Graded modal logic with a single modality

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Abstract

Graded modal logic is an extension of classical modal logic with modalities $\Diamond_n$, for $n \in \mathbb{N}$, that allows to count the number of successors of a state in a Kripke model. In this article we study the logics obtained by restricting the language to a single modality $\Diamond_n$ for a fixed natural number n, where $\Diamond_n \varphi$ is satisfied on a point w of a Kripke model exactly when w has at least n successors satisfying φ. We compare the logics $\mathcal{L}_n$ and $\mathcal{L}_m$ for $n \neq m$. We provide concrete axiomatizations in cases $n = 2$ and $n = 3$ and provide a method for generating axiomatizations for every n.

Keywords: Graded modal logic, Kripke models, completeness, monotone modal logic.

1 Introduction

Graded modal logic is an extension of classical modal logic with graded modalities $\Diamond_n (n \in \mathbb{N}^+)$ that allows to count the number of successors of a given state in a Kripke model. Intuitively, the formula $\Diamond_n A$ is satisfied at a point w of a Kripke frame if and only if w has at least n successors satisfying A.

Graded modal logic was originally introduced in Goble [10]. Kaplan [12] studied graded modal logic as an extension of **S5**. The completeness of graded modal logic and its extensions was investigated in [9,7,2]. Van der Hoek [15] and Cerrato [3] used filtrations to obtain the finite model property and decidability of graded modal logic. Van der Hoek [15] also studied the expressibility, definability and correspondence theory. Bisimulations for graded modal logic were introduced in [8], and used to provide an alternative proof of the finite model property, and show that a first-order formula is invariant under graded bisimulation iff it is equivalent to a graded modal formula. Aceto, Ingolfsdottir and Sack [1] showed that resource bisimulation and graded bisimulation coincide over image-finite Kripke frames. Finally, various notions of epistemic and dynamic graded modal logics have been investigated in [16] and [13].

Even though the modality $\Diamond_1$ corresponds to the standard classical modal logic connective, and therefore retains all its properties, the modalities $\Diamond_n$

for $n \geq 2$ do not. In particular, the modalities $\Diamond_n$ are monotone, i.e. they satisfy the rule $\vdash \varphi \to \psi / \vdash \Diamond_n \varphi \to \Diamond_n \psi$, and satisfy $\Diamond_n \bot \leftrightarrow \bot$, but are not *additive*, that is, the implication $\Diamond_n(p \vee q) \to (\Diamond_n p \vee \Diamond_n q)$ fails for $n \geq 2$. Modal logics with monotone modalities have been extensively studied [4,11,14]. However, not much work has been done regarding the connections between monotonic modal logics and graded modal logic. In [6], building on the proof-theoretic and algebraic analysis of non-normal modal logics of [5], a line of research studying these connection was initiated, where an elementary but not modally definable class of neighbourhood frames was shown to exactly correspond to graded Kripke frames, and the notion of graded bisimulation was recasted through the lens of neighbourhood bisimulations.

This article adds to the study of connections between monotonic modal logic and graded modal logic, albeit towards a different direction. Specifically, the standard axiomatization of graded modal logic relies on the *interaction* of the different graded modalities, and captures the properties of addition of natural numbers. However, when viewed as monotone modalities, each graded modality can also be studied in isolation. Accordingly, for every $n \in \mathbb{N}^+$, we introduce the logic $\mathcal{L}_n$, whose language contains a single modal operation, $\Diamond$, and whose theory is defined as the set of validities on Kripke frames, where $\Diamond$ is interpreted as the graded modality $\Diamond_n$ described in the first paragraph. We show the relationship between these logics, and that they have the finite model property and are decidable. Moreover, we investigate their axiomatizations.

This article is structured as follows: In Section 2, we present the basic definition of the logic $\mathcal{L}_n$ for each $n \in \mathbb{N}$ via its semantics. In Section 3 we show that if $n \neq m$ the logics $\mathcal{L}_n$ and $\mathcal{L}_m$ are distinct, we identify the relationship between them, and we observe that they are decidable and enjoy the strong finite model property. In Sections 4 we discuss possible axiomatizations and their completeness. In particular, we introduce axiomatizations of $\mathcal{L}_2$ and $\mathcal{L}_3$, and we present a method to generate axioms for $\mathcal{L}_n$, showing that all logics $\mathcal{L}_n$ are finitely axiomatizable. Finally, in Section 5 we suggest avenues for future research.

2 Preliminaries

In this section, we introduce the languages and semantics for the logics, and define some key notions that will be useful throughout this paper.

For every natural number $n \geq 2$, the language for graded modal logic restricted to the n-th modality will be the same Φ, generated by the following grammar:

$$\varphi ::= p \mid \neg\varphi \mid \varphi \wedge \varphi \mid \Diamond\varphi,$$

where p ranges over a countable collection of propositional variables AtProp. We define $\to$ and $\vee$, as usual.

Given some $n \geq 2$, the semantics of the language is given in terms of Kripke frames $\mathbb{X} = (X, R)$. For any valuation function $v : \mathsf{AtProp} \to \mathcal{P}(X)$ and any Kripke frame $\mathbb{X}$, $M = (\mathbb{X}, v)$ is a model for the graded modal logic restricted to the n-th modality. Truth in a model M at a state $x \in X$ is defined inductively as follows:

$$\begin{array}{lll} M, x \vDash_n p & \text{iff} & x \in v(p) \\ M, x \vDash_n \varphi \wedge \psi & \text{iff} & M, x \vDash_n \varphi \text{ and } M, x \vDash_n \psi \\ M, x \vDash_n \neg\varphi & \text{iff} & M, x \nvDash_n \varphi \\ M, x \vDash_n \Diamond\varphi & \text{iff} & |\{y \in R[x] : M, y \vDash_n \varphi\}| \geq n, \end{array}$$

where $R[z]$ with $z \in X$ indicates the direct image of $\{z\}$ through R. We write $M \vDash_n \varphi$ if $M, x \vDash_n \varphi$ for each $x \in X$ and we write $\mathbb{X} \vDash_n \varphi$ if $(\mathbb{X}, v) \vDash_n \varphi$ for every valuation v. Finally, we define

$$\mathcal{L}_n := \{\varphi \in \Phi \mid \forall \mathbb{X} \quad \mathbb{X} \vDash_n \varphi\}.$$

In what follows, to help the reader identify the intended interpretation of formulas, we will sometimes slightly abuse notation, and use the modality $\Diamond_n$ instead of $\Diamond$, when the formula is to be interpreted in $\mathcal{L}_n$. Seeing this formally, given Φ_G, the language of graded modal logic defined as

$$\Phi_G \ni \varphi ::= p \mid \neg\varphi \mid \varphi \wedge \varphi \mid \Diamond_n\varphi, \quad n \in \mathbb{N}^+,$$

we can define an embedding $\epsilon : \mathbb{N}^+ \times \Phi \to \Phi_G$ recursively by letting $\epsilon(n, \Diamond\varphi) = \Diamond_n \epsilon(n, \varphi)$.

