An Epistemic Interpretation of Tensor Disjunction

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Abstract

This paper aims to give an epistemic interpretation to the *tensor disjunction* in dependence logic, through a rather surprising connection to the so-called *weak disjunction* in Medvedev's early work on intermediate logic under the Brouwer-Heyting-Kolmogorov (BHK)-interpretation. We expose this connection in the setting of inquisitive logic with tensor $\mathbf{InqB}^{\otimes}$ [7], but from an epistemic perspective. More specifically, we translate the propositional formulas of $\mathbf{InqB}^{\otimes}$ into modal formulas in a powerful epistemic language of knowing how following the proposal by [21,18]. We give a complete axiomatization of the logic of our full language based on Fine's axiomatization of $\mathsf{S5}$ modal logic with propositional quantifiers. Finally we generalize the tensor operator with parameters k and n, which intuitively captures the epistemic situation that one has n potential answers to n questions and knows that at least k of them must be correct. The original tensor disjunction is the special case when $k = 1$ and $n = 2$. We show that adding the generalized tensor operators do not increase the expressive power of our logic, inquisitive logic and propositional dependence logic, though most of these generalized tensors are not uniformly definable in these logics, except in ours.

1 Introduction

As a rapidly growing field of research, *Dependence Logic* studies reasoning patterns expressed by logical languages extended with (in)dependence atoms (cf. e.g., [11] for a survey). The intuitive meaning of the atomic formula is best fleshed out formally by the *team semantics* capturing the (in)dependence between variables, where a *team* can be viewed as a collection of assignments or possible worlds. In defining the truth conditions for logical connectives, one guideline is to keep the property of *flatness*, i.e., for any formula α without the (in)dependence atoms, it is true w.r.t. a team X ($X \vDash \alpha$) if it is true on each singleton team $\{w\}$ such that $w \in X$. To some extent, flatness preserves the intuition of the classical logical connectives. In particular, the semantics of the distinct tensor disjunction $\otimes$ in dependence logic can be viewed as a natural lifting of the world-based semantics for classical disjunction to teams, viewed as *sets* of possible worlds:

$$X \vDash \alpha \otimes \beta \text{ iff there are } U, V \subseteq X \text{ such that } X \subseteq U \cup V,\ U \vDash \alpha \text{ and } V \vDash \beta$$

Note that a disjunction $\alpha \vee \beta$ is classically true on each world in a set X of possible worlds if and only if there are two subsets jointly covering the whole space of possible worlds such that one subset satisfies α homogeneously and the other satisfies β homogeneously. This lifting may also give the impression that $\otimes$ can be read more or less as a classical disjunction. However, it is not so straightforward. For example, the truth of the propositional dependence formula $(=(p, q) \otimes =(p, q))$ over a team is not equivalent to $=(p, q)$. In fact, $(=(p, q) \otimes =(p, q))$ is valid technically but $=(p, q)$, which says that the truth value of q depends on the truth value of p, is clearly *not* valid. A natural question arises: how to understand $\otimes$ intuitively and precisely? [1] Our work proposes a possible epistemic understanding of $\otimes$ (and its generalizations) from a Brouwer-Heyting-Kolmogorov (BHK)-like perspective to be explained below.

The initial idea is based on an unexpected connection between the tensor disjunction and the so-called *weak disjunction* in Medvedev's early work [14] on the *problem semantics* of intuitionistic logic, following Kolmogorov's problem-solving interpretation [13]. This connection is best exposed in the setting of *inquisitive logic* with tensor disjunction discussed in [7], since inquisitive logic has intimate connections with both the propositional dependence logic [25,4] and Medvedev's logic [9]. More specifically, various versions of propositional dependence logic can be viewed as disguised inquisitive logic, e.g., the dependence atom $=(p, q)$ becomes $(p \vee \neg p) \to (q \vee \neg q)$ in inquisitive logic [23,25,6]. On the other hand, Medvedev's logic is the substitution-closed core of inquisitive logic **InqB** that also admits a BHK-like interpretation via resolutions [3,9]. [2] Another advantage of using inquisitive logic as the "medium" is that we can put classical, intuitionistic, and tensor disjunctions in the same picture to reveal their differences. The last missing piece for an intuitive reading of tensor is an epistemic interpretation that can incorporate the BHK-interpretation. Wang proposed to capture intuitionistic truth using a modality Kh to express *knowing how to prove/solve* [21], which reflects Heyting's often-overlooked early view of intuitionistic logic as an epistemic logic [12]. This also led to an *alternative* epistemic interpretation of inquisitive logic [18], where a state supports a formula α is rendered as *it is known how α is resolved* when viewing the state as a set of possible worlds capturing the epistemic uncertainty. This can give us alternative epistemic readings of inquisitive formulas, e.g., the excluded middle $p \vee \neg p$ in inquisitive logic is first rendered as $\mathsf{Kh}(p \vee \neg p)$ (knowing how $p \vee \neg p$ is true), then it can be reduced to $\mathsf{Kh}p \vee \mathsf{Kh}\neg p$ (knowing how p is true or knowing

[1] In [17], it is suggested that the (in)dependence formulas can be viewed as *types* of teams, i.e., each formula specifies a property of the team. The truth conditions of connectives also have their roots in the game semantics for classical and IF logic [17].

[2] In recent literature, inquisitive logic is also viewed as an extension of classical logic [4].

how $\neg p$ is true), and finally it is equivalent to the intuitively invalid $\mathsf{K}p \vee \mathsf{K}\neg p$ [18]. We will also see these reductions later in this work.

Now we are ready to give the epistemic interpretation of tensor disjunction. According to Medvedev's problem semantics [14], the *weak disjunction* $\alpha \sqcup \beta$ captures a composite problem where the solutions are pairs of *potential* solutions to the problems of α and β respectively such that *at least one* solution in each pair is correct. [3] From our epistemic perspective, Medvedev's truth concept for a formula γ means it is known how to solve γ. In particular, a weak disjunction $\alpha \sqcup \beta$ is true w.r.t. a set of possible worlds (i.e., a team or a *state* in inquisitive logic) iff there are two solutions r_1 and r_2 such that it is known that one of r_1 and r_2 is a correct solution to the corresponding problems. In the setting of inquisitive logic, instead of problems, we take formulas as statements or questions, which may have *resolutions* instead of solutions. We will show such a truth condition amounts to exactly the team semantics for tensor.

We first summarize what we are going to do in the paper before diving into the technical details. After introducing inquisitive logic with tensor disjunction $\mathbf{InqB}^{\otimes}$ in Section 2, we first propose in Section 3 a logical language of know-that and know-how, with extra machinery of announcements and propositional quantifiers, interpreted over epistemic models that are essentially states/teams in the literature. The semantics of the know-how operator Kh is given by using the $\exists x\mathsf{K}$ schema as in know-wh logics [20], based on a BHK-like interpretation following Medvedev's idea. The intention is to capture the alternative epistemic meaning of an $\mathbf{InqB}^{\otimes}$ formula α as knowing how to resolve α. In Section 4, we show that valid know-how formulas correspond exactly the theorems in $\mathbf{InqB}^{\otimes}$. Moreover, we also show that the announcements and propositional quantifiers facilitate a recursive process to "open up" the know-how formulas, in particular to decode the $\otimes$, and eventually translate them into classical ones free of the know-how operator. Based on such a process we give a complete axiomatization of our full dynamic epistemic logic in Section 5. Finally, in Section 6 we generalize the idea of the tensor, from our epistemic interpretation, to obtain a spectrum of n-ary disjunctions $\otimes^k_n$, which captures the interesting epistemic situation of knowing n potential answers to n questions and being sure at least k of them must be correct. We show that adding the generalized tensor operators does not increase the expressive power of our logic, the inquisitive logic and propositional dependence logic, though most of these generalized tensors are not uniformly definable in these logics. In contrast, we can uniformly define the generalize tensors in our epistemic language.

2 Inquisitive logic with tensor

Following [7], we introduce the language and semantics of *Inquisitive Logic with Tensor Disjunction* ($\mathbf{InqB}^{\otimes}$). In contrast with [7], we use the symbol $\vee$ for the inquisitive disjunction and adopt the model-based semantics as in [5]. Throughout the paper, we fix a countable set $\mathbf{P}$ of proposition letters.

[3] See [2], for the corresponding Kripke semantics of weak disjunction.

Definition 2.1 (Language $\mathbf{PL}^{\otimes}$) *The language of propositional logic with tensor disjunction ($\mathbf{PL}^{\otimes}$) is defined as follows:*

$$\alpha ::= p \mid \bot \mid (\alpha \wedge \alpha) \mid (\alpha \vee \alpha) \mid (\alpha \to \alpha) \mid (\alpha \otimes \alpha)$$

where $p \in \mathbf{P}$. We write $\neg\alpha$ for $\alpha \to \bot$, $\top$ and $\alpha \leftrightarrow \beta$ are defined as usual.

Definition 2.2 (Model and state) *A model is a pair $\mathcal{M} = \langle W, V \rangle$ where:*

- *W is a non-empty set of possible worlds;*[4]
- *$V : \mathbf{P} \to \wp(W)$ is a valuation function.*

A state *(or, say, a* team*) s in $\mathcal{M}$ is a subset of W.*

We will also view these models as ***epistemic models*** for our dynamic epistemic language to be introduced in Section 3.

Given $\mathcal{M}$, we refer to its components as $W_{\mathcal{M}}$ and $V_{\mathcal{M}}$. We write $w \in \mathcal{M}$ in case that $w \in W_{\mathcal{M}}$, and $\mathcal{M}' \subseteq \mathcal{M}$ in case that $W'_{\mathcal{M}} \subseteq W_{\mathcal{M}}$. The semantics is defined through the *support relation* between *states* (in models) and formulas.

Definition 2.3 (Support [7]) *The* support *relation $\Vdash$ is defined inductively:*

$\mathcal{M}, s \Vdash p$	*iff*	$\forall w \in s, w \in V(p)$
$\mathcal{M}, s \Vdash \bot$	*iff*	$s = \varnothing$
$\mathcal{M}, s \Vdash (\alpha \wedge \beta)$	*iff*	$\mathcal{M}, s \Vdash \alpha$ *and* $\mathcal{M}, s \Vdash \beta$
$\mathcal{M}, s \Vdash (\alpha \vee \beta)$	*iff*	$\mathcal{M}, s \Vdash \alpha$ *or* $\mathcal{M}, s \Vdash \beta$
$\mathcal{M}, s \Vdash (\alpha \to \beta)$	*iff*	$\forall t \subseteq s :$ *if* $\mathcal{M}, t \Vdash \alpha$ *then* $\mathcal{M}, t \Vdash \beta$
$\mathcal{M}, s \Vdash (\alpha \otimes \beta)$	*iff*	*there exist two sets* $t \subseteq s$ *and* $t' \subseteq s$ *such that* $\mathcal{M}, t \Vdash \alpha$, $\mathcal{M}, t' \Vdash \beta$, *and* $t \cup t' = s$.

A formula α is valid *if it is supported by any state in any model.*

Here are some simple properties.

