Strong Completeness of the Closed Fragment of GLP

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Abstract

The polymodal provability logic GLP is known to be Kripke incomplete w.r.t. Kripke semantics. However, it was shown by Ignatiev that the closed (i.e. variable free) fragment of GLP is Kripke complete w.r.t. a certain Kripke frame, which is now known as the Ignatiev frame, moreover he showed that it is complete w.r.t. to a specific set of its points, which are called the main axis. We show that the closed fragment of GLP is strongly complete w.r.t. to an extended variant of the Ignatiev frame, as well is that it is not strongly complete w.r.t. to the main diagonal and moreover w.r.t. to neither Icard not Beklemishev-Gabelaia topological spaces, which are known to provide topological completeness for the closed fragment of GLP and GLP itself correspondingly.

Keywords: Provability logic, strong completeness, Ignatiev frame, GLP.

1 Introduction

The interest in the provability logic stems from the investigations of Gödel's incompleteness theorems. Löb [24] formulated three conditions on the provability predicate of Peano Arithmetic that form a useful modification of the conditions that Hilbert and Bernays [19] introduced for their proof of Gödel's second incompleteness theorem. Friedman [18] posed the problem of axiomatizing the set of valid arithmetical formulæ built from expressions of the form "φ is provable" by means of Boolean connectives and provability assertions. Boolos [13] (and independently Bernardi, Montagna, and van Benthem) proved that Löb's axiomatization was complete when restricting to closed (i.e., variable-free) formulæ, building on work of Segerberg [25] on the Kripke semantics of Löb's logic

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GL. Solovay [27] later extended Boolos' theorem to a completeness theorem of GL for its arithmetical interpretation.

We consider a modal logic with infinitely many modalities ($[n]$ and $\langle n\rangle$ for $n < \omega$). The intended interpretation of $[n]\varphi$ is "φ is provable in the formal system T_n," where T_n is an arithmetical or set-theoretic system such that T_{n+1} is stronger than T_n for each n. We shall not concern ourselves with the arithmetical interpretation of this polymodal provability logic here, however, and instead focus mainly on its relational semantics. This polymodal extension of GL – called GLP – was introduced by Japaridze [23] and is complete with respect to various choices of T_n; see e.g., Japaridze [23] or Fernandez-Duque and Joosten [17]. Beklemishev [7] and Beklemishev and Pakhomov [15] show how GLP can be applied for the purposes of ordinal analysis and other proof-theoretic results. The first author and Pakhomov [5] have recently proved the completeness of the extension GLP.3 of GLP for a set-theoretic interpretation, extending a result of Solovay [27] for the unimodal case.

Much work has been carried out on the models of GL and GLP, in part due to the fact that these are generally complicated, as we shall recall below. For instance, it is not difficult to show that GLP has no nontrivial Kripke frames and in particular is not Kripke-complete. Nonetheless, Ignatiev [22] proved a relational completeness theorem for the closed fragment of GLP with respect to what is nowadays called the *Ignatiev frame* $\mathfrak{I}$. The purpose of this article is to investigate the question of whether the closed fragment of GLP is strongly complete with respect to this semantics.

Definition 1.1 A modal logic L is *strongly Kripke-complete* with respect to a class of Kripke frames $\mathcal{C}$ if every consistent set of L-formulæ has a model in $\mathcal{C}$.

Here recall that a set of formulæ Γ is consistent relative to a logic L if $\mathsf{L} \nvdash \neg \bigwedge \Delta$ for any finite $\Delta \subset \Gamma$.

We shall prove:

Theorem 1.2 *The closed fragment of the provability logic* GLP *is strongly Kripke-complete with respect to* $\{\mathfrak{I}\}$*, where* $\mathfrak{I}$ *is the Ignatiev frame.*

Strictly speaking, the frame we use is a slight extension of Ignatiev's original frame, and this is necessary, as we shall see below. The idea to establish the result is to use maximal consistent sets of formulæ to inductively extract coordinates for the unique point in the Ignatiev frame which satisfies the given set of formulæ. Indeed, in addition to Theorem 1.2, we mention various counterexamples to strong completeness with respect to other semantics. In particular, we give a counterexample to the strong completeness of the closed fragment of GLP with respect to so-called *Icard spaces*. These were studied by Icard [20,21] as a variant of the blow-up constructions of Beklemishev [8] and provide a simple semantics for the closed fragment of GLP. The same argument shows that GLP is not strongly complete with respect to the topological spaces constructed by Beklemishev-Gabelaia [10]. The first author [4] had previously observed that GLP is also not strongly complete with respect to its so-called

"standard topological models" (see Beklemishev and Gabelaia [11] and Bagaria [6] for more on these) and so at the current stage it is not clear whether there are any natural candidates for strongly complete models. Nonetheless, Shamkanov [26] has proved a nice *global completeness* result making use of an illfounded proof system for GLP and the global consequence relation, from which strong completeness for topological models follows, however it is open whether strong completeness holds for any nice collection of ordinal or otherwise easily described spaces, and we shall see below that it fails for what might otherwise be the first natural candidates. A strong completeness theorem of GL for a slight variation of its relational semantics was obtained by the first author and Fernández-Duque [3]. This result extends to GLP – a proof of this shall appear in a forthcoming article.

2 Preliminaries

2.1 The logic GLP

We begin with some preliminary notions, definitions, and recall some relevant results. For general background on provability logic, we refer the reader to Boolos [14].

Definition 2.1 We consider the modal language $\mathcal{L}$ generated according to the following rules:

$$\mathcal{L} = \bot \mid p \mid \varphi \wedge \psi \mid \neg\varphi \mid [n]\varphi$$

where p is a propositional variable and $n < \omega$. The closed (variable free) fragment of $\mathcal{L}$ is defined by

$$\mathcal{L}_0 = \bot \mid \varphi \wedge \psi \mid \neg\varphi \mid [n]\varphi$$

Finally, the language $\mathcal{L}_{[k;l)}$ is defined by restricting the modalities in $\mathcal{L}$ to the interval $[k, l)$. We conventionally use $\langle n\rangle\varphi = \neg[n]\neg\varphi$.

Definition 2.2 The logic I is the minimal set of $\mathcal{L}$-formulæ closed under modus ponens and necessitation and containing the following axioms:

(i) All Boolean tautologies

(ii) $[n](\varphi \to \psi) \to ([n]\varphi \to [n]\psi)$;

(iii) $[n]([n]\varphi \to \varphi) \to [n]\varphi$ for all $n < \omega$;

(iv) $[m]\varphi \to [n][m]\varphi$ for all $m \leq n < \omega$;

(v) $\langle m\rangle\varphi \to [n]\langle m\rangle\varphi$ for all $m < n < \omega$;

The logic GLP is obtained from I by adding the monotonicity axiom:

(vi) $[m]\varphi \to [n]\varphi$ for all $m < n < \omega$.

Definition 2.3 A *Kripke frame* is a tuple $F = (W, R_0, R_1, \dots)$, where W is a set and $R_i \subset W \times W$ for each $i < \omega$. Given a Kripke frame F and a function $v : \mathsf{Vars} \to \mathcal{P}(W)$, we say that $M = (F, v)$ is a *Kripke model*, which yields the following interpretation $[\![\cdot]\!]$ of modal formulæ:

- $[\![\bot]\!] = \emptyset$;
- $[\![p]\!] = v(p)$, where $p \in \mathsf{Vars}$;
- $[\![\varphi \wedge \psi]\!] = [\![\varphi]\!] \cap [\![\psi]\!]$;
- $[\![\neg\varphi]\!] = W \setminus [\![\varphi]\!]$;
- $[\![\langle n\rangle\varphi]\!] = \{x : \exists y \in [\![\varphi]\!]\, xR_n y\}$;

We say that a formula φ holds at a point x in a model M if $x \in [\![\varphi]\!]$, in which case we write $M, x \Vdash \varphi$. We write $M \Vdash \varphi$ to mean $M, x \Vdash \varphi$ for some $x \in M$ and $M \models \varphi$ to mean $M, x \Vdash \varphi$ for all $x \in F$ and $F \models \varphi$ to mean $M \models \varphi$ for all models of the form $M = \langle F, v\rangle$.

We say that a Kripke frame is an I-*frame*, if each R_i is converse well-founded and transitive and the following holds:

$$\forall x, y\big(xR_n y \to \forall z(xR_m z \leftrightarrow yR_m z)\big)$$

if $m < n$.