Even though the language and semantics of $\mathcal{L}_n$ cannot express the classical normal modality [1], it turns out that the notion that a point w of a Kripke frame has at least one successor satisfying a formula φ can sometimes be captured with a help of an auxiliary formula. In particular, consider the following formula

$$\Diamond_1^{\psi}\varphi := \Diamond(\varphi \vee \psi) \wedge \neg\Diamond\psi \tag{1}$$

It is easy to see that if $M, w \models_n \Diamond_1^{\psi}\varphi$, then there exists some u, such that wRu, and $M, u \models_n \varphi$. Formulas of this form will be key in the construction of the canonical model used in the proofs of completeness

Another important convention that we will follow in this article is the following: We will reserve small letters from the Greek alphabet to denote sequences of mutually contradictory formulas. In particular, when we write $\alpha_1, \ldots, \alpha_n$ we understand that $\alpha_i \wedge \alpha_j \to \perp$ is provable in classical logic for $i \neq j$. For example, $\alpha_i = p_i \wedge \neg(\bigvee_{j \neq i} p_j)$.

Finally, throughout this paper we write $\mathbb{N}^+$ to denote the set of positive natural numbers; we will also slightly abuse notation and identify $n \in \mathbb{N}^+$ with the set $\{1, \ldots, n\}$.

3 Basic properties

In this section we discuss the basic properties of the logics $\mathcal{L}_n$ and we compare their validities.

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[1] For instance, a model with one point that is reflexive and and a model with a point which is not reflexive are indistinguishable for all the logics $\mathcal{L}_n$ for $n \geq 2$.

3.1 Decidability and Finite model property

A formula $\varphi \in \Phi$ is a validity in $\mathcal{L}_n$ if and only if $\epsilon(n, \varphi)$ is a validity in graded modal logic. Since graded modal logic is decidable, it immediately follows that $\mathcal{L}_n$ is decidable. For the strong finite model property, the filtration construction in [15, Section 6.1] works in this case verbatim, given that $\epsilon[\{n\}, \Phi] \subseteq \Phi_G$.

3.2 Comparing the logics $\mathcal{L}_n$

Lemma 3.1 *If $n < m$, there exists a formula ζ_n, such that $\zeta_n \in \mathcal{L}_n$ but $\zeta_n \notin \mathcal{L}_m$.*

Proof. Consider the formula

$$\zeta_n \coloneqq \left(\bigwedge_{i=1}^{n} \Diamond_1^{q_i} \alpha_i \right) \to \Diamond \left(\bigvee_{i=1}^{n} \alpha_i \right).$$

Given any model M, if $M, x \vDash_n \bigwedge_{i=1}^{n} \Diamond_1^{q_i} \alpha_i$, then for each $i \in n$, there exists a y_i, such that xRy_i and $M, y_i \vDash_n \alpha_i$. Since α_i are mutually contradictory formulas, the y_i are all distinct, and hence $M, x \vDash_n \Diamond \left(\bigvee_{i=1}^{n} \alpha_i \right)$.

On the other hand, let $m > n$ and consider the model

$$M = (m + n, R, v),$$

where $R = \{(1, k) \mid 2 \leq k \leq m + n\}$, $v(q_i) = m \setminus 1$, and $v(\alpha_i) = \{m + i\}$. Then $M, 1 \vDash_m \Diamond_1^{q_i} \alpha_i$, since $m \setminus 1 \cup \{m + i\} \subseteq R[1]$. However, there are only n points that satisfy $\bigvee_{i=1}^{n} \alpha_i$, and therefore $M, 1 \nvDash_m \Diamond \left(\bigvee_{i=1}^{n} \alpha_i \right)$. $\square$

Lemma 3.2 *Assume $n < m$ such that $m - 1 = (n - 1) \cdot k + r$ where $r < n - 1$. Then, $\mathcal{L}_m \subseteq \mathcal{L}_n$ if and only if $r < k$.*

Proof. First, let's assume that $r < k$. To show that $\mathcal{L}_m \subseteq \mathcal{L}_n$ it is enough to show that for every formula $\varphi \in \Phi$ and every model M, there exists a model M' such that $M, w \vDash_n \varphi$ if and only if $M, w' \vDash_m \varphi$. Given a model $M = (X, R, v)$, we define $M' = (X \times k, R', v')$, where $(x, i)R'(y, j)$ if and only if xRy, and $v'(p) = v(p) \times k$. We will show that for any formula $\varphi \in \Phi$,

$$M, w \vDash_n \varphi \quad \Longleftrightarrow \quad M', (w, j) \vDash_m \varphi$$

by induction on the complexity of φ. All cases are immediate, except for the case where $\varphi = \Diamond\psi$. Let's assume that $M, w \nvDash_n \Diamond\psi$. Then there are $z \leq n - 1$ successors of w satisfying ψ, so, by the induction hypothesis, exactly $z \cdot k \leq (n - 1) \cdot k \leq m - 1$ successors of (w, j) satisfy ψ, hence $M, (w, j) \nvDash_m \Diamond\psi$. Now assume that $M, (w, j) \nvDash_m \Diamond\psi$. By definition of M', it follows that (w, j) has $z \cdot k$ successors satisfying ψ where $z \cdot k \leq m - 1$. By induction hypothesis, w has z successors satisfying φ. Since $z \cdot k \leq m - 1$ and $r < k$ we have

$$z \cdot k \leq m - 1 = (n - 1) \cdot k + r < (n - 1) \cdot k + k = n \cdot k,$$

which implies that $z < n$. Hence $M, w \nvDash_n \Diamond\psi$.

Now let us assume that $k \leq r$. We consider the formula

$$\theta_n := \bigwedge_{i=1}^{n} \bigwedge_{j=1}^{n-1} \left(\diamond_1^{q_i^j} \alpha_i^j \wedge \bigwedge_{i=1}^{n} \neg\diamond(\bigvee_{j=1}^{n-1} \alpha_i^j) \right) \to \bigvee_{s:n \to n-1} \neg\diamond \bigvee_{i=1}^{n} \alpha_i^{s(i)}.$$

Let us show that $\theta_n \notin \mathcal{L}_n$. Recall that α_i^j are mutually contradictory. Let

$$M = (n \times n \sqcup \{x_1, \ldots x_{n-1}\}, R, v)$$

where $R[(1,1)] = n \times n \sqcup \{x_1, \ldots x_{n-1}\}$, $v(q_i^j) = \{x_1 \ldots, x_{n-1}\}$, and $v(\alpha_i^j) = \{(i, j+1)\}$. It is routine to check that $M, (1,1) \nvDash_n \theta_n$.

