Analytic Cut and Mints' Symmetric Interpolation Method for Bi-intuitionistic Tense Logic

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Abstract

This paper establishes the Craig interpolation theorem of bi-intuitionistic stable tense logic **BiSKt**, which is proposed by Stell et al. (2016). First, we define a sequent calculus $\mathsf{G}(\mathbf{BiSKt})$ with the cut rule for the logic and establish semantically that applications of the cut rule can be restricted to analytic ones, i.e., applications such that the cut formula is a subformula of the conclusion of the cut rule. Second, we apply a symmetric interpolation method, originally proposed by Mints (2001) for multi-succedent calculus for intuitionistic logic, to obtain the Craig interpolation theorem of the calculus $\mathsf{G}(\mathbf{BiSKt})$. Our argument also provides a simplification of Kowalski and Ono (2017)'s argument for the Craig interpolation theorem of bi-intuitionistic logic.

Keywords: Analytic Cut, Craig Interpolation, Subformula Property, Bi-intuitionistic Logic, Bi-intuitionistic Tense Logic.

1 Introduction

The Craig interpolation theorem for classical (first-order) logic states that if $\varphi \to \psi$ is a theorem then there exists a formula γ, which is called an *interpolant*, such that both $\varphi \to \gamma$ and $\gamma \to \psi$ are theorems and the set of all free variables (constant symbols or predicate symbols) occurring in γ is a subset of the set of all free variables (constant symbols or predicate symbols, respectively) occurring in both φ and ψ (we do not consider function symbols and equality

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here). Maehara [12] showed that an interpolant can be effectively calculated from a cut-free derivation of a sequent calculus **LK** of classical first-order logic by considering a partition of a sequent. This is now called Maehara's method.

Since then, Maehara's method has been applied to various non-classical logics, e.g., intuitionistic first-order logic [28,34] and propositional modal logics (see, e.g., [20,21]), when the target logic has a cut-free sequent calculus in terms of an ordinary notion of sequent (i.e., a pair of finite multisets or lists). For a single succedent sequent calculus of intuitionistic propositional logic **Int**, we restrict the form of a partition where to apply Maehara's method (see, e.g., [21, Theorem 3.7]). For a multi-succedent sequent calculus of **Int**, Mints [16] did not restrict the form of a partition but generalized the notion of Craig interpolant to calculate such a generalized interpolant inductively. Mints' method can be regarded as a generalization of Maehara's method. One remarkable point of Mints' method is that a generalized interpolant implies the existence of a Craig interpolant for the right rule of implication.

Since Maehara's method assumes a cut-free sequent calculus (based on the ordinary notion of sequent) for a logic, if the sequent calculus is not cut-free, it becomes challenging to establish the Craig interpolation theorem proof-theoretically. Well-known examples of the failure of cut-elimination theorem for proposed sequent calculi are modal logic **S5** [18], basic tense logic **Kt** [17], bi-intuitionistic logic [22], etc. For such cases, there are at least two proof-theoretic approaches to obtain the Craig interpolation theorem (if it holds). The first approach is to restrict applications of cut to ensure that such restricted applications are still compatible with Maehara's method. The second approach is to extend the notion of sequent to get a cut-free sequent calculus to apply Maehara's method. For **S5**, the reader is referred to [31,33,20] for the first approach and to [8] for the second approach which is based on the notion of hypersequent [1]. It is noted that we can restrict all applications of cut to analytic ones for **S5** [31,33] (for the relationship between a derivation with analytic cut of the sequent calculus for **S5** [18] and a cut-free derivation of a hypersequent calculus for **S5**, the reader is referred to [2]).

For bi-intuitionistic logic [23,24,25] (see [6] for one of the recent studies), the first approach is taken in [7]. Kowalski and the first author semantically established that applications of cut can be restricted to analytic ones and then showed that Maehara's method can be applied to a restricted form of a partition of a sequent. Maehara's partition argument for analytic cut (actually, analytic *mix* rule, see [7, Lemma 5.4]), however, becomes more involved than for **S5** [20, pp.245-6]. Later, the second approach is taken in [11] by Lyon et al. They employed nested sequent calculi and generalized the notion of interpolant to get the interpolation theorem. As far as the authors can see, this generalized notion of interpolant in [11] is exactly the same one as Mints' [16], but [11] has no reference to [16]. They also applied the same approach based on nested sequents to extensions of basic tense logic. As for the first approach to basic tense logic, the reader is referred to [27].

This paper takes the first approach above to bi-intuitionistic stable tense logic **BiSKt** [30], a combination of bi-intuitionistic logic and basic tense logic, where we choose a residuated pair of past possibility operator $\blacklozenge$ and future necessity operator $\Box$ as primitive symbols for tense logic. Semantically, it is a tense logic with a Kripke semantics where worlds in a frame are equipped with a pre-order $\leqslant$ as well as with an accessibility relation R which is *stable* with respect to this pre-order, i.e., $\leqslant \circ R \circ \leqslant \subseteq R$ where $\circ$ is the composition of two relations. This logic arose in the semantic context of hypergraphs because a special case of the pre-order can represent the incidence structure of a hypergraph (see [30] for the detail). A labelled tableau calculus [30] and Hilbert system [26] of the logic **BiSKt** have already been studied and it also enjoys the finite model property via filtration technique [26]. A sequent calculus for **BiSKt** and the Craig interpolation, however, have not been studied in the literature.

Our sequent calculus $\mathsf{G}(\mathbf{BiSKt})$ for **BiSKt** is a sequent calculus $\mathbf{LBJ}_1$ for bi-intuitionistic logic [7] expanded with two inference rules for tense operators, which are a reformulation of Nishimura [17]'s rules in terms of $\blacklozenge$ and $\Box$. It is noted that the left rule for coimplication and rules for $\blacklozenge$ and $\Box$ as well as the right rule for implication has context restrictions. We first establish semantically that every application of cut in a derivation of $\mathsf{G}(\mathbf{BiSKt})$ can be replaced with an analytic one. A key notion for this aim is: Ξ-*partial valuation*, which is also employed in [15,27] (the original idea in [15] is due to Mitio Takano). Then, we follow Mints' symmetric interpolation method to calculate an interpolant by induction on derivation, similarly to Maehara's method. A novelty of our argument is to demonstrate that Mints' symmetric interpolation method [16] works more properly with an analytic cut rule than the ordinary Maehara's method in [7]. For inference rules with context restrictions mentioned above, we can construct a Craig interpolant from a generalized interpolant.

We proceed as follows. Section 2 introduces the syntax and Kripke semantics of bi-intuitionistic stable tense logic **BiSKt**. In Section 3, we define three sequent calculi for **BiSKt** and prove the soundness of a sequent calculus $\mathsf{G}(\mathbf{BiSKt})$ for **BiSKt** with no restriction on rule applications (Theorem 3.3). In Section 4, we establish that a sequent calculus $\mathsf{G}^a(\mathbf{BiSKt})$, whose rule applications are always analytic, is semantically complete for Kripke semantics (Theorem 4.7) and conclude that $\mathsf{G}(\mathbf{BiSKt})$ enjoys the subformula property (Corollary 4.8). In Section 5, we introduce Mints' notion of interpolant [16] (a generalization of a Craig interpolant) to show the Craig interpolation theorem for $\mathsf{G}(\mathbf{BiSKt})$ (Theorem 5.19). Section 5 is the most important contribution of this paper. Section 6 concludes the paper.

2 Syntax and Kripke Semantics

We introduce the syntax and semantics for bi-intuitionistic stable tense logic [30,26]. Let Prop be a countably infinite set of propositional variables. Our syntax $\mathcal{L}$ for bi-intuitionistic stable tense logic consists of all logical connectives of bi-intuitionistic logic, i.e., two constant symbols $\bot$ and $\top$, disjunction $\vee$, conjunction $\wedge$, implication $\to$, coimplication $\prec$, as well as two modal operators $\{\blacklozenge, \Box\}$. The set of all formulas in $\mathcal{L}$ is defined in a standard way as follows:

$$\varphi ::= \top \mid \bot \mid p \mid \varphi \wedge \varphi \mid \varphi \vee \varphi \mid \varphi \to \varphi \mid \varphi \prec \varphi \mid \blacklozenge \varphi \mid \Box\varphi \quad (p \in \mathsf{Prop}).$$

Given a finite set Γ of formulas, we define $\bigwedge \Gamma$ and $\bigvee \Gamma$ as the conjunction and disjunction of all formulas in Γ, respectively, where $\bigwedge \varnothing := \top$ and $\bigvee \varnothing := \bot$. Given any formula φ, we define $\mathsf{Sub}(\varphi)$ as the set of all subformulas of φ. Moreover, for any set (or multiset) Γ of formulas, we define $\mathsf{Sub}(\Gamma) = \bigcup_{\varphi\in\Gamma} \mathsf{Sub}(\varphi)$. We say that a set (or multiset) Γ is *subformula closed* if $\mathsf{Sub}(\Gamma) \subseteq \Gamma$. We define the set $\mathsf{Prop}(\Gamma)$ of propositional variables occurring in Γ as $\mathsf{Sub}(\Gamma) \cap \mathsf{Prop}$. We often write $\mathsf{Prop}(\varphi)$ instead of $\mathsf{Prop}(\{\varphi\})$. We use $\varphi[\psi_1/p_1, \ldots, \psi_n/p_n]$ as the result of uniformly substituting each propositional variable p_i in φ with a formula ψ_i simultaneously for all $1 \leqslant i \leqslant n$, where all p_is are pairwise distinct. For a set Γ of formulas, we also naturally define $\Gamma[\psi_1/p_1, \ldots, \psi_n/p_n]$ as $\{\varphi[\psi_1/p_1, \ldots, \psi_n/p_n] \mid \varphi \in \Gamma\}$.

