllogistic provides the foundation for the entire syllogistic system" [35, p. 11]. In Section 3.1, we cursorily introduce core notions at the basis of the whole Aristotelian syllogistic. Then, in Section 3.2, we present a formalization for the assertoric language and primitive rules that constitute assertoric syllogistic. Finally, in Section 3.3, we consider each perfecting proof by Aristotle and show them correct. In this way we provide a *plain* and *faithful* presentation of these reduction proofs, departing from the original source only by moving from linear to tree form and by using the nowadays standard (Medieval) names for syllogisms. [9]

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[6] For an overview of Medieval theories of syllogisms, see [43].

[7] In *An.Pr.* I, 1 Aristotle introduced basic notions related to syllogistic in general, here summarized in Section 3.1. Assertoric primitive syllogisms are presented in *An.Pr.* I, 2. Then, *An.Pr.* I, 4, 5 and 6 deal with the first, second and third figures, respectively. In *An.Pr.* I, 7, Aristotle shows that perfect `Darii` and `Ferio` (and related imperfect moods) can actually be perfected due to `Barbari` and `Celarent`, introducing what we define as the minimal system $\mathbf{SYL}^*$, see Appendix A.

[8] In fact, controversies concern its semantical interpretation.

[9] For space reasons, we do not deal with the historical development of assertoric syllogistic and related mnemonics. For further details on the topic, see [43].

3.1 Aristotle's Assertoric Syllogisms

As seen, the definition of assertoric syllogistic has changed over two millennia. According to Aristotle, there are fourteen assertoric syllogisms, i.e. concluding arguments defined by two categorical premisses. They were originally divided into three *figures*, based on the relation linking the middle term and the extremes. He doesn't consider a fourth figure, or the notion of mood. As we shall see, Aristotle's theory is perfectly intelligible, and incoherence appears only in later versions of it (see, for instance, [17]).

Assertoric syllogisms are usually defined as valid arguments constituted by two assertoric premisses and an assertoric conclusion. In fact, when inspecting the original source, it emerges that a syllogism should probably be identified with its *pair of productive premisses* [34].[10] Aristotle establishes syllogistic validity either by 'semantic justification'[11] or by one 'method of proof'. Accordingly, syllogisms are divided into *perfect* or *complete* ones, which are somehow self-evident, and *imperfect* or *incomplete* ones, which are reducible to the first class [4, 24b23-27]. The core of *An.Pr.* I is the 'perfection' or 'reduction' of *all* imperfect syllogisms to perfect ones: by systematically considering every combination of premisses for each figure, Aristotle proves that either a given conclusion follows from them or that no conclusion can. In the latter case, the aforementioned premisses are said not to 'syllogize' and a counterexample is (usually) provided.

Three main 'kinds' of perfecting proofs have been recognized in the literature: ostensive, indirect and by *ecthesis*.[12] In fact, our reconstruction deals with the first two 'methods' only [4, 19a30-33]. Whenever possible, Aristotle constructs ostensive proofs, that is, after assuming the premisses of the syllogism he wants to 'perfect', he applies first-figure and conversion 'rules'. In this way, imperfect syllogisms are reduced to perfect (self-evident) ones [4, 29a33-35]. Indirect or "through impossibility" proofs are needed for syllogisms that are not directly provable, i.e., `Bocardo` and `Baroco`: the conclusion is assumed not to follow from the premisses, and given the premisses and the contradictory of the conclusion, falsity is derived.[13] We will show that, given the primitive syllogistic system, 'perfecting' proofs for imperfect syllogisms are nothing but

[10] Otherwise, assertoric syllogisms would be more than fourteen; in particular, subaltern moods should also be counted (see Appendix A).

[11] It is usually done by invoking the meaning of the given logical expression and appealing to the so-called *dictum de omni et nullo*, see for example [23,46,24].

[12] The intuition behind *ecthesis* seems not far from the one at the basis of semantical justification for perfect syllogisms, see [47, pp. 332-333]. However, its nature remains ambiguous [34, p. 13], lying somewhere in between a semantic explanation and an informal proof device. Being it a redundant or an essential part of the system, *ecthesis* appears in several passages of *An.Pr.*, e.g. it is mentioned as an alternative proof to complete indirect moods [4, 28a22-26, 28b13-15] and to justify perfect (non-mixed) apodictic [`Baro`] and [`Boc`] (several commentators also consider Π^{-}- and Σ^{-}-conversion rules as obtained through it [4, 25a14-26]).

[13] Indirect proofs are also used to obtain (more complicated) alternatives to direct ones, see, for instance, [4, 27a14-15] (for `Cesare`) or [4, 28a28-30] (for `Felapton`) or [4, 28b13-15] (for `Disamis`) (see also Appendix A). For further details see [1].

derivability proofs. Indeed, in each of the aforementioned cases, Aristotle assumes the premisses of the imperfect syllogisms that he wants to perfect and, by applying primitive rules, obtains the conclusion.

3.2 The Primitive System SYL

We introduce the assertoric language and two primitive proof systems, **SYL** and **SYL***, the rules of which are justified by original quotation reference. Actually, although the two systems are equivalent (see Appendix A), only the less economical **SYL** is used by Aristotle in his perfecting proofs.

3.2.1 The Assertoric Language

Assertoric propositions are either universal or particular and either affirmative or negative [4, 24a16-20].[14] Subjects and predicates are called *terms*. All propositions express the belonging of a term, the predicate, to another term, the subject. We abbreviate these four forms following the notation by [47, p. 327]:

- $\Pi^+(A, B)$ for affirmative universal propositions: *Every A is B*
- $\Pi^-(A, B)$ for negative universal propositions: *No A is B*
- $\Sigma^+(A, B)$ for affirmative particular propositions: *Some A is B*
- $\Sigma^-(A, B)$ for negative particular propositions: *Some A is not B.*[15]

We also introduce a syntactical negation symbol so that, for any given assertoric proposition (P), also its negation ($P^\neg$) is part of our formal language.[16]

Definition 3.1 Let A, B be terms and Q be a proposition in the assertoric language. Then, the language of assertoric syllogistic $\mathcal{L}_{\mathbf{SYL}}$ is defined as follows:

$$P ::= \Pi^+(A, B) \mid \Pi^-(A, B) \mid \ \Sigma^+(A, B) \mid \Sigma^-(A, B) \mid \ Q^\neg.$$

3.2.2 Introducing the Rules of SYL

Here we present the primitive rules of the assertoric syllogistic system **SYL**. Most of them are introduced by quoting the corresponding source text. As we shall see, in a few cases, no explicit justification is given in *An.Pr.*, but can be found in other works of Aristotle's *corpus.*

The Rule of Contradictory Pairs. In *De Int.* 7, Aristotle introduces the contradictory pairs, $\Pi^+(A, B)$-$\Sigma^-(A, B)$ and $\Pi^-(A, B)$-$\Sigma^+(A, B)$ [2, 17a33-b19]. The corresponding rules are defined accordingly, by using the negation symbol $(\cdot)^\neg$: as seen, given an assertoric proposition P, its contradictory opposite is denoted by $P^\neg$. We follow Whitaker's terminology calling them "rules of contradictory pairs" [49].