Let's show now that $\theta_n \in \mathcal{L}_m$. Let M be a model. If w satisfies the antededent of the implication, then, for each i, there exists some j (let's call it $s(i)$), such that w has at most k successors satisfying α_i^j. Indeed, otherwise for each j, w has at least $k+1$ successors satisfying α_i^j and hence it has at least $(n-1) \cdot (k+1)$ successors satisfying $\bigvee_{j=1}^{n-1} \alpha_i^j$ by the fact that the α_i^j are mutually contradictory. Since

$$(n-1) \cdot (k+1) = (n-1) \cdot k + (n-1) > k(n-1) + r = m - 1,$$

it follows that $M, w \vDash_m \diamond \bigvee_{j=1}^{n-1} \alpha_i^j$, a contradiction. Now, since w has at most k successors satisfying $\alpha_i^{s(i)}$ for every i, it follows that w has at most $n \cdot k$ successors satisfying $\bigvee_{i=1}^{n} \alpha_i^{s(i)}$. Since

$$n \cdot k = (n-1) \cdot k + k \leq (n-1) \cdot k + r = m - 1,$$

it follows that $M, w \vDash_m \neg\diamond \bigvee_{i=1}^{n} \alpha_i^{s(i)}$. Hence $M, w \vDash_m \theta_n$. $\square$

Summarizing the above results, we obtain a complete description of the relation between the logics $\mathcal{L}_n$:

Theorem 3.3 *Let $n < m$ such that $m - 1 = (n-1) \cdot k + r$ where $r < n - 1$. Then, if $r < k$ it follows that $\mathcal{L}_m \subsetneq \mathcal{L}_n$. If $k \leq r$, then there exists $\zeta_n, \theta_n \in \Phi$, such that $\zeta_n \in \mathcal{L}_n$ while $\zeta_n \notin \mathcal{L}_m$ and $\theta_n \in \mathcal{L}_m$ while $\theta_n \notin \mathcal{L}_n$.*

4 Axiomatizations and completeness

In this section we will discuss the completeness of the logics $\mathcal{L}_n$ with respect to the proposed axiomatizations. Before going into the various cases, we will present the axioms that are present in the axiomatization of every $\mathcal{L}_n$, as well as a key result that will allow us to construct enough distinct points in the canonical model.

For every logic $\mathcal{L}_n$, their corresponding axiomatization GP_n will include all the propositional tautologies (or an axiomatization of them), the formula

$$(\bot) \quad \diamond\bot \to \bot,$$

and it will be closed under modus ponens, uniform substitution, and the monotonicity rule

$$\frac{\vdash p \to q}{\vdash \Box p \to \Box q} \quad \text{(M)}.$$

We call this basic system GP_0, which we will augment with further axioms (depending on n) in the following sections.

Lemma 4.1 *Let* $(\mathbb{B}, \Diamond)$ *be a Boolean algebra with a monotone operation satisfying* $\Diamond\bot = \bot$. *Let* u *be an ultrafitler on* $\mathbb{B}$ *and let*

$$Z_u = \{a \in \mathbb{B} \mid \forall b \in \mathbb{B}(\Diamond(a \vee b) \in u \Rightarrow \Diamond b \in u)\}.$$

Then Z_u *is an ideal on* $\mathbb{B}$ *such that* $\Diamond a \in u$ *implies that* $a \notin Z_u$.

Proof. Clearly $\Diamond(c \vee \bot) = \Diamond(c)$ so $\bot \in Z_u$. Assume that $b \in Z_u$ and $a \leq b$. Then if $\Diamond(c \vee a) \in u$, by monotonicity it follows that $\Diamond(c \vee b) \in u$, which implies $\Diamond c \in u$, so $a \in Z_u$. Finally assume that $a, b \in Z_u$, and $\Diamond(c \vee (a \vee b)) \in u$. Then since $a \in Z_u$, it follows that $\Diamond(c \vee b) \in u$ and since $b \in Z_u$ it follows that $\Diamond c \in u$. Finally, assume that $\Diamond a \in u$. Then $\Diamond\bot \notin u$, while $\Diamond(a \vee \bot) \in u$. Hence $a \notin Z_u$. $\Box$

Remark 4.2 Notice that $Z_u = \{a \in \mathbb{B} \mid \forall b \in \mathbb{B}\ (\Diamond_1^b a \notin u)\}$.

4.1 Case n=2

The system GP_2 is obtained by adding the following axiom-schema to GP_0:

$$\text{(G2)} \quad [\Diamond_1^{q_1}(\alpha_1) \wedge \Diamond_1^{q_2}(\alpha_2)] \to \Diamond(\alpha_1 \vee \alpha_2);$$

where, as discussed in Section 2, α_1 and α_2 contradict each other. This axiom intuitively states if w has at least one successor satisfying α_1 (witnessed using q_1) and at least one successor satisfying α_2 (witnessed using q_2), then w has at least two successors satisfying $\alpha_1 \vee \alpha_2$. It is easy to check, also given this explanation, that G2 is sound.

Completeness. To show completeness we will construct a canonical model using, as usual, the ultrafilters of the free Boolean algebra generated by the axiomatic system. Let $\mathbb{B}$ be a Boolean algebra with a monotone operation satisfying the axioms and rules of GP_2. Let u be an ultrafilter on $\mathbb{B}$. Let us define $e_u : \mathbb{B} \to \{0, 1, 2\}$ as follows:

$$e_u(a) = \begin{cases} 2 & \text{if } \Diamond a \in u, \\ 1 & \text{if } \Diamond a \notin u \text{ and } (\exists b \in \mathbb{B})(\Diamond b \notin u \text{ and } \Diamond(a \vee b) \in u), \\ 0 & \text{otherwise.} \end{cases}$$

Intuitively, this function roughly represents the number of permissible successors of u satisfying the statement a. Indeed, if $\Diamond a \in u$, u must have at least 2 successors satisfying a, while if $e_u(a) = 1$, u must have exactly one successor satisfying a. In particular notice that $e_u(a) = 1$, if $\Diamond a \notin u$ and $\Diamond_1^b a \in u$, for some $b \in \mathbb{B}$.

Corollary 4.3 *The set* $Z_u = \{a \in \mathbb{B} \mid e_u(a) = 0\}$ *is an ideal.*

Proof. Follows immediately from Lemma 4.1 and the definition of e_u. □

In the remainder of the paper, we write $\mathbf{n}$ to denote the set $\{1, 2, \ldots, n\}$ given any natural number n.