Definition 2.1 We say that $F = (W, \leqslant, R)$ is a *frame* if W is a nonempty set, $\leqslant$ is a preorder on W, and R is a *stable* binary relation on W, i.e., R satisfies the following condition: $\leqslant \circ R \circ \leqslant\ \subseteq R$, where "$\circ$" is the relational composition.

Since $\leqslant$ is reflexive, it is easy to see that R is stable iff $\leqslant \circ R \circ \leqslant\ = R$. This condition is also employed in [35] for interpreting $\Box$ on the intuitionistic setting.

Definition 2.2 Given a frame $F = (W, \leqslant, R)$, $X \subseteq W$ is said to be *$\leqslant$-closed* (or *up-closed*) if $u \in X$ and $u \leqslant v$ jointly imply $v \in X$ for all $u, v \in W$. A *valuation* on a frame $F = (W, \leqslant, R)$ is a mapping V from Prop to the set of all $\leqslant$-closed sets on W. A pair $M = (F, V)$ is a *model* if $F = (W, \leqslant, R)$ is a frame and V is a valuation. Given a model $M = (W, \leqslant, R, V)$, a state $u \in W$ and a formula φ, the *satisfaction relation* $M, u \models \varphi$ is defined inductively as follows:

$$
\begin{array}{lll}
M, u \models p & \text{iff} & u \in V(p), \\
M, u \models \top, & & \\
M, u \not\models \bot, & & \\
M, u \models \varphi \vee \psi & \text{iff} & M, u \models \varphi \text{ or } M, u \models \psi, \\
M, u \models \varphi \wedge \psi & \text{iff} & M, u \models \varphi \text{ and } M, u \models \psi, \\
M, u \models \varphi \to \psi & \text{iff} & \text{For all } v \in W\ ((u \leqslant v \text{ and } M, v \models \varphi) \text{ imply } M, v \models \psi), \\
M, u \models \varphi \prec \psi & \text{iff} & \text{For some } v \in W\ (v \leqslant u \text{ and } M, v \models \varphi \text{ and } M, v \not\models \psi), \\
M, u \models \blacklozenge \varphi & \text{iff} & \text{For some } v \in W\ (vRu \text{ and } M, v \models \varphi), \\
M, u \models \Box\varphi & \text{iff} & \text{For all } v \in W\ (uRv \text{ implies } M, v \models \varphi).
\end{array}
$$

Proposition 2.3 *For every model $M = (W, \leqslant, R, V)$ and formula φ, the truth set $[\![\varphi]\!]_M := \{u \in W \mid M, u \models \varphi\}$ is $\leq$-closed.*

3 Sequent Calculi

In what follows in this section, we use Γ, Δ, etc. to denote finite multisets of formulas. A *sequent* is a pair (Γ, Δ) of finite multisets and it is denoted

Table 1
Sequent Calculi $\mathsf{G}(\mathbf{BiSKt})$, $\mathsf{G}^*(\mathbf{BiSKt})$, and $\mathsf{G}^a(\mathbf{BiSKt})$

Sequent Calculus $\mathsf{G}(\mathbf{BiSKt})$

Initial Sequents:

$$\Rightarrow \top \qquad \varphi \Rightarrow \varphi \qquad \bot \Rightarrow$$

Structural Rules:

$$\frac{\Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \varphi}\ (\Rightarrow w) \qquad \frac{\Gamma \Rightarrow \Delta}{\varphi, \Gamma \Rightarrow \Delta}\ (w \Rightarrow)$$

$$\frac{\Gamma \Rightarrow \Delta, \varphi, \varphi}{\Gamma \Rightarrow \Delta, \varphi}\ (\Rightarrow c) \qquad \frac{\varphi, \varphi, \Gamma \Rightarrow \Delta}{\varphi, \Gamma \Rightarrow \Delta}\ (c \Rightarrow)$$

Logical Rules:

$$\frac{\Gamma \Rightarrow \Delta, \varphi_1 \quad \Gamma \Rightarrow \Delta, \varphi_2}{\Gamma \Rightarrow \Delta, \varphi_1 \wedge \varphi_2}\ (\Rightarrow \wedge) \qquad \frac{\varphi_i, \Gamma \Rightarrow \Delta}{\varphi_1 \wedge \varphi_2, \Gamma \Rightarrow \Delta}\ (\wedge \Rightarrow)$$

$$\frac{\Gamma \Rightarrow \Delta, \varphi_i}{\Gamma \Rightarrow \Delta, \varphi_1 \vee \varphi_2}\ (\Rightarrow \vee) \qquad \frac{\varphi_1, \Gamma \Rightarrow \Delta \quad \varphi_2, \Gamma \Rightarrow \Delta}{\varphi_1 \vee \varphi_2, \Gamma \Rightarrow \Delta}\ (\vee \Rightarrow)$$

$$\frac{\varphi, \Gamma \Rightarrow \psi}{\Gamma \Rightarrow \varphi \to \psi}\ (\Rightarrow \to) \qquad \frac{\Gamma \Rightarrow \Delta, \varphi \quad \psi, \Pi \Rightarrow \Sigma}{\varphi \to \psi, \Gamma, \Pi \Rightarrow \Delta, \Sigma}\ (\to \Rightarrow)$$

$$\frac{\Gamma \Rightarrow \Delta, \varphi \quad \psi, \Pi \Rightarrow \Sigma}{\Gamma, \Pi \Rightarrow \Delta, \Sigma, \varphi \prec \psi}\ (\Rightarrow \prec) \qquad \frac{\varphi \Rightarrow \Delta, \psi}{\varphi \prec \psi \Rightarrow \Delta}\ (\prec \Rightarrow)$$

Modal Rules:

$$\frac{\blacklozenge \Theta, \Gamma \Rightarrow \varphi}{\Theta, \Box\Gamma \Rightarrow \Box\varphi}\ (\Box) \qquad \frac{\varphi \Rightarrow \Delta, \Box\Theta}{\blacklozenge \varphi \Rightarrow \blacklozenge \Delta, \Theta}\ (\blacklozenge)$$

Cut Rule:

$$\frac{\Gamma \Rightarrow \Delta, \varphi \quad \varphi, \Pi \Rightarrow \Sigma}{\Gamma, \Pi \Rightarrow \Delta, \Sigma}\ (Cut)$$

$\mathsf{G}^*(\mathbf{BiSKt})$: the same as $\mathsf{G}(\mathbf{BiSKt})$ except the following analytic cut rule.

$$\frac{\Gamma \Rightarrow \Delta, \varphi \quad \varphi, \Pi \Rightarrow \Sigma}{\Gamma, \Pi \Rightarrow \Delta, \Sigma}\ (Cut)^a, \text{ where } \varphi \in \mathsf{Sub}(\Gamma, \Pi, \Delta, \Sigma).$$

$\mathsf{G}^a(\mathbf{BiSKt})$: the same as $\mathsf{G}^*(\mathbf{BiSKt})$ except the following modal rules.

$$\frac{\blacklozenge \Theta, \Gamma \Rightarrow \varphi}{\Theta, \Box\Gamma \Rightarrow \Box\varphi}\ (\Box)^a, \text{ where } \blacklozenge \Theta \subseteq \mathsf{Sub}(\Gamma, \varphi).$$

$$\frac{\varphi \Rightarrow \Delta, \Box\Theta}{\blacklozenge \varphi \Rightarrow \blacklozenge \Delta, \Theta}\ (\blacklozenge)^a, \text{ where } \Box\Theta \subseteq \mathsf{Sub}(\varphi, \Delta).$$

by $\Gamma \Rightarrow \Delta$, where Γ is an *antecedent*, Δ is a *succedent* (or a *consequent*) and $\Gamma \Rightarrow \Delta$ is read as "if all the formulas of Γ hold, then some formula in Δ holds".