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[14] Aristotle introduces indeterminate propositions as well, but these have no role in assertoric syllogistic.

[15] Observe that the order of subject and predicate is the reverse of the original one.

[16] Notice that different choices are possible. For instance, in [47], the assertoric language is 'minimal' since negation is defined as a meta-logical symbol. Following this approach, our `RCP`s and `RAA` would "collapse" into a unique rule.

Definition 3.2 The *rules of contradictory pairs* (RCPs) are defined below: [17]

$$\frac{\Pi^-(A,B)}{\Sigma^+(A,B)^\neg}\ \texttt{RCP}_1 \qquad \frac{\Pi^+(A,B)}{\Sigma^-(A,B)^\neg}\ \texttt{RCP}_2$$

$$\frac{\Sigma^+(A,B)^\neg}{\Pi^-(A,B)}\ \texttt{RCP}_3 \qquad \frac{\Sigma^-(A,B)^\neg}{\Pi^+(A,B)}\ \texttt{RCP}_4$$

Intuitively, these are justified by the fact that "of every contradictory pair, one member is true and the other false" [49, p. 79].

The rule of non-contradiction. According to Aristotle's principle of non-contradiction, contradictory propositions cannot be both true at the same time [3, 1005b19-20, 1005b25-30, 1011b13-14].

Definition 3.3 The *rule of non-contradiction* is the schematic rule below:

$$\frac{P \qquad P^\neg}{\perp}\ \perp\texttt{I}$$

Reductio ad Absurdum. As said, together with probative or direct deduction, Aristotle considers indirect proofs [4, 41a23-32, 29a35-29b1]. [18] In fact, he variously uses *reductio ad absurdum* (RAA) or 'proofs through impossibility' in syllogistic derivations by absurdly assuming that the conclusion to be proved does not hold, so that (RCPs given) its contradictory opposite does. But then absurdity follows, and the desired conclusion comes out to hold.

Definition 3.4 The *schematic rule of reductio ad absurdum* ($\texttt{RAA}$) is defined as follows:

$$\begin{array}{c} [P^\neg]^1 \\ \vdots \\ \dfrac{\perp}{P}\ \texttt{RAA}, 1 \end{array}$$

Since Aristotle does not give a unique and formal definition of RAA, the given schematic representation is not the only possible one. In [47], the second author introduces a more economical rule, encapsulating both our $\texttt{RAA}$ and $\texttt{RCP}_{1\text{-}4}$.

Rules of Conversion. Aristotle presents three one-premiss rules of conversion: universal negative conversion [4, 25a6–7], universal affirmative conversion [4, 25a7–8], and particular affirmative conversion [4, 25a9–10]. [19]

Definition 3.5 The *rule of negative, universal* ($\Pi^-\texttt{C}$), of *affirmative, universal* ($\Pi^+\texttt{C}$), and of *affirmative, particular conversion* ($\Sigma^+\texttt{C}$) are defined as follows:

[17] Actually, we should consider eight RCPs, but, in the presence of $\texttt{RAA}$, $\texttt{RCP}_1$-$\texttt{RCP}_4$ are enough to derive all imperfect syllogisms as done in *An.Pr.* I, 1-7, see [2, 17b19-26]. RCPs are also at the basis of the development of the so-called square of the opposition, see e.g. [41].

[18] Remarkably, when reading [4, 41a23-32] through the lenses of modern logic, it looks striking that the example of 'proof through impossible' introduced by Aristotle is the paradigmatic example of a proof of negation. Namely, the given proof of the incommensurability of the diagonal and the square is not a genuine indirect proof, but a direct proof deriving falsity.

[19] For their (semantical) justification, see [47, pp. 329–331].

$$\frac{\Pi^-(P,S)}{\Pi^-(S,P)}\ \Pi^-\mathtt{C}. \qquad \frac{\Pi^+(P,S)}{\Sigma^+(S,P)}\ \Pi^+\mathtt{C} \qquad \frac{\Sigma^+(P,S)}{\Sigma^+(S,P)}\ \Sigma^+\mathtt{C}$$

Rules $\Pi^-\mathtt{C}$ and $\Sigma^+\mathtt{C}$ are commonly called (negative and positive) *conversio simplex*, whereas the Latin name for $\Pi^+\mathtt{C}$ is *conversio per accidens*. Observe also that $\Sigma^+\mathtt{C}$ is derivable in a 'more economical' system **SYL***, where $\Sigma^+\mathtt{C}$ is not primitive, see Appendix A.

Syllogistic Rules. Finally, Aristotle introduces *complete* or *perfect* syllogisms, namely four first-figure, two-premisses rules, such that the middle term is the subject of one of the premisses, later labelled as major premiss, and the predicate of the other, the so-called minor premiss [4, 25b31-38]. As said, differently from the notion of mood, that of figure is original to Aristotle [4, 26b25-34]. He starts with perfect `Barbara` in [4, 25b37–40], and `Celarent` [4, 26a1–2]. Then, after considering a couple of premisses which do not syllogize, he introduces `Darii` [4, 26a23–25] and `Ferio` [4, 26a25–26].

Definition 3.6 *First-figure syllogisms* are defined as follows:

$$\frac{\Pi^+(B,A) \qquad \Pi^+(C,B)}{\Pi^+(C,A)}\ \mathtt{Barbara} \qquad \frac{\Pi^-(B,A) \qquad \Pi^+(C,B)}{\Pi^-(C,A)}\ \mathtt{Celarent}$$

$$\frac{\Pi^+(B,A) \qquad \Sigma^+(C,B)}{\Sigma^+(C,A)}\ \mathtt{Darii} \qquad \frac{\Pi^-(B,A) \qquad \Sigma^+(C,B)}{\Sigma^-(C,A)}\ \mathtt{Ferio}$$

Aristotle also shows that `Darii` and `Ferio` are not 'really primitive', as they can be derived using (minimal) first-figure `Barbara` and `Celarent` together with `RAA`, `RCP`s and conversion rules. [20]

3.2.3 The System SYL (and SYL*)

In [4], Aristotle presents two syllogistic systems: **SYL** corresponds to the system which is effectively used in *An.Pr.* to 'perfect' imperfect moods, whereas **SYL*** is more economical as it does not include `Darii`, `Ferio` and $\Sigma^+\mathtt{C}$.

Definition 3.7 The minimal system **SYL*** is made of the four rules of contradictory pairs $\mathtt{RCP}_1$-$\mathtt{RCP}_4$, the two conversion rules $\Pi^-\mathtt{C}, \Pi^+\mathtt{C}$, the rule of *reductio* `RAA`, the rule of non-contradiction $\bot\mathtt{I}$, and the two syllogistic rules `Barbara` and `Celarent`. The system **SYL** is made of all rules defining **SYL*** plus the rule of positive *conversio simplex* $\Sigma^+\mathtt{C}$, and first-figure `Darii` and `Ferio`.