Proposition 2.4 (Downward closure) *For any $\alpha \in \mathbf{PL}^{\otimes}$, if $\mathcal{M}, s \Vdash \alpha$ then $\mathcal{M}, t \Vdash \alpha$ for any $t \subseteq s$. Moreover, $\mathcal{M}, \varnothing \Vdash \alpha$ for all $\alpha \in \mathbf{PL}^{\otimes}$.*

Proposition 2.5 *For any $\alpha \in \mathbf{PL}^{\otimes}$, $\mathcal{M}, s \Vdash \alpha$ implies $\mathcal{M}', s \Vdash \alpha$ for any $\mathcal{M}' \subseteq \mathcal{M}$ such that $s \subseteq \mathcal{M}'$. Conversely, if $\mathcal{M}', s \Vdash \alpha$ then $\mathcal{M}, s \Vdash \alpha$ given $\mathcal{M}' \subseteq \mathcal{M}$. Namely, the support relation only depends on the state.*

Definition 2.6 *Inquisitive Logic with Tensor Disjunction ($\mathbf{InqB}^{\otimes}$) is the set of valid $\mathbf{PL}^{\otimes}$ formulas under the support relation.*

3 A dynamic epistemic language

Definition 3.1 (Language $\mathbf{PALKh\Pi}$) *The language of* Public Announcement Logic with Know-how Operator and Propositional Quantifier *is:*[5]

$$\varphi ::= p \mid \bot \mid (\varphi \wedge \varphi) \mid (\varphi \vee \varphi) \mid (\varphi \otimes \varphi) \mid (\varphi \to \varphi) \mid \mathsf{K}\varphi \mid \mathsf{Kh}\alpha \mid \forall p\varphi \mid [\varphi]\varphi$$

where $p \in \mathbf{P}$ and $\alpha \in \mathbf{PL}^{\otimes}$. We write $\neg\varphi$ for $\varphi \to \bot$, $\widehat{\mathsf{K}}$ for $\neg\mathsf{K}\neg$, $\exists p$ for $\neg\forall p\neg$ for all $p \in \mathbf{P}$ and $\langle\varphi\rangle$ for $\neg[\varphi]\neg$ for all $\varphi \in \mathbf{PALKh\Pi}$.

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[4] In [8], the world set W could be empty. The distinction is not technically significant.

[5] Π in the name $\mathbf{PALKh\Pi}$ denotes propositional quantifiers as in the literature [10].

Intuitively, $\mathsf{K}\varphi$ expresses "the agent *knows that* φ", $\mathsf{Kh}\alpha$ says that "the agent *knows how* to resolve α" or simply "the agent *knows how* α is true", $\forall p\varphi$ says that "for any proposition p, φ holds" and $[\varphi]\psi$ means that "after announcing φ, ψ holds". Note that Kh only allows $\mathbf{PL}^{\otimes}$-formula α in its scope. For instance, we can express $\mathsf{K}\neg\mathsf{Kh}\alpha$ but not $\mathsf{Kh}\mathsf{K}\alpha$ in **PALKhΠ**. We write $\varphi[\psi/\chi]$ for any formula obtained by replacing one or several occurrences of ψ with χ in φ.

We view the models in Definition 2.2 as *epistemic models* where the implicit epistemic relation is the total relation. The semantics of **PALKhΠ** is given on such models, with the notions of *resolution space* and *resolution* as below.

Definition 3.2 (Resolution space) *S is a function assigning each $\alpha \in \mathbf{PL}^{\otimes}$ its (non-empty) set of potential resolutions:*

$$S(p) = \{p\},\ \textit{for } p \in \mathbf{P} \qquad S(\bot) = \{\bot\}$$
$$S(\alpha \vee \beta) = (S(\alpha) \times \{0\}) \cup (S(\beta) \times \{1\}) \qquad S(\alpha \to \beta) = S(\beta)^{S(\alpha)}$$
$$S(\alpha \wedge \beta) = S(\alpha) \times S(\beta) \qquad S(\alpha \otimes \beta) = S(\alpha) \times S(\beta)$$

Resolution spaces reflect the BHK-interpretation, e.g., a possible resolution of an implication is a function transforming resolutions of the antecedent into resolutions of the consequent. [6] Note that resolution spaces for atomic propositions are singletons, based on the assumption in inquisitive semantics that atomic propositions are statements without inquisitiveness. The set of *actual* resolutions of each formula on each world in a given model is a (possibly empty) subset of the corresponding resolution space, as defined below.

Definition 3.3 (Resolution in model) *Given $\mathcal{M}$, $R : W_{\mathcal{M}} \times \mathbf{PL}^{\otimes} \to \bigcup_{\alpha \in \mathbf{PL}^{\otimes}} S(\alpha)$ gives the (actual) resolutions for each formula on each world:*

$$R(w,\bot) = \varnothing \qquad R(w,p) = \begin{cases} \{p\} & \textit{if } w \in V_{\mathcal{M}}(p) \\ \varnothing & \textit{otherwise} \end{cases}$$
$$R(w, \alpha \vee \beta) = (R(w,\alpha) \times \{0\}) \cup (R(w,\beta) \times \{1\})$$
$$R(w, \alpha \wedge \beta) = R(w,\alpha) \times R(w,\beta)$$
$$R(w, \alpha \to \beta) = \{f \in S(\beta)^{S(\alpha)} \mid f[R(w,\alpha)] \subseteq R(w,\beta)\}$$
$$R(w, \alpha \otimes \beta) = (R(w,\alpha) \times S(\beta)) \cup (S(\alpha) \times R(w,\beta))$$

Important notation: *For $U \subseteq W_{\mathcal{M}}$, we write $R(U,\alpha)$ for $\bigcap_{w \in U} R(w,\alpha)$.*

While $S(\bot) = \{\bot\}$ is non-empty, it never has any actual resolution on specific worlds. For any $p \in \mathbf{P}$, p has itself as its resolution iff it is true on w. For any implication $\alpha \to \beta \in \mathbf{PL}^{\otimes}$, each of its resolution on w is a function in $S(\alpha \to \beta)$ which maps an actual resolution of α to an actual resolution of β on w. Following the idea of the weak disjunction introduced in [14], each resolution for $\alpha \otimes \beta \in \mathbf{PL}^{\otimes}$ on w is a pair of resolutions in $S(\alpha \otimes \beta)$, such that *at least one* in the pair is actual on w for the corresponding formula.

[6] See [18] for a more detailed explaination (without $\otimes$). The definition of resolutions is based on Medvedev's problem semantics [14]. A similar definition can be found in [15].

Note that by definition, $R(w, \alpha) \subseteq S(w, \alpha) \neq \varnothing$.

Proposition 3.4 *For any $\alpha \in \mathbf{PL}^{\otimes}$, $S(\alpha) \neq \varnothing$ and $S(\alpha)$ is finite.*

Below is a useful observation on the resolution of negations ($\neg\alpha := \alpha \to \bot$).

Proposition 3.5 ([18]) *For any $\mathcal{M}, w$, any α, $R(w, \neg\alpha)$ is either $\varnothing$ or a fixed singleton set independent from w, and $R(w, \neg\alpha) = \varnothing$ iff $R(w, \alpha) \neq \varnothing$.*

Let $\mathbf{P}(\alpha)$ be the set of propositional letters occurring in α and let $V^{\alpha}_{\mathcal{M}}(w)$ be the collection of $p \in \mathbf{P}(\alpha)$ that are true on w in $\mathcal{M}$. Proposition 3.6 says that $R(w, \alpha)$ only depends on the relevant valuation on w itself.

Proposition 3.6 *For any $\mathcal{M}, w$ and $\mathcal{N}, v$, for all $\alpha \in \mathbf{PL}^{\otimes}$, if $V^{\alpha}_{\mathcal{M}}(w) = V^{\alpha}_{\mathcal{N}}(v)$, then $R(w, \alpha) = R(v, \alpha)$.*

Now we define the satisfaction relation of **PALKhΠ** on *pointed models*, i.e, a model with a designated world, in contrast with the state-based support-semantics. Note that the connectives *outside* the scope of Kh are classical, e.g., $\otimes$ just functions as a classical disjunction. K is the standard epistemic modality of know-that. The semantics for $\mathsf{Kh}\alpha$ following the $\exists x\mathsf{K}$ schema in [20,19] via resolutions, and is intended to capture the know-how interpretation of $\mathbf{InqB}^{\otimes}$. $\forall p$ is a propositional quantifier over the full power set of $W_{\mathcal{M}}$. The semantics of the dynamic operator $[\psi]$ is as in public announcement logic [16].

Definition 3.7 (Semantics) *For $\varphi, \psi \in$ **PALKhΠ**, $\alpha \in \mathbf{PL}^{\otimes}$ and $\mathcal{M}, w$ where $\mathcal{M} = \langle W, V\rangle$, the semantics is defined as below ($\bigcirc \in \{\vee, \otimes\}$):*

$\mathcal{M}, w \nvDash \bot$		
$\mathcal{M}, w \vDash p$	$\iff$	$w \in V(p)$
$\mathcal{M}, w \vDash (\varphi \bigcirc \psi)$	$\iff$	$\mathcal{M}, w \vDash \varphi$ *or* $\mathcal{M}, w \vDash \psi$
$\mathcal{M}, w \vDash (\varphi \wedge \psi)$	$\iff$	$\mathcal{M}, w \vDash \varphi$ *and* $\mathcal{M}, w \vDash \psi$
$\mathcal{M}, w \vDash (\varphi \to \psi)$	$\iff$	$\mathcal{M}, w \vDash \varphi$ *implies* $\mathcal{M}, w \vDash \psi$
$\mathcal{M}, w \vDash \mathsf{K}\varphi$	$\iff$	*for any* $v \in \mathcal{M}, \mathcal{M}, v \vDash \varphi$
$\mathcal{M}, w \vDash \mathsf{Kh}\alpha$	$\iff$	*there is an* $x \in S(\alpha)$ *s.t. for any* $v \in \mathcal{M}, x \in R(v, \alpha)$
$\mathcal{M}, w \vDash \forall p\varphi$	$\iff$	*for any* $U \in \wp(W), \mathcal{M}[p \mapsto U], w \vDash \varphi$
$\mathcal{M}, w \vDash [\psi]\varphi$	$\iff$	$\mathcal{M}, w \vDash \psi$ *implies* $\mathcal{M}\vert_{[\![\psi]\!]}, w \vDash \varphi$

where:

- *Given $U \in \wp(W_{\mathcal{M}})$ and $p \in \mathbf{P}$, recall that $\mathcal{M}[p \mapsto U] = \langle W, V'\rangle$, where the assignment V' assigns U to p and coincides with V on all other atoms; and*
- *$[\![\psi]\!]_{\mathcal{M}} = \{w \in W_{\mathcal{M}} \mid \mathcal{M}, w \vDash \psi\}$ and $\mathcal{M}\vert_X$ is the submodel of $\mathcal{M}$ by restricting to $\varnothing \neq X \subseteq W_{\mathcal{M}}$. Thus $\mathcal{M}\vert_{[\![\psi]\!]_{\mathcal{M}}}$ is the submodel restricted to the worlds satisfying ψ in $\mathcal{M}$. We also write $\mathcal{M}\vert_{[\![\psi]\!]_{\mathcal{M}}}$ as $\mathcal{M}\vert_{\psi}$ for brevity.*

Validity and entailment are defined as usual.

In [18], to handle the implication in the know-how scope, we have a dynamic operator $\Box$ such that $\Box\varphi$ says that "given any information updates, φ holds". This can be expressed by $\forall p[p]\varphi$ in our language, given that p is not free in φ.

We write $\mathcal{M} \vDash \varphi$ iff $\mathcal{M}, w \vDash \varphi$ for all $w \in W_{\mathcal{M}}$. Clearly, $\mathcal{M}, w \vDash \mathsf{Kh}\alpha$ iff $\mathcal{M} \vDash \mathsf{Kh}\alpha$ and $\mathcal{M}, w \vDash \mathsf{K}\varphi$ iff $\mathcal{M} \vDash \varphi$. The rephrased truth condition of Kh

below says that $\mathsf{Kh}\alpha$ holds on a (pointed) model as long as there is a uniform resolution for α on that model, where we define $R(U, \alpha)$ as $\bigcap_{w \in U} R(w, \alpha)$.

$$\boxed{\mathcal{M} \vDash \mathsf{Kh}\alpha \iff \mathcal{M}, w \vDash \mathsf{Kh}\alpha \iff R(W_{\mathcal{M}}, \alpha) \neq \varnothing}$$

As in [18], we can give a uniform alternative truth condition for $\mathbf{PL}^{\otimes}$-formulas via the existence of actual resolutions.