For each point $x \in W$ we assign an *i-degree* $\deg_i(x)$, which is 0 for each element that has no i-successors, and otherwise it is

$$\deg_i(x) = \sup\{\deg_i(y) + 1 : y \text{ is an immediate } R_i\text{-successor of } x\}.$$

The study of models of GLP is complicated, as these are hard to produce.

Example 2.4 Let $(W, R_0, R_1, R_2, \ldots)$ be a Kripke frame such that $W \models \mathsf{GLP}$. From the validity of axiom (v) it follows that if wR_0v and wR_1u, then uR_0v.

Suppose thus that wR_1v. From the validity of axiom (vi) it follows that wR_0v and thus that vR_0v, by the comment above. However, it is well known that the validity of Löb's axiom requires that frames be wellfounded and in particular irreflexive. Thus we conclude all GLP frames satisfy $R_1 = R_2 = \cdots = \varnothing$. In particular, GLP is not Kripke-complete.

This example in fact shows that in I-frames there cannot be more than one arrow between any two points.

The problem of Kripke-incompleteness is sometimes circumvented by considering other kinds of models, such as topological models; we shall return to these later. It also motivates the study of its closed fragment, which is sufficient for many of its applications in particular to proof theory and ordinal analysis. As was shown by Ignatiev, the closed fragment of GLP coincides with the closed fragment of I:

Theorem 2.5 ([22]) *Let* $\mathsf{GLP}_0 = \mathsf{GLP} \cap \mathcal{L}_0$ *and* $\mathsf{I}_0 = \mathsf{I} \cap \mathcal{L}_0$. *Then* $\mathsf{GLP}_0 = \mathsf{I}_0$.

Ignatiev showed that I is complete with respect to the class of I-frames. In particular $\mathsf{I}_0 = \mathsf{GLP}_0$ is Kripke-complete. In addition, it is complete with respect to a single certain frame, called the *Ignatiev frame*, which we shall recall below. Let us first remind ourselves some basic facts about ordinal numbers.

2.2 On ordinals $\leq \varepsilon_0$

Definition 2.6 ε_0 is the least ordinal α that satisfies the equation $\omega^\alpha = \alpha$.

The ordinal ε_0 satisfies

$$\varepsilon_0 = \sup\left\{ \underbrace{\omega^{\omega^{\cdot^{\cdot^{\cdot^{\omega}}}}}}_{n \text{ times}} : n < \omega \right\} = \omega^{\omega^{\cdot^{\cdot^{\cdot^{\omega^{\cdot^{\cdot^{\cdot}}}}}}}}.$$

Definition 2.7 Given an ordinal $\alpha > 0$, it can be uniquely represented in its *Cantor normal form* as

$$\alpha = \omega^{\lambda_0} + \omega^{\lambda_1} + \cdots + \omega^{\lambda_n}$$

where $\lambda_0 \geq \lambda_1 \geq \cdots \geq \lambda_n$. Assuming $\alpha > 0$ we define $\log(\alpha) = \lambda_n$ and its iteration, $\log^0(\alpha) = \alpha$ and $\log^{n+1}(\alpha) = \log(\log^n(\alpha))$. Similarly, we put $\log(0) = 0$.

Note that for any ordinal $\alpha < \varepsilon_0$, there is large enough $m < \omega$, such that $\log^n(\alpha) = 0$ for any $n > m$, however $\log^n(\varepsilon_0) = \varepsilon_0$ for any $n < \omega$.

2.3 The frame $\mathfrak{I}$

Definition 2.8 Let $\iota \leq \varepsilon_0$. The Ignatiev frame $\mathfrak{I}_{\leq\iota}$ consists of functions

$$\boldsymbol{\alpha} : \omega \to \iota + 1$$

with the property that $\alpha_{i+1} \leq \log(\alpha_i)$. Here and below, we write α_i to mean $\alpha(i)$ in order to regard $\boldsymbol{\alpha}$ as a sequence. For $\boldsymbol{\alpha}, \boldsymbol{\beta} \in \mathfrak{I}_{\leq\iota}$, we define $\boldsymbol{\alpha} R_k \boldsymbol{\beta}$ if and only if the following hold:

- $\forall i < k\ \alpha_i = \beta_i$;
- $\alpha_k > \beta_k$.

Definition 2.9 Let ι be an ordinal. We define $\mathfrak{I}_{<\iota}$ to be the union of $\{\mathfrak{I}_{\leq\eta} : \eta < \iota\}$. We also use $\mathfrak{I}$ to denote the union of $\{\mathfrak{I}_{<\eta} : \eta \in \mathsf{Ord}\}$, though this shall only be needed occasionally for notational convenience.

Note that the structures $\mathfrak{I}_{\leq\iota}$ strictly extend each other as ι increases, so $\mathfrak{I}_{<\iota}$ makes sense model-theoretically. In the past, typically only the model $\mathfrak{I}_{<\varepsilon_0}$ has been considered, but here we shall find use for the slightly larger model $\mathfrak{I}_{\leq\varepsilon_0}$. For sequences $\boldsymbol{\alpha} \in \mathfrak{I}_{<\varepsilon_0}$, we always find some $n < \omega$ such that $\alpha_k = 0$ for all $k > n$, i.e., elements of $\mathfrak{I}_{<\varepsilon_0}$ are finitely supported. We may occasionally identify elements of $\mathfrak{I}$ with their restriction to their support. In $\mathfrak{I}_{\leq\varepsilon_0}$ the only sequence with infinite support is the constant-ε_0 sequence. For all finitely supported sequences we write $|\boldsymbol{\alpha}|$ to denote the least $i > 0$ such that $\alpha_{i-1} = 0$.

Let us introduce some notational conventions. First, given an ordinal α, we define a sequence $\hat{\alpha}$ by

$$\hat{\alpha}_i = \log^i(\alpha).$$

For all $k < \omega$ we always denote $\boldsymbol{\alpha}_k = \langle \alpha_0, \dots, \alpha_k \rangle$, $\langle \alpha, \boldsymbol{\beta} \rangle = \langle \alpha, \beta_i \rangle_{i<|\boldsymbol{\beta}|}$, $\langle \boldsymbol{\alpha}, \beta \rangle = \langle \alpha_i, \beta \rangle_{i<|\boldsymbol{\alpha}|}$ and $\langle \boldsymbol{\alpha}, \boldsymbol{\beta} \rangle = \langle \alpha_i, \beta_j \rangle_{i<|\boldsymbol{\alpha}|, j<|\boldsymbol{\beta}|}$. We also define pointwise addition, that is $\boldsymbol{\alpha} + \boldsymbol{\beta}$ is a sequence given by $\alpha_i + \beta_i$ for all $i < \omega$, as well as pointwise comparison which is denoted by $\preceq$. Finally, we define a useful operator that glues the first n elements of a sequence with the rest of another sequence, we define a sequence $(\boldsymbol{\alpha} :_n \boldsymbol{\beta})$ by:

$$(\boldsymbol{\alpha} :_n \boldsymbol{\beta})_i = \begin{cases} \alpha_i, & i < n; \\ \beta_i, & i \geq n; \end{cases}$$

The motivation behind it is the following, for any $\boldsymbol{\alpha}, \boldsymbol{\beta} \in \mathfrak{I}$, $\boldsymbol{\alpha} \succeq \boldsymbol{\beta}$ implies $(\boldsymbol{\alpha} :_n \boldsymbol{\beta}) \in \mathfrak{I}$, since $\log(\alpha_{n-1}) \geq \alpha_n \geq \beta_n$.

It can be verified that $\mathfrak{I}$ is an I-frame. The following theorem shows that considering this very frame is enough for establishing completeness.

Theorem 2.10 ([22]) $\mathsf{GLP}_0 \vdash \varphi$ *if and only if* $\mathfrak{I}_{<\varepsilon_0} \models \varphi$.

In fact, Ignatiev showed a slightly stronger fact:

Definition 2.11 Let $\boldsymbol{\alpha} = \langle \alpha_0, \dots, \alpha_n \rangle \in \mathfrak{I}$. We say that $\boldsymbol{\alpha}$ belongs to the *main axis* of $\mathfrak{I}$ if it satisfies $\log(\alpha_i) = \alpha_{i+1}$ for all $i < n$, i.e., if it is of the form $\hat{\gamma}$ for some γ. We denote by $\mathrm{ma}(\mathfrak{I})$ the main axis of $\mathfrak{I}$.