The two systems are equivalent in the sense that exactly the same set of (imperfect) syllogisms is derivable in them, see Appendix A.

3.3 Imperfect Syllogisms

Aristotle considers syllogisms in the second and third figures as *incomplete.* Most of *An.Pr.* I, 1-6 is devoted to "perfecting" *each of them.* [21] In modern

[20] For a brief presentation of the mentioned derivability proofs, see Appendix A. Full details can be found in [1].

[21] Actually, in some cases proofs are only sketched, for example the ones of second-figure `Camestres` or third-figure `Felapton` and `Datisi`.

terms, after presenting the primitive rule system(s) **SYL** (and **SYL***), Aristotle proves the derivability of imperfect syllogisms in it. He also claims that these 'perfectable' moods represent the totality of the two-premisses rules which are derivable in **SYL**, so providing a proof of syntactical completeness.

3.3.1 The Second Figure

Second-figure syllogisms are syllogisms in which both terms in the conclusion appear in subject position in the premisses [4, 26a4-27a3]. In presenting this figure, variables A, B, C, used for perfect syllogisms, are substituted by N, M, X, respectively. There are four second-figure syllogisms: for each of them we quote Aristotle's original proof and show its correctness – namely, that any rule applied is a proper instance of one in **SYL** – by simply presenting it in tree form.

Cesare. Aristotle considers `Cesare` first and shows it derivable in [4, 27a5–8]:

$$\dfrac{\dfrac{\Pi^-(N,M)}{\Pi^-(M,N)}\,\Pi^-\texttt{C} \qquad \Pi^+(X,M)}{\Pi^-(X,N)}\,\texttt{Celarent}$$

Festino. Then, `Festino` is presented and derived in [4, 27a32–37]:

$$\dfrac{\dfrac{\Pi^-(N,M)}{\Pi^-(M,N)}\,\Pi^-\texttt{C} \qquad \Sigma^+(X,M)}{\Sigma^-(X,N)}\,\texttt{Ferio}$$

Baroco. The subsequent derivability proof is that of `Baroco` [4, 27a37–b2], and it is not ostensive:

$$\dfrac{\dfrac{\Sigma^-(X,M) \qquad \dfrac{\dfrac{\Pi^+(N,M) \qquad \dfrac{[(\Sigma^-(X,N))^\neg]^1}{\Pi^+(X,N)}\,\texttt{RCP}_4}{\Pi^+(X,M)}\,\texttt{Barbara}}{(\Sigma^-(X,M))^\neg}\,\texttt{RCP}_2}{\bot}\,\bot\texttt{I}}{\Sigma^-(X,N)}\,\texttt{RAA},1$$

Camestres. Imperfect `Camestres` is also presented in [4, 27b–4], but its proof is omitted as close to `Cesare`'s one:

$$\dfrac{\dfrac{\dfrac{\Pi^-(X,M)}{\Pi^-(M,X)}\,\Pi^-\texttt{C} \qquad \Pi^+(N,M)}{\Pi^-(N,X)}\,\texttt{Celarent}}{\Pi^-(X,N)}\,\Pi^-\texttt{C}$$

3.3.2 The Third Figure

Finally, Aristotle moves to the third figure, in which the subject and predicate of the conclusion are the predicates of the premisses [4, 28a10–14]. Here, he uses the variables S, R, P for middle term, conclusions' subject and predicate, respectively.

Darapti. First, he introduces `Darapti` and shows it derivable in [4, 28a22–26]:

$$\dfrac{\Pi^+(S,P) \qquad \dfrac{\Pi^+(S,R)}{\Sigma^+(R,S)}\,\Pi^+\mathtt{C}}{\Sigma^+(R,P)}\,\mathtt{Darii}$$

Felapton. Then, `Felapton` is considered and ostensively derived [4, 28a26-30]:

$$\dfrac{\Pi^-(S,P) \qquad \dfrac{\Pi^+(S,R)}{\Sigma^+(R,S)}\,\Pi^+\mathtt{C}}{\Sigma^-(R,P)}\,\mathtt{Ferio}$$

Disamis. Next, `Disamis` is proved derivable in [4, 29b8–11]:

$$\dfrac{\dfrac{\Pi^+(S,R) \qquad \dfrac{\Sigma^+(S,P)}{\Sigma^+(P,S)}\,\Sigma^+\mathtt{C}}{\Sigma^+(P,R)}\,\mathtt{Darii}}{\Sigma^+(R,P)}\,\Sigma^+\mathtt{C}$$

Datisi. `Datisi` is presented as perfectable, but no explicit derivability proof is given. Yet, this can be easily established by applying $\Sigma^+\mathtt{C}$ and perfect `Darii`:

$$\dfrac{\Pi^+(S,P) \qquad \dfrac{\Sigma^+(S,R)}{\Sigma^+(R,S)}\,\Sigma^+\mathtt{C}}{\Sigma^+(R,P)}\,\mathtt{Darii}$$

Bocardo. The proof of third-figure `Bocardo` is indirect [4, 28b17–20]:

$$\dfrac{\dfrac{\Sigma^-(S,P) \qquad \dfrac{\dfrac{\dfrac{[(\Sigma^-(R,P))^\neg]^1}{\Pi^+(R,P)}\,\mathtt{RCP}_4 \qquad \Pi^+(S,R)}{\Pi^+(S,P)}\,\mathtt{Barbara}}{(\Sigma^-(S,P))^\neg}\,\mathtt{RCP}_2}{\bot}\,\bot\mathtt{I}}{\Sigma^-(R,P)}\,\mathtt{RAA},1$$

Ferison. Finally, `Ferison` is defined and shown derivable in [4, 28b31–36]:

$$\dfrac{\Pi^-(S,P) \qquad \dfrac{\Sigma^+(S,R)}{\Sigma^+(R,S)}\,\Sigma^+\mathtt{C}}{\Sigma^-(R,P)}\,\mathtt{Ferio}$$

4 On Apodictic Syllogistic

The reconstruction of Aristotle's modal syllogistic is controversial and a variety of attempts for a consistent interpretation have appeared in the literature. Most of them focus on the reconstruction of semantical interpretations. On the contrary, as for the assertoric fragment, we treat apodictic syllogistic as a deductive system (that is, as it is actually presented in the source) without suggesting any interpretation. Again, our rule system is obtained directly from *An.Pr.*, and proofs are presented in tree form. Also in this case, we show that Aristotle always applies his rules coherently. Concretely, we start by introducing the apodictic language (Section 4.1). Then, we briefly consider the purely-apodictic system (Section 4.2). Finally, we deal with the mixed-apodictic fragment (Section 4.3).