Definition 4.4 Let $\mathbb{B}$ be the free Boolean algebra generated by the axiomatic system above. The *canonical frame* of GP_2 is the Kripke frame $\mathbb{X}_{\mathbb{B}} = (\mathcal{U}(\mathbb{B}) \times \mathbf{2}, R)$, where $\mathcal{U}(\mathbb{B})$ denotes the collection of ultrafilters on $\mathbb{B}$, and R is such that for any $u, w \in \mathcal{U}(\mathbb{B})$, and $i, j \in \mathbf{2}$,

$$(u, j)R(w, i) \qquad \text{iff} \qquad e(w, u) \geq i,$$

where $e(w, u) = \min\{e_u(a) \mid a \in w\}$. The *canonical model* of GP_2 is the Kripke model $M_{\mathbb{B}} = (\mathbb{X}_{\mathbb{B}}, v)$ such that, for any $u \in \mathcal{U}(\mathbb{B})$ and $p \in \mathsf{AtProp}$,

$$(u, i) \in v(p) \qquad \text{iff} \qquad p \in u.$$

Lemma 4.5 (Truth lemma for GP_2) *For any Φ-formula φ, $u \in \mathcal{U}(\mathbb{B})$, and $i \in 2$,*

$$M_{\mathbb{B}}, (u, i) \vDash_2 \varphi \qquad \textit{iff} \qquad \varphi \in u.$$

Proof. We proceed by induction on the complexity of φ. All the cases are trivial, except the one in which $\varphi = \Diamond\psi$ for some $\psi \in \Phi$.

Assume that $M_{\mathbb{B}}, (u, i) \vDash_2 \Diamond\psi$. Then, there exist $(w, j), (r, k) \in \mathcal{U}(\mathbb{B}) \times \mathbf{2}$ such that $(u, i)R(w, j)$, $(u, i)R(r, k)$, and $\psi \in w \cap r$ by the induction hypothesis. Consider two cases:

If $w = r$, then we can assume without loss of generality that $j = 1$ and $k = 2$. So $e(w, u) \geq 2$, and therefore for all $a \in w$, $e_u(a) = 2$ holds; hence $e_u(\psi) = 2$, i.e., $\Diamond\psi \in u$.

Now suppose that $w \neq r$ and $e(w, u) = e(r, u) = 1$, as otherwise, if $e(w, u) = 2$ or $e(r, u) = 2$, we proceed as above. Since $w \neq r$, there is some $\theta \in w \setminus r$, and therefore $\neg\theta \in r \setminus w$. Since $e(w, u) = e(r, u) = 1$, we can assume without loss of generality that $e_u(\psi \wedge \theta) = e_u(\psi \wedge \neg\theta) = 1$. By definition of e_u, there are a and b such that

$$\neg\Diamond a, \quad \neg\Diamond b, \quad \Diamond((\psi \wedge \theta) \vee a), \quad \Diamond((\psi \wedge \neg\theta) \vee b) \quad \in u.$$

Hence by (G2) and modus ponens $\Diamond((\psi \wedge \theta) \vee (\psi \wedge \neg\theta)) \in u$, i.e., $\Diamond\psi \in u$.

For the converse direction assume that $\Diamond\psi \in u$. If there is an ultrafilter w such that $e(w, u) = 2$ and $\psi \in w$, then we are done since both $(w, 1)$ and $(w, 2)$ are R-successors of (u, i). Suppose that there is no such ultrafilter. Since, Z_u is an ideal by Corollary 4.3, and $\psi \notin Z_u$ since $e_u(\psi) = 2$, by the prime ideal theorem (PIT), there exists some ultrafilter w such that $\psi \in w$ and $w \cap Z_u = \varnothing$; thus $(u, i)R(w, 1)$ as $e(w, u) \geq 1$. Having ruled out ultrafilters containing ψ and such that $e(w, u) = 2$, it must be $e(w, u) = 1$. Therefore, there exists some $\zeta \in w$ such that $e_u(\zeta) = 1$, implying $e_u(\psi \wedge \zeta) = 1$, so $\Diamond(\psi \wedge \zeta) \notin u$. By hypothesis $\Diamond\psi \in u$, thus $e_u(\psi \wedge \neg\zeta) \geq 1$. Using PIT again, there exists some ultrafilter w' such that $\psi \wedge \neg\zeta \in w'$ and $w' \cap Z_u = \varnothing$; hence $(u,i)R(w',1)$ as $e_u(w,u) \geq 1$. Since $\psi \in w \cap w'$ and $\psi \wedge \neg\zeta \notin w$, it follows that $w \neq w'$. Therefore, $M_{\mathbb{B}}, (u,i) \vDash_2 \varphi$ holds. This concludes the proof. $\Box$

From the lemma above, using the standard argument, the following theorem holds.

Theorem 4.6 *The system GP_2 is strongly complete with respect to the logic $\mathcal{L}_2$.*

4.2 Case n=3

The system GP_3 is obtained by adding the following axiom-schemata to GP_0:

(G3$_1$) $[\Diamond_1^{q_1}(\alpha_1) \wedge \Diamond_1^{q_2}(\alpha_2) \wedge \Diamond_1^{q_3}(\alpha_3)] \to \Diamond(\alpha_1 \vee \alpha_2 \vee \alpha_3)$,

(G3$_2$) $[\Diamond_1^{q_1}(\alpha_2) \wedge \Diamond_1^{q_2}(\beta_2) \wedge \Diamond(\alpha_1 \vee \beta_1) \wedge \neg\Diamond(\alpha_1 \vee \alpha_2)] \to \Diamond(\beta_1 \vee \beta_2)$;

where, as discussed in Section 2, the α_i contradict each other, and likewise the β_i. The axiom (G3$_1$) states that if w has at least one successor satisfying each of α_1, α_2 and α_3, then w must satisfy $\Diamond(\alpha_1 \vee \alpha_2 \vee \alpha_3)$. The axiom (G3$_2$) intuitively expresses the idea that if w has at most one successor satisfying α_1 (which is captured by the fact that w doesn't satisfy $\Diamond(\alpha_1 \vee \beta_1)$), while satisfying $\Diamond(\alpha_1 \vee \beta_1)$, then w must have at least 2 successors satisfying β_1. It is routine to verify the soundness of these axioms.