For any sequent calculus in Table 1, we define the notion of derivation (from initial sequents of the calculus) and derivable sequent in the calculus as usual. The bi-intuitionistic fragment of $\mathsf{G}(\mathbf{BiSKt})$ is the same as the system $\mathbf{LBJ}_1$ in [7]. We use the same modal rules as in [27] for basic tense logic $\mathbf{Kt}$ (over classical logic) but these are a reformulation in terms of $\blacklozenge$ and $\Box$ of Nishimura's rules [17] for tense operators G and H over classical logic. The calculus $\mathsf{G}^*(\mathbf{BiSKt})$ is the same as $\mathsf{G}(\mathbf{BiSKt})$ except (Cut) is replaced with its analytic variant $(Cut)^a$ but the rules $(\Box)$ and $(\blacklozenge)$ do not satisfy the subformula property, i.e., each formula in the premise(s) of a rule may not be a subformula of the conclusion of the rule (we use this calculus for establishing the Craig interpolation theorem). Finally, $\mathsf{G}^a(\mathbf{BiSKt})$ is a fully analytic calculus all of whose rules enjoy the subformula property, where the side syntactic conditions for $(\blacklozenge)$ and $(\Box)$ were originally proposed by Takano [31].

Remark 3.1 For the bi-intuitionistic fragment of $\mathsf{G}(\mathbf{BiSKt})$ (i.e., $\mathbf{LBJ}_1$ of [7]), since all rules except (Cut) are analytic, the restriction of applications of (Cut) to analytic ones ensures the subformula property of the whole calculus and vice versa. However, it is not the case for the full calculus $\mathsf{G}(\mathbf{BiSKt})$. To establish the subformula property, we also need to restrict applications of the modal rules $\Box$ and $\blacklozenge$ to analytic ones. Since the Craig interpolation theorem is concerned with propositional variables, however, $\mathsf{G}^*(\mathbf{BiSKt})$ will do the job in Section 5.

Definition 3.2 A sequent $\Gamma \Rightarrow \Delta$ is *valid* in a model $M = (W, \leqslant, R, V)$ if, for every $u \in W$, whenever $M, u \models \gamma$ for all $\gamma \in \Gamma$, then $M, u \models \delta$ for some $\delta \in \Delta$.

Theorem 3.3 *If a sequent $\Gamma \Rightarrow \Delta$ is derivable in $\mathsf{G}(\mathbf{BiSKt})$, then it is valid in all models.*

Proof. We only prove that $(\Box)$ preserves validity on a model $M = (W, \leqslant, R, V)$. Suppose that $\blacklozenge\Theta, \Gamma \Rightarrow \varphi$ is valid on M. To show that $\Theta, \Box\Gamma \Rightarrow \Box\varphi$ is valid on M, let us fix any state $u \in W$ such that $M, u \models \bigwedge(\Theta, \Box\Gamma)$. Fix any state $v \in W$ such that uRv. Our goal is to show $M, v \models \varphi$. By uRv and $M, u \models \bigwedge\Theta$, we get $M, v \models \bigwedge\blacklozenge\Theta$. It follows also from uRv and $M, u \models \bigwedge\Box\Gamma$ that $M, v \models \bigwedge\Gamma$. By our initial supposition, we obtain $M, v \models \varphi$, as desired.$\Box$

The bi-intuitionistic fragment $\mathbf{LBJ}_1$ in [7] is known to be not cut-free (cf. [7, Theorem 2.3]) where a counterexample is a sequent $p \Rightarrow q, r \rightarrow (p \prec q) \wedge r$, which was pointed out by Uustalu for Dragalin-style sequent calculus of bi-intuitionistic logic (see [22, Section 2]):

$$\dfrac{\dfrac{p \Rightarrow p \quad q \Rightarrow q}{p \Rightarrow q, p \prec q}(\Rightarrow \prec) \qquad \dfrac{\dfrac{\dfrac{\dfrac{p \prec q \Rightarrow p \prec q}{r, p \prec q \Rightarrow p \prec q}(w\Rightarrow) \quad \dfrac{r \Rightarrow r}{r, p \prec q \Rightarrow r}(w\Rightarrow)}{r, p \prec q \Rightarrow (p \prec q) \wedge r}(\Rightarrow\wedge)}{p \prec q \Rightarrow r \rightarrow ((p \prec q) \wedge r)}(\Rightarrow\rightarrow)}{p \Rightarrow q, r \rightarrow ((p \prec q) \wedge r)}(Cut)\ .$$

Therefore we cannot also eliminate (Cut) from the sequent calculi $\mathsf{G}(\mathbf{BiSKt})$. Even if we remove the coimplication from our syntax, the resulting system is not cut-free, either. The sequent $p, \blacklozenge \Box(p \to \bot) \Rightarrow$ is derivable in $\mathsf{G}(\mathbf{BiSKt})$ with the help of (Cut) as in the following, but the application of (Cut) is indispensable for the purpose:

$$\dfrac{\dfrac{\Box(p \to \bot) \Rightarrow \Box(p \to \bot)}{\blacklozenge \Box(p \to \bot) \Rightarrow p \to \bot}\,(\blacklozenge) \quad \dfrac{p \Rightarrow p \quad \bot \Rightarrow}{p \to \bot, p \Rightarrow}\,(\to\Rightarrow)}{p, \blacklozenge \Box(p \to \bot) \Rightarrow}\,(Cut)$$

(this kind of phenomena are well-known for a sequent calculus of modal logic **S5**, see, e.g., [20, p.222]). It is noted in the above derivation that the cut formula $p \to \bot$ is a subformula of the conclusion of (Cut) and moreover $\Box(p \to \bot)$ is also a subformula of the conclusion of the rule $(\blacklozenge)$. Therefore, all the applications of the inference rules in the derivation are *analytic*, i.e., they satisfy the subformula property locally. A similar argument also holds for the above derivation of Uustalu's sequent $p \Rightarrow q, r \to (p \prec q) \wedge r$. These motivate us to consider if a derivation of $\mathsf{G}(\mathbf{BiSKt})$ implies the existence of a derivation of $\mathsf{G}^a(\mathbf{BiSKt})$.

4 Subformula Property

In this section, we follow Takano [32,33]'s semantic approach to establish the subformula property of $\mathsf{G}(\mathbf{BiSKt})$. That is, by Theorem 3.3, it suffices for our purpose to show that the fully analytic variant $\mathsf{G}^a(\mathbf{BiSKt})$ is semantically complete for the semantics (see Theorem 4.7). A key notion for our proof is: Ξ-*partial valuation* [15,27]. In what follows in this section, we use Γ, Δ, Σ, etc. to denote finite sets of formulas.

Definition 4.1 [Ξ-partial valuation] Let Ξ be a subformula closed finite set. We say that a pair (Γ, Δ) of finite sets of formulas is a Ξ-*partial valuation* in $\mathsf{G}^a(\mathbf{BiSKt})$ if the following three conditions are satisfied: (i) $\Gamma \Rightarrow \Delta$ is underivable in $\mathsf{G}^a(\mathbf{BiSKt})$, (ii) $\Gamma \cup \Delta = \mathsf{Sub}(\Gamma \cup \Delta)$, and (iii) $\mathsf{Sub}(\Gamma \cup \Delta) \subseteq \Xi$.

A Ξ-partial valuation can be constructed from an unprovable sequent in $\mathsf{G}^a(\mathbf{BiSKt})$ by the next lemma.

Lemma 4.2 *Let* $\Gamma \Rightarrow \Delta$ *be underivable in* $\mathsf{G}^a(\mathbf{BiSKt})$*. For any subformula closed finite set* Ξ *such that* $\mathsf{Sub}(\Gamma, \Delta) \subseteq \Xi$*, there exists a* Ξ*-partial valuation* (Γ^+, Δ^+) *in* $\mathsf{G}^a(\mathbf{BiSKt})$ *such that* (i) $\Gamma \subseteq \Gamma^+$, (ii) $\Delta \subseteq \Delta^+$*, and* (iii) $\Gamma^+ \cup \Delta^+ = \mathsf{Sub}(\Gamma, \Delta)$.

Proof. We need to use $(Cut)^a$ in the proof. Suppose that $\Gamma \Rightarrow \Delta$ is underivable in $\mathsf{G}^a(\mathbf{BiSKt})$ and that $\mathsf{Sub}(\Gamma, \Delta) \subseteq \Xi$. Let $\varphi_1, \ldots, \varphi_n$ be an enumeration of all formulas in $\mathsf{Sub}(\Gamma, \Delta)$. In what follows, we inductively construct a sequence $(\Gamma_l, \Delta_l)_{1 \leqslant l \leqslant n}$ such that $\Gamma_l \Rightarrow \Delta_l$ is underivable in $\mathsf{G}^a(\mathbf{BiSKt})$, $\Gamma_l \subseteq \Gamma_{l+1}$ and $\Delta_l \subseteq \Delta_{l+1}$ for all $1 \leqslant l < n$.