For each $\alpha_0 < \varepsilon_0$, there is precisely one point in the main axis of $\mathfrak{I}$ whose first coordinate is α_0, namely $\hat{\alpha}_0$. Ignatiev proved that every closed formula consistent with GLP is satisfied somewhere along the main axis of $\mathfrak{I}_{<\varepsilon_0}$.

The goal of this article is to study the question of whether $\mathfrak{I}$ is strongly complete for the closed fragment of GLP. We shall show:

Theorem 2.12 (Strong completeness)

(A) *The closed fragment of* GLP *is strongly complete with respect to* $\mathfrak{I}_{\leq\varepsilon_0}$. *More precisely: let* Γ *be a set of closed* $\mathcal{L}$*-formulæ. Then the following are equivalent:*
   (i) Γ *is consistent with* GLP*; and*
   (ii) $\mathfrak{I}_{\leq\varepsilon_0}, \boldsymbol{\alpha} \Vdash \Gamma$ *for some* $\boldsymbol{\alpha} \in \mathfrak{I}_{\leq\varepsilon_0}$.

(B) *Moreover,*
   (i) *The closed fragment of* GLP *is not strongly complete with respect to* $\mathfrak{I}_{<\varepsilon_0}$*; and*
   (ii) *The closed fragment of* GLP *is not strongly complete with respect to the main axis of* $\mathfrak{I}_{\leq\varepsilon_0}$.

3 Proof of Theorem 2.12

3.1 Proof of strong completeness

In this section, we prove Theorem 2.12. We will find it convenient to introduce some further notation. Given a finite tuple of ordinals $\boldsymbol{\alpha}$, we denote

$$\mathfrak{I}_{\boldsymbol{\alpha}} = \{\boldsymbol{\gamma} \in \mathfrak{I} : \forall i < |\boldsymbol{\alpha}|\, \gamma_i = \alpha_i\}.$$

The intuition behind the notation is that when $\boldsymbol{\alpha}$ is a singleton α, then $\mathfrak{I}_\alpha$ consists of precisely the points in $\mathfrak{I}$ of "level" α, and $\mathfrak{I}_{\leq\iota}$ gathers precisely all the levels $\mathfrak{I}_\alpha$ with $\alpha \leq \iota$.

Definition 3.1 Let $\varphi \in \mathcal{L}$. We define a formula $\uparrow\varphi$ by induction on the syntactical complexity of φ:

$$\uparrow\varphi = \begin{cases} \bot, & \varphi = \bot; \\ p, & \varphi = p; \\ \uparrow\psi \wedge \uparrow\pi, & \varphi = \psi \wedge \pi; \\ \neg\uparrow\psi, & \varphi = \neg\psi; \\ [k+1]\psi, & \varphi = [k]\varphi,\ k < \omega; \\ \langle k+1\rangle\psi, & \varphi = \langle k\rangle\varphi,\ k < \omega. \end{cases}$$

Iterations of $\uparrow$ are defined by $\uparrow^0\varphi = \varphi$ and $\uparrow^{(k+1)}\varphi = \uparrow(\uparrow^k\varphi)$. For any $\Gamma \subset \mathcal{L}$ we write $\uparrow^k\Gamma = \{\uparrow^k\varphi : \varphi \in \Gamma\}$.

The next lemma shows how formulæ reflect the similarity of modal relations.

Claim 3.2 *For all $\varphi \in \mathcal{L}_0$, $\mathfrak{I}, \boldsymbol{\gamma} \Vdash \varphi$ if and only if for all $\boldsymbol{\alpha}$ and $k := |\boldsymbol{\alpha}|$, we have $\mathfrak{I}, \langle\boldsymbol{\alpha}, \boldsymbol{\gamma}\rangle \Vdash \uparrow^k\varphi$ whenever $\langle\boldsymbol{\alpha}, \boldsymbol{\gamma}\rangle \in \mathfrak{I}$.*

Proof. Fix $\boldsymbol{\alpha}$. We reason by induction on the complexity of φ:

- Booleans and $\top$, $\bot$ are clear;
- Suppose $\varphi = \langle i\rangle\psi$. First, we make the observation that if $\boldsymbol{\gamma} R_i \boldsymbol{\beta}$, then $\langle\boldsymbol{\alpha}, \boldsymbol{\beta}\rangle \in \mathfrak{I}$ and, moreover $\langle\boldsymbol{\alpha}, \boldsymbol{\gamma}\rangle R_{i+k} \langle\boldsymbol{\alpha}, \boldsymbol{\beta}\rangle$. Now, we see that $\mathfrak{I}, \boldsymbol{\gamma} \Vdash \varphi$ if and only if there exists $\boldsymbol{\beta} \in \mathfrak{I}$ such that $\boldsymbol{\gamma} R_i \boldsymbol{\beta}$ and $\boldsymbol{\beta} \Vdash \psi$. By the observation above and the induction hypothesis, for such a $\boldsymbol{\beta}$ we have $\langle\boldsymbol{\alpha}, \boldsymbol{\beta}\rangle \Vdash \uparrow^k\psi$. By the second part of the observation above, we see that $\mathfrak{I}, \boldsymbol{\gamma} \Vdash \varphi$ if and only if there is $\boldsymbol{\beta}$ such that $\langle\boldsymbol{\alpha}, \boldsymbol{\gamma}\rangle R_{i+k} \langle\boldsymbol{\alpha}, \boldsymbol{\beta}\rangle$ and $\langle\boldsymbol{\alpha}, \boldsymbol{\beta}\rangle \Vdash \uparrow^k\psi$. Using the definition of R_{i+k} we see that this is equivalent to $\langle\boldsymbol{\alpha}, \boldsymbol{\gamma}\rangle \Vdash \uparrow^k\varphi$, as desired;
- $\varphi = [i]\psi$ follows from duality using the fact that $\uparrow$ commutes with negation.

This proves the claim. □

Definition 3.3 We say that a formula $\varphi \in \mathcal{L}$ is a *worm* if it has the form $\langle k_0\rangle \dots \langle k_n\rangle\top$.

For any worm $\varphi = \langle k_0\rangle \dots \langle k_n\rangle\top$ and any $\boldsymbol{\alpha} \in \mathfrak{I}$ such that $\boldsymbol{\alpha} \Vdash \varphi$, we can always find a sequence $(\boldsymbol{\alpha}^1, \dots, \boldsymbol{\alpha}^n)$ witnessing that $\boldsymbol{\alpha} \Vdash \varphi$, i.e., such that $\boldsymbol{\alpha}^0 R_{k_1} \boldsymbol{\alpha}^1 \dots R_{k_n} \boldsymbol{\alpha}^n$. The next lemma says that if a worm is satisfied at some $\boldsymbol{\alpha} \in \mathfrak{I}$, then it is also satisfied at $\langle\beta\rangle + \boldsymbol{\alpha}$ for each $\beta \leq \varepsilon_0$ (recall that $\langle\beta\rangle = \langle\beta, 0, 0, \dots\rangle$ and "+" is a poinwise addition, so $\langle\beta\rangle + \boldsymbol{\alpha} = \langle\beta + \alpha_0, \alpha_1, \dots\rangle$).

Claim 3.4 *For any sequence $(\boldsymbol{\alpha}^1, \dots, \boldsymbol{\alpha}^n) \subset \mathfrak{I}$, $\boldsymbol{\alpha}^0 R_{k_1} \boldsymbol{\alpha}^1 \dots R_{k_n} \boldsymbol{\alpha}^n$ implies that for any $\beta \leq \varepsilon_0$ it is true that $(\langle\beta\rangle + \boldsymbol{\alpha}^0) R_{k_1} (\langle\beta\rangle + \boldsymbol{\alpha}^1) \dots R_{k_n} (\langle\beta\rangle + \boldsymbol{\alpha}^n)$.*

Proof. One can see $\alpha_1^i \leq \log(\beta + \alpha_0^i)$ for each $1 \leq i \leq n$, since $\log(\beta + \alpha_0^i) = \log \alpha_0^i$. Hence, $\langle\beta\rangle + \boldsymbol{\alpha}^i \in \mathfrak{I}$ for each $i \leq n$. The fact that $(\langle\beta\rangle + \boldsymbol{\alpha}^i) R_{k_{i+1}} (\langle\beta\rangle + \boldsymbol{\alpha}^{i+1})$ for each $i < n$ is straightforward. □

The next lemma relies upon a construction which is very similar to one introduced and analyzed in [7], however we aim to introduce an explicit proof particular to the Ignatiev frame. Indeed, [7, Lemma 12] relies on assigning ordinals to worms in order to compare them, while we will need the opposite procedure of assigning worms to ordinals. It is possible that the lemma is well known but we were unable to locate a reference. We also refer the reader to Fernández-Duque and Joosten [16] or to Borges and Joosten [1] for some similar arguments.