Completeness. Let $\mathbb{B}$ be a Boolean algebra with a monotone operation satisfying the axioms of GP_3. Let u be an ultrafilter on $\mathbb{B}$. Let us define $e_u : \mathbb{B} \to \{0,1,2,3\}$ as follows:

$$e_u(a) = \begin{cases} 3 & \text{if } \Diamond a \in u, \\ 2 & \text{if } e_u(a) \neq 3 \text{ and } (\forall b,c \in \mathbb{B})\big(\Diamond c \notin u \text{ and } \Diamond(b \vee c) \in u, \\ & \qquad\qquad\qquad a \wedge b = \bot \implies \Diamond(a \vee b) \in u\big) \\ 1 & \text{if } e_u(a) \notin \{3,2\} \text{ and } (\exists b \in \mathbb{B})(\Diamond b \notin u \text{ and } \Diamond(a \vee b) \in u), \\ 0 & \text{otherwise.} \end{cases}$$

Notice that we can write the condition for 2 as

$$\forall b,c \in \mathbb{B}\ ((\Diamond_1^c b \in u \text{ and } a \wedge b = \bot) \Rightarrow \Diamond(a \vee b) \in u).$$

Corollary 4.7 *The set $Z_u = \{a \in \mathbb{B} \mid e_u(a) = 0\}$ is an ideal.*

Proof. Follows immediately from Lemma 4.1 and the definition of e_u. $\Box$

Definition 4.8 Let $\mathbb{B}$ be the free Boolean algebra of GP_3. The *canonical frame* of GP_3 is the Kripke frame $\mathbb{X}_{\mathbb{B}} = (\mathcal{U}(\mathbb{B}) \times \mathbf{3}, R)$, where $\mathcal{U}(\mathbb{B})$ denotes the collection of ultrafilters of $\mathbb{B}$, and R is such that for any $u, w \in \mathcal{U}(\mathbb{B})$, and $i, j \in \mathbf{3}$,

$$(u,j)R(w,i) \qquad \text{iff} \qquad e(w,u) \geq i,$$

where $e(w,u) = \min\{e_u(a) \mid a \in w\}$. The *canonical model* of GP_3 is the Kripke model $M_{\mathbb{B}} = (\mathbb{X}_{\mathbb{B}}, v)$ such that, for any $u \in \mathcal{U}(\mathbb{V})$ and $p \in \mathsf{AtProp}$,

$$u \in v(p) \qquad \text{iff} \qquad p \in u.$$

Lemma 4.9 (Truth lemma for GP_3**)** *For any* Φ*-formula* φ*,* $u \in \mathcal{U}(\mathbb{B})$*, and* $i \in 3$*:*

$$M_{\mathbb{B}}, (u, i) \vDash_3 \varphi \qquad \textit{iff} \qquad \varphi \in u.$$

Proof. We proceed by induction on the complexity of φ. The only non-trivial case is when $\varphi = \Diamond\psi$ for some $\psi \in \Phi$.

Assume $M_{\mathbb{B}}, (u, i) \vDash_3 \varphi$. Then there are $(w_1, j_1), (w_2, j_2), (w_3, j_3) \in \mathcal{U}(\mathbb{B}) \times \mathbf{3}$ such that $(u, i)R(w_1, j_1)$, $(u, i)R(w_2, j_2)$, $(u, i)R(w_3, j_3)$, and $\psi \in w_1 \cap w_2 \cap w_3$. Without loss of generality, we assume $j_1 \leq j_2 \leq j_3$. There are three possible cases:

(1) $w_1 = w_2 = w_3$ and $j_1 = 1$, $j_2 = 2$, and $j_3 = 3$. In this case, $e(w, u) = 3$, thus $e_u(\psi) = 3$, i.e., $\varphi = \Diamond\psi \in u$.

(2) $w_1 \neq w_2 = w_3$, $j_1 = j_2 = 1$, and $j_3 = 2$. In this case, there is $\theta \in w_1 \setminus w_2$, hence $\neg\theta \in w_2 \setminus w_1$. If $e_u(\psi \wedge \theta) = 3$, then by monotonicity $e_u(\Diamond\psi) = 3$ and we proceed as the case above. So let us suppose that $e_u(\psi \wedge \theta) = 2$. Since $\neg\theta \in w_2$, it must be that $e_u(\psi \wedge \neg\theta) \geq 1$, i.e. $\Diamond_1^b(\psi \wedge \neg\theta) \in u$. By the definition of $e_u(\cdot)$, since $e_u(\psi \wedge \theta) = 2$ it follows that if $\Diamond_1^b c \in u$, then $\Diamond((\psi \wedge \theta) \vee c) \in u$. Hence, $\Diamond((\psi \wedge \theta) \vee (\psi \wedge \neg\theta)) \in u$, i.e. $\Diamond\psi \in u$.

(3) $w_1 \neq w_2 \neq w_3 \neq w_1$, and $j_1 = j_2 = j_3 = 1$. Clearly, there are

$$\theta_1 \in w_1 \setminus (w_2 \cup w_3), \qquad \theta_2 \in w_2 \setminus (w_1 \cup w_3), \qquad \theta_3 \in w_3 \setminus (w_1 \cup w_2),$$

such that $e_u(\theta_1) = e_u(\theta_2) = e_u(\theta_3) = 1$, and $\theta_1, \theta_2, \theta_3 \leq \psi$ [2], which are w.l.o.g. contradictory. By the definition of $e_u(\cdot)$, there are $\zeta_1, \zeta_2, \zeta_3$ such that

$$\begin{array}{ccc} \neg\Diamond\zeta_1, & \neg\Diamond\zeta_2, & \neg\Diamond\zeta_3, \\ \Diamond(\theta_1 \vee \zeta_1) \in u, & \Diamond(\theta_2 \vee \zeta_2) \in u, & \Diamond(\theta_3 \vee \zeta_3) \in u. \end{array}$$

By axiom (G3$_1$) and modus ponens, $\Diamond(\theta_1 \vee \theta_2 \vee \theta_3) \in u$. By monotonicity of $\Diamond$ (axiom (M)), as $\theta_1 \vee \theta_2 \vee \theta_3 \leq \psi$, then $\Diamond\psi \in u$.

For the converse direction, assume $\Diamond\psi \in u$. There are three possible cases:

(1) There is an ultrafilter w such that $e(w, u) = 3$ and $\psi \in w$. In this case we are done since $(w, 1)$, $(w, 2)$, and $(w, 3)$ are (distinct) successors of (u, i) (for any $i \in 3$), i.e. $M_{\mathbb{B}}, (u, i) \vDash_3 \Diamond\psi$.

(2) There is an ultrafilter w such that $e(w, u) = 2$ and $\psi \in w$. Since $e(w, u) = 2$, there is $\theta \in w$ such that $e_u(\theta) = 2$; hence $e_u(\psi \wedge \theta) = 2$ and, since $\Diamond\psi \in u$, $e_u(\psi \wedge \neg\theta) \geq 1$. By the prime ideal theorem, there exists an ultrafilter w' such that $\psi \wedge \neg\theta \in w'$ and $w' \cap Z_u = \varnothing$, i.e. $e(w', u) \geq 1$, and thus $(u, 1)R(w', 1)$. Since $e(w, u) = 2$, we have that $(u, 1)R(w, 1)$ and $(u, 1)R(w, 2)$. Because $\psi \wedge \theta \in w$, $w \neq w'$. Hence there are at least three different successors of (u, i) that satisfy ψ, i.e. $M_{\mathbb{B}}, (u, i) \vDash_3 \Diamond\psi$.