For the base step, we define $(\Gamma_0, \Delta_0) := (\Gamma, \Delta)$, where $\Gamma \Rightarrow \Delta$ is clearly underivable in $\mathsf{G}^a(\mathbf{BiSKt})$. For the inductive step, let us suppose that we have constructed pairs $(\Gamma_l, \Delta_l)_{1 \leqslant l \leqslant k}$ such that each corresponding sequent $\Gamma_l \Rightarrow \Delta_l$ is underivable in $\mathsf{G}^a(\mathbf{BiSKt})$, $\Gamma_l \subseteq \Gamma_{l+1}$ and $\Delta_l \subseteq \Delta_{l+1}$ for all $1 \leqslant l < k$. We show that either $\Gamma_k \Rightarrow \Delta_k, \varphi_k$ or $\varphi_k, \Gamma_k \Rightarrow \Delta_k$ is underivable in $\mathsf{G}^a(\mathbf{BiSKt})$. Suppose otherwise, i.e., both sequents are derivable in $\mathsf{G}^a(\mathbf{BiSKt})$. Then we obtain the following derivation:

$$\dfrac{\dfrac{\Gamma_k \Rightarrow \Delta_k, \varphi_k \quad \varphi_k, \Gamma_k \Rightarrow \Delta_k}{\Gamma_k, \Gamma_k \Rightarrow \Delta_k, \Delta_k}\ (Cut)^a}{\Gamma_k \Rightarrow \Delta_k}\ (c \Rightarrow, \Rightarrow c) \quad ,$$

which implies a contradiction with the unprovability of $\Gamma_k \Rightarrow \Delta_k$. We note that the side condition of $(Cut)^a$ in the above derivation is satisfied because $\varphi_k \in \mathsf{Sub}(\Gamma, \Delta) \subseteq \mathsf{Sub}(\Gamma_k, \Delta_k)$. So, we define one of underivable sequents as $(\Gamma_{k+1}, \Delta_{k+1})$.

Finally, we define $(\Gamma^+, \Delta^+) := (\Gamma_{n+1}, \Delta_{n+1})$, which is easily shown to be a Ξ-partial valuation in $\mathsf{G}^a(\mathbf{BiSKt})$ satisfying the conditions from (i) to (iii) of the statement. □

Definition 4.3 Let Ξ be a subformula closed finite set. We define the *derived model* $M^\Xi = (W^\Xi, \leqslant^\Xi, R^\Xi, V^\Xi)$ *from* Ξ by:

- $W^\Xi := \{ (\Pi, \Sigma) \mid (\Pi, \Sigma)$ is a Ξ-partial valuation in $\mathsf{G}^a(\mathbf{BiSKt}) \}$.
- $(\Gamma, \Delta) \leqslant^\Xi (\Pi, \Sigma)$ iff $\Gamma \subseteq \Pi$ and $\Sigma \subseteq \Delta$.
- $(\Gamma, \Delta) R^\Xi (\Pi, \Sigma)$ iff the following two conditions hold:
  - (i) if $\Box\psi \in \Gamma$ then $\psi \in \Pi$, for all formulas ψ,
  - (ii) if $\blacklozenge\psi \in \Sigma$ then $\psi \in \Delta$, for all formulas ψ.
- $(\Gamma, \Delta) \in V^\Xi(p)$ iff $p \in \Gamma$.

Lemma 4.4 *Let* Ξ *be a subformula closed finite set. Then,* M^Ξ *is a model.*

Proof. It is easy to see that $\leqslant^\Xi$ is reflexive and transitive and that V^Ξ is $\leqslant$-closed. So, we show that R^Ξ is stable, i.e., $\leqslant^\Xi \circ R^\Xi \circ \leqslant^\Xi \subseteq R^\Xi$. Suppose that $(\Gamma, \Delta) \leqslant^\Xi (\Gamma_1, \Delta_1) R^\Xi (\Gamma_2, \Delta_2) \leqslant^\Xi (\Pi, \Sigma)$. To show our goal of $(\Gamma, \Delta) R^\Xi (\Pi, \Sigma)$, we need to verify two conditions (i) and (ii) for R^Ξ of Definition 4.3. However, these are easy to establish. □

Lemma 4.5 *Let* $(\Gamma, \Delta) \in W^\Xi$*. Then, all of the following hold.*

(i) *If* $\varphi \wedge \psi \in \Gamma$*, then* $\varphi \in \Gamma$ *and* $\psi \in \Gamma$*.*

(ii) *If* $\varphi \wedge \psi \in \Delta$*, then* $\varphi \in \Delta$ *or* $\psi \in \Delta$*.*

(iii) *If* $\varphi \vee \psi \in \Gamma$*, then* $\varphi \in \Gamma$ *or* $\psi \in \Gamma$*.*

(iv) *If* $\varphi \vee \psi \in \Delta$*, then* $\varphi \in \Delta$ *and* $\psi \in \Delta$*.*

(v) *If* $\varphi \to \psi \in \Gamma$*, then* $((\Gamma, \Delta) \leqslant^\Xi (\Pi, \Sigma)$ *and* $\varphi \in \Pi$ *imply* $\psi \in \Pi)$ *for all* $(\Pi, \Sigma) \in W^\Xi$*.*

(vi) *If* $\varphi \to \psi \in \Delta$*, then* $((\Gamma, \Delta) \leqslant^\Xi (\Pi, \Sigma)$ *and* $\varphi \in \Pi$ *and* $\psi \in \Sigma)$ *for some* $(\Pi, \Sigma) \in W^\Xi$*.*

(vii) *If* $\varphi \prec \psi \in \Gamma$*, then* $((\Pi, \Sigma) \leqslant^{\Xi} (\Gamma, \Delta)$ *and* $\varphi \in \Pi$ *and* $\psi \in \Sigma)$ *for some* $(\Pi, \Sigma) \in W^{\Xi}$*.*

(viii) *If* $\varphi \prec \psi \in \Delta$*, then* $((\Pi, \Sigma) \leqslant^{\Xi} (\Gamma, \Delta)$ *and* $\varphi \in \Pi$ *imply* $\psi \in \Pi)$ *for all* $(\Pi, \Sigma) \in W^{\Xi}$*.*

(ix) *If* $\Box\varphi \in \Gamma$*, then* $((\Gamma, \Delta)R^{\Xi}(\Pi, \Sigma)$ *implies* $\varphi \in \Pi)$ *for all* $(\Pi, \Sigma) \in W^{\Xi}$*.*

(x) *If* $\Box\varphi \in \Delta$*, then* $((\Gamma, \Delta)R^{\Xi}(\Pi, \Sigma)$ *and* $\varphi \in \Sigma)$ *for some* $(\Pi, \Sigma) \in W^{\Xi}$*.*

(xi) *If* $\blacklozenge\varphi \in \Gamma$*, then* $((\Pi, \Sigma)R^{\Xi}(\Gamma, \Delta)$ *and* $\varphi \in \Pi)$ *for some* $(\Pi, \Sigma) \in W^{\Xi}$*.*

(xii) *If* $\blacklozenge\varphi \in \Delta$*, then* $((\Pi, \Sigma)R^{\Xi}(\Gamma, \Delta)$ *implies* $\varphi \in \Sigma)$ *for all* $(\Pi, \Sigma) \in W^{\Xi}$*.*

Proof. It is easy to establish items from (i) to (iv), and items (ix) and (xii) are immediate from the definition of R^{Ξ}. Moreover, we can prove items (v) and (vi) similarly to (vii) and (viii), respectively, by duality. So, we prove the remaining items below.

(vii) Suppose that $\varphi \prec \psi \in \Gamma$. We show that $\varphi \Rightarrow \Delta, \psi$ is underivable in $\mathsf{G}^a(\mathbf{BiSKt})$. Suppose otherwise. Then, we obtain the following derivation:

$$\dfrac{\varphi \Rightarrow \Delta, \psi}{\varphi \prec \psi \Rightarrow \Delta}\ (\prec\Rightarrow),$$

which implies the derivability of $\Gamma \Rightarrow \Delta$, a contradiction. So, by Lemma 4.2, there exists $(\Pi, \Sigma) \in W^{\Xi}$ such that $\varphi \in \Pi$, $\{\psi\} \cup \Delta \subseteq \Sigma$ and $\Pi \cup \Sigma = \mathsf{Sub}(\varphi, \Delta, \psi)$. To finish our proof, it suffices to show that $(\Pi, \Sigma) \leqslant^{\Xi} (\Gamma, \Delta)$. Since $\Delta \subseteq \Sigma$ is easy, we show that $\Pi \subseteq \Gamma$. Suppose that $\pi \in \Pi$. We show that $\pi \in \Gamma$. We observe that $\pi \in \Pi \subseteq \Pi \cup \Sigma = \mathsf{Sub}(\varphi, \Delta, \psi) \subseteq \mathrm{Sub}(\Gamma, \Delta)$ $= \Gamma \cup \Delta$. Suppose for contradiction that $\pi \notin \Gamma$, i.e., $\pi \in \Delta \subseteq \Sigma$. This implies the derivability of $\Pi \Rightarrow \Sigma$ in $\mathsf{G}^a(\mathbf{BiSKt})$. A contradiction. We conclude $\pi \in \Gamma$, as desired.