Lemma 3.5 *For each $k < \omega$ and $\alpha < \varepsilon_0$ there exists a worm φ_k^α, such that whenever $\boldsymbol{\beta} \in \mathfrak{I}_{\leq \varepsilon_0}$ we have $\boldsymbol{\beta} \Vdash \langle k\rangle \varphi_k^\alpha$ if and only if $\beta_k > \alpha$. Moreover, for each $\alpha \leq \varepsilon_0$ there exists a set of formulæ T_k^α such that for each $\boldsymbol{\gamma}$, $\boldsymbol{\gamma} \Vdash T_k^\alpha$ if and only if $\gamma_k = \alpha$.*

Proof. For any two worms $\varphi = \langle k_0\rangle \dots \langle k_n\rangle \top, \psi = \langle l_0\rangle \dots \langle l_m\rangle \top$, we write $\varphi\psi$ for $\langle k_0\rangle \dots \langle k_n\rangle \langle l_0\rangle \dots \langle l_m\rangle \top$. We build φ_0^α inductively:

- $\alpha = 0$ then $\varphi_0^\alpha = \top$;
- $\alpha = \beta + 1 + \gamma$ for some $\beta \geq \gamma$, then $\varphi_0^\alpha = \varphi_0^\gamma \langle 0\rangle \varphi_0^\beta$;
- $\alpha = \omega^\beta$ then $\varphi_0^\alpha = \uparrow \varphi_0^\beta$;

First, note that this definition of φ_0^α is exhaustive, in the sense that every $\alpha < \varepsilon_0$ is indeed assigned a formula φ_0^α. However, the assignment is not unique; the following claim shows that this is not a problem.

Claim 3.6 *For each α, φ_0^α is well defined in the sense that if $\alpha = \beta + 1 + \gamma = \delta + 1 + \zeta$, then $\mathsf{GLP} \vdash \varphi_0^\gamma \langle 0\rangle \varphi_0^\beta \leftrightarrow \varphi_0^\zeta \langle 0\rangle \varphi_0^\delta$. Moreover, for each $\alpha > \beta$, $\mathsf{GLP} \vdash \varphi_0^\alpha \to \langle 0\rangle \varphi_0^\beta$.*

The proof of Claim 3.6 can be found in the appendix. Now for $\alpha < \varepsilon_0$ we set $T_0^\alpha = \{\langle 0\rangle \varphi^{\alpha'} : \alpha' < \alpha\} \cup \{\neg \langle 0\rangle \varphi^\alpha\}$ and $T_0^{\varepsilon_0} = \{\langle 0\rangle \varphi^{\alpha'} : \alpha' < \varepsilon_0\}$. Let us first assume that the first part of the lemma is true for $k = 0$. Then, the "moreover" part follows immediately from the definition, for $k = 0$. In addition, if we inductively let $\varphi_{k+1}^\alpha = \uparrow \varphi_k^\alpha$ and $T_{k+1}^\alpha = \uparrow T_k^\alpha$, then we see that the lemma holds for all k by Claim 3.2.

It remains to verify that the formulæ work as desired for $k = 0$. We start by showing that φ_0^α holds at $\hat{\alpha}$. Afterwards we will show that φ_0^α fails at all $\boldsymbol{\beta} \in \mathfrak{I}_{\leq \varepsilon_0}$ with $\beta_0 < \alpha$. We proceed by induction:

- Suppose $\alpha = 0$. Then $\top$ definitely holds at $\hat{0}$.
- Suppose $\alpha = \beta + 1 + \gamma$ with $\beta \geq \gamma$. Write $\varphi_0^\gamma = \langle n_1\rangle \dots \langle n_m\rangle \top$ and choose a sequence $\hat{\gamma} R_{n_1} \boldsymbol{\delta}^1 \dots R_{n_m} \boldsymbol{\delta}^m$ witnessing $\hat{\gamma} \Vdash \varphi_0^\gamma$. Noting that $\langle \beta + 1\rangle + \hat{\gamma} = \widehat{\beta + 1 + \gamma}$, we get $\widehat{\beta + 1 + \gamma} R_{n_1} (\langle \beta + 1\rangle + \boldsymbol{\delta}^1) \dots R_{n_m} (\langle \beta + 1\rangle + \boldsymbol{\delta}^m)$ by Claim 3.4. This sequence witnesses $\widehat{\beta + 1 + \gamma} \Vdash \langle n_1\rangle \dots \langle n_m\rangle \psi$ for any ψ that holds in $\langle \beta + 1\rangle + \boldsymbol{\delta}^m$; in particular $\langle \beta + 1\rangle + \boldsymbol{\delta}^m \Vdash \langle 0\rangle \varphi_0^\beta$, since

$(\langle \beta+1\rangle + \boldsymbol{\delta}^m)R_0\hat{\beta}$.

- Suppose $\alpha = \omega^\beta$. We have $\hat{\alpha} = \langle \alpha, \beta, \log\beta, \dots\rangle = \langle \alpha, \hat{\beta}\rangle$. By induction hypothesis we have $\hat{\beta} \Vdash \varphi_0^\beta$, so $\hat{\alpha} \Vdash \varphi_1^\beta$ by Claim 3.2. Since $\varphi_1^\beta = \varphi_0^\alpha$, the claim follows.

We have shown that φ_0^α holds at $\hat{\alpha}$. Before moving on, we state a claim.

Claim 3.7 *For any worm $\psi = \langle n_1\rangle \dots \langle n_m\rangle \top$ and for any $\boldsymbol{\gamma} \in \mathfrak{I}_{\leq \varepsilon_0}$, $\boldsymbol{\gamma} \Vdash \psi$ implies $\hat{\gamma}_0 \Vdash \psi$.*

Proof. Let $(\boldsymbol{\alpha}^1, \dots, \boldsymbol{\alpha}^m) \subset \mathfrak{I}_{\leq\varepsilon_0}$ be such that $\boldsymbol{\gamma} R_{n_1} \boldsymbol{\alpha}^1 \dots R_{n_m}\boldsymbol{\alpha}^m$. Take $\boldsymbol{\delta}^1 = (\hat{\gamma}_0 :_{n_1} \boldsymbol{\alpha}^1)$ and $\boldsymbol{\delta}^{i+1} = (\boldsymbol{\delta}^i :_{n_{i+1}} \boldsymbol{\alpha}^{i+1})$ (recall the definition on p. 46). One can see that $\boldsymbol{\delta}^i \succeq \boldsymbol{\alpha}^i$ for each $1 \leq i \leq m$, hence $\boldsymbol{\delta}^i \in \mathfrak{I}_{\leq\varepsilon_0}$ for each $1 \leq i \leq n$ (recall that $\succeq$ denotes pointwise comparison), hence $\hat{\gamma}_0 \Vdash \psi$ is witnessed by $\hat{\gamma}_0 R_{n_1}\boldsymbol{\delta}^1 \dots R_{n_m}\boldsymbol{\delta}^m$. □

Now, assume towards a contradiction that for some $\gamma < \alpha$ there exists $\boldsymbol{\beta}$ such that $\gamma = \beta_0$ and $\boldsymbol{\beta} \Vdash \varphi_0^\alpha$. By Claim 3.7, we have $\hat{\gamma} \Vdash \varphi_0^\alpha$. Assuming without loss of generality that γ is least, it follows from the induction hypothesis on α that $\hat{\gamma} \Vdash \varphi_0^\alpha \wedge \neg\langle 0\rangle\varphi_0^\gamma$. However, $\mathsf{GLP} \vdash \varphi_0^\alpha \to \langle 0\rangle\varphi_0^\gamma$ by Claim 3.6, which is a contradiction. This finishes the proof of Lemma 3.5. □

Below, we use the formulæ φ_k^α and the sets T_k^α defined in the proof of Lemma 3.5.