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[2] In general, given two distinct filters f_1 and f_2 of some lattice L, and an element $x \in f_1 \cap f_2$, there is an element $y \in f_1 \setminus f_2$ such that $y \leq x$. Indeed, without loss of generality, there is $z \in f_1 \setminus f_2$. As f_1 is a filter, $y := x \wedge z \in f_1$. Clearly $y \leq x$, and $y \notin f_2$, as otherwise also x would be in f_2.

(3) Every ultrafilter w that contains ψ is such that $e(w,u) \leq 1$. By the PIT, there is some ultrafilter w_1 such that $\psi \in w_1$ and $w_1 \cap Z_u = \varnothing$ (as $e_u(\psi) = 3$), hence $e(w_1, u) = 1$, yielding $(u,i)R(w_1,1)$. It follows that there exists $\theta \in w$ such that $e_u(\theta) = 1$, and so $e(\psi \wedge \theta) = 1$. Therefore, there are b and c such that $\Diamond_1^c b \in u$, $(\psi \wedge \theta) \wedge b = \bot$, and $\Diamond((\psi \wedge \theta) \vee b) \notin u$. Hence $\neg\Diamond((\psi \wedge \theta) \vee b) \in u$, and hence, by axiom (G3$_2$), for any $\Diamond_1^e d \in u$, with $d \wedge \psi \wedge \neg\theta = \bot$, it follows that $\Diamond(\psi \wedge \neg\theta \vee d) \in u$, i.e. $e_u(\psi \wedge \neg\theta) \geq 2$. Hence there exists ζ such that $e_u(\psi \wedge \neg\theta \wedge \zeta) = 1$ and an ultrafilter w_2, such that $\psi \wedge \neg\theta \wedge \zeta \in w_2$ and $w_2 \cap Z_u = \varnothing$, i.e. $(u,i)R(w_2,1)$. Now, if $\Diamond((\psi \wedge \theta) \vee (\psi \wedge \neg\theta \wedge \zeta)) \in u$, axiom (G3$_2$) implies (arguing as above) that $e_u(\psi \wedge \neg\theta \wedge \zeta)) \geq 2$, a contradiction. Hence $\neg\Diamond((\psi \wedge \theta) \vee (\psi \wedge \neg\theta \wedge \zeta)) \in u$ and, as $\Diamond\psi \in u$, it follows that $e_u(\psi \wedge \neg\theta \wedge \neg\zeta) \geq 1$. By the PIT there is an ultrafilter w_3 containing $\psi \wedge \neg\theta \wedge \neg\zeta$ and such that $w_2 \cap Z_u = \varnothing$, i.e. $(u,i)R(w_3,1)$. Clearly w_1, w_2, w_3 are all distinct and hence we have $M_{\mathbb{B}}, (u,i) \vDash_3 \Diamond\psi$.

This concludes the proof. □

From Lemma 4.9, using the standard argument, the following theorem holds.

Theorem 4.10 *The system GP_3 is strongly complete with respect to the logic $\mathcal{L}_3$.*

4.3 General case

Through the remainder of this section we will fix a natural number n and its corresponding logic $\mathcal{L}_n$. For the general case, we will not provide an explicit axiomatization but connect axioms with inequalities of positive natural numbers.

The logic. We intend to introduce axioms that encode, in the same way as the axioms of GP_2 and GP_3 did, several implications regarding inequalities of natural numbers. In particular, we want to express for each $m_i, m_j \leq n$, $I = \{1, 2, \ldots, m_i\}$, $J = \{1, \ldots, m_j\}$, and positive natural numbers $x_i^j \in \mathbb{N}$ (with $i \in I$ and $j \in J$),

$$\bigwedge_{j \in J} (\sum_{i \in I} x_i^j < n) \to \bigvee_{k \in K \subset J} (\sum_{h \in H \subset I} x_h^k < n). \tag{2}$$

Clearly, as there can be just a finite number of sets I and J since their size is bounded, there is a finite amount of such implications, and each implication has a finite number of inequalities on both sides.

Definition 4.11 Let $GP_n \subseteq \Phi$ be the collection of Φ-formulas that contains GP_0 and for each *true* implication as in (2), an axiom

$$\left(\Diamond_1^{q_i^j} (\alpha_i^j) \wedge \bigwedge_{j \in J} \neg\Diamond \bigvee_{i \in I} \alpha_i^j \right) \to \bigvee_{k \in K \subset J} \neg\Diamond \bigvee_{h \in H \subset I} \alpha_h^k.$$

As discussed above, the number of possible such inequalities is finite and hence also the axiomatization proposed here is finite. Furthermore, knowing whether the implication of inequalities is true or not is a decidable procedure: these are statements in Presburger arithmetic which is a decidable theory.

Completeness. Let $\mathbb{B}$ be a Boolean algebra with a monotone operation satisfying the axioms of GP_n. Let u be an ultrafilter on $\mathbb{B}$. Similar to the previous cases, we want to define a function $e_u : \mathbb{B} \to \mathbb{N}$ which satisfies the following conditions for every $\varphi \in \Phi$:

(i) $e_u(\varphi) \geq n$ if $\Diamond(\varphi) \in u$,

(ii) $e_u(\varphi) < n$ if $\Diamond(\varphi) \notin u$,

(iii) $e_u(\varphi) + e_u(\psi) = e_u(\varphi \wedge \psi) + e_u(\varphi \vee \psi)$, for every $\psi \in \Phi$,

(iv) $e_u(\varphi) = 0$ whenever for every $\psi \in \Phi$, $\Diamond(\psi \vee \varphi) \to \Diamond(\psi) \in u$.

Lemma 4.12 *Such an e_u exists for each ultrafilter $u \in \mathcal{U}(\mathbb{B})$.*

Proof. These restrictions on e_u define a system of equations, Δ, that has a solution if and only if $\Delta \cup P$ is satisfiable, where P is the set of the axioms of Presburger arithmetic. This set, in turn, is satisfiable if and only if every finite subset $Y \subseteq \Delta \cup P$ of it is satisfiable, by the compactness of first-order logic. Finally Y is satisfiable if and only if $X = Y \cap \Delta$ has a solution. Hence, let Z be a finite subset of Δ. The system Z contains as parameters formulas of Φ. Let's call the set of these formulas Φ_Z We will strengthen the system Z, by adding extra condition: We stipulate that for $\varphi \in \Phi_Z$, $e_u(\varphi) = 0$ whenever for every $\psi \in \Phi_Z$, $\Diamond(\psi \vee \varphi) \to \Diamond(\psi) \in u$. Let's call this new system X. Notice that $\Phi_Z = \Phi_X$. Clearly if X has a solution, so does Z. So let's show that X has a solution.