(viii) Assume that $\varphi \prec \psi \in \Delta$. Fix any $(\Pi, \Sigma) \in W^{\Xi}$ such that $(\Pi, \Sigma) \leqslant^{\Xi} (\Gamma, \Delta)$ and $\varphi \in \Pi$. We show $\psi \in \Pi$. Since $\Delta \subseteq \Sigma$, we have $\varphi \prec \psi \in \Sigma$. This implies that $\psi \in \mathsf{Sub}(\Pi, \Sigma) = \Pi \cup \Sigma$. To obtain our goal, it suffices to show that $\psi \notin \Sigma$. So, suppose for contradiction that $\psi \in \Sigma$. Since $\varphi \Rightarrow \varphi \prec \psi, \psi$ is derivable in $\mathsf{G}^a(\mathbf{BiSKt})$, this implies that $\Pi \Rightarrow \Sigma$ is derivable in $\mathsf{G}^a(\mathbf{BiSKt})$, a contradiction. Therefore, $\psi \notin \Sigma$ hence $\psi \in \Pi$.

(x) Assume that $\Box\varphi \in \Delta$. Let us define Π and Θ as follows:

$$\Pi := \{\,\psi \mid \Box\psi \in \Gamma\,\},\ \Theta := \{\,\psi \mid \psi \in \Gamma \text{ and } \blacklozenge\psi \in \mathsf{Sub}(\Pi, \varphi)\,\},$$

where it is noted that $\mathsf{Sub}(\Pi, \varphi) \subseteq \mathsf{Sub}(\Gamma, \Delta)$. We show that $\blacklozenge\Theta, \Pi \Rightarrow \varphi$ is underivable in $\mathsf{G}^a(\mathbf{BiSKt})$. Suppose not. Then, we can consider the following derivation:

$$\dfrac{\dfrac{\blacklozenge\Theta, \Pi \Rightarrow \varphi}{\Theta, \Box\Pi \Rightarrow \Box\varphi}\ (\Box)^a}{\Gamma \Rightarrow \Delta}\ (w\Rightarrow, \Rightarrow w),$$

where $\blacklozenge\Theta \subseteq \mathsf{Sub}(\Pi, \varphi)$ holds by definition of Θ and note that $\Theta \cup \Box\Pi \subseteq \Gamma$ and $\Box\varphi \in \Delta$. But, this is a contradiction with the underivability of $\Gamma \Rightarrow \Delta$. By Lemma 4.2, there exists $(\Pi^+, \Sigma^+) \in W^\Xi$ such that $\blacklozenge\Theta \cup \Pi \subseteq \Pi^+$, $\varphi \in \Sigma^+$, and $\Pi^+ \cup \Sigma^+ = \mathsf{Sub}(\blacklozenge\Theta, \Pi, \varphi)$. Let us establish $(\Gamma, \Delta)R^\Xi(\Pi^+, \Sigma^+)$. We verify two conditions (i) and (ii) for R^Ξ of Definition 4.3. Condition (i) is easy to verify, so we focus on condition (ii). Suppose that $\blacklozenge\sigma \in \Sigma^+$. Our goal is to show $\sigma \in \Delta$. We first observe that $\blacklozenge\sigma \in \Sigma^+ \subseteq \Pi^+ \cup \Sigma^+ = \mathsf{Sub}(\blacklozenge\Theta, \Pi, \varphi) = \mathsf{Sub}(\blacklozenge\Theta) \cup \mathsf{Sub}(\Pi, \varphi) \subseteq \mathsf{Sub}(\Gamma, \Delta)$. It follows that $\sigma \in \mathsf{Sub}(\Gamma, \Delta) = \Gamma \cup \Delta$. To show our goal of $\sigma \in \Delta$, suppose for contradiction that $\sigma \notin \Delta$. It follows from $\sigma \in \Gamma \cup \Delta$ that $\sigma \in \Gamma$. From our supposition of $\blacklozenge\sigma \in \Sigma^+$, $\blacklozenge\sigma \in \mathsf{Sub}(\Pi, \varphi)$ holds. So, by our definition of Θ, we have $\sigma \in \Theta$ hence $\blacklozenge\sigma \in \blacklozenge\Theta \subseteq \Pi^+$. Since $\blacklozenge\sigma \in \Sigma^+$, we showed that $\Pi^+ \Rightarrow \Sigma^+$ is derivable in $\mathsf{G}^a(\mathbf{BiSKt})$, but this is a contradiction with the underivability of $\Pi^+ \Rightarrow \Sigma^+$ in $\mathsf{G}^a(\mathbf{BiSKt})$. Therefore, $\sigma \in \Delta$, as desired.

(xi) Assume that $\blacklozenge\varphi \in \Gamma$. Define Σ and Θ as follows:

$$\Sigma := \{\, \psi \mid \blacklozenge\psi \in \Delta \,\},\ \Theta := \{\, \psi \mid \psi \in \Delta \text{ and } \Box\psi \in \mathsf{Sub}(\varphi, \Sigma) \,\},$$

where we note that $\mathsf{Sub}(\varphi, \Sigma) \subseteq \mathsf{Sub}(\Gamma, \Delta)$. We show that $\varphi \Rightarrow \Sigma, \Box\Theta$ is underivable in $\mathsf{G}^a(\mathbf{BiSKt})$. Suppose not. Then, we can obtain the following derivation:

$$\dfrac{\dfrac{\varphi \Rightarrow \Sigma, \Box\Theta}{\blacklozenge\varphi \Rightarrow \blacklozenge\Sigma, \Theta}\ (\blacklozenge)^a}{\Gamma \Rightarrow \Delta}\ (w \Rightarrow, \Rightarrow w),$$

where we note that $\Box\Theta \subseteq \mathsf{Sub}(\varphi, \Sigma)$ and $\blacklozenge\varphi \in \Gamma$ and $\blacklozenge\Sigma \cup \Theta \subseteq \Delta$. But, this is a contradiction with the underivability of $\Gamma \Rightarrow \Delta$, So, by Lemma 4.2, there exists $(\Pi^+, \Sigma^+) \in W^\Xi$ such that: $\varphi \in \Pi^+$, $\Sigma \cup \Box\Theta \subseteq \Sigma^+$, and $\Pi^+ \cup \Sigma^+ = \mathsf{Sub}(\varphi, \Sigma, \Box\Theta)$. Finally, let us establish $(\Pi^+, \Sigma^+)R^\Xi(\Gamma, \Delta)$. We check two conditions one by one. As for condition (ii), we proceed as follows. Assume that $\blacklozenge\sigma \in \Delta$. Our goal is to show $\sigma \in \Sigma^+$. By assumption, $\sigma \in \Sigma$. We deduce from $\Sigma \subseteq \Sigma^+$ that $\sigma \in \Sigma^+$, as desired.

Let us move to condition (i). Assume that $\Box\gamma \in \Pi^+$. We show $\gamma \in \Gamma$. Observe that $\Box\gamma \in \Pi^+ \cup \Sigma^+ = \mathsf{Sub}(\varphi, \Sigma, \Box\Theta) \subseteq \mathsf{Sub}(\varphi, \Sigma) \subseteq \mathsf{Sub}(\Gamma, \Delta)$ by $\Box\Theta \subseteq \mathsf{Sub}(\varphi, \Sigma)$. This implies that $\gamma \in \mathsf{Sub}(\Gamma, \Delta) = \Gamma \cup \Delta$. Suppose for contradiction that $\gamma \notin \Gamma$. Then $\gamma \in \Delta$. By $\Box\gamma \in \mathsf{Sub}(\varphi, \Sigma)$, now we obtain $\gamma \in \Theta$ by definition of Θ, which implies $\Box\gamma \in \Box\Theta \subseteq \Sigma^+$. Together with $\Box\gamma \in \Pi^+$, we obtain the derivability of $\Pi^+ \Rightarrow \Sigma^+$, a contradiction.

$\Box$

It is immediate from Lemma 4.5 to obtain the following.