4.1 The Apodictic Language

Syllogistic propositions are characterized not only by quality and quantity, but also by *modality*: assertoric, necessary, and possible [4, 25a1–2].[22] We abbreviate apodictic predications of the forms *All/No S is/is not necessarily P* as: $\Pi^+[S,P]$, $\Pi^-[S,P]$, $\Sigma^+[S,P]$ and $\Sigma^-[S,P]$. Observe that modalities are not treated as logical operators, but as part of the structure of atomic formulas to which the four quantifiers are applied. In addition, our notation for apodictic predication is modeled upon the standard box operator coming from modal logics.

4.2 Purely Apodictic Syllogistic

We start by presenting the language of purely apodictic syllogistic in Section 4.2.1. Then, in Section 4.2.2, its rule system $\mathbf{AP}_1$ is introduced. Finally, in Section 4.2.3, we briefly consider imperfect purely-apodictic moods.

4.2.1 The Language of Purely Apodictic Syllogistic

The language of purely apodictic syllogistic is defined by universal and particular, positive and negative apodictic propositions.

Definition 4.1 Let A, B be terms. Then, the language of purely apodictic syllogistic $\mathcal{L}_{\mathbf{AP}_1}$ is defined as follows:

$$P ::= \Pi^+[A,B] \mid \Pi^-[A,B] \mid \ \Sigma^+[A,B] \mid \Sigma^-[A,B].$$

4.2.2 The Purely Apodictic System $\mathbf{AP}_1$

Aristotle's treatment of purely apodictic syllogistic is extremely coincise. Apodictic conversions [4, 25a26–36] and perfect moods [4, 29b35–30a3] are defined analogously to the corresponding assertoric ones.

Definition 4.2 *Purely apodictic conversion rules* are defined below:

$$\frac{\Pi^-[P,S]}{\Pi^-[S,P]}\ \Pi^-[\mathtt{C}] \qquad \frac{\Pi^+[P,S]}{\Sigma^+[S,P]}\ \Pi^+[\mathtt{C}] \qquad \frac{\Sigma^+[P,S]}{\Sigma^+[S,P]}\ \Sigma^+[\mathtt{C}]$$

Definition 4.3 *Perfect purely apodictic syllogisms* are the following six ones:

$$\frac{\Pi^+[B,A] \quad \Pi^+[C,B]}{\Pi^+[C,A]}\ [\mathtt{Bar}] \qquad \frac{\Pi^-[B,A] \quad \Pi^+[C,B]}{\Pi^-[C,A]}\ [\mathtt{Cel}]$$

$$\frac{\Pi^+[B,A] \quad \Sigma^+[C,B]}{\Sigma^+[C,A]}\ [\mathtt{Dar}] \qquad \frac{\Pi^-[B,A] \quad \Sigma^+[C,B]}{\Sigma^-[C,A]}\ [\mathtt{Fer}]$$

$$\frac{\Pi^+[A,B] \quad \Sigma^-[C,B]}{\Sigma^-[C,A]}\ [\mathtt{Baro}] \qquad \frac{\Sigma^-[B,A] \quad \Pi^+[B,C]}{\Sigma^-[C,A]}\ [\mathtt{Boc}]$$

Indeed, differently from **SYL**, in $\mathbf{AP}_1$ [Baro] and [Boc] are primitive.[23]

Definition 4.4 The *purely-apodictic system* $\mathbf{AP}_1$ is made of the three purely apodictic rules of conversion and six perfect purely apodictic syllogisms above.

[22] Actually, Aristotle seems to use two different notions of possibility, but we leave the systematic analysis of the possible fragments to future work.

[23] Actually, they are presented using the second- and third-figure term variables, namely (N, M, X) and (S, P, R), respectively, and are justified by *echtesis*.

4.2.3 Imperfect Purely Apodictic Syllogisms

Derivability proofs in $\mathbf{AP}_1$ are obtained precisely as the corresponding assertoric (ostensive) ones, see Section 3.3. There are four first-figure, four second- and six third-figure purely apodictic syllogisms. For space reasons, we consider just two examples of derivability in $\mathbf{AP}_1$. Second-figure [`Festino`] can be perfected by Π^-[`C`] and [`Fer`] as follows,

$$\dfrac{\dfrac{\Pi^-[N,M]}{\Pi^-[M,N]}\,\Pi^-[\mathtt{C}] \qquad \Sigma^+[X,M]}{\Sigma^-[X,N]}\,[\mathtt{Fer}]$$

Third-figure [`Disamis`] is shown derivable in $\mathbf{AP}_1$ by Σ^+[`C`] and [`Dar`],

$$\dfrac{\dfrac{\Pi^+[S,R] \qquad \dfrac{\Sigma^+[S,P]}{\Sigma^+[P,S]}\,\Sigma^+[\mathtt{C}]}{\Sigma^+[P,R]}\,[\mathtt{Dar}]}{\Sigma^+[R,P]}\,\Sigma^+[\mathtt{C}]$$

First Figure	Second Figure	Third Figure
[`Bar`]	[`Cesare`]	[`Darapti`]
[`Cel`]	[`Festino`]	[`Felapto`]
[`Dar`]	[`Baro`]	[`Disamis`]
[`Fer`]	[`Camestres`]	[`Datisi`]
		[`Boc`]
		[`Ferison`]

A proof of syntactical completeness can also be provided by a systematic inspection that shows these 14 syllogisms to represent the *totality* of the purely-apodictic ones.[24] In fact, this is true only when defining a syllogism as a *pair* of productive premisses [34]. Otherwise, three further imperfect moods have to be added, namely, first-figure [`Barbari`] and [`Celaront`] and second-figure [`Cesaro`]. On the other hand, subaltern [`Camestros`] is not derivable (at least ostensively) in $\mathbf{AP}_1$.[25]

4.3 Mixed Apodictic Syllogistic AP

Then, Aristotle introduces *mixed apodictic* syllogistic. Mixed apodictic syllogisms are made of an assertoric and an apodictic premiss and from them an apodictic conclusion is inferred.

4.3.1 The Mixed Apodictic Language

The mixed apodictic language is obtained by enriching that for **SYL** with apodictic predications.

[24] For further details, see Appendix B.

[25] Indeed, differently from the other subaltern moods, its derivation would require negative subalternation conversion, which is not, in turn, derivable in $\mathbf{AP}_1$. The investigation of (possible) 'apodictic RAA' is left to future study.

Definition 4.5 Let A, B be terms, and Q be a proposition in the assertoric language. Then, the language of mixed apodictic syllogistic $\mathcal{L}_{\mathbf{AP}}$ is as below:

$$P ::= \Pi^{+/-}(A,B) \mid \Pi^{+/-}[A,B] \mid \Sigma^{+/-}(A,B) \mid \Sigma^{+/-}[A,B] \mid \ Q^{\perp}.$$

4.3.2 The Mixed Apodictic System AP

Aristotle presents the first-figure mixed moods accompanying them with explanatory remarks in [4, 30a16–b2].