Let $\mathbb{B}_X$ be the finite Boolean algebra generated by Φ_X. Since $\mathbb{B}_X$ is finite, it is routine to verify that the subsystem X has a solution if and only if there exists an assignment $s : A \to \mathbb{N}$ on the atoms A of $\mathbb{B}_X$ such that $\sum_{a \in C \subseteq A} s(a) \geq n$ if and only if $\Diamond(\bigvee_{a \in C \subseteq A} a) \in u$ for every $C \subseteq A$.

First notice that, by monotonicity of $\Diamond$, for every $a_1, \ldots, a_m$,

$$e_u(\bigvee_{i=1}^{m} a_i) = 0 \quad \text{iff} \quad (\forall i \in m) s(a_i) = 0.$$

Indeed, for $\varphi \in \Phi_X$ if $\Diamond(a_i \vee \varphi) \in u$ then $\Diamond(\bigvee_{i=1}^{m} a_i \vee \varphi) \in u$; therefore $\Diamond\varphi \in u$. For the other direction assume that $\Diamond((a_1 \vee \cdots \vee a_m) \vee \varphi) \in u$. Then $\Diamond(a_2 \vee \cdots \vee a_m \vee \varphi) \in u$. Continuing recursively on m, we conclude that $\Diamond\varphi \in u$.

Now let us show that such an s exists. If no such s exists, then this is a true statement about inequalities

$$\bigwedge_{j \in J} (\sum_{i \in I} x_i^j < n) \to \bigvee_{k \in K \subset J} (\sum_{h \in H \subset I} x_h^k < n),$$

where J corresponds to the set of inequalities in X of the form $e_u(\varphi) < n$ and K to the set of inequalities in X of the form $e_u(\varphi) \geq n$. But then GP_n contains an axiom of the form

$$\left(\Diamond_1^{q_i^j}(\alpha_i^j) \wedge \bigwedge_{j\in J} \neg\Diamond \bigvee_{i\in I} \alpha_i^j\right) \to \bigvee_{k\in K\subset J} \neg\Diamond \bigvee_{h\in H\subset I} \alpha_h^k.$$

This is a contradiction, since by definition on the conditions of e_u $\neg\Diamond(\bigvee_{i\in I}\alpha_i^j) \in u$ and $\Diamond(\bigvee_{h\in H}\alpha_h^k) \in u$ for every $k \in K$. Therefore s exists, and so e_u also exists. □

By Lemma 4.12, for each ultrafilter u, there is some map $e_u : \mathbb{B} \to \mathbb{N}$ satisfying the conditions above. By the axiom of choice, we choose one of such e_u for every ultrafilter and define

$$e(w, u) = \min\{e_u(a) \mid a \in w\}.$$

Notice that for each e_u, also the inverse of condition (iii) holds: that is if there exists ψ such that $\Diamond(\varphi \vee \psi) \in u$ and $\Diamond(\psi) \notin u$, then $e_u(\varphi \vee \psi) \geq n$, while $e_u(\psi) < n$. So

$$s(\varphi) = s(\varphi \vee \psi) - s(\psi) + s(\varphi \wedge \psi) > 0.$$

Corollary 4.13 *The set* $Z_u = \{a \in \mathbb{B} \mid e_u(a) = 0\}$ *is an ideal.*

Proof. Immediately by Lemma 4.1. □

Definition 4.14 Let $\mathbb{B}$ be the free Boolean algebra of GP_n. The *canonical frame* of GP_n is the Kripke frame $\mathbb{X}_{\mathbb{B}} = (\mathcal{U}(\mathbb{B}) \times \mathbf{n}, R)$, where $\mathcal{U}(\mathbb{B})$ denotes the collection of ultrafilters of $\mathbb{B}$, and R is such that for any $u, w \in \mathcal{U}(\mathbb{B})$, and $i, j \in \mathbf{n}$,

$$(u, j)R(w, i) \quad \text{iff} \quad e(w, u) \geq i.$$

The *canonical model* of GP_n is the Kripke model $M_{\mathbb{B}} = (\mathbb{X}_{\mathbb{B}}, v)$ such that, for any $u \in \mathcal{U}(\mathbb{V})$ and $p \in \mathsf{AtProp}$,

$$u \in v(p) \quad \text{iff} \quad p \in u.$$

Lemma 4.15 (Truth lemma for GP_n) *For any* Φ*-formula* φ*,* $u \in \mathcal{U}(\mathbb{B})$*, and* $j \in n$*,*

$$M_{\mathbb{B}}, (u, j) \vDash_n \varphi \quad \textit{iff} \quad \varphi \in u.$$

Proof. We prove the statement by induction on the complexity of φ. We check only the case $\varphi = \Diamond\psi$ for some $\phi \in \Phi$, the other cases being trivial.

Assume that $M, (u, j) \vDash_n \Diamond\psi$. Then there exist k different ultrafilters $w_1, \ldots, w_k \in \mathcal{U}(\mathbb{B})$, and $m_1, \ldots, m_k \in \mathbf{n}$ such that $m_1 + \cdots + m_k \geq n$ and $(u, j)R(w_i, m_i)$ and such that $M, (w_i, m_i) \vDash_n \psi$, and so $\psi \in w_i$ for each $i \in k$ by the induction hypothesis. Since all the w_i are distinct ultrafilters, there exist $\theta_1, \ldots, \theta_k \in \mathbb{B}$ such that for all $i \in \{1, \ldots, k\}$,

$$\theta_i \in w_i \quad \text{and} \quad \neg\theta_j \in w_i \text{ for all } j \in \{1, \ldots, k\} \setminus \{i\}.$$

Without loss of generality we can assume $\theta_1, \dots, \theta_k$ are mutually disjoint [3] and by the argument in Footnote 2, we can also assume that $\theta_1, \dots, \theta_k \leq \psi$. For each i, $e_u(\psi \wedge \theta_i) \geq m_i$ since $\mathrm{e}_u(w_i, u) \geq m_i$. Hence,

$$e_u(\psi) = e_u(\psi \wedge (\theta_1 \vee \cdots \vee \theta_k)) \geq \sum_{1 \leq i \leq k} m_i \geq n.$$

By the conditions that e_u satisfies, it follows that $\Diamond\psi \in u$.