Lemma 4.6 *Let Ξ be a subformula closed finite set. Then, for all $(\Gamma, \Delta) \in W^\Xi$ and for all $\chi \in \Gamma \cup \Delta$, the following hold:*

(i) *If* $\chi \in \Gamma$*, then* $M^{\Xi}, (\Gamma, \Delta) \models \chi$*,* (ii) *If* $\chi \in \Delta$*, then* $M^{\Xi}, (\Gamma, \Delta) \not\models \chi$*.*

Theorem 4.7 *If a sequent* $\Gamma \Rightarrow \Delta$ *is valid in all finite models, then it is derivable in* $\mathsf{G}^a(\mathbf{BiSKt})$*.*

Proof. We prove the contrapositive implication. Suppose that a sequent $\Gamma \Rightarrow \Delta$ is underivable in $\mathsf{G}^a(\mathbf{BiSKt})$. Define $\Xi := \mathsf{Sub}(\Gamma, \Delta)$, which is a subformula closed finite set. It follows from Lemma 4.2 that there exists a Ξ-partial valuation $(\Gamma^+, \Delta^+) \in W^{\Xi}$ such that $\Gamma \subseteq \Gamma^+ \subseteq \Xi$, $\Delta \subseteq \Delta^+ \subseteq \Xi$ and $\Gamma^+ \cup \Delta^+ = \mathsf{Sub}(\Gamma, \Delta)$. By Lemma 4.6, we can conclude that $\Gamma \Rightarrow \Delta$ is not valid in a finite model M^{Ξ}, where it is noted that the domain W^{Ξ} of M^{Ξ} is finite. $\Box$

Corollary 4.8 *The following are all equivalent: for any sequent* $\Gamma \Rightarrow \Delta$*,* (i) *it is valid in all models,* (ii) *it is valid in all finite models,* (iii) *it is derivable in* $\mathsf{G}^a(\mathbf{BiSKt})$*,* (iv) *it is derivable in* $\mathsf{G}^*(\mathbf{BiSKt})$*,* (v) *it is derivable in* $\mathsf{G}(\mathbf{BiSKt})$*. Therefore,* $\mathsf{G}(\mathbf{BiSKt})$ *enjoys the finite model property and the subformula property. Moreover,* $\mathsf{G}(\mathbf{BiSKt})$ *is decidable.*

Proof. The direction from (ii) to (iii) is due to Theorem 4.7 and the direction from (v) to (i) is due to Theorem 3.3. The remaining directions from (i) to (ii), from (iii) to (iv), and from (iv) to (v), are immediate. $\Box$

Corollary 4.8 also provides an alternative proof of the finite model property of **BiSKt** (see [26, Theorem 5]), which was originally established via filtration technique in [26].

5 Craig Interpolation Theorem

This section establishes the Craig Interpolation Theorem of **BiSKt** by Mints' symmetric interpolation method [16], which is a generalization of Maehara's method [12]. For this purpose, it suffices to make use of $\mathsf{G}^*(\mathbf{BiSKt})$ from Table 1, instead of the fully analytic calculus $\mathsf{G}^a(\mathbf{BiSKt})$. This is because the Craig interpolation theorem is concerned with propositional variables and we do not need to consider analytic applications of modal rules of ($\blacklozenge$) and ($\Box$). Also, we do not restrict partitions of a sequent to a particular form, e.g., a *normal partition* as in [7]. Instead, we use the same unrestricted notion of partition of a sequent as for classical propositional logic (this is why the method is called *symmetric* in [16]) and follow Mints [16] to generalize the notion of Craig interpolant.

5.1 Mints' Notion of Interpolant

In what follows, we use S, T, U, S_i, etc. to denote sequents. Given any sequent $S = \Gamma \Rightarrow \Delta$, the following abbreviation is used: $\mathsf{Ant}(S) := \Gamma$ and $\mathsf{Suc}(S) := \Delta$. When we list sequents, we use the semicolon ";" (instead of the comma) to separate sequents such as $S_1; S_2; S_3$. Given any finite list $S_1; \ldots; S_n$ of sequents, we say that a sequent S is *derivable* in a system from $S_1; \ldots; S_n$ if there is a finite tree generated by inference rules of the system from initial sequents of the system and sequents among $S_1; \ldots; S_n$. It is not difficult to see that a sequent S is derivable in a system iff S is derivable from the empty list in the system.

When S is a sequent $\Gamma \Rightarrow \Delta$, we define $S[\psi_1/p_1, \ldots, \psi_n/p_n]$ as the sequent $\Gamma[\psi_1/p_1, \ldots, \psi_n/p_n] \Rightarrow \Delta[\psi_1/p_1, \ldots, \psi_n/p_n]$.

The following simple notation introduced in [16] makes the essence of an argument in this section more explicit.

Definition 5.1 Given sequents $S := \Gamma \Rightarrow \Delta$ and $S' := \Gamma' \Rightarrow \Delta'$, the notation SS' is defined as $SS' := \Gamma, \Gamma' \Rightarrow \Delta, \Delta'$.

Definition 5.2 ([16, Definition 1]) Let $p_1, \ldots, p_l$ be distinct propositional variables and $S_1; \ldots; S_k$ a finite list of sequents such that $\mathsf{Ant}(S_i) \cup \mathsf{Suc}(S_i) \subseteq \{p_1, \ldots, p_l\}$ and all elements in $\mathsf{Ant}(S_i)$ and $\mathsf{Suc}(S_i)$ are distinct $(1 \leqslant i \leqslant k)$. We say that the list $S_1; \ldots; S_k$ is *closed* if the empty sequent $\Rightarrow$ is derivable from the list by applying (Cut) and contraction rules alone.

For example, both $(p \Rightarrow\ ;\ \Rightarrow p)$ and $((p \Rightarrow q); (p, q \Rightarrow\ ); (\ \Rightarrow p))$ are closed.

Lemma 5.3 *Let $S_1; \ldots; S_m; S_{m+1}; \ldots; S_k$ $(m \geqslant 1)$, T and U be sequents. Suppose that the empty sequent $\Rightarrow$ is derivable from $S_1; \ldots; S_m; S_{m+1}; \ldots; S_k$ only by the rule of cut and contraction rules. Then, TU is derivable from $S_1T; \ldots; S_mT; S_{m+1}U; \ldots; S_kU$ by the rule of cut and contraction rules.*

Proof. It suffices to consider the following transformation of derivations:

$$\begin{array}{c} S_1 \quad \cdots \quad S_m \quad S_{m+1} \quad \cdots \quad S_k \\ \vdots\ (Cut), (c) \\ \Rightarrow \end{array} \quad \rightsquigarrow \quad \begin{array}{c} S_1T \quad \cdots \quad S_mT \quad S_{m+1}U \quad \cdots \quad S_kU \\ \vdots\ (Cut), (c) \\ \underbrace{T \cdots T}_{m}\underbrace{U \cdots U}_{k-m} \\ \vdots\ (c) \\ TU \end{array} .$$

□

It is noted that $S_1; \ldots; S_m; S_{m+1}; \ldots; S_k$ of Lemma 5.3 may not be atomic sequents and that T or U could be an empty sequent.

Definition 5.4 A *partition* of a sequent S is an arbitrary pair $(S_1; S_2)$ of sequents such that $S = S_1S_2$. A partition $(\Gamma_1 \Rightarrow \Delta_1; \Gamma_2 \Rightarrow \Delta_2)$ of a sequent S is also denoted by $(\Gamma_1 : \Delta_1); (\Gamma_2 : \Delta_2)$ (cf. [7]).

Definition 5.5 Let $S = \Gamma \Rightarrow \Delta$ be a sequent. The *formulaic translation* $\mathtt{f}(S)$ and *dual formulaic translation* $\mathtt{d}(S)$of S is defined as:

$$\mathtt{f}(S) := \bigwedge \Gamma \rightarrow \bigvee \Delta \quad \text{and} \quad \mathtt{d}(S) := \bigwedge \Gamma \prec \bigvee \Delta.$$

It is noted that $(\mathtt{f}(S) \Rightarrow)S$ is derivable, because the sequent $\bigwedge \Gamma \rightarrow \bigvee \Delta, \Gamma \Rightarrow \Delta$ is derivable. Similarly, $(\Rightarrow \mathtt{d}(S))S$ is derivable, because the sequent $\Gamma \Rightarrow \Delta, \bigwedge \Gamma \prec \bigvee \Delta$ is derivable.

Definition 5.6 We say that γ is a *Craig interpolant* for a partition $(S; S')$ if both $S(\Rightarrow \gamma)$ and $S'(\gamma \Rightarrow)$ are derivable and $\mathsf{Prop}(\gamma) \subseteq \mathsf{Prop}(S) \cap \mathsf{Prop}(S')$.

In other words, γ is a Craig interpolant for $(\Gamma_1 : \Delta_1); (\Gamma_2 : \Delta_2)$ if both $\Gamma_1 \Rightarrow \Delta_1, \gamma$ and $\gamma, \Gamma_2 \Rightarrow \Delta_2$ are derivable and $\mathsf{Prop}(\gamma) \subseteq \mathsf{Prop}(\Gamma_1, \Delta_1) \cap \mathsf{Prop}(\Gamma_2, \Delta_2)$.