Proposition 3.8 *For any* $\alpha \in \mathbf{PL}^{\otimes}$ *and* $\mathcal{M}, w$*,* $\mathcal{M}, w \vDash \alpha \iff R(w, \alpha) \neq \varnothing$*.*

Proof. We prove by induction on the structure of α. We only show the cases for $\otimes$. The other cases can be found in [18].

$$\begin{aligned}
\mathcal{M}, w \vDash (\alpha \otimes \beta) &\iff \mathcal{M}, w \vDash \alpha \text{ or } \mathcal{M}, w \vDash \beta \\
&\iff R(w, \alpha) \neq \varnothing \text{ or } R(w, \beta) \neq \varnothing \\
&\iff \text{ there is an } x \in R(w, \alpha) \text{ or there is a } y \in R(w, \beta) \\
&\iff \exists \langle x, y' \rangle \in R(w, \alpha \otimes \beta) \text{ s.t. } x \in R(w, \alpha) \wedge y' \in S(\beta) \\
&\qquad \text{or } \exists \langle x', y \rangle \in R(w, \alpha \otimes \beta) \text{ s.t. } x' \in S(\alpha) \wedge y \in R(w, \beta) \\
&\iff R(w, \alpha \otimes \beta) \neq \varnothing
\end{aligned}$$

□

From Proposition 3.8, we see that in propositional formulas, both $\vee$ and $\otimes$ collapse to the classical disjunction outside the scope of Kh. Yet $\otimes$ is weaker than $\vee$ in the way that we can construct a resolution of $\alpha \otimes \beta$ from that of $\alpha \vee \beta$. It also follows from Proposition 3.8 that for any $\alpha \in \mathbf{PL}^{\otimes}$, $\mathcal{M}, w \vDash \mathsf{K}\alpha$ iff for each $v \in \mathcal{M}$, there is *some* resolution for α on v, in the shape of $\mathsf{K}\exists x$. In contrast, $\mathcal{M}, w \vDash \mathsf{Kh}\alpha$ iff there is a *uniform* resolution for α on $\mathcal{M}$ in the shape of $\exists x \mathsf{K}$. The following is then immediate.

Proposition 3.9 $\mathsf{Kh}\alpha \to \mathsf{K}\alpha$ *is valid for all* $\alpha \in \mathbf{PL}^{\otimes}$*.*

Since each $p \in \mathbf{P}$ only has one possible resolution, when each point has a resolution for p, the model has a uniform one. Thus we have Proposition 3.10

Proposition 3.10 $\mathsf{Kh}p \leftrightarrow \mathsf{K}p$ *is valid for all* $p \in \mathbf{P}$*.*

Based on the semantics, we can have more intuitive readings of the formulas in inquisitive logic, e.g., $\mathsf{Kh}(p \vee \neg p)$ is equivalent to $\mathsf{Kh}p \vee \mathsf{Kh}\neg p$ and $\mathsf{K}p \vee \mathsf{K}\neg p$. This also explains the failure of *excluded middle* in inquisitive logic (cf. [18] for more discussions).

The rule of *replacement of equals* is not valid in general, for instance, although $(p \vee \neg p) \leftrightarrow (p \to p)$ is valid, $\mathsf{Kh}(p \vee \neg p) \leftrightarrow \mathsf{Kh}(p \to p)$ is not. However, if we only allow substitution to happen outside the scope of Kh operators, the rule becomes valid. It is not hard to verify the following:

Proposition 3.11 *For any* $\varphi, \psi, \chi \in \mathbf{PALKh\Pi}$*, the validity of* $\varphi \leftrightarrow \psi$ *implies the validity of* $\chi[\varphi/\psi] \leftrightarrow \chi$*, given that the substitution does not happen in the scope of* Kh*.*

4 Expressivity

Let **PALΠ** be the Kh-free fragment of **PALKhΠ**, **ELΠ** be the $[\varphi]$-free fragment of **PALΠ** and **EL** be the $\forall p$-free fragment of **ELΠ**. In Subsection 4.1, we show Kh and $[\varphi]$ can be eliminated, thus making **PALKhΠ**, **PALΠ** and **ELΠ** equally expressive. In Subsection 4.2, we show that the valid Kh formulas of **PALKhΠ** correspond to $\mathbf{InqB}^{\otimes}$ precisely.

4.1 Reduction

We introduce the reduction schemata to eliminate the Kh modality, which will also be used in the proof system to be introduced later. First, we have the following observation.

Proposition 4.1 *For any $\alpha_1, \alpha_2 \in \mathbf{PL}^{\otimes}$ where p_1 and p_2 do not occur free, for any pointed model $\mathcal{M}, w$, $\mathcal{M}, w \vDash \exists p_1 \exists p_2 \mathsf{K}((p_1 \otimes p_2) \wedge [p_1]\mathsf{Kh}\alpha_1 \wedge [p_2]\mathsf{Kh}\alpha_2)$ iff there are $U_1, U_2 \subseteq W_{\mathcal{M}}$ s.t. $U_1 \cup U_2 = W_{\mathcal{M}}$ and $U_i \neq \varnothing$ implies $R(U_i, \alpha_i) \neq \varnothing$ for $i = 1, 2$.*

Proof. Given a $U \subseteq W_{\mathcal{M}}$, for any $w \in \mathcal{M}$, $\mathcal{M}[p \mapsto U], w \vDash p \iff w \in U$ $(\star)$. For brevity, we write $\exists U$ for there exists $U \in W_{\mathcal{M}}$. Recall that $\mathcal{M}|_U$ denotes the submodel of $\mathcal{M}$ restricted to U, if U is non-empty (otherwise undefined).

$\mathcal{M}, w \vDash \exists p_1 \exists p_2 \mathsf{K}((p_1 \otimes p_2) \wedge [p_1]\mathsf{Kh}\alpha_1 \wedge [p_2]\mathsf{Kh}\alpha_2)$

$\iff$ $\exists U_1 \exists U_2, \mathcal{M}[p_1, p_2 \mapsto U_1, U_2], w \vDash \mathsf{K}((p_1 \otimes p_2) \wedge \bigwedge_{i=1}^{2}[p_i]\mathsf{Kh}\alpha_i)$

$\iff$ $\exists U_1 \exists U_2, \forall v \in W_{\mathcal{M}}, \mathcal{M}[p_1, p_2 \mapsto U_1, U_2], v \vDash (p_1 \otimes p_2) \wedge \bigwedge_{i=1}^{2}[p_i]\mathsf{Kh}\alpha_i$
(by $(\star)$ and the fact that $[\varphi]\psi$ holds trivially if φ is false)

$\iff$ $\exists U_1 \exists U_2, \forall v \in W_{\mathcal{M}}, \mathcal{M}[p_1, p_2 \mapsto U_1, U_2], v \vDash p_1 \vee p_2$
and $v \in U_i$ implies $\mathcal{M}[p_1, p_2 \mapsto U_1, U_2], v \vDash [p_i]\mathsf{Kh}\alpha_i$ for $i = 1, 2$

$\iff$ $\exists U_1 \exists U_2, U_1 \cup U_2 = W_{\mathcal{M}}$ and $\forall v \in W_{\mathcal{M}}, v \in U_i$ implies
$\mathcal{M}[p_1, p_2 \mapsto U_1, U_2]|_{p_i}, v \vDash \mathsf{Kh}\alpha_i$ for $i = 1, 2$
(since p_1 and p_2 do not occur free in α_1 and α_2, we have:)

$\iff$ $\exists U_1 \exists U_2, U_1 \cup U_2 = W_{\mathcal{M}}$ and $\forall v \in W_{\mathcal{M}}, v \in U_i$ implies
$\mathcal{M}|_{U_i}, v \vDash \mathsf{Kh}\alpha_i$ for $i = 1, 2$

$\iff$ $\exists U_1 \exists U_2, U_1 \cup U_2 = W_{\mathcal{M}}$ and $U_i \neq \varnothing$ implies $R(U_i, \alpha_i) \neq \varnothing$ for $i = 1, 2$ □

Together with Proposition 3.10 and 3.11, Proposition 4.2 helps us to first eliminate the Kh modality without changing the expressive power, i.e., each **PALKhΠ**-formula is equivalent to a **PALΠ**-formula.

Proposition 4.2 *The following formulas and schemata are valid:*

`KKhp` :	$\mathsf{K}p \to \mathsf{Kh}p$	$\mathtt{Kh}_{\bot}$:	$\mathsf{Kh}\bot \leftrightarrow \bot$
$\mathtt{Kh}_{\vee}$:	$\mathsf{Kh}(\alpha \vee \beta) \leftrightarrow \mathsf{Kh}\alpha \vee \mathsf{Kh}\beta$	$\mathtt{Kh}_{\wedge}$:	$\mathsf{Kh}(\alpha \wedge \beta) \leftrightarrow \mathsf{Kh}\alpha \wedge \mathsf{Kh}\beta$

$\mathtt{Kh}_{\to}$: $\mathsf{Kh}(\alpha \to \beta) \leftrightarrow \mathsf{K}\forall p[p](\mathsf{Kh}\alpha \to \mathsf{Kh}\beta)$, *$p$ does not occur free in α or β*

$\mathtt{Kh}_{\otimes}$: $\mathsf{Kh}(\alpha \otimes \beta) \leftrightarrow \exists p_1 \exists p_2 \mathsf{K}((p_1 \otimes p_2) \wedge [p_1]\mathsf{Kh}\alpha \wedge [p_2]\mathsf{Kh}\beta)$,
where p_1, p_2 do not occur free in α or β

Proof. We only show $\mathtt{Kh}_{\otimes}$ and $\mathtt{Kh}_{\to}$. The other cases can be found in [18].

For $\mathtt{Kh}_{\otimes}$:

$\Longrightarrow$: Suppose $\mathcal{M}, w \vDash \mathsf{Kh}(\alpha \otimes \beta)$, then by the semantics, there is some $\langle x, y\rangle \in R(W_{\mathcal{M}}, \alpha \otimes \beta)$. Let $U = \{u \in W_{\mathcal{M}} \mid x \in R(u, \alpha)\}$ and $V = \{v \in W_{\mathcal{M}} \mid x \in R(v, \beta)\}$. It is not hard to see that $U \cup V = W_{\mathcal{M}}$, $U \neq \varnothing$ implies $R(U, \alpha) \neq \varnothing$ and $V \neq \varnothing$ implies $R(V, \beta) \neq \varnothing$. By Proposition 4.1, $\mathcal{M}, w \vDash \exists p_1 \exists p_2 \mathsf{K}((p_1 \otimes p_2) \wedge [p_1]\mathsf{Kh}\alpha \wedge [p_2]\mathsf{Kh}\beta)$.

$\Longleftarrow$: Suppose $\mathcal{M}, w \vDash \exists p_1 \exists p_2 \mathsf{K}((p_1 \otimes p_2) \wedge [p_1]\mathsf{Kh}\alpha \wedge [p_2]\mathsf{Kh}\beta)$, by Proposition 4.1, there are U_1, U_2 satisfying the desired property. If $U_1 \neq \varnothing$ and $U_2 \neq \varnothing$, pick $\langle x, y\rangle$ as the witness for $R(W_{\mathcal{M}}, \alpha \otimes \beta)$ s.t. $x \in R(U_1, \alpha)$ and $y \in R(U_2, \beta)$. If $U_1 = \varnothing$ then $U_2 \neq \varnothing$ since $W_{\mathcal{M}}$ is non-empty, then we pick $\langle x, y\rangle$ such that $y \in R(U_2, \beta)$ and $x \in S(\alpha)$. Similar for the case when $U_2 = \varnothing$. This suffices to show $\mathcal{M}, w \vDash \mathsf{Kh}(\alpha \otimes \beta)$.