Definition 4.6 The four *perfect mixed-apodictic syllogisms* are as follows:

$$\frac{\Pi^{+}[B,A] \qquad \Pi^{+}(C,B)}{\Pi^{+}[C,A]}\ \mathtt{B[a]rbar[a]} \qquad\qquad \frac{\Pi^{-}[B,A] \qquad \Pi^{+}(C,B)}{\Pi^{-}[C,A]}\ \mathtt{C[e]lar[e]nt}$$

$$\frac{\Pi^{+}[B,A] \qquad \Sigma^{+}(C,B)}{\Sigma^{+}[C,A]}\ \mathtt{D[a]ri[i]} \qquad\qquad \frac{\Pi^{-}[B,A] \qquad \Sigma^{+}(C,B)}{\Sigma^{-}[C,A]}\ \mathtt{F[e]ri[o]}$$

By adding these rules to the ones of **SYL** and $\mathbf{AP}_1$, the mixed apodictic system **AP** is obtained.

Definition 4.7 The *mixed apodictic system* **AP** is made of the rules of **SYL** plus those of $\mathbf{AP}_1$.

4.3.3 Imperfect Mixed Apodictic Syllogisms

Aristotle shows that there are nine imperfect mixed-apodictic syllogisms: three in the second figure and six in the third.

Mixed-Apodictic Second Figure. The three mixed-apodictic, second-figure moods are `C[e]sar[e]`, `Cam[e]str[e]s` and `F[e]stin[o]`. `C[e]sar[e]` is considered first and shown derivable by $\Pi^{-}[\mathtt{C}]$ and mixed-apodictic `C[e]lar[e]nt`:

> For first let the privative [26] be necessary and let it not be possible for A to belong to any B, but let A merely belong to C. Then, since the privative converts, neither is it possible for B to belong to any A. But A belongs to every C; consequently, it is not possible for B to belong to any C, for C is below A. [4, 30b6–14]

$$\cfrac{\cfrac{\Pi^{-}[B,A]}{\Pi^{-}[A,B]}\ \Pi^{-}[\mathtt{C}] \qquad \Pi^{+}(C,A)}{\Pi^{-}[C,B]}\ \mathtt{C[e]lar[e]nt}$$

Then, `Cam[e]str[e]s` is derived in a similar way:

> And likewise also if the privative is put in relation to C. For if it is not possible for A to belong to any C, then neither will it be possible for C to belong to any A. But A belongs to every B; consequently, it will not be possible for C to belong to any of the Bs, for it becomes the first figure again. Therefore, neither will it be possible for B to belong to C, for it converts similarly. [4, 30b14–9]

$$\cfrac{\cfrac{\cfrac{\Pi^{-}[C,A]}{\Pi^{-}[A,C]}\ \Pi^{-}[\mathtt{C}] \qquad \Pi^{+}(B,A)}{\Pi^{-}[B,C]}\ \mathtt{C[e]lar[e]nt}}{\Pi^{-}[C,B]}\ \Pi^{-}[\mathtt{C}]$$

[26] This corresponds to our notion of negative proposition.

Finally, F[e]stin[o] is introduced and shown perfectable by purely apodictic $\Pi^-[\mathtt{C}]$ and mixed apodictic F[e]r[i]o:

> The situation will also be similar in the case of the particular decisions. For when the privative premise is both universal and necessary. First, then, let the privative premise be both universal and necessary, and let it not be possible for A to belong to any B, but let A belong to some C. Then, since the privative converts, neither would it be possible for B to belong to any A. But A belongs to some one of the Cs, so that B will of necessity not belong to some one of the Cs. [4, 31a1–11]

$$\dfrac{\dfrac{\Pi^-[B,A]}{\Pi^-[A,B]}\,\Pi^-[\mathtt{C}] \qquad \Sigma^+(C,A)}{\Sigma^-[C,B]}\,\mathtt{F[e]ri[o]}$$

There is no mixed-apodictic syllogism corresponding to Baroco [4, 31a11–16].

Mixed-Apodictic Third Figure. Aristotle presents completion proofs for third-figure D[a]rapt[i], Dar[a]pt[i], F[e]lapt[o]n, D[a]tis[i], Dis[a]m[i]s and F[e]ris[o]n. He starts with D[a]rapt[i], which is derived due to $\Pi^-\mathtt{C}$ and perfect D[a]ri[i]:

> For let both premises first be positive, and let both A and B belongs to every C, but let AC be necessary. Then, since B belongs to every C, C will also belong to some B because the universal converts into a particular. Consequently, if A belongs to every C of necessity, and C belongs to some B, then it is also necessary for A to belong to some B (for B is under C). This first figure therefore comes about. [4, 31a24-31]

$$\dfrac{\Pi^+[C,A] \qquad \dfrac{\Pi^+(C,B)}{\Sigma^+(B,C)}\,\Pi^+\mathtt{C}}{\Sigma^+[B,A]}\,\mathtt{D[a]ri[i]}$$

Dar[a]pt[i] is shown derivable in a similar way, but also using apodictic $\Sigma^+[\mathtt{C}]$:

> And it will also be proved in the same way if BC is necessary. For C converts to some A; consequently, if B belongs to every C of necessity, then it will also belong to some A of necessity. [4, 31a31–35]

$$\dfrac{\dfrac{\Pi^+[C,B] \qquad \dfrac{\Pi^+(C,A)}{\Sigma^+(A,C)}\,\Pi^+\mathtt{C}}{\Sigma^+[A,B]}\,\mathtt{D[a]ri[i]}}{\Sigma^+[B,A]}\,\Sigma^+[\mathtt{C}]$$

Then, F[e]lapt[o]n is perfected relying on $\Pi^+\mathtt{C}$ and mixed-apodictic F[e]ri[o]:

> Next, let AC be privative, BC affirmative, and the privative necessary. Then, since C converts to some B and A belongs of necessity to no C, A will also of necessity not belong to some B: for B is below C. [4, 31a35–38]

$$\dfrac{\Pi^-[C,A] \qquad \dfrac{\Pi^+(C,B)}{\Sigma^+(B,C)}\,\Pi^+\mathtt{C}}{\Sigma^-[B,A]}\,\mathtt{F[e]ri[o]}$$

Subsequently, mixed-apodictic D[a]tis[i] is proved derivable in $\mathbf{AP}_1$ due to $\Sigma^+\mathtt{C}$ and mixed-apodictic D[a]ri[i]:

> But if one term is universal, the other is particular, and both are positive, then the conclusion will be necessary whenever the universal is necessary. The demonstration is the same as the previous one, for the positive particular also converts: Thus if it is necessary for B to belong to every C and A is below C, then it is necessary for B to belong to some A. [4, 31b12–17]

$$\dfrac{\Pi^+[C,B] \qquad \dfrac{\Sigma^+(C,A)}{\Sigma^+(A,C)}\,\Sigma^+\texttt{C}}{\Sigma^+[A,B]}\,\texttt{D[a]ri[i]}$$