Definition 5.7 (Interpolant [16, Definition 3]) Let S and S' be arbitrary sequents, $\gamma_1, \ldots, \gamma_l$ be formulas, $p_1, \ldots, p_l$ be distinct propositional variables, and $T_1; \ldots; T_k$ ($k \geqslant 2$) be sequents such that $\mathsf{Ant}(T_i) \cup \mathsf{Suc}(T_i) \subseteq \{ p_1, \ldots, p_l \}$ and all elements in $\mathsf{Ant}(T_i)$ and $\mathsf{Suc}(T_i)$ are distinct. We say that $(\gamma_1, \ldots, \gamma_l, (T_1; \ldots; T_k))$ is an *interpolant* for a partition $(S; S')$ if the following are satisfied:

(i) $\mathsf{Prop}(\gamma_i) \subseteq \mathsf{Prop}(S) \cap \mathsf{Prop}(S')$ for all i such that $1 \leqslant i \leqslant l$,

(ii) $(T_1; \ldots; T_k)$ is closed,

(iii) there exists m such that all sequents $ST_1^*; \ldots; ST_m^*; S'T_{m+1}^*; \ldots; S'T_k^*$ are derivable, where $T_i^* := T_i[\gamma_1/p_1, \ldots, \gamma_l/p_l]$.

The notion of interpolant in Definition 5.7 is a generalization of the notion of Craig interpolant (Definition 5.6). This is because $(\gamma, (\Rightarrow p; p \Rightarrow))$ is an interpolant for $(\Gamma_1 \Rightarrow \Delta_1; \Gamma_2 \Rightarrow \Delta_2)$ iff $\mathsf{Prop}(\gamma) \subseteq \mathsf{Prop}(\Gamma_1, \Delta_1) \cap \mathsf{Prop}(\Gamma_2, \Delta_2)$, $(\Rightarrow p; p \Rightarrow)$ is closed (this is trivial) and both $\Gamma_1 \Rightarrow \Delta_1, \gamma$ and $\gamma, \Gamma_2 \Rightarrow \Delta_2$ are derivable.

Let us say that a partition $(\Gamma_1, \Delta_1); (\Gamma_2, \Delta_2)$ is *normal* if Δ_1 or Γ_2 is empty ([7, p.11]). The following two lemmas tell us that a Craig interpolant can be constructed from an interpolant for a normal partition. Lemma 5.8 is needed for calculating interpolants for rules $(\Rightarrow\to)$ and $(\Box)$ and Lemma 5.9 is for rules $(\prec \Rightarrow)$ and $(\blacklozenge)$. The following lemma generalizes [16, Lemma 1(b)].

Lemma 5.8 *From an interpolant for* $(\Gamma_1 : \varnothing); (\Gamma_2 : \Delta)$*, a Craig interpolant for* $(\Gamma_1 : \varnothing); (\Gamma_2 : \Delta)$ *can be constructed.*

Proof. Let $(\gamma_1, \ldots, \gamma_l, (T_1; \ldots; T_k))$ be an interpolant for $(\Gamma_1 : \varnothing); (\Gamma_2 : \Delta)$. We can find an m such that $1 \leqslant m \leqslant k$ and all of the following sequents are derivable:

$$(\Gamma_1 \Rightarrow)T_1^*; \ldots; (\Gamma_1 \Rightarrow)T_m^*; (\Gamma_2 \Rightarrow \Delta)T_{m+1}^*; \ldots; (\Gamma_2 \Rightarrow \Delta)T_k^*,$$

where recall that $T^* := T[\gamma_1/p_1, \ldots, \gamma_l/p_l]$. It is noted that $(T_1; \ldots; T_k)$ is closed and $\mathsf{Prop}(\gamma_i) \subseteq \mathsf{Prop}(\Gamma_1) \cap \mathsf{Prop}(\Gamma_2, \Delta)$. Let us define $\gamma := \bigwedge_{1 \leqslant i \leqslant m} \mathtt{f}(T_i^*)$. We show that γ is a Craig interpolant for $(\Gamma_1 : \varnothing); (\Gamma_2 : \Delta)$. The variable condition is easily verified. So, we focus on checking the derivability condition. First, let us check the derivability of $\Gamma_1 \Rightarrow \gamma$ is derivable. The empty succedent of $(\Gamma_1 : \varnothing)$ becomes crucial here. It suffices to show that $\Gamma_1 \Rightarrow \mathtt{f}(T_i^*)$ for every $1 \leqslant i \leqslant m$. By recalling $\mathtt{f}(T_i^*) := \bigwedge \mathsf{Ant}(T_i^*) \to \bigvee \mathsf{Suc}(T_i^*)$, this is shown as follows:

$$\frac{\Gamma_1, \bigwedge \mathsf{Ant}(T_i^*) \Rightarrow \bigvee \mathsf{Suc}(T_i^*)}{\Gamma_1 \Rightarrow \bigwedge \mathsf{Ant}(T_i^*) \to \bigvee \mathsf{Suc}(T_i^*)} \ (\Rightarrow\to),$$

where the upper sequent is derivable since $(\Gamma_1 \Rightarrow)T_i^*$ is derivable by assumption.

Second, let us establish that $\gamma, \Gamma_2 \Rightarrow \Delta$ is derivable. It suffices to derive $\mathtt{f}(T_1^*), \ldots, \mathtt{f}(T_m^*), \Gamma_2 \Rightarrow \Delta$. Since $(T_1; \ldots; T_k)$ is closed, we can derive the empty sequent $\Rightarrow$ from $T_1^*, \ldots, T_k^*$ by applying the rule of cut and contraction rules *alone*. It follows from Lemma 5.3 that $\mathtt{f}(T_1^*), \ldots, \mathtt{f}(T_m^*), \Gamma_2 \Rightarrow \Delta$ is derivable from $(\mathtt{f}(T_1^*), \ldots, \mathtt{f}(T_m^*) \Rightarrow)T_i^*$ $(1 \leqslant i \leqslant m)$ and $(\Gamma_2 \Rightarrow \Delta)T_j^*$ $(m+1 \leqslant j \leqslant k)$ by the rule of cut and contraction rules alone. Each $(\Gamma_2 \Rightarrow \Delta)T_j^*$ is derivable by assumption. Since $(\mathtt{f}(T_i^*) \Rightarrow)T_i^*$ is derivable, all the sequents $(\mathtt{f}(T_1^*), \ldots, \mathtt{f}(T_m^*) \Rightarrow)T_i^*$ are also derivable by weakening rules. This finishes establishing that $\mathtt{f}(T_1^*), \ldots, \mathtt{f}(T_m^*), \Gamma_2 \Rightarrow \Delta$ is derivable.

Therefore, γ is a Craig interpolant for $(\Gamma_1 : \varnothing); (\Gamma_2 : \Delta)$. □

While we use a formulaic translation $\mathtt{f}(S)$ of a sequent S in the proof of Lemma 5.8, we need to use its dual variant $\mathtt{d}(S)$ in the following proof.

Lemma 5.9 *From an interpolant for* $(\Gamma : \Delta_1); (\varnothing : \Delta_2)$*, a Craig interpolant for* $(\Gamma : \Delta_1); (\varnothing : \Delta_2)$ *can be constructed.*

Proof. Let $(\gamma_1, \ldots, \gamma_l, (T_1; \ldots; T_k))$ be an interpolant for $(\Gamma : \Delta_1); (\varnothing : \Delta_2)$. We can find an m such that $1 \leqslant m \leqslant k$ and all of the following sequents are derivable:

$$(\Gamma \Rightarrow \Delta_1)T_1^*; \ldots; (\Gamma \Rightarrow \Delta_1)T_m^*; (\Rightarrow \Delta_2)T_{m+1}^*; \ldots; (\Rightarrow \Delta_2)T_k^*,$$

where we recall that $T^* := T[\gamma_1/p_1, \ldots, \gamma_l/p_l]$. It is noted that every γ_i satisfies $\mathsf{Prop}(\gamma_i) \subseteq \mathsf{Prop}(\Gamma, \Delta_1) \cap \mathsf{Prop}(\Delta_2)$. Define

$$\rho := \bigvee\nolimits_{m+1 \leqslant j \leqslant k} \mathtt{d}(T_j^*).$$

We show that ρ is a Craig interpolant for $(\Gamma : \Delta_1); (\varnothing : \Delta_2)$. The variable condition is easily verified. So, we focus on checking the derivability condition in what follows. First, let us check the derivability of $\rho \Rightarrow \Delta_2$. The empty antecedent of $(\varnothing : \Delta_2)$ becomes crucial here. It suffices to show that $\mathtt{d}(T_j^*) \Rightarrow \Delta_2$ for every $m+1 \leqslant j \leqslant k$. By recalling $\mathtt{d}(T_j^*) := \bigwedge \mathsf{Ant}(T_j^*) \prec \bigvee \mathsf{Suc}(T_j^*)$, this is shown as follows:

$$\frac{\bigwedge \mathsf{Ant}(T_j^*) \Rightarrow \Delta_2, \bigvee \mathsf{Suc}(T_j^*)}{\bigwedge \mathsf{Ant}(T_j^*) \prec \bigvee \mathsf{Suc}(T_j^*) \Rightarrow \Delta_2}\ (\prec\Rightarrow),$$

where the upper sequent is derivable since $(\Rightarrow \Delta_2)T_j^*$ is derivable by assumption.