For $\mathtt{Kh}_{\to}$: In [18], we showed the validity of $\mathsf{Kh}(\alpha \to \beta) \leftrightarrow \mathsf{K}\Box(\mathsf{Kh}\alpha \to \mathsf{Kh}\beta)$ where $\Box$ is the informational update operator such that $\mathcal{M}, w \vDash \Box\varphi \iff$ for any $\mathcal{M}' \subseteq \mathcal{M}$ s.t. $w \in \mathcal{M}'$, $\mathcal{M}', w \vDash \varphi$. Note that $\Box\varphi$ can be defined by $\forall p[p]\varphi$ where p does not occur free in φ. The rest is the same as in [18]. □

By Proposition 4.3 we can further eliminate the announcement operator in a formula without Kh.[7]

Proposition 4.3 *The following formulas and schemata are valid:*

$$
\begin{array}{ll}
[\,]_{\mathtt{p}} & [\chi]p \leftrightarrow (\chi \to p),\ p \in \mathbf{P} \cup \{\bot\} \\
[\,]_{\bigcirc} & [\chi](\varphi \bigcirc \psi) \leftrightarrow [\chi]\varphi \bigcirc [\chi]\psi, \bigcirc \in \{\wedge, \vee, \otimes, \to\} \\
[\,]_{\mathsf{K}} & [\chi]\mathsf{K}\varphi \leftrightarrow (\chi \to \mathsf{K}([\chi]\varphi)) \\
[\,]_{\forall} & [\chi]\forall p\varphi \leftrightarrow \forall p[\chi]\varphi, p \textit{ is not in } \chi
\end{array}
$$

Proof. We only show $[\,]_{\vee}$ and $[\,]_{\forall}$ as examples.

For $[\,]_{\vee}$: $\mathcal{M}, w \vDash [\chi](\varphi \vee \psi)$ iff $\mathcal{M}, w \vDash \chi$ implies $\mathcal{M}|_{\chi}, w \vDash \varphi \vee \psi$ iff $\mathcal{M}, w \vDash \chi$ implies ($\mathcal{M}|_{\chi}, w \vDash \varphi$ or $\mathcal{M}|_{\chi}, w \vDash \psi$) iff ($\mathcal{M}, w \vDash \chi$ implies $\mathcal{M}|_{\chi}, w \vDash \varphi$) or ($\mathcal{M}, w \vDash \chi$ implies $\mathcal{M}|_{\chi}, w \vDash \psi$) iff $\mathcal{M}, w \vDash [\chi]\varphi$ or $\mathcal{M}, w \vDash [\chi]\psi$ iff $\mathcal{M}, w \vDash [\chi]\varphi \vee [\chi]\psi$.

For $[\,]_{\forall}$: $\mathcal{M}, w \vDash [\chi]\forall p\varphi$ iff $\mathcal{M}, w \vDash \chi$ implies $\mathcal{M}|_{\chi}, w \vDash \forall p\varphi$ iff $\mathcal{M}, w \vDash \chi$ implies for all $U' \subseteq W_{\mathcal{M}|_{\chi}}$, $(\mathcal{M}|_{\chi})[p \mapsto U'], w \vDash \varphi$. If $U \subseteq W_{\mathcal{M}}$ then $U' = U \cap W_{\mathcal{M}|_{\chi}} \subseteq W_{\mathcal{M}|_{\chi}}$. Conversely, if $U' \subseteq W_{\mathcal{M}|_{\chi}}$ then for some $U \subseteq W_{\mathcal{M}}$, $U' = U \cap W_{\mathcal{M}|_{\chi}}$. Hence, $\mathcal{M}, w \vDash [\chi]\forall p\varphi$ iff $\mathcal{M}, w \vDash \chi$ implies for all $U \subseteq W_{\mathcal{M}}$, $(\mathcal{M}[p \mapsto U])|_{\chi}, w \vDash \varphi$ iff for all $U \subseteq W_{\mathcal{M}}$, $\mathcal{M}, w \vDash \chi$ implies $(\mathcal{M}[p \mapsto U])|_{\chi}, w \vDash \varphi$. Since p is not in χ, it is easy to see that $\mathcal{M}, w \vDash \chi$ iff $\mathcal{M}[p \mapsto U], w \vDash \chi$. Hence, $\mathcal{M}, w \vDash [\chi]\forall p\varphi$ iff for all $U \subseteq W_{\mathcal{M}}$, $\mathcal{M}[p \mapsto U], w \vDash \chi$ implies $(\mathcal{M}[p \mapsto U])|_{\chi}, w \vDash \varphi$ iff for all $U \subseteq W_{\mathcal{M}}$, $\mathcal{M}[p \mapsto U], w \vDash [\chi]\varphi$ iff $\mathcal{M}, w \vDash \forall p[\chi]\varphi$. □

Without loss of generality, we can always rename the bound variable in case it occurs in χ. Then for any Kh-free formula φ, by repeatedly applying

[7] An alternative set of reduction formulas for the announcement operator is presented in Lemma 12 of [1] on top of reduction axioms in [16]. See [22] for more detailed discussions on reduction axioms for **PAL**.

Proposition 4.3, we can get rid of all $[\cdot]$ operators and find an equivalent **EL**Π-formula for each **PAL**Π-formula. We will give a formal presentation of this result in Theorem 5.8 as a natural consequence of Theorem 5.2 (Soundness).

4.2 $\mathbf{Inq}^{\otimes}\mathbf{Kh} = \mathbf{InqB}^{\otimes}$

Now we show that $\mathbf{Inq}^{\otimes}\mathbf{Kh} = \{\alpha \in \mathbf{PL}^{\otimes} \mid \vDash \mathsf{Kh}\alpha\}$ is exactly $\mathbf{InqB}^{\otimes}$.

Lemma 4.4 *For any $\alpha \in \mathbf{PL}^{\otimes}$, $\mathcal{M}, w \vDash \mathsf{Kh}\alpha$ iff $\mathcal{M}, W_{\mathcal{M}} \Vdash \alpha$. As a consequence, for any non-empty state s in $\mathcal{M}$, $\mathcal{M}, s \Vdash \alpha$ iff $\mathcal{M}|_s \vDash \mathsf{Kh}\alpha$.*

Proof. Note that $\mathcal{M}, w \vDash \mathsf{Kh}\alpha$ iff $\mathcal{M} \vDash \mathsf{Kh}\alpha$ by the semantics, so we simply show $\mathcal{M} \vDash \mathsf{Kh}\alpha$ iff $\mathcal{M}, W_{\mathcal{M}} \Vdash \alpha$ inductively on the structure of α. We only prove the case of $\otimes$ since the rest are the same as in [18]. By Proposition 4.1 and 4.2, $\mathcal{M} \vDash \mathsf{Kh}(\alpha \otimes \beta)$ amounts to $\exists U, V$ s.t. $U \cup V = W_{\mathcal{M}}$, $U \neq \varnothing$ implies $R(U, \alpha) \neq \varnothing$ and $V \neq \varnothing$ implies $R(V, \beta) \neq \varnothing$. We show this is exactly $\mathcal{M}, W_{\mathcal{M}} \Vdash \alpha \otimes \beta$.

$\Longrightarrow$: If both U and V are non-empty, then $\mathcal{M} \vDash \mathsf{Kh}(\alpha \otimes \beta)$ amounts to $\mathcal{M}|_U \vDash \mathsf{Kh}\alpha$ and $\mathcal{M}|_V \vDash \mathsf{Kh}\beta$. By IH, it is equivalent to $\mathcal{M}|_U, U \Vdash \alpha$ and $\mathcal{M}|_V, V \Vdash \beta$, which implies $\mathcal{M}, W_{\mathcal{M}} \Vdash \alpha \otimes \beta$ since $U \cup V = W_{\mathcal{M}}$. If one of U and V is empty, suppose w.l.o.g. $U = \varnothing$, then we can also show $\mathcal{M}, V \Vdash \beta$ (as before), and $\mathcal{M}, U \Vdash \alpha$, for the empty state support all formulas by Proposition 2.4. Thus $\mathcal{M}, W_{\mathcal{M}} \Vdash \alpha \otimes \beta$.

$\Longleftarrow$: Suppose $\mathcal{M}, W_{\mathcal{M}} \Vdash \alpha \otimes \beta$, then there are substates t and t' such that $t \cup t' = W_{\mathcal{M}}$, $\mathcal{M}, t \Vdash \alpha$ and $\mathcal{M}, t' \Vdash \beta$. Take $U = t$ and $V = t'$, by IH, we have $U \cup V = W_{\mathcal{M}}$, $U \neq \varnothing$ implies $\mathcal{M}|_U \vDash \mathsf{Kh}\alpha$ and $V \neq \varnothing$ implies $\mathcal{M}|_V \vDash \mathsf{Kh}\beta$. Hence, $U \neq \varnothing$ implies $R(U, \alpha) \neq \varnothing$ and $V \neq \varnothing$ implies $R(V, \beta) \neq \varnothing$. By Proposition 4.1 and 4.2, we have $\mathcal{M}, W_{\mathcal{M}} \vDash \mathsf{Kh}(\alpha \otimes \beta)$.

For the consequence, $\mathcal{M}|_s, w \vDash \mathsf{Kh}\alpha$ iff $\mathcal{M}|_s, s \Vdash \alpha$ iff $\mathcal{M}, s \Vdash \alpha$, and the last step is due to Proposition 2.5. □

Remark 4.5 Note that the proof for the $\otimes$ case above actually established the equivalence between our semantics inspired by Medvedev's weak disjunction and the team/support semantics in dependence/inquisitive logics. As mentioned in the introduction, the formula $=\!(p, q) \otimes =\!(p, q)$ in propositional dependence logic is equivalent to the following formula in inquisitive logic

$$((p \vee \neg p) \to (q \vee \neg q)) \otimes ((p \vee \neg p) \to (q \vee \neg q)).$$

In our setting, it says that there is a pair of dependence functions $\langle f_1, f_2 \rangle$ s.t. you know that one of these functions captures how q depends on p actually. Note that such a pair of functions always exists: every state can be split into two substates such that one collects all the worlds where q is true, and the other one collects the rest. Then in the substate where q is homogeneously true we can define a constant function essentially assigning any resolution of $p \vee \neg p$ to the fixed resolution q. Similarly for the other substate where q is homogeneously false. Therefore this formula is valid.

Based on the lemma above, we can establish the relation between $\mathbf{InqB}^{\otimes}$ and $\mathbf{Inq}^{\otimes}\mathbf{Kh}$, where $\mathsf{Kh}\Gamma = \{\mathsf{Kh}\alpha \mid \alpha \in \Gamma\}$.

Theorem 4.6 *Given any* $\{\alpha\} \cup \Gamma \subseteq \mathbf{PL}^{\otimes}$*,* $\Gamma \Vdash \alpha$ *iff* $\mathsf{Kh}\Gamma \vDash \mathsf{Kh}\alpha$*. As a consequence when* $\Gamma = \varnothing$*,* $\mathbf{InqB}^{\otimes} = \mathbf{Inq}^{\otimes}\mathbf{Kh}$*.*

Proof. Suppose $\Gamma \Vdash \alpha$ and $\mathcal{M}, w \vDash \mathsf{Kh}\Gamma$. Now we have $\mathcal{M}, W_{\mathcal{M}} \Vdash \Gamma$ by Lemma 4.4 thus $\mathcal{M}, W_{\mathcal{M}} \Vdash \alpha$, therefore $\mathcal{M}, w \vDash \alpha$. For the other way around, if $\mathsf{Kh}\Gamma \vDash \mathsf{Kh}\alpha$ and $\mathcal{M}, s \Vdash \Gamma$, then $\mathcal{M}|_s \vDash \mathsf{Kh}\Gamma$ by Lemma 4.4, thus $\mathcal{M}|_s \vDash \mathsf{Kh}\alpha$. By Lemma 4.4 again, $\mathcal{M}, s \Vdash \alpha$. □

5 Axiomatization of PALKhΠ

In this section, we introduce the proof system $\mathsf{SPALKh}\Pi^{+}$ as below. The axioms can help us to "open up" the Kh-formulas step by step, and eventually eliminate all the Kh operator and the announcement operators.