Next, Dis[a]m[i]s is perfected in a similar way, by using also $\Sigma^+[\texttt{C}]$:

> But if it is necessary for B to belong to some A, then it is also necessary for A to belong to some B (for it converts). [4, 31b17-20]

$$\dfrac{\dfrac{\Pi^+[C,B] \qquad \dfrac{\Sigma^+(C,A)}{\Sigma^+(A,C)}\,\Sigma^+\texttt{C}}{\Sigma^+[A,B]}\,\texttt{D[a]ri[i]}}{\Sigma^+[B,A]}\,\Sigma^+[\texttt{C}]$$

Finally, F[e]ris[o]n is proved perfectable via $\Sigma^+\texttt{C}$ and perfect F[e]ri[o]:

> But if one of the terms is positive and the other privative, then when the universal is both privative and necessary, the conclusion will also be necessary. For if it is not possible for A to belong to any C and B belongs to some C, then it is necessary for A not to belong to some B. [4, 31b33–36]

$$\dfrac{\Pi^-[C,A] \qquad \dfrac{\Sigma^+(C,B)}{\Sigma^+(B,C)}\,\Sigma^+\texttt{C}}{\Sigma^-[B,A]}\,\texttt{F[e]ri[o]}$$

First Figure	Second Figure	Third Figure
B[a]rbar[a]	C[e]sar[e]	D[a]rapt[i]
		Dar[a]pt[i]
C[e]lar[e]nt	Cam[e]str[e]s	F[e]lapt[o]n
D[a]ri[i]	F[e]stin[o]	Dis[a]m[i]s
F[e]ri[o]		D[a]tis[i]
		F[e]ris[o]n

5 Related Works

Reconstructions of Aristotle's logic via modern tools date back at least to the 1950s [19], whereas first interpretations of syllogistic as a natural deduction (not axiomatic) systems are due to Corcoran's [9,10] and Smiley's [37] 1973 works. [27] Other proposals following this line were then presented, for instance, in [21] and in [41]. Yet, none of these proof systems is made of rules in arboreal form. A turning point is instead represented by [47,48], in which syllogistic derivations

[27] In particular, Corcoran introduced a perfecting system where *reductio* is not an inference rule and, as for **SYL**, positive *conversio simplex* and perfect *Darii* and *Ferio* correspond to primitive rules. On the contrary, as for **SYL***, Smiley does not consider these three rules (but the contradictory symbol is treated as a meta-linguistic one).

are arranged as trees. [28] Another related, original reconstruction was developed by Dyckhoff in [12], as based on the notion of inverses of syllogisms. [29] The present work is strongly inspired by [47]. In fact, Section 3 can be seen as a systematic application of the second author's approach to *An.Pr.* I, 1-6, and Section 4 as its generalization to the apodictic fragment.

On the other hand, investigations of Aristotle's modal syllogistic started to appear already in the 1960s [22]. Later on, different reconstructive proposals for this system have been presented in the literature, sometimes even in a formal setting or based on contemporary logical tools [15,44,32,30,42,20,35]. Differently from our study, most of these analyses are somehow 'semantically-oriented' and do not provide a plain reconstruction, to be directly checked by the reader.

Notably, all the mentioned works are historically-driven, that is, aim at reconstructing Aristotle's original system. In addition to that, we hope that our study could point at new (modal) research directions in the framework of Natural Logic. Indeed, one of the goals of this discipline is to define systems which, differently from FOL, are tailored on human reasoning. From the 1970s on, it has led to the revival of syllogistic logic, [30] and, more recently, implementations and applications of (extended) syllogistic systems in AI and NLP have also been provided [28,29,27]. Yet, to the best of our knowledge, no apodictic natural logic system has been developed so far.

6 Conclusion

This paper aims at 'rehabilitating' syllogistic logic by challenging the widespread idea of its "lack of sophistication" [18, p. ix] and at showing its potential appeal for contemporary logic. In particular, we have systematically applied to assertoric and extended to apodictic syllogistic the proof-theoretical approach by [47]. When treating syllogistic as a natural-deduction rule system and presenting perfecting derivations in tree form, it is strikingly straightforward to see that they are all correct, and that the apodictic syllogistic is not a "realm of darkness" [35, p. 1] anymore. In the future, we wish to complete our investigation of Aristotle's modal syllogistic by taking the contingent and mixed apodictic-contingent fragments into account. We hope that, as a byproduct, our reconstruction would also shed new light on the semantics associated with Aristotle's modal logic. In addition, we plan to start a comprehensive study of the proof-theoretical properties of these new syllogistic rule systems.

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[28] In fact, a pioneering use of tree form to investigate Aristotle's perfection proofs already appears in two examples by [17].

[29] Slightly different reconstructions are those focussing on *ecthesis*, see e.g. in [38,16].

[30] The history of Natural Logic can be divided into three main phases: in the 1970s, a renewed interest emerges for term logic; in the 1980s-90s, the discipline acquires a defined status, and first foundational works are written [45,36]; more recently (non-)monotonic aspects of extended syllogistic and related implementations or applications in AI have also been investigated [25,14].

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Appendix

A The system $\mathbf{SYL}^*$

The language of $\mathbf{SYL}^*$ is the same as that of $\mathbf{SYL}$ (Definition 3.1).
The derivability of $\Sigma^+\mathtt{C}$ can be obtained by $\mathtt{RCP}_{1,3}$ together with $\bot\mathtt{I}$ and $\mathtt{RAA}$:

$$\dfrac{\dfrac{\Sigma^+(A,B) \qquad \dfrac{\dfrac{\dfrac{[(\Sigma^+(B,A))^\neg]^1}{\Pi^-(B,A)}\,\mathtt{RCP}_3}{\Pi^-(A,B)}\,\Pi^-\mathtt{C}}{(\Sigma^+(A,B))^\neg}\,\mathtt{RCP}_1}{\bot}\,\bot\mathtt{I}}{\Sigma^+(B,A)}\,\mathtt{RAA},1$$

First-figure `Darii` and `Ferio` are also derivable in $\mathbf{SYL}^*$. Aristotle himself proves this after concluding his presentation of assertoric syllogistic [4, 29b6–9]. Then, in [4, 29b9-11], `Darii` is shown to be derivable by `RAA`, second-figure `Camestres`, and $\bot\mathtt{I}$:

$$\dfrac{\dfrac{\Sigma^+(C,B) \qquad \dfrac{\dfrac{\Pi^+(B,A) \qquad \dfrac{[(\Sigma^+(C,A))^\neg]^1}{\Pi^-(C,A)}\,\mathtt{RCP}_3}{\Pi^-(C,B)}\,\mathtt{Camestres}}{(\Sigma^+(C,B))^\neg}\,\mathtt{RCP}_1}{\bot}\,\bot\mathtt{I}}{\Sigma^+(C,A)}\,\mathtt{RAA},1$$