For the other direction, assume that $\Diamond\psi \in u$, and so $e_u(\psi) \geq n$. We will recursively define a sequence of distinct ultrafilters $w_1, \dots, w_k$, such that $0 < e(w_i, u) = m_i$, $m_1 + \cdots + m_k \geq n$, and $\psi \in w_i$ for all i. For the base case, by the PIT, there exists an ultrafilter w_1 disjoint from Z_u such that $\psi \in w_1$. Since it's disjoint from Z_u, $e(w_1, u) > 0$. Assume now that we already have ℓ distinct ultrafilters $w_1, \dots, w_\ell$ and $m_1 + \cdots + m_\ell < n$ such that $\psi \in \bigcap_{i=1}^{\ell} w_i$, and such that $m_1, \dots, m_\ell > 0$. Since the ultrafilters are distinct we have that there exist mutually disjoint $\theta_i \in w_i$ for $1 \leq i \leq \ell$, such that $\neg\theta_i \in w_1, \dots, w_{i-1}, w_{i+1}, \dots, w_\ell$, $\theta_i \leq \psi$ and $e_u(\psi \wedge \theta_i) = m_i$. Then,

$$\sum_{1=1}^{\ell} e_u(\psi \wedge \theta_i) = m_1 + \cdots + m_\ell < n,$$

Hence, since $\theta_1, \dots, \theta_\ell$ are mutually disjoint,

$$\begin{array}{rcll}
e_u(\psi) & = & e_u\left((\psi \wedge \bigvee_{i=1}^{\ell} \theta_i) \vee (\psi \wedge \neg \bigvee_{i=1}^{\ell} \theta_i)\right) & \mathbb{B} \text{ Boolean} \\
& = & e_u\left(\bigvee_{i=1}^{\ell} (\psi \wedge \theta_i) \vee (\psi \wedge \neg \bigvee_{i=1}^{\ell} \theta_i)\right) & \mathbb{B} \text{ distributive} \\
& = & e_u\left(\bigvee_{i=1}^{\ell} (\psi \wedge \theta_i)\right) + e_u\left(\psi \wedge \neg \bigvee_{i=1}^{\ell} \theta_i\right) & \text{property of } e_u \\
& = & \sum_{1=1}^{\ell} e_u(\psi \wedge \theta_i) + e_u\left(\psi \wedge \neg \bigvee_{i=1}^{\ell} \theta_i\right) & \theta_1, \dots, \theta_\ell \text{ disjoint} \\
& = & \sum_{1=1}^{\ell} m_i + e_u\left(\psi \wedge \neg \bigvee_{i=1}^{\ell} \theta_i\right) & \text{definition of } m_i \\
& \text{iff} & e_u\left(\psi \wedge \neg \bigvee_{i=1}^{\ell} \theta_i\right) = e_u(\psi) - \sum_{1=1}^{\ell} m_i & \\
& \text{implies} & e_u\left(\psi \wedge \neg \bigvee_{i=1}^{\ell} \theta_i\right) \geq n - \sum_{1=1}^{\ell} m_i & \Diamond\psi \in u \\
& \text{implies} & e_u\left(\psi \wedge \neg \bigvee_{i=1}^{\ell} \theta_i\right) > 0 & \sum_{1=1}^{\ell} m_i < n
\end{array}$$

Thus, by the PIT, there exists some ultrafilter, $w_{\ell+1}$ that contains $\psi \wedge \neg(\theta_1 \vee \cdots \vee \theta_\ell)$ (and so distinct from $w_1, \dots, w_\ell$) that is disjoint from Z_u, and hence $e(w_{\ell+1}, u) > 0$.

Finally, given that this process will terminate after a finite number of steps we will obtain a sequence of distinct ultrafilters $w_1, \dots, w_k$ such that

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[3] From any $\theta_1, \dots \theta_k$ such as the ones above, one could consider for each i,

$$\theta_i' \coloneqq \theta_i \wedge \bigwedge_{j \neq i} \neg\theta_j \in w_i.$$

Clearly, for each $i, j \in \{1, \dots, k\}$, $\theta_i' \wedge \theta_j' = \bot$ whenever $i \neq j$.

$0 < e(w_i, u) = m_i$, $m_1 + \cdots + m_k \geq n$, and $\psi \in w_i$ for all i. Therefore $(u, j)R(w_i, k_i)$, for $k_i \leq m_i$, i.e. $M, (u, j) \vDash_n \Diamond\psi$. □

The following theorem follows from Lemma 4.15 using standard completeness arguments.

Theorem 4.16 (AC) *The system GP_n is strongly complete with respect to the logic $\mathcal{L}_n$.*

5 Conclusions

Contributions. In this article we studied the family of the monotonic modal logics $\mathcal{L}_n$ for any $n \geq 2$, each with a single modality which is interpreted semantically as a graded modality $\Diamond_n$. We observed that all these logics are decidable and have the strong finite model property. We compared these logics with each other showing that, if $m - n$ is small, the logics $\mathcal{L}_n$ and $\mathcal{L}_m$ might be incomparable, while if $m - n$ is large enough, then $\mathcal{L}_m \subsetneq \mathcal{L}_n$. We also provided concrete complete axiomatizations for $\mathcal{L}_2$ and $\mathcal{L}_3$, and we showed that each $\mathcal{L}_n$ for $n \geq 4$ is finitely axiomatizable, by showing that each axiom corresponds to a "rule" in a finite set.

The maps e_u. The construction of the canonical model for classical graded modal logic (see e.g. [7]) pivots on the map $e : \mathcal{U}(\mathcal{B}) \times \mathcal{U}(\mathcal{B}) \to n$ (which we also used in Definition 4.14). However, thanks to the fact that the language of graded modal logic is much more expressive, in [7], the map e_u can be obtained immediately as $e_u(\varphi) = \sup\{k \in \mathbb{N} \mid \Diamond_k\varphi \in u\}$, for every formula $\varphi \in \Phi_G$. In the case of $\mathcal{L}_n$, since the language is restricted, defining the map e_u becomes much more intricate and complicated. We showed that, if $n = 2$ or $n = 3$, the language is expressive enough to define e_u in a unique and uniform way. However, this is no longer possible already for $n = 4$. To see this, consider the frame $(\{u\} \sqcup 4, u \times 4)$, and the valuation $v(p) = \{1\}$, $v(q) = \{2\}$ and $v(r) = \{3, 4\}$. Then under any permutation of $\{p, q, r\}$, the theory of u remains unchanged, meaning that u cannot "tell apart" p, q, r. If a uniform way of defining e_u existed, then $e_u(p) = e_u(q) = e_u(r)$, but this is impossible, since exactly one of them needs value 2, and the other two should have value 1. Therefore, when defining e_u for $n \geq 4$, arbitrary choices need to be made.

Future directions. Even though, as discussed above, some steps towards complete axiomatization for $\mathcal{L}_n$ for $n \geq 4$ were taken, the question of identifying concrete axiomatic systems for $\mathcal{L}_n$ remains open. From the discussion in the paragraph above, it is clear that such axiomatizations need to be more complex than the ones presented for $n = 2$ and $n = 3$.

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