System $\mathsf{SPALKh}\Pi^{+}$

Axioms

TAUT	Propositional tautologies
Rd⊗	$(\varphi \otimes \psi) \leftrightarrow (\varphi \vee \psi)$
$\mathtt{DIST}_{\mathsf{K}}$	$\mathsf{K}(\varphi \to \psi) \to (\mathsf{K}\varphi \to \mathsf{K}\psi)$
$[\,]_{\mathtt{p}}$	$[\chi]p \leftrightarrow (\chi \to p),\ p \in \mathbf{P} \cup \{\bot\}$
$[\,]_{\bigcirc}$	$[\chi](\varphi \bigcirc \psi) \leftrightarrow [\chi]\varphi \bigcirc [\chi]\psi$
$[\,]_{\mathsf{K}}$	$[\chi]\mathsf{K}\varphi \leftrightarrow \chi \to \mathsf{K}[\chi]\varphi$
$[\,]_{\forall}$	$[\chi]\forall p\varphi \leftrightarrow \forall p[\chi]\varphi$, p is not in χ
$\mathtt{DIST}_{\forall}$	$\forall p(\varphi \to \psi) \to (\forall p\varphi \to \forall p\psi)$
$\mathtt{SUB}_{\forall}$	$\forall p\varphi \to \varphi[\psi/p]$, ψ is free for p in φ
SU	$\exists p(p \wedge \forall q(q \to \mathsf{K}(p \to q)))$
BC	$\forall p\mathsf{K}\varphi \to \mathsf{K}\forall p\varphi$
KhK	$\mathsf{Kh}\alpha \to \mathsf{K}\alpha$
KKhp	$\mathsf{K}p \to \mathsf{Kh}p$
$\mathtt{Kh}_{\bot}$	$\mathsf{Kh}\bot \leftrightarrow \bot$
$\mathtt{Kh}_{\vee}$	$\mathsf{Kh}(\alpha \vee \beta) \leftrightarrow \mathsf{Kh}\alpha \vee \mathsf{Kh}\beta$
$\mathtt{Kh}_{\wedge}$	$\mathsf{Kh}(\alpha \wedge \beta) \leftrightarrow \mathsf{Kh}\alpha \wedge \mathsf{Kh}\beta$
$\mathtt{Kh}_{\to}$	$\mathsf{Kh}(\alpha \to \beta) \leftrightarrow \mathsf{K}\forall p[p](\mathsf{Kh}\alpha \to \mathsf{Kh}\beta)$
$\mathtt{Kh}_{\otimes}$	$\mathsf{Kh}(\alpha \otimes \beta) \leftrightarrow \exists p_1 \exists p_2 \mathsf{K}((p_1 \otimes p_2) \wedge [p_1]\mathsf{Kh}\alpha \wedge [p_2]\mathsf{Kh}\beta)$
$\mathtt{T}_{\mathsf{K}}$	$\mathsf{K}\varphi \to \varphi$
4_{K}	$\mathsf{K}\varphi \to \mathsf{K}\mathsf{K}\varphi$
5_{K}	$\neg\mathsf{K}\varphi \to \mathsf{K}\neg\mathsf{K}\varphi$
4_{Kh}	$\mathsf{Kh}\alpha \to \mathsf{K}\mathsf{Kh}\alpha$
5_{Kh}	$\neg\mathsf{Kh}\alpha \to \mathsf{K}\neg\mathsf{Kh}\alpha$

Rules

MP	$\dfrac{\varphi, \varphi \to \psi}{\psi}$
$\mathtt{NEC}_{\mathsf{K}}$	$\dfrac{\vdash \varphi}{\vdash \mathsf{K}\varphi}$
$\mathtt{GEN}_{\forall}$	$\dfrac{\vdash \varphi \to \psi}{\vdash \varphi \to \forall p\psi}$ p not free in φ
rRE	$\dfrac{\vdash \varphi \leftrightarrow \psi}{\vdash \chi[\varphi/\psi] \leftrightarrow \chi}$, given that the substitution does not happen in the scope of Kh

where $p \in \mathbf{P}$*,* $\alpha, \beta \in \mathbf{PL}^{\otimes}$*,* $\varphi, \psi, \chi \in \mathbf{PALKh}\Pi$*,* $\bigcirc \in \{\wedge, \vee, \otimes, \to\}$*;* p, p_1, p_2 *do not occur free in* α *and* β *in* $\mathtt{Kh}_{\to}$ *and* $\mathtt{Kh}_{\otimes}$*.*

Together with rRE, Rd⊗ states the fact that ⊗ behaves exactly like ∨ when it occurs outside Kh. S5 axiom schemata/rules for the know-that modality K together with TAUT, $\mathtt{DIST}_{\forall}$, $\mathtt{SUB}_{\forall}$, SU and rule $\mathtt{GEN}_{\forall}$ form a complete axiomatization $\mathsf{S5}\Pi^{+}$ of S5 logic with propositional quantifiers [10], where SU states the existence of *atoms*, crucial to capture the powerset domain for the propositional quantifier. Axioms $[\,]_{\mathtt{p}}$, $[\,]_{\bigcirc}$, $[\,]_{\mathsf{K}}$ and $[\,]_{\forall}$ are reduction axioms for the announcement operator $[\cdot]$ [16,1].[8] KKhp, $\mathtt{Kh}_{\bot}$, $\mathtt{Kh}_{\vee}$, $\mathtt{Kh}_{\wedge}$, $\mathtt{Kh}_{\to}$ and $\mathtt{Kh}_{\otimes}$ are the reduction axioms decoding the $\mathbf{PL}^{\otimes}$ formulas, whose usages are shown in

[8] The original form of $[\,]_{\forall}$ in [1] is $[\chi]\forall p\varphi \leftrightarrow (\chi \to \forall p[\chi]\varphi)$ (p is not in χ).

Lemma 5.3. Barcan formula BC, introspection schemata 4_{K}, 4_{Kh} and 5_{Kh} can be proved from the rest of the system. In particular, 4_{Kh} requires an inductive proof on the structure of α. We include them for their intuitive meanings.

Remark 5.1 Compared to the proof system SDELKh in [18] for the standard propositional inquisitive logic, there are a few notable differences:

- In order to capture tensor, we need a more powerful language **PALKh**Π than the language **DELKh** in [18]. Since we have the public announcement operator $[\cdot]$ and propositional quantifier $\forall p$ in **PALKh**Π, the informational update operator $\Box$ in **DELKh** can be expressed by $\forall p[p]$. The axioms and rules of $[\cdot]$ and $\forall p$ can handle $\Box$ implicitly. In Section 6, we will see our language can uniformly define various generalised versions of tensor as well.
- In the proof system SDELKh of [18], there is a set of axiom schemata $\{\mathsf{EU}_k \mid k \in \mathbb{N}\}$, which captures the idea (roughly) that given a definable finite set of worlds, we can have an updated submodel with it as the set of possible worlds. These axioms are also necessary in the process of eliminating $\Box$ in [18] and have an intimate connection with the axioms $\mathtt{ND}_k$ in inquisitive logic. However, these axioms are no longer needed here, as the same function of postulating the existence of certain updated models can be realized by concrete announcements in our language **PALKh**Π.

The next subsection explores reductions of Kh systematically.

5.1 Provable equivalence

In Section 4.1, we showed that **PALKh**Π is expressively equivalent to **EL**Π. Now we can show that each **PALKh**Π-formula φ is *provably equivalent* to an **EL**Π-formula φ' (Lemma 5.7) in SPALKhΠ^+. Meanwhile we provide a translation from φ to φ'.

Theorem 5.2 (Soundness) SPALKhΠ^+ *is sound over the class of all models.*

Proof. The validity of $[\,]_{\mathtt{p}}$, $[\,]_{\bigcirc}$, $[\,]_{\mathsf{K}}$ and $[\,]_{\forall}$ are given in Proposition 4.3. $\mathtt{DIST}_{\forall}$, $\mathtt{SUB}_{\forall}$, SU and rule $\mathtt{GEN}_{\forall}$ are given in [10]. KKhp, $\mathtt{Kh}_{\bot}$, $\mathtt{Kh}_{\vee}$, $\mathtt{Kh}_{\wedge}$, $\mathtt{Kh}_{\rightarrow}$, and $\mathtt{Kh}_{\otimes}$ are shown to be valid in Proposition 4.2. The validity of rRE is given in Proposition 3.11. The rest are trivial. $\Box$

To prove the completeness we first prove Lemmata 5.3 and 5.6 with two sets of reduction axioms for Kh and $[\cdot]$ respectively. Recall that **PAL**Π is the Kh-free fragment of **PALKh**Π, and **EL**Π is the $[\cdot]$-free fragment of **PAL**Π.

Lemma 5.3 *Each* **PALKh**Π*-formula is provably equivalent to a* Kh*-free* **PAL**Π *formula in* SPALKhΠ^+*.*

Proof. We use rRE and axioms $\mathtt{Kh}_{\bot}, \mathtt{Kh}_{\wedge}, \mathtt{Kh}_{\vee}, \mathtt{Kh}_{\rightarrow}$, $\mathtt{Kh}_{\otimes}$ repeatedly to reduce $\mathsf{Kh}\alpha$ to some formula with $\mathsf{Kh}p$ only. With $\vdash \mathsf{Kh}p \leftrightarrow \mathsf{K}p$ from KhK and KKhp, we can eliminate all Kh modalities. $\Box$

To eliminate the announcement operator, we need a notion of complexity.

Definition 5.4 (Announcement rank) *For each* $\varphi \in$ **PAL**Π*, we define its announcement rank* $\mathbf{ar}(\varphi)$ *inductively as follows:*

- *If* $\varphi = p$ *or* $\varphi = \bot$*, then* $\mathbf{ar}(\varphi) = 0$.
- *If* $\varphi = \psi_1 \bigcirc \psi_2$ *where* $\bigcirc = \wedge, \vee, \otimes$ *or* $\to$*, then* $\mathbf{ar}(\varphi) = \max\{\mathbf{ar}(\psi_1), \mathbf{ar}(\psi_2)\}$.
- *If* $\varphi = \mathsf{K}\psi$*, then* $\mathbf{ar}(\varphi) = \mathbf{ar}(\psi)$.
- *If* $\varphi = \forall p\psi$*,* $p \in \mathbf{P}$*, then* $\mathbf{ar}(\varphi) = \mathbf{ar}(\psi)$.
- *If* $\varphi = [\chi]\psi$*, then* $\mathbf{ar}(\varphi) = \mathbf{ar}(\psi) + \mathbf{ar}(\chi) + 1$.

Lemma 5.5 *Each* **PAL**Π*-formula of the form* $[\chi]\psi$ *is provably equivalent to a* **PAL**Π*-formula* φ *in* $\mathsf{SPALKh}\Pi^+$ *such that* $\mathbf{ar}(\varphi) < \mathbf{ar}([\chi]\psi)$.

Proof. We prove by induction on ψ:

(i) If $\psi = p$ or $\psi = \bot$, then by axiom $[]_\mathtt{p}$, $[\chi]\psi \leftrightarrow (\chi \to \psi)$ and $\mathbf{ar}(\chi \to \psi) = \max\{\mathbf{ar}(\chi), \mathbf{ar}(\psi)\} < \mathbf{ar}([\chi]\psi)$. Hence $\varphi = \chi \to \psi$ is what we need.