Since `Camestres` is derivable from `Celarent`, `Darii` comes out to be derivable in primitive $\mathbf{SYL}^*$ as well:

$$\dfrac{\dfrac{\Sigma^+(C,B) \qquad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{[(\Sigma^+(C,A))^\neg]^1}{\Pi^-(C,A)}\,\mathtt{RCP}_3}{\Pi^-(A,C)}\,\Pi^-\mathtt{C} \qquad \Pi^+(B,A)}{\Pi^-(B,C)}\,\mathtt{Celarent}}{\Pi^-(C,B)}\,\Pi^-\mathtt{C}}{(\Sigma^+(C,B))^\neg}\,\mathtt{RCP}_1}{\bot}\,\bot\mathtt{I}}{\Sigma^+(C,A)}\,\mathtt{RAA},1$$

On the other hand, `Ferio` is proved to be perfectable relying on second-figure `Cesare` [4, 29b11-15]: [31]

$$\dfrac{\dfrac{\Sigma^+(C,B) \qquad \dfrac{\dfrac{\Pi^-(B,A) \qquad \dfrac{[(\Sigma^-(C,A))^\neg]^1}{\Pi^+(C,A)}\,\mathtt{RCP}_4}{\Pi^-(C,B)}\,\mathtt{Cesare}}{(\Sigma^+(C,B))^\neg}\,\mathtt{RCP}_1}{\bot}\,\bot\mathtt{I}}{\Sigma^-(C,A)}\,\mathtt{RAA},1$$

[31] Actually, in the original formulation, this proof is quite elliptical.

Given that `Cesare` is derivable in $\mathbf{SYL}^*$ via `Celarent`, also `Ferio` comes out as provable in $\mathbf{SYL}^*$.

Therefore, as noticed by Aristotle in [4, 29b15-20], the basic assertoric system can alternatively (but equally) be defined using primitive `Barbara` and `Celarent` only.

B Syntactical Completeness (Proof Sketch)

The proof of syntactical completeness for $\mathbf{AP}_1$ and $\mathbf{AP}$ is established by systematically inspecting all possible cases. For space reasons and since proofs are tedious but not complicated, only a few cases are considered here. Overall, it emerges that, except for the subaltern moods, the set of syllogisms presented by Aristotle represents the totality of syllogisms derivable in $\mathbf{AP}_1$ and $\mathbf{AP}$ systems (which, as seen, are in turn supposed to actually represent Aristotle's primitive rule systems). Remarkably, subaltern moods are proper moods only when syllogisms are defined by their sets of premisess *together with their conclusion*. On the other hand, when, following [34], syllogisms are defined as pairs of *productive* premisses, a subaltern syllogism is nothing but a (somehow weaker) version of its superaltern, corresponding 'mood'; indeed, for instance, [`Barbara`] and its subaltern [`Barbari`] are actually the same syllogism, i.e. a (perfect) first-figure, purely apodictic syllogism defined by productive premisses of the form $\Pi^+[B, A]$ and $\Pi^+[C, B]$.

B.1 Syntactical Completeness of $\mathbf{AP}_1$

We start by considering the only detour conversions involved, namely that for $\Pi^-[\mathtt{C}]$:

$$\dfrac{\dfrac{\begin{array}{c}\vdots\\ \Pi^-[P,S]\end{array}}{\Pi^-[S,P]}\ \Pi^-[\mathtt{C}]}{\Pi^-[P,S]}\ \Pi^-[\mathtt{C}] \qquad \rightsquigarrow \qquad \begin{array}{c}\vdots\\ \Pi^-[S,P]\end{array}$$

and the one for $\Sigma^+[\mathtt{C}]$:

$$\dfrac{\dfrac{\begin{array}{c}\vdots\\ \Sigma^+[P,S]\end{array}}{\Sigma^+[S,P]}\ \Sigma^+[\mathtt{C}]}{\Sigma^+[P,S]}\ \Sigma^+[\mathtt{C}] \qquad \rightsquigarrow \qquad \begin{array}{c}\vdots\\ \Sigma^+[S,P]\end{array}$$

Then, syntactical completeness of $\mathbf{AP}_1$ is established by tedious, but systematical, inspection of all possible combinations of premisses. The whole proof crucially relies on the following three propositions.

Proposition B.1 *In $\mathbf{AP}_1$, no syllogistic conclusion follows from two particular premisses.*

Proof Sketch. $\mathbf{AP}_1$ contains no primitive rule with two particular premisses nor a conversion rule to derive a universal proposition from a primitive one. $\square$

Proposition B.2 *In $\mathbf{AP}_1$, no syllogistic conclusion follows from two negative premisses.*

Proof Sketch. $\mathbf{AP}_1$ contains no primitive rule with two negative premisses, nor a conversion rule to derive a positive proposition from a negative one. □

Proposition B.3 *If the major premiss of a first-figure pair is particular, then the couple does not syllogize.*

Proof Sketch. By straightforward inspection. □

Theorem B.4 *The totality of purely apodictic syllogisms (primitive or) derivable in* $\mathbf{AP}_1$ *is presented in Figure B.1.*

Proof Sketch. For space reason, we explicitly consider the first-figure case only. We start noticing that by Proposition B.1, the pairs $\big(\Sigma^+[B,A],\Sigma^+[C,B]\big)$, $\big(\Sigma^+[B,A],\Sigma^-[C,B]\big)$, $\big(\Sigma^-[B,A],\Sigma^+[C,B]\big)$ and $\big(\Sigma^-[B,A],\Sigma^-[C,B]\big)$ do not syllogize. Similarly, by Proposition B.2, the pairs $\big(\Pi^-[B,A],\Pi^-[C,B]\big)$, $\big(\Pi^-[B,A],\Sigma^-[C,B]\big)$, $\big(\Sigma^-[B,A],\Pi^-[C,B]\big)$ and, as seen, $\big(\Sigma^-[B,A],\Sigma^-[C,B]\big)$ are not syllogisms.

Let us consider positive premisses. First of all, we have already seen that the pairs $\big(\Pi^+[B,A],\Pi^+[C,B]\big)$ and $\big(\Pi^+[B,A],\Sigma^+[C,B]\big)$ do syllogize.[32] On the other hand, no syllogism is derivable from the pair $\big(\Sigma^+[B,A],\Pi^+[C,B]\big)$ by Proposition B.3. As said, $\big(\Sigma^+[B,A],\Sigma^+[C,B]\big)$ cannot syllogize (Prop. B.1).

We can now consider positive and negative premisses together. Again, the pairs $\big(\Pi^-[B,A],\Pi^+[C,B]\big)$ and $\big(\Pi^-[B,A],\Sigma^+[C,B]\big)$ do syllogize as, respectively, [`Cel`] and [`Fer`] are primitive rule.[33] On the other hand, by Prop. B.3, $\big(\Sigma^-[B,A],\Pi^+[C,B]\big)$ and $\big(\Sigma^-[B,A],\Sigma^-[C,B]\big)$ do not conclude syllogistically. For the same reason, the pairs $\big(\Sigma^+[B,A],\Pi^-[C,B]\big)$ and $\big(\Sigma^+[B,A],\Sigma^-[C,B]\big)$ do not lead to syllogistic conclusions. That the pairs $\big(\Pi^+[B,A],\Pi^-[C,B]\big)$ and $\big(\Pi^+[B,A],\Sigma^-[C,B]\big)$ do not syllogize is proved by inspection.[34] Since, as seen, a negative pair of premisses never syllogizes (Prop. B.2), this concludes our proof for the first figure.