Claim 3.8 *Let $\beta < \gamma < \varepsilon_0$ and $k < \omega$. Then, we have $\mathsf{I} \vdash \uparrow^k\langle 0\rangle\varphi_0^\gamma \to \uparrow^k\langle 0\rangle\varphi_0^\beta$.*

Proof. Instead of arguing syntactically, we appeal to Ignatiev's completeness theorem (Theorem 2.10). Let $\boldsymbol{\alpha} \in \mathfrak{I}_{<\varepsilon_0}$ be such that $\mathfrak{I}_{<\varepsilon_0}, \boldsymbol{\alpha} \Vdash \uparrow^k\langle 0\rangle\varphi_0^\gamma$. By Lemma 3.5, we must have $\alpha_k \geq \gamma$ and thus $\alpha_k \geq \beta$, so that by Lemma 3.5 once more we obtain $\mathfrak{I}_{<\varepsilon_0}, \boldsymbol{\alpha} \Vdash \uparrow^k\langle 0\rangle\varphi_0^\beta$. By the completeness of $\mathfrak{I}_{<\varepsilon_0}$, we thus have $\mathsf{I} \vdash \uparrow^k\langle 0\rangle\varphi_0^\gamma \to \uparrow^k\langle 0\rangle\varphi_0^\beta$, as desired. □

Claim 3.9 *Let Γ be a maximal set of closed $\mathcal{L}$-formulæ consistent with GLP. Then for each $k < \omega$, there exists a unique $\alpha \leq \varepsilon_0$ such that $T_k^\alpha \subset \Gamma$.*

Proof. We let

$$\alpha = \sup\Big\{\beta + 1 : \beta < \varepsilon_0 \wedge \uparrow^k\langle 0\rangle\varphi_0^\beta \in \Gamma\Big\}.$$

By maximality, we must then have $\neg\uparrow^k\langle 0\rangle\varphi_0^\alpha \in \Gamma$ or $\alpha = \varepsilon_0$. Similarly, by maximality, Γ contains all consequences of I, thus, by Claim 3.8 the set of β such that $\uparrow^k\langle 0\rangle\varphi_0^\beta \in \Gamma$ forms an initial segment of the ordinals. Since

$$\begin{aligned} T_k^\alpha &= \{\uparrow^k\varphi : \varphi \in T_0^\alpha\} \\ &= \{\uparrow^k\langle 0\rangle\varphi_0^{\alpha'} : \alpha' < \alpha\} \cup \{\neg\uparrow^k\langle 0\rangle\varphi_0^\alpha\}, \end{aligned}$$

it follows that $T_k^\alpha \subset \Gamma$. □

As a consequence of this claim, it follows that, for such a Γ, there is at most one $w \in \mathfrak{I}_{\leq\varepsilon_0}$ such that $\mathfrak{I}_{\leq\varepsilon_0}, w \Vdash \Gamma$. We are now ready to prove the first part of the strong completeness theorem, which we restate for convenience.

Theorem 2.12(A) *Let $\Gamma \subset L_0$ be a maximal set of closed $\mathcal{L}$-formulæ consistent with* GLP. *Then there is $w \in \mathfrak{I}_{\leq \varepsilon_0}$ such that $w \Vdash \Gamma$.*

Proof. In this proof we assume that all formulæ of Γ are in the language:

$$\mathcal{L}' = \bot \mid \top \mid \varphi \wedge \psi \mid \varphi \vee \psi \mid [n]\varphi \mid \langle n\rangle\varphi$$

That is, we only consider formulas without negation, which we can do without loss of generality. Nonetheless, we may use the notation $\neg\eta$ for some η as shorthand for the negation normal form of $\neg\eta$. For each $i < \omega$, we let

$$\Gamma_i = \Big\{[i]\varphi : [i]\varphi \in \Gamma\Big\} \cup \Big\{\langle i\rangle\psi : \langle i\rangle\psi \in \Gamma\Big\}.$$

Our plan is to treat these strata consecutively.

We show inductively that for each $k < \omega$ there exists α_k such that:

(i) if $k \neq 0$, then $\alpha_i \leq \log(\alpha_{i-1})$ for each $0 < i < k$, and

(ii) $\mathfrak{I}_{\leq\varepsilon_0}, \boldsymbol{\gamma} \Vdash \bigcup_{i\leq k} \Gamma_i$ whenever $\gamma_i = \alpha_i$ for each $i \leq k$.

We denote by $(\#)_k$ the sentence asserting the existence of such an α_k.

Inductively, suppose that $(\#)_i$ has been proved for all $i < k$. We prove $(\#)_k$. Let $\boldsymbol{\alpha}_{k-1} = \langle \alpha_0, \ldots, \alpha_{k-1}\rangle$. The next claim asserts that it is enough to consider only formulæ that contain only boxes and diamonds with index $\geq k$.

Lemma 3.10 *Let $\Gamma'_k = \Gamma_k \cap \mathcal{L}_{[k;\omega)}$ and suppose $\beta \leq \log(\alpha_{k-1})$. Then $\langle \boldsymbol{\alpha}_{k-1}, \beta\rangle \Vdash \Gamma'_k$ if and only if $\langle \boldsymbol{\alpha}_{k-1}, \beta\rangle \Vdash \Gamma_k$.*

Proof. For all $i < k$ and any $\mathcal{L}'$-formula $\varphi \in \Gamma_k$ we modify it as follows: substitute each subformula of φ of the form $\langle i\rangle\psi$ or $[i]\psi$ by $\top$ if it belongs to Γ or by $\bot$ if it does not belong to Γ. We call the result of such a substitution $\varphi^{(k)}$. Formally,

$$\begin{array}{rll}
\bot^{(k)} = \bot; & \top^{(k)} = \top; & \\
(\psi \wedge \varphi)^{(k)} = \psi^{(k)} \wedge \varphi^{(k)}; & (\psi \vee \varphi)^{(k)} = \psi^{(k)} \vee \varphi^{(k)}; & \\
(\langle i\rangle\varphi)^{(k)} = \langle i\rangle(\varphi^{(k)}); & ([i]\varphi)^{(k)} = [i](\varphi^{(k)}) & \text{for } j \geq k; \\
(\langle i\rangle\varphi)^{(k)} = \top \text{ if } \langle i\rangle\varphi \in \Gamma; & (\langle i\rangle\varphi)^{(k)} = \bot \text{ otherwise} & \text{for } i < k; \\
([i]\varphi)^{(k)} = \top \text{ if } [i]\varphi \in \Gamma; & ([i]\varphi)^{(k)} = \bot \text{ otherwise} & \text{for } i < k;
\end{array}$$

Let $\Gamma_k^{(k)} = \{\varphi^{(k)} : \varphi \in \Gamma_k\}$. Then, $\Gamma_k^{(k)}$ is consistent provided Γ_k is; indeed:

Claim 3.11 *For any $\psi \in \mathcal{L}_0$ and $k \leq n < \omega$, $\Gamma \vdash [n](\psi \leftrightarrow \psi^{(k)})$.*

Here, entailment refers to provability from the axioms of GLP.

Proof. It is sufficient to show that $\Gamma \vdash [k](\psi \leftrightarrow \psi^{(k)})$. We prove this by induction on the complexity of ψ.

- Suppose $\psi = \eta \wedge \zeta$. By induction hypothesis, $\Gamma \vdash [k](\eta \leftrightarrow \eta^{(k)})$ and $\Gamma \vdash [k](\zeta \leftrightarrow \zeta^{(k)})$; hence $\Gamma \vdash [k](\eta \wedge \zeta \leftrightarrow \eta^{(k)} \wedge \zeta^{(k)})$ by normality. The case $\psi = \eta \vee \zeta$ is similar.

- Suppose $\xi = \langle i\rangle\eta$ with $i < k$. If $\langle i\rangle\eta$ belongs to Γ, then $\Gamma \vdash [k]\langle i\rangle\eta$, so $\Gamma \vdash [k](\langle i\rangle\eta \leftrightarrow \top)$; otherwise if $\langle i\rangle\eta$ does not belong to Γ, then $[i]\neg\eta \in \Gamma$ and thus by GLP we have $\Gamma \vdash [k][i]\neg\eta$, so that $\Gamma \vdash [k](\langle i\rangle\eta \leftrightarrow \bot)$. The case $\xi = [i]\eta$ with $i < k$ is analogous.
- Finally, suppose $\xi = \langle i\rangle\eta$ with $i \geq k$. By induction hypothesis, $\Gamma \vdash [k](\eta \leftrightarrow \eta^{(k)})$. This implies $\Gamma \vdash [k][k](\eta \leftrightarrow \eta^{(k)})$, and hence $\Gamma \vdash [k][i](\eta \leftrightarrow \eta^{(k)})$ by GLP. It is easy to deduce from this, using normality, that $\Gamma \vdash [k](\langle i\rangle\eta \leftrightarrow \langle i\rangle\eta^{(k)})$. The case $\xi = [i]\eta$ with $i \geq k$ is similar.