(ii) If $\psi = \psi_1 \bigcirc \psi_2$ where $\bigcirc = \wedge, \vee, \otimes, \to$, then by $[]_\bigcirc$, $[\chi]\psi \leftrightarrow [\chi]\psi_1 \bigcirc [\chi]\psi_2$. By IH, there are $\varphi_1 \leftrightarrow [\chi]\psi_1$ and $\varphi_2 \leftrightarrow [\chi]\psi_2$ such that $\mathbf{ar}(\varphi_1) < \mathbf{ar}([\chi]\psi_1)$ and $\mathbf{ar}(\varphi_1) < \mathbf{ar}([\chi]\psi_1)$. Hence, $\varphi = \varphi_1 \bigcirc \varphi_2$ is what we need.

(iii) If $\psi = \mathsf{K}\psi'$, then by $[]_\mathsf{K}$, $[\chi]\psi \leftrightarrow (\chi \to \mathsf{K}[\chi]\psi')$. By IH, there is a φ' such that $\varphi' \leftrightarrow [\chi]\psi'$ and that $\mathbf{ar}(\varphi') < \mathbf{ar}([\chi]\psi')$. Then $\mathbf{ar}(\chi \to \mathsf{K}\varphi') = \max\{\mathbf{ar}(\chi), \mathbf{ar}(\mathsf{K}\varphi')\} = \max\{\mathbf{ar}(\chi), \mathbf{ar}(\varphi')\} < \max\{\mathbf{ar}(\chi), \mathbf{ar}([\chi]\psi')\} = \mathbf{ar}([\chi]\mathsf{K}\psi')$. Hence, $\varphi = \chi \to \mathsf{K}\varphi'$ is what we need.

(iv) If $\psi = \forall p\psi'$ where $p \in \mathbf{P}$, we consider two subcases. 1) If p is not in χ, we use $[]_\forall$ and the proof is similar to the above cases. 2) If p is in χ, replace p with the first letter $q \in \mathbf{P}$ which is not in χ (such relettering can be done in the system), and then go to case 1).

(v) If $\psi = [\chi']\psi'$, by IH, there is a φ' such that $\varphi' \leftrightarrow [\chi']\psi'$ and that $\mathbf{ar}(\varphi') < \mathbf{ar}([\chi']\psi')$. So $[\chi][\chi']\psi' \leftrightarrow [\chi]\varphi'$ and $\mathbf{ar}([\chi]\varphi') < \mathbf{ar}([\chi][\chi']\psi')$. Hence $\varphi = [\chi]\varphi'$ is what we need.

□

Given the above lemma, we can eliminate the announcement operators eventually. The idea is that each formula in the shape of $[\chi]\varphi$ has a finite announcement rank, and can be reduced to an equivalent formula φ' with a lower rank. In case φ' still has subformulas with the announcement operators, we can replace each of these subformulas with a provably equivalent one with a lower rank. Note that by definition, a subformula's rank is no greater than the whole formula. Eventually, by repeating this process, we can decrease the rank to zero and obtain an equivalent formula without the announcement operators.

Lemma 5.6 *Each* **PAL**Π*-formula is provably equivalent to an* **EL**Π*-formula in* $\mathsf{SPALKh}\Pi^+$.

Combining Lemmata 5.3 and 5.6 we immediately have.

Lemma 5.7 *Each* **PALKh**Π*-formula is provably equivalent to an* **EL**Π*-formula in* $\mathsf{SPALKh}\Pi^+$.

Theorem 5.8 follows naturally from Lemma 5.7 and Theorem 5.2.

Theorem 5.8 **PALKhΠ** *is equally expressive as* **ELΠ** *over all models.*

Note that **ELΠ** is more expressive than **EL** [10].

5.2 Completeness

With Lemma 5.7 and Theorem 5.8, the completeness of System $\mathsf{SPALKh}\Pi^+$ can be reduced to that of $\mathsf{S5}\Pi^+$, which is given in [10]. $\mathsf{S5}\Pi^+$ is a variation of second-order modal logic, containing all the axiom schemata/rules of $\mathsf{S5}$ as well as those concerning propositional quantifiers in $\mathsf{SPALKh}\Pi^+$.

Theorem 5.9 (Completeness of $\mathsf{S5}\Pi^+$ [10]) $\mathsf{S5}\Pi^+$ *is a complete axiomatization with regard to the class of models.*

Theorem 5.10 (Completeness) *System* $\mathsf{SPALKh}\Pi^+$ *is complete over the class of all models.*

Proof. We first use Lemma 5.3 and Lemma 5.6 to translate each **PALKhΠ**-formula φ into an equivalent **ELΠ**-formula φ' and then use the completeness of $\mathsf{S5}\Pi^+$. Note that $\vdash \varphi$ below means φ is in $\mathsf{SPALKh}\Pi^+$.

$$\vDash \varphi \underset{\text{Theorem 5.8}}{\overset{\text{expressive equivalence}}{\Longleftrightarrow}} \vDash \varphi' \underset{\text{Theorem 5.9}}{\overset{\text{completeness of } \mathsf{S5}\Pi^+}{\Longleftrightarrow}} \vdash_{\mathsf{S5}\Pi^+} \varphi'$$

$$\overset{\mathsf{S5}\Pi^+ \subseteq \mathsf{SPALKh}\Pi^+}{\Longrightarrow} \vdash \varphi' \underset{\text{Lemma 5.7}}{\overset{\text{provable equivalence}}{\Longleftrightarrow}} \vdash \varphi$$

□

6 Generalization of Tensor Disjunction

Inspired by our epistemic interpretation, we generalize the binary $\otimes$ to n-ary operators for any $n \geq 2$ with another parameter $k \leq n$. Due to the lack of space, most of the proofs are omitted, except for a few crucial ones.

6.1 Generalizing the tensor operator

Consider the following scenario: You completed an exam with n questions, for which you need at least k correct answers to pass. Now you only know you have passed the exam. What is your epistemic state about your answers? For any $n \geq 2$ and $1 \leq k \leq n$, we now define an n-ary connective $\otimes^k_n$ capturing that you are sure at least k of your n answers must be correct, but may not know which ones are correct. The original tensor actually captures the special case when $k = 1$ and $n = 2$.

Definition 6.1 (Language $\mathbf{PL}^{\otimes^k_n}$) *The* propositional language with general tensor *($\mathbf{PL}^{\otimes^k_n}$) is defined as follows:*

$$\alpha ::= p \mid \bot \mid (\alpha \wedge \alpha) \mid (\alpha \vee \alpha) \mid (\alpha \to \alpha) \mid \otimes^k_n(\underbrace{\alpha, \cdots, \alpha}_{n})$$

where $p \in \mathbf{P}$ and $n \geq 2$, $1 \leq k \leq n$.

Definition 6.2 (Language PALKh$\Pi_{\mathbf{G}}$) *The* Public Announcement Logic with Know-how and General Tensor *(*$\mathbf{PALKh\Pi_G}$*) is as follows:*

$$\varphi ::= p \mid \bot \mid (\varphi \wedge \varphi) \mid (\varphi \vee \varphi) \mid (\varphi \to \varphi) \mid \otimes_n^k(\underbrace{\varphi, \cdots, \varphi}_{n}) \mid \mathsf{K}\varphi \mid \mathsf{Kh}\alpha \mid \forall p\varphi \mid [\varphi]\varphi$$

where $p \in \mathbf{P}$ *and* $\alpha \in \mathbf{PL}^{\otimes_n^k}$*.*

Now, we introduce the semantics of new connectives $\otimes_n^k$ via resolutions.

Definition 6.3 *For any positive integer* $n \geq 2$ *and* $1 \leq k \leq n$*, we define the resolution space and resolution of* $\otimes_n^k$ *as follow:*

$$S(\otimes_n^k(\alpha_1, \cdots, \alpha_n)) = S(\alpha_1) \times \cdots \times S(\alpha_n)$$
$$R(w, \otimes_n^k(\alpha_1, \cdots, \alpha_n)) = \{(r_1, \cdots, r_n) \mid k \leq |\{i \in [1, n] \mid r_i \in R(w, \alpha_i)\}|\}$$

The truth condition for Kh is as before in Definition 3.7. In particular, $\mathcal{M}, w \vDash \mathsf{Kh} \otimes_n^k (\alpha_1, \cdots, \alpha_n)$ iff $R(W_{\mathcal{M}}, \otimes_n^k(\alpha_1, \cdots, \alpha_n)) \neq \varnothing$.

By Definition 3.7 and 6.3, it is not hard to see the following.

Proposition 6.4 $\mathcal{M}, w \vDash \mathsf{Kh} \otimes_n^k (\alpha_1, \cdots, \alpha_n)$ *if and only if there is an n-tuple* $\langle r_1, \cdots, r_n\rangle$ *such that for any* $v \in W_{\mathcal{M}}$*,* $|\{i \mid r_i \in R(v, \alpha_i)\}| \geq k$*, i.e., there are at least* k *indexes* $i \in [1, n]$ *such that* $r_i \in R(v, \alpha_i)$*.*

Based on the above proposition, the truth condition for $\otimes_2^1$ is exactly as the one for the standard $\otimes$ defined earlier. Note that $\otimes_n^k$ can also appear outside the scope of Kh in our language $\mathbf{PALKh\Pi_G}$ and we define its semantics below.

Definition 6.5 (Semantics)

$$\mathcal{M}, w \vDash \otimes_n^k(\varphi_1, \cdots, \varphi_n) \iff \mathcal{M}, w \vDash \bigvee_{\substack{I \subseteq \{1,2,\cdots,n\} \\ |I|=k}} \bigwedge_{i \in I} \varphi_i$$

The semantics is guided by Proposition 3.8, with the desired property below.

Proposition 6.6 *For any* $\alpha \in \mathbf{PL}^{\otimes_n^k}$ *and* $\mathcal{M}, w$*,* $\mathcal{M}, w \vDash \alpha$ *iff* $R(w, \alpha) \neq \varnothing$*.*

Next, we show how to reduce the general tensors in $\mathbf{PALKh\Pi_G}$.

Proposition 6.7 *The following schemata are valid:*

$$\mathtt{Rd}\otimes_{\mathtt{n}}^{\mathtt{k}} \quad \otimes_n^k (\varphi_1, \cdots, \varphi_n) \leftrightarrow \bigvee_{\substack{I \subseteq \{1,2,\cdots,n\} \\ |I|=k}} \bigwedge_{i \in I} \varphi_i$$

$$\mathtt{Kh}_{\otimes_{\mathtt{n}}^{\mathtt{k}}} \quad \mathsf{Kh} \otimes_n^k (\alpha_1, \cdots, \alpha_n) \leftrightarrow \exists p_1 \cdots \exists p_n \mathsf{K}(\otimes_n^k(p_1, \cdots, p_n) \wedge \bigwedge_{i=1}^{n} [p_i]\mathsf{Kh}\alpha_i)$$

(where all the p_i *do not occur free in all the* α_i*)*

Proof. $\mathtt{Rd}\otimes_{\mathtt{n}}^{\mathtt{k}}$ is valid by the truth condition of $\otimes_n^k$ in Definition 6.5.