[32] Indeed, [`Bar`] and [`Fer`] are primitive rule. Also subaltern [`Barbari`] can be shown derivable in $\mathbf{AP}_1$. On the contrary, it can be easily proved that no negative conclusion follows from positive premisses so pseudo-moods [`Barbare`], [`Barbaro`], [`Darie`] and [`Dario`] are not syllogisms. Also [`Daria`] is not a syllogism as it can be shown by inspection that from $\big(\Pi^+[B,A],\Sigma^+[C,B]\big)$ no universal conclusion follows.

[33] The former pair would possibly lead also to a particular negative conclusion as subaltern [`Celaront`] is derivable in $\mathbf{AP}_1$. It can also be easily proved by inspection that from $\Pi^-[B,A]$ and $\Sigma^+[C,B]$ no universal conclusion follows.

[34] About the former, (i) there is no primitive, first-figure syllogism with the given premisses, (ii) converting the minor premiss only (via Π^-[`C`]) would lead to $(\Pi^+[B,A],\Pi^-[B,C])$, but there is no third-figure syllogism in this form, (iii) converting the major premiss only (via Π^+[`C`]) would lead to $(\Sigma^+[A,B],\Pi^-[C,B])$, and there is no second-figure syllogism in this form; if converting it again via Σ^+[`C`], no conclusion can follow by Prop. B.1 (no other non-detour conversion is possible), (iv) by converting both premisses, the pair $(\Pi^-[B,C],\Sigma^+[A,B])$ is obtained, which can be easily shown not to give any conclusion with subject C and predicate A. Concerning the latter, (i) it is clear that there is no primitive syllogism with these premisses, (ii) the major premiss converts only via Π^+[`C`], leading to a particular proposition and, (iii) by Prop. B.1, no syllogistic conclusion follows from two particular premisses.

First Figure	Second Figure	Third Figure
[Barbara]		
[Barbari]		[Darapti]
	[Camestres]	
		[Disamis]
[Celarent]	[Cesare]	
[Celaront]	[Cesaro]	[Felapton]
		[Bocardo]
[Darii]		[Datisi]
	[Baroco]	
[Ferio]	[Festino]	[Ferison]

Fig. B.1. Primitive and Derivable Purely Apodictic Syllogisms

From the systematic analysis of all the three syllogistic figures, it emerges that precisely the syllogisms in Figure B.1 are those derivable in $\mathbf{AP}_1$. In particular, the derivability of subaltern [Barbari] is established by perfect [Bar] and apodictic conversions $\Pi^+[\mathtt{C}]$ and $\Sigma^+[\mathtt{C}]$:

$$\dfrac{\dfrac{\dfrac{\Pi^+[B,A] \qquad \Pi^+[C,B]}{\Pi^+[C,A]}\,[\mathtt{Bar}]}{\Sigma^+[A,C]}\,\Pi^+[\mathtt{C}]}{\Sigma^+[C,A]}\,\Sigma^+[\mathtt{C}]$$

First-figure subaltern [Celaront] is derived in $\mathbf{AP}_1$ via perfect [Fer] and apodictic conversions $\Pi^+[\mathtt{C}]$ and $\Sigma^+[\mathtt{C}]$:

$$\dfrac{\Pi^-[B,A] \qquad \dfrac{\dfrac{\Pi^+[C,B]}{\Sigma^+[B,C]}\,\Pi^+[\mathtt{C}]}{\Sigma^+[C,B]}\,\Sigma^+[\mathtt{C}]}{\Sigma^-[C,A]}\,[\mathtt{Fer}]$$

Finally, second-figure [Cesaro] can be perfected by means of perfect [Fer] and all apodictic conversion rules:

$$\dfrac{\dfrac{\Pi^-[A,B]}{\Pi^-[B,A]}\,\Pi^-[\mathtt{C}] \qquad \dfrac{\dfrac{\Pi^+[C,B]}{\Sigma^+[B,C]}\,\Pi^+[\mathtt{C}]}{\Sigma^+[C,B]}\,\Sigma^+[\mathtt{C}]}{\Sigma^-[C,A]}\,[\mathtt{Fer}]$$

□

B.2 Syntactical Completeness of AP

By a similar systematical inspection it emerges that, as sketched by Aristotle himself, there are 4 (perfect) first-figure 3 (imperfect) second-figure and 6 (imperfect) third-figure, mixed apodictic syllogisms. When defining syllogisms in terms of a pair of premisses *together with* their conclusion, two subaltern first-figure syllogisms and one subaltern second-figure syllogism have to be added.

First Figure	Second Figure	Third Figure
B[a]rbar[a]		
B[a]rbar[i]		D[a]rapt[i]
		Dar[a]pt[i]
	Cam[e]str[e]s	
C[e]lar[e]nt	C[e]sar[e]	F[e]lapt[o]n
C[e]lar[o]nt	C[e]sar[o]	
D[a]ri[i]		D[a]tis[i]
		Dis[a]m[i]s
F[e]ri[o]	F[e]stin[o]	
		F[e]ris[o]n

Fig. B.2. Primitive and Derivable Mixed Apodictic Syllogisms

In particular, subaltern B[a]rbar[i] is derived in **AP** as follows:

$$\cfrac{\cfrac{\cfrac{\Pi^+[B,A] \qquad \Pi^+(C,B)}{\Pi^+[C,A]}\ \texttt{B[a]rbar[a]}}{\Sigma^+[A,C]}\ \Pi^+[\texttt{C}]}{\Sigma^+[C,A]}\ \Sigma^+[\texttt{C}]$$

Perfection proof for subaltern C[e]lar[e]nt is below:

$$\cfrac{\Pi^-[B,A] \qquad \cfrac{\cfrac{\Pi^+(C,B)}{\Sigma^+(B,C)}\ \Pi^+\texttt{C}}{\Sigma^+(C,B)}\ \Sigma^+\texttt{C}}{\Sigma^-[C,A]}\ \texttt{F[e]ri[o]}$$

Finally, second-figure, subaltern C[e]sar[o] is shown to be derivable in **AP** as:

$$\cfrac{\cfrac{\cfrac{\Pi^-[C,B]}{\Pi^-[B,C]}\ \Pi^-[\texttt{C}] \qquad \Pi^+(A,B)}{\Pi^-[A,C]}\ \texttt{C[e]lar[e]nt}}{\Pi^-[C,A]}\ \Pi^-[\texttt{C}]$$