This proves the claim. □

Claim 3.12 $\Gamma \vdash \Gamma_k^{(k)}$.

Proof. We show that $\Gamma \vdash \varphi$ implies $\Gamma \vdash \varphi^{(k)}$ for each $\varphi \in \Gamma_k$. This is proved by induction. The Boolean cases, as well as the base cases $\top, \bot$ are straightforward. We consider the case where $\varphi = \langle n\rangle\psi$; the boxes can be treated similarly. There are two subcases:

- Suppose $n \geq k$. By the previous claim we have $\Gamma \vdash [n](\psi \to \psi^{(k)})$. By normality, $\Gamma \vdash [n](\psi \to \psi^{(k)}) \to (\langle n\rangle\psi \to \langle n\rangle\psi^{(k)})$, so that $\Gamma \vdash \langle n\rangle\psi$ implies $\Gamma \vdash \langle n\rangle\psi^{(k)}$.
- Suppose $n < k$. Then $\varphi^{(k)} = \top$, so the conclusion follows immediately.

This proves the claim. □

Claim 3.13 *For any $w \in \mathfrak{I}_{\leq\varepsilon_0}$ such that $w = \langle \boldsymbol{\alpha}_{k-1}, \boldsymbol{\beta}\rangle$, $\mathfrak{I}_{\leq\varepsilon_0}, w \Vdash \Gamma_k^{(k)}$ if and only if $\mathfrak{I}_{\leq\varepsilon_0}, w \Vdash \Gamma_k$.*

Proof. Let $\varphi \in \mathcal{L}'$. We reason by induction on the syntactic construction of φ. We have the following three cases:

(i) Booleans are clear, as are the constants $\bot$ and $\top$.

(ii) Suppose $\varphi = \langle i\rangle\psi$ or $\varphi = [i]\psi$ with $i < k$. By the induction hypothesis $(\#)_i$ for $i < k$, we have that $\mathfrak{I}_{\leq\varepsilon_0}, w \Vdash \varphi$ if and only if $\varphi \in \Gamma_i$. By definition of $\Gamma^{(k)}$, we thus have

$$\varphi \in \Gamma_i \leftrightarrow \varphi^{(k)} = \top \leftrightarrow \varphi^{(k)} \neq \bot \leftrightarrow \mathfrak{I}_{\leq\varepsilon_0}, w \Vdash \varphi^{(k)}.$$

(iii) Suppose $\varphi = \langle i\rangle\psi$ with $i \geq k$. Then, we have $\mathfrak{I}_{\leq\varepsilon_0}, w \Vdash \varphi$ if and only if there $u \in \mathfrak{I}_{\leq\varepsilon_0}$ such that wR_iu and $\mathfrak{I}_{\leq\varepsilon_0}, u \Vdash \psi$. By the inductive hypothesis, this occurs if and only if $\mathfrak{I}_{\leq\varepsilon_0}, u \Vdash \psi^{(k)}$ and thus if and only if $\mathfrak{I}_{\leq\varepsilon_0}, w \Vdash \varphi^{(k)}$.

(iv) Suppose $\varphi = [i]\psi$ with $i \geq k$. This is analogous to the previous case.

This completes the proof of Claim 3.13. □

Now, one can also see that $\Gamma_k^{(k)} = \Gamma'_k$. Obviously, $\Gamma'_k \subset \Gamma_k^{(k)}$, moreover from the previous claim we have that $\Gamma_k^{(k)} \subset \Gamma$, thus $\Gamma_k^{(k)} \subset \Gamma'_k$ follows. This completes the proof of Lemma 3.10. □

One can see that having fixed α_i for all $i < k$ the set of worlds accessible by R_k depends only on the α_k in $\boldsymbol{\alpha} = \langle \alpha_0, \ldots, \alpha_k \ldots\rangle$. We must choose an

ordinal α_k such that $\langle \boldsymbol{\alpha}_k, \boldsymbol{\beta}\rangle \Vdash \Gamma'_k$ whenever $\langle \boldsymbol{\alpha}_k, \boldsymbol{\beta}\rangle \in \mathfrak{I}_{\leq \varepsilon_0}$. The following lemma asserts the existence of such an ordinal.

Lemma 3.14 *There exists a unique ordinal α such that $\langle \boldsymbol{\alpha}_{k-1}, \alpha\rangle \Vdash \Gamma'_k$.*

Proof. Fix an enumeration $\{\varphi'_i : i < \omega\}$ of all formulæ φ such that $[k]\varphi \in \Gamma'_k$ and an enumeration $\{\psi'_i : i < \omega\}$ of all formulæ ψ such that $\langle k\rangle\psi \in \Gamma'_k$. Let

$$\Gamma'_k(i) = \{\langle k\rangle\varphi'_j : j < i\}.$$

Since $\Gamma'_k(i) \subset \mathcal{L}_{[k;\omega)}$ there is $\Sigma_i \subset \mathcal{L}$ for each $i < \omega$ such that $\Gamma'_k(i) = \uparrow^k \Sigma_i$ (so that $\Gamma'_k = \uparrow^k \Sigma$ where $\Sigma = \bigcup_i \Sigma_i$). By Ignatiev's Completeness Theorem 2.10, there is a point $\langle\gamma\rangle \in \mathfrak{I}$ that $\langle\gamma\rangle \Vdash \Sigma_i$ (note that each formula in Σ is of the form $\langle 0\rangle\psi$ or $[0]\varphi$); let α^i_k be the least such γ and let $\alpha = \sup\{\alpha^i_k : i < \omega\}$. Evidently, $\langle\alpha\rangle \Vdash \langle 0\rangle\psi$ for each $\langle 0\rangle\psi \in \Sigma$. Suppose towards a contradiction that $\langle\alpha\rangle \Vdash \langle 0\rangle\neg\varphi$ for some φ with $[0]\varphi \in \Sigma$. That is, there exists $\boldsymbol{\gamma} \in \mathfrak{I}_{<\alpha}$ such that $\boldsymbol{\gamma} \Vdash \neg\varphi$. Let $m < \omega$ be large enough so that $\alpha^n_k \Vdash \langle 0\rangle\neg\varphi$ for every $n > m$. Since each α^i_k was chosen minimal, we have

$$\mathfrak{I} \models \bigwedge_{i<m+1} \Sigma_i \to \langle 0\rangle\neg\varphi.$$

By Theorem 2.10, we have

$$\mathsf{I} \vdash \bigwedge_{i<m+1} \Sigma_i \to \langle 0\rangle\neg\varphi,$$

contradicting the consistency of Σ and thus of Γ. This shows that $\mathfrak{I}_{\leq\varepsilon_0}, \langle\alpha\rangle \Vdash \Sigma$. Now we are to show that $\alpha < \log(\alpha_{k-1})$.

Claim 3.15 $\alpha \leq \log(\alpha_{k-1})$.

Proof. Let $\gamma < \alpha$. Thus, $\langle\alpha\rangle \Vdash \varphi^\gamma_0$. we want to show $\alpha < \gamma$. Since $T^{\alpha_{k-1}}_{k-1} \subset \Gamma$, we have $\langle k-1\rangle\varphi^{\omega^\gamma}_{k-1} \in \Gamma$.

Suppose towards a contradiction that $\alpha_{k-1} = \xi + \omega^\gamma$, where $\gamma < \alpha$. Since $T^{\alpha_{k-1}}_{k-1} \subset \Gamma$, we have in particular that $\neg\langle k-1\rangle\varphi^{\omega^\gamma}_{k-1}\langle 0\rangle\varphi^\xi_{k-1} \in \Gamma$, following the construction in the proof of Lemma 3.5. According to this construction, we have $\varphi^{\omega^\gamma}_{k-1} = \uparrow\varphi^\gamma_{k-1} = \varphi^\gamma_k$. From this, we get $\neg\langle k\rangle(\uparrow\varphi^\gamma_{k-1})\langle 0\rangle\varphi^\xi \in \Gamma$. It follows that $\neg\langle k\rangle\varphi^\gamma_k \in \Gamma$, since $\varphi^\xi \in \Gamma$, hence $\alpha < \gamma$. □

Thus, by Claim 3.2 we have $\langle\alpha\rangle \Vdash \Sigma$, hence $\langle \boldsymbol{\alpha}_{k-1}, \alpha\rangle \Vdash \Gamma'_k$. This proves the lemma. □

With it, we finish the proof of # for arbitrary k. Therefore, setting $\boldsymbol{\alpha} = \langle\alpha_k\rangle_{k<\omega}$ we get $\boldsymbol{\alpha} \Vdash \Gamma$.