For $\mathtt{Kh}_{\otimes_{\mathtt{n}}^{\mathtt{k}}}$:

$\Longrightarrow$: By Proposition 6.4 $\mathcal{M}, w \vDash \mathsf{Kh} \otimes_n^k (\alpha_1, \cdots, \alpha_n)$ iff there is an n-tuple $\langle r_1, \cdots, r_n \rangle$ s.t. for any $v \in W_\mathcal{M}$, there are at least k indexes $i \in [1, n]$ such that $r_i \in R(v, \alpha_i)$. Let $U_i = \{v \in W_\mathcal{M} \mid r_i \in R(v, \alpha_i)\}$, then consider $\mathcal{M}[\bar{p} \mapsto \bar{U}] = \langle W, V' \rangle$ such that V' assigns U_i to p_i for $i \in \{1, \ldots, n\}$ and coincides with V on all other atoms. Then, for any $v \in W_\mathcal{M}$, there are at least k indexes $i \in [1, n]$ s.t. $\mathcal{M}[\bar{p} \mapsto \bar{U}], v \vDash p_i$, so $\mathcal{M}[\bar{p} \mapsto \bar{U}], v \vDash \otimes_n^k(p_1, \cdots, p_n)$. And since for any $v \in U_i$ we have $r_i \in R(v, \alpha_i)$, so $R(U_i, \alpha_i) \neq \varnothing$. Hence, for any $v \in W_\mathcal{M}$, $\mathcal{M}[\bar{p} \mapsto \bar{U}], v \vDash [p_i]\mathsf{Kh}\alpha_i$. In sum, $\mathcal{M}[\bar{p} \mapsto \bar{U}], w \vDash \mathsf{K}(\otimes_n^k(p_1, \cdots, p_n) \wedge \bigwedge_{i=1}^n [p_i]\mathsf{Kh}\alpha_i)$, which is equivalent to $\mathcal{M}, w \vDash \exists p_1 \cdots \exists p_n \mathsf{K}(\otimes_n^k(p_1, \cdots, p_n) \wedge \bigwedge_{i=1}^n [p_i]\mathsf{Kh}\alpha_i)$.

$\Longleftarrow$: Suppose $\mathcal{M}, w \vDash \exists p_1 \cdots \exists p_n \mathsf{K}(\otimes_n^k(p_1, \cdots, p_n) \wedge \bigwedge_{i=1}^n [p_i]\mathsf{Kh}\alpha_i)$, then there are $U_i \subseteq W_\mathcal{M}$ such that $\mathcal{M}[\bar{p} \mapsto \bar{U}], w \vDash \mathsf{K}(\otimes_n^k(p_1, \cdots, p_n) \wedge \bigwedge_{i=1}^n [p_i]\mathsf{Kh}\alpha_i)$, which is equivalent to $\mathcal{M}[\bar{p} \mapsto \bar{U}], w \vDash (\mathsf{K} \otimes_n^k (p_1, \cdots, p_n)) \wedge \bigwedge_{i=1}^n \mathsf{K}[p_i]\mathsf{Kh}\alpha_i$.

For the first conjunct: $\mathcal{M}[\bar{p} \mapsto \bar{U}], w \vDash \mathsf{K} \otimes_n^k (p_1, \cdots, p_n)$ means that for any $v \in W_\mathcal{M}$ we have $\mathcal{M}[\bar{p} \mapsto \bar{U}], v \vDash \otimes_n^k(p_1, \cdots, p_n)$. So at least k of p_i is true in v, which means v belongs to at least k of the corresponding U_i. For the second conjunct: $\mathcal{M}[\bar{p} \mapsto \bar{U}], w \vDash \bigwedge_{i=1}^n \mathsf{K}[p_i]\mathsf{Kh}\alpha_i$ means that for any $v \in W_\mathcal{M}$ and $i \in [1, n]$, $v \in U_i$ implies that $R(U_i, \alpha_i) \neq \varnothing$. So, if $U_i \neq \varnothing$, pick a r_i from $R(U_i, \alpha_i)$. If $U_i = \varnothing$, pick an arbitrary r_i from $S(\alpha_i)$. Hence, we have for any $i \in [1, n]$, $U_i \neq \varnothing$ implies $r_i \in R(U_i, \alpha_i)$.

Combining the meaning of the two conjuncts, we know that for any $v \in W_\mathcal{M}$, there are at least k indexes $i \in [1, n]$ such that $v \in U_i$ and $U_i \neq \varnothing$ implies $r_i \in R(U_i, \alpha_i)$ for any $i \in [1, n]$. Hence, $\langle r_1, \cdots, r_n \rangle$ is a n-tuple satisfying the desired property, by Proposition 6.4, we have $\mathcal{M}, w \vDash \mathsf{Kh} \otimes_n^k (\alpha_1, \cdots, \alpha_n)$. □

By using the reduction axioms above, *all* the general tensors can be eliminated semantically, and thus $\mathbf{PALKh\Pi_G}$ and $\mathbf{PALKh\Pi}$ are equally expressive.

Let $\mathsf{SPALKh\Pi_G^+}$ be $\mathsf{SPALKh\Pi^+}$ extended with $\mathtt{Rd}\otimes^{\mathtt{k}}_{\mathtt{n}}$ and $\mathtt{Kh}_{\otimes^{\mathtt{k}}_{\mathtt{n}}}$ for any $n \geq 2$ and $1 \leq k \leq n$. Similar to Theorem 5.10, it is straightforward to show:

Theorem 6.8 (Soundness and completeness) *Proof system* $\mathsf{SPALKh\Pi_G^+}$ *is sound and complete over the class of all models.*

6.2 Support semantics for $\otimes_n^k$

We can now go back to define the support semantics for $\otimes_n^k$.

Definition 6.9 (Support for $\otimes_n^k$) $\mathcal{M}, s \Vdash \otimes_n^k(\alpha_1, \cdots, \alpha_n)$ *iff there exist n subsets $t_1, \cdots, t_n$ of s such that for any $i \in [1, n]$, $\mathcal{M}, t_i \Vdash \alpha_i$ and for any $w \in s$, there are at least k indexes $i \in [1, n]$ such that $w \in t_i$.*

The support semantics for other connectives stays the same as in Definition 2.3. Let $\mathbf{InqB}^{\otimes^{\mathbf{k}}_{\mathbf{n}}}$ be the set of valid $\mathbf{PL}^{\otimes_n^k}$ formulas by the support semantics. We can show $\mathbf{Inq}^{\otimes^{\mathbf{k}}_{\mathbf{n}}}\mathbf{Kh} = \{\alpha \in \mathbf{PL}^{\otimes_n^k} \mid \vDash \mathsf{Kh}\alpha\}$ is exactly $\mathbf{InqB}^{\otimes^{\mathbf{k}}_{\mathbf{n}}}$, based on the following generalization of Lemma 4.4.

Proposition 6.10 *For any $\alpha \in \mathbf{PL}^{\otimes_n^k}$, $\mathcal{M}, w \vDash \mathsf{Kh}\alpha \iff \mathcal{M}, W_\mathcal{M} \Vdash \alpha$.*

Proof. Based on Lemma 4.4, we only consider the case of $\otimes_n^k(\alpha_1, \cdots, \alpha_n)$ and write $\exists U$ for $\exists U \subseteq W_\mathcal{M}$ for brevity, similarly for $\exists t$.

$$\begin{aligned}
&\mathcal{M}, w \vDash \mathsf{Kh}(\otimes_n^k(\alpha_1, \cdots, \alpha_n)) \\
\iff &\mathcal{M}, w \vDash \exists p_1 \cdots \exists p_n \mathsf{K}(\otimes_n^k(p_1, \cdots, p_n) \wedge \bigwedge_{i=1}^{n} [p_i]\mathsf{Kh}\alpha_i) \text{ (by Proposition 6.7).} \\
\iff &\exists U_1, \cdots, U_n, \forall v \in W_{\mathcal{M}}, \text{ there are at least } k \text{ indexes } i \in [1,n] \text{ s.t. } v \in U_i \\
&\quad \text{and for any } i, U_i \neq \varnothing \text{ implies } R(U_i, \alpha_i) \neq \varnothing. \text{(similar to Proposition 4.1)} \\
\iff &\exists t_1, \cdots, t_n, \forall i \in [1,n], t_i \Vdash \alpha_i \text{ and } \forall v \in W_{\mathcal{M}}, k \leq |\{i \in [1,n] \mid v \in U_i\}| \\
\iff &\mathcal{M}, W_{\mathcal{M}} \Vdash \otimes_n^k(\alpha_1, \cdots, \alpha_n).
\end{aligned}$$

□

As shown in [24], adding tensor does not increase the expressive power of inquisitive logic. In fact, adding all the general tensors also does not increase the expressive power of inquisitive logic.

First, we extend the definition of realization in [6] to our new connectives.

Definition 6.11 (Realizations)

- $RL(p) = \{p\}$ *for* $p \in \mathbf{P}$
- $RL(\bot) = \{\bot\}$
- $RL(\alpha \vee \beta) = RL(\alpha) \cup RL(\beta)$
- $RL(\alpha \wedge \beta) = \{\rho \wedge \sigma \mid \rho \in RL(\alpha)$ *and* $\sigma \in RL(\beta)\}$
- $RL(\alpha \to \beta) = \{\bigwedge_{\rho \in RL(\alpha)}(\rho \to f(\rho)) \mid f : RL(\alpha) \to RL(\beta)\}$
- $RL(\otimes_n^k(\alpha_1, \cdots, \alpha_n)) = \{\neg \bigwedge_{\substack{I \subseteq \{1,2,\cdots,n\} \\ |I|=k}} \neg \bigwedge_{i \in I} \rho_i \mid$ *for all* $i, \rho_i \in RL(\alpha_i)\}$

Then we can generalize the inquisitive *normal form* in [6,9].

Proposition 6.12 *For any* $\alpha \in \mathbf{PL}^{\otimes_n^k}$, $s \Vdash \alpha$ *iff* $s \Vdash \bigvee_{\rho \in RL(\alpha)} \rho$.

Theorem 6.13 **PL** *and* $\mathbf{PL}^{\otimes_n^k}$ *are equally expressive w.r.t. the support semantics.*

Proof. By Proposition 6.12, for any $\alpha \in \mathbf{PL}^{\otimes_n^k}$, α is equivalent to a disjunction of some ρ without general tensors. □

In [25], it is shown that the variants of propositional dependence logics **PD**, $\mathbf{PD}^{\vee}$, **PID**, **InqB** are all equally expressive. Similarly, adding general tensors to the languages of these logics will also not increase the expressive power.

Corollary 6.14 *Adding general tensors to the langugaes of* **PD**, $\mathbf{PD}^{\vee}$, **PID** *or* **InqB** *does not increase their expressive power.*

6.3 Interdefinability

In [7], it is proved that although adding $\otimes$ to **PL** does not increase the expressive power, $\otimes$ is not *uniformly definable* in **PL**, i.e., $\varphi \otimes \psi$ cannot be expressed by a uniform formula "template" with φ and ψ as its only parameters. It is also natural to ask whether general tensors are uniformly definable by the standard binary tensor $\otimes$.

First, on the positive side, we show that $\otimes_n^n$ is a trivial conjunction and $\otimes_n^1$ can be uniformly defined by $\otimes_2^1$. Moreover, by fixing some components as $\top$ or $\bot$, some general tensors can be uniformly defined by others with smaller parameters.

Proposition 6.15 *For any $\alpha_1, \cdots, \alpha_n \in \mathbf{PL}^{\otimes_n^k}$, the following hold:*

- *For any $n \geq 2$ and any state s, $s \Vdash \otimes_n^n(\alpha_1, \cdots, \alpha_n) \iff s \Vdash \bigwedge_{i=1}^n \alpha_i$.*
- *For any $n \geq 3$, $s \Vdash \otimes_n^1(\alpha_1, \cdots, \alpha_n) \iff s \Vdash \otimes_2^1(\otimes_{n-1}^1(\alpha_1, \cdots, \alpha_{n-1}), \alpha_n)$.*
- *For any $n \geq 3$, $1 \leq k \leq n$ and any state s, $s \Vdash \otimes_n^k(\alpha_1, \cdots, \alpha_{n-1}, \top) \iff s \Vdash \otimes_{n-1}^{k-1}(\alpha_1, \cdots, \alpha_{n-1})$.*
- *For any $n \geq 3$, $1 \leq k \leq n-1$ and any state s, $s \Vdash \otimes_n^k(\alpha_1, \cdots, \alpha_{n-1}, \bot) \iff s \Vdash \otimes_{n-1}^k(\alpha_1, \cdots, \alpha_{n-1})$.*

Inspired by the proof in [7], we can show that $\otimes_3^2$ is not uniformly definable by the connectives $\{\bot, \wedge, \rightarrow, \vee, \otimes_2^1\}$ in $\mathbf{PL}^{\otimes}$.

To show the negative results, we need the following definitions about uniform definability from [24].

Definition 6.16 (Context) *A context for a propositional logic $\mathbf{L}$ is an $\mathbf{L}$-formula $\varphi(p_1, \cdots, p_n)$ with distinguished atoms $p_1, \cdots, p_n$, and it is also allowed to contain other atoms besides $p_1, \cdots, p_n$. For any $\mathbf{L}$-formulas $\psi_1, \cdots, \psi_n$, we write $\varphi(\psi_1, \cdots, \psi_n)$ for the formula $\varphi(\psi_1/p_1, \cdots, \psi_n/p_n)$.*

Definition 6.17 (Uniform definability) *In a language $\mathbf{L}$, we say that an n-ary connective $\odot$ is* uniformly definable *if there exists a context $\zeta(p_1, \cdots, p_n)$ such that for all $\chi_1, \cdots, \chi_n \in \mathbf{L}$: $\odot(\chi_1, \cdots, \chi_n)$ is equivalent to $\zeta(\chi_1, \cdots, \chi_n)$.*

As the first negative result, we show $\otimes_3^2$ is not uniformly definable in $\mathbf{PL}^{\otimes}$. We consider relative equivalence with respect to a state s.

Definition 6.18 (Relative equivalence [7]) *Let s be a state in $\mathcal{M}$ and $\varphi, \psi \in \mathbf{PL}^{\otimes_n^k}$. We say that φ and ψ are relatively equivalent in s, $\varphi \equiv_s \psi$ iff for all states $t \subseteq s$, $t \Vdash \varphi \iff t \Vdash \psi$.*

It is easy to see that if φ and ψ are equivalent then they are relatively equivalent in any state s.

Consider $\psi = p_1 \vee p_2 \vee p_3 \vee p_4$ and $s = \{w_{12}, w_{13}, w_{14}, w_{23}, w_{24}, w_{34}\}$ where only p_i, p_j are true in w_{ij} and all the other propositional letters are false. Now, we show that with respect to this state s, $\otimes_3^2$ cannot be uniformly defined by any context in $\mathbf{PL}^{\otimes}$.

Lemma 6.19 *For any context $\varphi(p_0)$, with $\varphi \in \mathbf{PL}^{\otimes}$ not containing p_1, p_2, p_3, p_4, $\varphi(\psi/p_0)$ would be equivalent to $\bot, \psi, \otimes_2^1(\psi, \psi)$ or $\top$ in s.*

Proof. First we notice for any state t, $t \Vdash \bot \Rightarrow t \Vdash \psi \Rightarrow t \Vdash \otimes_2^1(\psi, \psi) \Rightarrow t \Vdash \top$ ($\star$). Then we prove by induction on φ. For short, we write φ^* for $\varphi(\psi/p_0)$:

- For $\varphi = \bot$ or $\varphi = p$ with $p \neq p_0$: Since we assume that p_1, p_2, p_3, p_4 are not in φ, so p is different from them. Hence, it is obvious that $\varphi^* \equiv_s \bot$.

- For $\varphi = p_0$: It is obvious that $\varphi^* \equiv_s \psi$.
- For $\varphi = \varphi_1 \wedge \varphi_2$: so $\varphi^* = \varphi_1^* \wedge \varphi_2^*$. By IH, φ_1^* and φ_2^* are both equivalent to $\bot, \psi, \otimes_2^1(\psi,\psi)$ or $\top$ in s. Since $(t \Vdash \chi_1 \Rightarrow t \Vdash \chi_2)$ implies $(t \Vdash \chi_1 \wedge \chi_2 \iff t \Vdash \chi_1)$, by $(\star)$, obviously φ^* is also equivalent to $\bot, \psi, \otimes_2^1(\psi,\psi)$ or $\top$ in s.
- For $\varphi = \varphi_1 \vee \varphi_2$: similar to the case of conjunction.
- For $\varphi = \varphi_1 \to \varphi_2$: so $\varphi^* = \varphi_1^* \to \varphi_2^*$, and for any state t, $t \Vdash \varphi_1^* \to \varphi_2^*$ iff for any $t' \subseteq t$, $t' \Vdash \varphi_1^*$ implies $t' \Vdash \varphi_2^*$. By $(\star)$, we could know that:
  · $\bot \to \bot$, $\bot \to \psi$, $\bot \to \otimes_2^1(\psi,\psi)$, $\bot \to \top$, $\psi \to \psi$, $\psi \to \otimes_2^1(\psi,\psi)$, $\psi \to \top$, $\otimes_2^1(\psi,\psi) \to \otimes_2^1(\psi,\psi)$, $\otimes_2^1(\psi,\psi) \to \top$, $\top \to \top$ are all equivalent to $\top$ in s.
  · $\psi \to \bot$, $\otimes_2^1(\psi,\psi) \to \bot$ and $\top \to \bot$ are all equivalent to $\bot$ in s.
  · $\otimes_2^1(\psi,\psi) \to \psi$ and $\top \to \psi$ are equivalent to ψ in s.
  · $\top \to \otimes_2^1(\psi,\psi)$ is equivalent to $\otimes_2^1(\psi,\psi)$ in s.
  Hence, φ^* is equivalent to $\bot, \psi, \otimes_2^1(\psi,\psi)$ or $\top$ in s.
- For $\varphi = \otimes_2^1(\varphi_1, \varphi_2)$: so $\varphi^* = \otimes_2^1(\varphi_1^*, \varphi_2^*)$. We consider the following cases:
  · $\varphi_1^* \equiv_s \top$. Then $\otimes_2^1(\varphi_1^*, \varphi_2^*) \equiv_s \top$.
  · $\varphi_1^* \equiv_s \bot$. Then $\otimes_2^1(\varphi_1^*, \varphi_2^*) \equiv_s \varphi_2^*$.
  · $\varphi_1^* \equiv_s \psi$. If $\varphi_2^* \equiv_s \top$ or $\varphi_2^* \equiv_s \bot$, it would be the same as former cases. Then we need to discuss two sub-cases:
    * $\varphi_2^* \equiv_s \psi$. Then $\otimes_2^1(\varphi_1^*, \varphi_2^*) \equiv_s \otimes_2^1(\psi,\psi)$.
    * $\varphi_2^* \equiv_s \otimes_2^1(\psi,\psi)$. Then $t \Vdash \varphi^* \iff$ there are $t_1, t_2 \subseteq t$ and $t_1 \cup t_2 = t$ such that $t_1 \Vdash \psi$ and $t_2 \Vdash \otimes_2^1(\psi,\psi) \iff$ there are $t_1, t_2 \subseteq t$, $t_1 \cup t_2 = t$ and $p_{i_1}, p_{i_2}, p_{i_3}$ such that p_{i_1} is true in any $w \in t_1$ and for any $w \in t_2$, p_{i_2} or p_{i_3} is true in $w \iff$ there are $p_{i_1}, p_{i_2}, p_{i_3}$ such that for any $w \in t$, p_{i_1}, p_{i_2} or p_{i_3} is true in w. However, there are only four propositional letters p_1, p_2, p_3, p_4 and in each $w \in s$, two of these propositional letters are true. So consider p_1, p_2 and p_3, we will know that for any $w \in t \subseteq s$, at least one of p_1, p_2 and p_3 is true in w. Hence, $\otimes_2^1(\psi, \otimes_2^1(\psi,\psi)) \equiv_s \top$.
  · $\varphi_1^* \equiv_s \otimes_2^1(\psi,\psi)$. Then if $\varphi_2^* \equiv_s \top$, $\varphi_2^* \equiv_s \bot$ or $\varphi_2^* \equiv_s \psi$, it would be the same as former cases. And if $\varphi_2^* \equiv_s \otimes_2^1(\psi,\psi)$, the proof is similar to the previous case and the result is that $\otimes_2^1(\otimes_2^1(\psi,\psi), \otimes_2^1(\psi,\psi)) \equiv_s \top$. □

Lemma 6.20 *$\otimes_3^2$ is not uniformly definable in* $\mathbf{PL}^{\otimes}$.

Proof. If $\otimes_3^2$ is uniformly definable in $\mathbf{PL}^{\otimes}$, there will be a context $\varphi(p)$ such that for any $\chi \in \mathbf{PL}^{\otimes}$: $\varphi(\chi)$ is equivalent to $\otimes_3^2(\chi,\chi,\chi)$.

However, as we proved in Lemma 6.19, for any context $\varphi(p_0) \in \mathbf{PL}^{\otimes}$, $\varphi(\psi/p_0)$ would be relatively equivalent to $\bot$, ψ, $\otimes_2^1(\psi,\psi)$ or $\top$ in s. But it is obvious that $\otimes_3^2(\psi,\psi,\psi)$ is not relatively equivalent to $\bot$, ψ, $\otimes_2^1(\psi,\psi)$ or $\top$ in s. Hence, $\otimes_3^2(\psi,\psi,\psi)$ and $\varphi(\psi/p_0)$ are not relatively equivalent in s, and hence not equivalent in general, which gives rise to a contradiction. □

Theorem 6.21 *All the $\otimes_n^k$ are **not** uniformly definable in $\mathbf{PL}^{\otimes}$ except $\otimes_n^1$ and $\otimes_n^n$, i.e., for any $2 \le k \le n-1$, $\otimes_n^k$ is not uniformly definable.*

Proof. Note that $n \ge 2$ by definition. When $2 \le k \le n-1$ (thus $n \ge 3$), by

Proposition 6.15, $\otimes_3^2$ can be uniformly defined by $\otimes_n^k$, so $\otimes_3^2$ is not uniformly definable in $\mathbf{PL}^{\otimes}$ implies that $\otimes_n^k$ is not uniformly definable in $\mathbf{PL}^{\otimes}$. Based on the first two items of Proposition 6.15, we have the desired result. □

7 Conclusions and future work

In this paper, we proposed an epistemic interpretation of the tensor disjunction in dependence logic. The interpretation is inspired by the notion of weak disjunction in Medvedev's early work in terms of the BHK-like semantics. The connection between the two disjunctions is exposed in inquisitive logic with tensor disjunction ($\mathbf{InqB}^{\otimes}$) studied in the literature. We introduce a powerful dynamic epistemic language which can turn each formula in the language of $\mathbf{InqB}^{\otimes}$ into a know-how formula, which can be further reduced into a know-how-free formula. Along the way we need to use announcement operators and the propositional quantifiers to capture the epistemic meaning of the tensor disjunction. We give the axiomatization of our full logic, and generalize the tensor disjunction to a family of n-ary operators parameterized by a $k \leq n$, which capture the intuitive epistemic situations that knowing a list of n possible answers to n questions such that one knows at least k of them are correct.

We have seen that the propositional quantifiers are playing an important role in our framework, i.e., in defining the tensor and its generalizations. However, technically speaking, it might be an overkill since the expressive power of **InqB** and $\mathbf{InqB}^{\otimes}$ are the same. It remains to see whether we can use a simpler machinery to capture the epistemic meaning of tensor discussion without using the full power of the propositional quantifiers.

Besides further technical questions regarding our logic, the generalized tensor clearly has a life of its own, and invites further investigations. Its obvious combinatorial features may find applications in cryptographic protocols and game theory. To see the connection with the latter, we end the paper with the following interesting scenario where $\otimes_3^2$ makes perfect sense. Consider a badminton match between two teams. Each team has one good player with two other less capable ones. We can measure the abilities of the players by numbers which will determine the result of the matches in the most natural way. For team A, it is $6, 2, 2$ for the three players, and for team B it is $5, 3, 3$. The battle between the two teams consists of three single matches, and the rule of the game does not prevent one player from playing two matches if not in a row, although the second time the player will lose $1/3$ of his or her ability due to tiredness. Now, with some reflection, we can see team B has a unique arrangement of the playing players to make sure they can win at least two out of the three matches no matter how team A will do. Do you know which one?

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