□

3.2 Counterexamples

In this subsection, we include the counterexamples for the two incompleteness results in the statement of Theorem 2.12.

Lemma 3.16 *Let* $\Gamma = \{\langle i\rangle\top : i < \omega\}$. *Then,* Γ *is consistent with* GLP*, but for all* $\boldsymbol{\alpha} \in \mathfrak{I}_{<\varepsilon_0}$*, we have* $\mathfrak{I}_{<\varepsilon_0}, \boldsymbol{\alpha} \nVdash \Gamma$.

Proof. It is easy to verify directly that $\mathfrak{I}_{\leq\varepsilon_0}, \langle \varepsilon_0, \varepsilon_0, \varepsilon_0, \ldots\rangle \Vdash \Gamma$, so that indeed Γ is consistent with GLP. However, each ordinal $\alpha < \varepsilon_0$ satisfies $\log^k \alpha = 0$ for some $k < \omega$ and thus each point in $\mathfrak{I}_{<\varepsilon_0}$ satisfies $[k]\bot$ for some k. □

Lemma 3.17 *Let* $\Gamma = \{\langle 0\rangle^k\top : k < \omega\} \cup \{[1]\bot\} \cup \{[0][1]\bot\}$. *Then,* Γ *is consistent with* GLP*, but for all* $\boldsymbol{\alpha} \in \mathrm{ma}(\mathfrak{I}_{\leq\varepsilon_0})$*, we have* $\mathfrak{I}_{<\varepsilon_0}, \boldsymbol{\alpha} \nVdash \Gamma$.

Proof. It is easy to verify directly that $\mathfrak{I}_{\leq\varepsilon_0}, \langle\omega, 0\rangle \Vdash \Gamma$, so that indeed Γ is consistent with GLP. Suppose $\mathfrak{I}_{<\varepsilon_0}, \boldsymbol{\alpha} \Vdash \Gamma$. Then by $\langle 0\rangle^k\top$, we must have $\alpha_0 > k$. By $[1]\bot$, we must have $\alpha_1 = 0$ and by $[0][1]\bot$ we must have $\alpha_0 \leq \omega$, so the only point in $\mathfrak{I}_{\leq\varepsilon_0}$ which satisfies Γ is $\langle\omega, 0\rangle$, which is not on the main axis of $\mathfrak{I}_{\leq\varepsilon_0}$. □

4 Counterexamples to strong completeness for topological spaces

In this section we are to show that strong completeness of GLP_0 (and therefore of GLP) fails for the hitherto studied topological (ordinal) models.

4.1 Topological semantics

We start with the definition of topological models for GL.

Definition 4.1 Given a topological space (X, τ) for each $A \subset X$ we denote $d_\tau A = \{x : \forall U \in \tau \exists y \neq x (y \in U \cap A)\}$. We call d_τ the derivative operator. We omit the index if there's no risk of confusion.

Definition 4.2 A *topological model* is a tuple (X, τ, v), where (X, τ) is a topological space and $v : \mathsf{Vars} \to \mathcal{P}(X)$ is an interpretation of variables. Such an interpretation is extended to arbitrarily formulæ by the following clauses:

- $[\![p]\!] = v(p)$;
- $[\![\neg\varphi]\!] = X \setminus [\![\varphi]\!]$;
- $[\![\varphi \wedge \psi]\!] = [\![\varphi]\!] \cap [\![\psi]\!]$;
- $[\![\Diamond\varphi]\!] = d_\tau[\![\varphi]\!]$;

One can show that GL is valid in a space (X, τ) if and only if it is *scattered*, i.e., if every $A \subset X$ has an isolated point. A natural example of such space is an ordinal with its order topology (also called interval topology). In fact, GL is complete for such semantics and indeed it suffices to restrict to a single ordinal:

Theorem 4.3 (Abashidze [2], Blass [12]) GL *is sound and complete with respect to every ordinal* $\Omega \geq \omega^\omega$ *when equipped with the order topology.*

Similarly, we define the semantics for the polymodal case:

Definition 4.4 A *topological model* is a tuple $(X, \tau_i, v)_{i<\omega}$, where $(X, \tau_i)_{i<\omega}$ is a polytopological space and $v : \mathsf{Vars} \to \mathcal{P}(X)$ is an interpretation. As before, v is readily extended to arbitrary formulæ:

- $[\![p]\!] = v(p)$;
- $[\![\neg\varphi]\!] = X \setminus [\![\varphi]\!]$;
- $[\![\varphi \wedge \psi]\!] = [\![\varphi]\!] \cap [\![\psi]\!]$;
- $[\![\langle i\rangle\varphi]\!] = d_{\tau_i}[\![\varphi]\!]$;

We say that topological space is a *GLP-space* if it is a model of GLP. Generally, GLP can be shown to be strongly complete with respect to GLP-spaces (cf. [26]). However, it is open whether we could obtain strong completeness to some more specific and tame spaces, in particular ordinal spaces. In particular, if we consider certain *canonical* topologies, then regular completeness is already non-trivial, namely independent from ZFC. For instance Beklemishev in [9] has shown consistency of completeness for the canonical topologies for GLB (GLP restricted to only two modalities) are studied, whereas from the results by Blass [12] it follows that it is consistent that GLP is incomplete to any canonical topologies.

4.2 Icard spaces

By Ignatiev's theorem, given a closed formula consistent with GLP, there is a point on the *main axis* of $\mathfrak{I}$ that satisfies it, i.e., for any closed formula φ consistent with GLP there is an ordinal $\alpha < \varepsilon_0$ such that $\hat{\alpha} \Vdash \varphi$. Since the main axis of $\mathfrak{I}$ is wellordered in length ε_0, this motivates the question of whether a natural topology on ε_0 yields completeness for GLP or its closed fragment and leads to the notion of *Icard spaces*:

Definition 4.5 Let Θ_λ denote the space $\langle \lambda, \tau_n : n < \omega\rangle$, where for each $n < \omega$, τ_n is generated by:

$$\{\alpha : \log^m(\alpha) < \beta\} \quad \{\alpha : \log^k \alpha > \beta\}$$

where $m \leq n$, $k < n$.

These spaces provide a complete semantics for the closed fragment of GLP.

Theorem 4.6 ([20]) $\mathsf{GLP}_0 \vdash \varphi$ *if and only if* $\Theta_{\varepsilon_0} \models \varphi$.

This result gives completeness of GLP_0 with respect to ordinal spaces. Icard spaces do not validate all the axioms of GLP, however. The first complete semantics for GLP introduced was given by what are nowadays called *Beklemishev-Gabelaia spaces*. We briefly recall their definition although it will not be used in an essential way in what follows. In order to introduce BG spaces we define certain operations on topologies:

Definition 4.7 Given a scattered space (X, τ) we call τ^+ the topology generated by $\tau \cup \{d_\tau A : A \subset X\}$.

The derivative operation $d = d_\tau$ can be iterated transfinitely by putting $d^{\alpha+1}(A) = d(d^\alpha(A))$ and $d^\lambda(A) = \bigcap_{\alpha<\lambda}(A)$ at limit stages.

Definition 4.8 Let (X, τ) be a scattered space. We define the rank function $\rho_\tau : X \to \mathsf{Ord}$ by $\rho_\tau(x) = \inf\{\alpha : x \notin d^{\alpha+1}X\}$.

Definition 4.9 Let (X, τ) be a scattered space. We say that σ is an ℓ-extension of τ if $\tau \subset \sigma$, σ and τ have the same rank function, but σ adds no new neighborhoods at points of successor rank.

Below, we speak of maximal ℓ-extensions of some τ: these are ℓ-extensions of τ which have no proper ℓ-extension.

Definition 4.10 Let $\lambda \in \mathsf{Ord}$. We call $(\lambda, \sigma_n : n < \omega)$ a Beklemishev-Gabelaia space (or BG-space) if σ_0 is a maximal ℓ-extension of the order topology on λ and, for each n, σ_{n+1} is a maximal ℓ-extension of σ_n^+.

It can be inductively checked for any BG-space, the rank functions are given by $\rho_{\sigma_n} = \log^{n+1}$. Hence, the ranks of points in BG-spaces are the same as in Icard spaces.

4.3 Counterexamples to strong completeness

Theorem 4.11 GLP_0 *is not strongly complete with respect to any Icard space.* GLP *is not strongly complete with respect to any Beklemishev-Gabelaia space.*

Proof. The counterexample is the same as before. We let

$$\Gamma = \{\langle 0\rangle^k \top : k < \omega\} \cup \{[1]\bot\} \cup \{[0][1]\bot\}.$$

We had seen in the proof of Theorem 2.12 that this set is consistent with GLP. Suppose towards a contradiction that $\alpha \Vdash \Gamma$ for some ordinal α in some Icard space (the proof for Beklemishev-Gabelaia spaces is the same). Write α in the form

$$\alpha = \beta + \omega^\gamma.$$

Since $\alpha \Vdash [0][1]\bot$, it follows that all ordinals α' sufficiently close to α satisfy $\log^2(\alpha') = 0$, which implies $\gamma = \log(\alpha) \leq \omega$. Since $\alpha \Vdash [1]\bot$, we must have $\log^2(\alpha) = 0$, so γ is finite. Hence, α is of the form $\beta + \omega^k$ for some k and thus satisfies $[0]^{k+1}\bot$, which is a contradiction. □

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A Proof of Claim 3.6

Towards proving Claim 3.6 we recall a certain construction that was introduced in [7]. Given a worm φ we one can assign an ordinal $o_n(\varphi) < \varepsilon_0$ with the property that for any pair of worms $\varphi, \psi \in \mathcal{L}_{[n;\omega)}$, $\mathsf{GLP} \vdash \varphi \to \langle n \rangle \psi$ implies $o_n(\varphi) > o_n(\psi)$. The definition is the following:

- $o_n(\langle n \rangle^k \top) = k$;
- $o_n(\varphi_0 \langle n \rangle \dots \langle n \rangle \varphi_n) = \omega^{o_{n+1}(\varphi_n)} + \dots + \omega^{o_{n+1}(\varphi_0)}$, where $\varphi_i \in \mathcal{L}_{[n+1;\omega)}$ for each $i \leq n$;

Note that $o_n(\varphi) = o_{n+1}(\uparrow\varphi)$.

Claim A.1 (Beklemishev) *For any pair of worms φ, ψ in the language $\mathcal{L}_{[n;\omega)}$, exactly one of the following holds:*

(i) $\mathsf{GLP} \vdash \varphi \to \langle n \rangle \psi$,

(ii) $\mathsf{GLP} \vdash \psi \to \langle n \rangle \varphi$,

(iii) $\mathsf{GLP} \vdash \varphi \leftrightarrow \psi$.

Moreover, which of the three alternatives occurs is determined by comparing the ordinals $o_n(\varphi)$ and $o_n(\psi)$.

Proof. In [7], Beklemishev identifies a particular collection of worms called *normal forms*. It is shown that each worm is equivalent to a unique normal form and that the set of normal forms is isomorphic to $(\varepsilon_0, <)$ under the ordering $\varphi <_0 \psi :\leftrightarrow \mathsf{GLP} \vdash \psi \to \langle 0 \rangle \varphi$. Moreover, the transformation to normal forms preserves the ordinal $o_0(\varphi)$. More generally, normal forms in the language $\mathcal{L}_{[n;\omega)}$ are isomorphic to $(\varepsilon_0, <)$ under the ordering $\varphi <_n \psi :\leftrightarrow \mathsf{GLP} \vdash \psi \to \langle n \rangle \varphi$ and moreover, the transformation of worms in $\mathcal{L}_{[n;\omega)}$ to normal forms preserves the ordinal $o_n(\varphi)$. The claim follows from this easily. □

The proof of Claim A.1 shows that if $o_n(\varphi) < o_n(\psi)$, then $\mathsf{GLP} \vdash \psi \to \langle n \rangle \varphi$, provided that φ and ψ are of the form $\uparrow^n \chi$ for some χ, i.e., that φ and ψ belong to $\mathcal{L}_{[n;\omega)}$. We now prove Claim 3.6, which we restate for convenience:

Claim 3.6 *For each $\alpha < \varepsilon_0$, φ_n^α is well-defined in the sense that for any $\beta, \gamma, \delta, \zeta$ such that $\alpha = \beta + 1 + \gamma = \delta + 1 + \zeta$ we have $\mathsf{GLP} \vdash \varphi_n^\gamma \langle n \rangle \varphi_n^\beta \leftrightarrow \varphi_n^\zeta \langle n \rangle \varphi_n^\delta$. Moreover, for each $\alpha > \beta$, $\mathsf{GLP} \vdash \varphi_n^\alpha \to \langle n \rangle \varphi_n^\beta$.*

Proof. It is sufficient to show that $o_n(\varphi_n^\alpha) = \alpha$ for each $\alpha < \varepsilon_0$. If so, then the claim immediately from Claim A.1. We do this by induction.

(i) When $\alpha < \omega$ or when $\alpha = \omega^\beta$, the identity follows immediately from the fact that $o_n(\varphi) = o_{n+1}(\uparrow\varphi)$ for all φ.

(ii) Suppose $\alpha = \beta + 1 + \gamma$ with $\beta \geq \gamma$. Write $\beta = \beta_1 + 1 + \dots + 1 + \beta_k$ and $\gamma = \gamma_1 + 1 + \dots + 1 + \gamma_m$ where each term is either infinite and additively

indecomposable or else equal to zero. Then,

$$\begin{aligned}
o_n\left(\varphi_n^\alpha\right) &= o_n\left(\varphi_n^{\beta+1+\gamma}\right) \\
&= o_n\left(\uparrow^n \varphi_0^{\beta+1+\gamma}\right) && \text{definition of } \varphi_n, \\
&= o_0\left(\varphi_0^{\beta+1+\gamma}\right) && o_n(\varphi) = o_{n+1}(\uparrow\varphi), \\
&= o_0\left(\varphi_0^{\beta_1+1+\cdots+1+\beta_k+1+\gamma_1+\cdots+1+\gamma_m}\right) \\
&= o_0\left(\varphi_0^{\gamma_m}\langle 0\rangle \ldots \langle 0\rangle \varphi_0^{\gamma_1}\langle 0\rangle \varphi_0^{\beta_k}\langle 0\rangle \ldots \langle 0\rangle \varphi_0^{\beta_1}\right) && \text{definition of } \varphi_0 \\
&= \omega^{o_1(\varphi_0^{\beta_1})} + \cdots + \omega^{o_1(\varphi_0^{\beta_k})} + \omega^{o_1(\varphi_0^{\gamma_1})} + \cdots + \omega^{o_1(\varphi_0^{\gamma_m})} && \text{definition of } o_0
\end{aligned}$$

Now, each term $\xi = \beta_i$ or $\xi = \gamma_i$ is either equal to 0, in which case $o_1(\varphi_0^\xi) = 0$ and thus $\omega^{o_1(\varphi_0^\xi)} = 1$; or otherwise is infinite and additively indecomposable ordinal and thus of the form ω^ζ for some ζ. If so, then

$$\omega^{o_1(\varphi_0^\xi)} = \omega^{o_1(\varphi_0^{\omega^\zeta})} = \omega^{o_1(\uparrow\varphi_0^\zeta)} = \omega^{o_0(\varphi_0^\zeta)} = \omega^\zeta = \xi = 1 + \xi$$

where the third-to-last equality makes use of the induction hypothesis. Applying this to each term, we have:

$$\begin{aligned}
o_n(\varphi_n^\alpha) &= \omega^{o_1(\varphi_0^{\beta_1})} + \cdots + \omega^{o_1(\varphi_0^{\beta_k})} + \omega^{o_1(\varphi_0^{\gamma_1})} + \cdots + \omega^{o_1(\varphi_0^{\gamma_m})} \\
&= \beta_1 + 1 + \cdots + 1 + \beta_k + 1 + \gamma_1 + 1 + \cdots + 1 + \gamma_n \\
&= \beta + 1 + \gamma = \alpha,
\end{aligned}$$

as desired.

This proves the claim. □