Second, let us establish that $\Gamma \Rightarrow \Delta_1, \rho$ is derivable. It suffices to derive $\Gamma \Rightarrow \Delta_1, \mathtt{d}(T_{m+1}^*), \ldots, \mathtt{d}(T_k^*)$. Since $(T_1; \ldots; T_k)$ is closed, we can derive the empty sequent $\Rightarrow$ from $T_1^*, \ldots, T_k^*$ by applying the rule of cut and contraction rules *alone*. It follows from Lemma 5.3 that $\Gamma \Rightarrow \Delta_1, \mathtt{d}(T_{m+1}^*), \ldots, \mathtt{d}(T_k^*)$ is derivable from $(\Gamma \Rightarrow \Delta_1)T_i^*$ $(1 \leqslant i \leqslant m)$ and $(\Rightarrow \mathtt{d}(T_{m+1}^*), \ldots, \mathtt{d}(T_k^*))T_j^*$ $(m+1 \leqslant j \leqslant k)$ by the rule of cut and contraction rules. Note that each $\Gamma \Rightarrow \Delta_1, \mathtt{d}(T_i^*)$ is derivable by assumption. Since $(\Rightarrow \mathtt{d}(T_j^*))T_j^*$ is derivable, all the sequents $(\Rightarrow \mathtt{d}(T_{m+1}^*), \ldots, \mathtt{d}(T_k^*))T_j^*$ are also derivable by weakening rules. This finishes establishing that $\Gamma \Rightarrow \Delta_1, \mathtt{d}(T_{m+1}^*), \ldots, \mathtt{d}(T_{k+1}^*)$ is derivable.

Therefore, ρ is a Craig interpolant for $(\Gamma : \Delta_1); (\varnothing : \Delta_2)$. □

5.2 Transfer of Interpolant Across Inference Rules

Lemma 5.10 *Every partition of an initial sequent of Table 1 has an interpolant.*

Proof. We only check the initial sequent of the form $\varphi \Rightarrow \varphi$ here (because the other two cases are easy). There are four possible partitions as follows:

(i) $(\varphi : \varphi);(\varnothing : \varnothing)$. An interpolant is $(\bot, (\Rightarrow p; p \Rightarrow))$,

(ii) $(\varphi : \varnothing);(\varnothing : \varphi)$. An interpolant is $(\varphi, (\Rightarrow p; p \Rightarrow))$,

(iii) $(\varnothing : \varphi);(\varphi : \varnothing)$. An interpolant is $(\varphi, (p \Rightarrow; \Rightarrow p))$,

(iv) $(\varnothing : \varnothing);(\varphi : \varphi)$. An interpolant is $(\top, (\Rightarrow p; p \Rightarrow))$,

where item 3 is most important because there is no Craig interpolant for a partition $(\varnothing : p);(p : \varnothing)$ in the multi-succedent calculus of intuitionistic propositional logic (see [16, p.226]). □

In what follows, we show that interpolation transfers across applications of all inference rules of $\mathsf{G}^*(\mathbf{BiSKt})$ in the following sense (cf. [7, p.12]).

Definition 5.11 We say that *interpolation transfers across applications of an inference rule* if from the assumption that an interpolant exists for each partition of the upper sequent(s) of the rule it follows that an interpolant exists for each partition of the lower sequent of the rule.

Lemma 5.12 *Interpolation transfers across applications of weakening and contraction rules,* $(\wedge \Rightarrow)$ *and* $(\Rightarrow \vee)$.

Proof. For these one premise rules, the same interpolant can be used from a partition of the premise to the corresponding partition of the conclusion. □

We go back to the remaining one premise rules later. To deal with two premise rules, we introduce the following notion of *composition* of lists of sequents from [16, p.231] (but we use the different notation).

Definition 5.13 Given two finite lists $(S_1; \dots; S_m)$ and $(T_1; \dots; T_n)$ of sequents, the composition $(S_1; \dots; S_m) \circ (T_1; \dots; T_n)$ of them is defined as:

$$(S_1; \dots; S_m) \circ (T_1; \dots; T_n) := (S_i T_j | 1 \leqslant i \leqslant m \text{ and } 1 \leqslant j \leqslant n),$$

where we use the lexicographic order for the composition (but the order will not matter below).

It is noted that the resulting composition has $m \cdot n$ sequents.

The following lemma witnesses that our combination of analytic cut and Mints' notion of interpolant work neatly (compare the following proof of Lemma 5.14 with [7, Lemma 5.4] which states that a Craig interpolant transfers across (essential) applications of the analytic mix rule, i.e., an extended version of (Cut)).

Lemma 5.14 *Interpolation transfers across applications of* $(Cut)^a$.

Proof. Let us write the application of the rule as:

$$\frac{\Gamma_1, \Gamma_2 \Rightarrow \Delta_1, \Delta_2, \varphi \quad \varphi, \Pi_1, \Pi_2 \Rightarrow \Sigma_1, \Sigma_2}{\Gamma_1, \Gamma_2, \Pi_1, \Pi_2 \Rightarrow \Delta_1, \Delta_2, \Sigma_1, \Sigma_2}\ (Cut)^a,$$

where $\varphi \in \mathsf{Sub}(\Gamma_1, \Gamma_2, \Pi_1, \Pi_2, \Delta_1, \Delta_2, \Sigma_1, \Sigma_2)$. Let $(\Gamma_1, \Pi_1 : \Delta_1, \Sigma_1); (\Gamma_2, \Pi_2 : \Delta_2, \Sigma_2)$ be a partition. We divide our argument into the following two cases: (i) $\varphi \in \mathsf{Sub}(\Gamma_1, \Pi_1, \Delta_1, \Sigma_1)$; (ii) $\varphi \in \mathsf{Sub}(\Gamma_2, \Pi_2, \Delta_2, \Sigma_2)$. We consider case (i) alone because case (ii) is shown similarly. By induction hypothesis, there is an interpolant $(\gamma_1, \dots, \gamma_a, (T_1; \dots; T_m; T_{m+1}; \dots; T_b))$ for $(\Gamma_1 : \Delta_1, \varphi); (\Gamma_2 : \Delta_2)$. That is,

- $\mathsf{Prop}(\gamma_i) \subseteq \mathsf{Prop}(\Gamma_1, \Delta_1, \varphi) \cap \mathsf{Prop}(\Gamma_2, \Delta_2)$,
- $(T_1; \dots; T_m; T_{m+1}; \dots; T_b)$ is closed,
- all of the following sequents are derivable:

$$(\Gamma_1 \Rightarrow \Delta_1, \varphi)T_1^*; \dots; (\Gamma_1 \Rightarrow \Delta_1, \varphi)T_m^*; (\Gamma_2 \Rightarrow \Delta_2)T_{m+1}^*; \dots; (\Gamma_2 \Rightarrow \Delta_2)T_b^*$$

Again, by induction hypothesis, let $(\rho_1, \dots, \rho_c, (U_1; \dots; U_n; U_{n+1}; \dots; U_d))$ be an interpolant for the partition $(\varphi, \Pi_1 : \Sigma_1); (\Pi_2 : \Sigma_2)$. That is,

- $\mathsf{Prop}(\rho_j) \subseteq \mathsf{Prop}(\varphi, \Pi_1, \Sigma_1) \cap \mathsf{Prop}(\Pi_2, \Sigma_2)$,
- $(U_1; \dots; U_n; U_{n+1}; \dots; U_d)$ is closed,
- all of the following sequents are derivable:

$$(\varphi, \Pi_1 \Rightarrow \Sigma_1)U_1^\star; \dots; (\varphi, \Pi_1 \Rightarrow \Sigma_1)U_n^\star; (\Pi_2 \Rightarrow \Sigma_2)U_{n+1}^\star; \dots; (\Pi_2 \Rightarrow \Sigma_2)U_d^\star.$$

We can assume without loss of generality that $(T_1; \dots; T_b)$ and $(U_1; \dots; U_d)$ have no common propositional variable. In what follows, we prove that our interpolant for the original partition is:

$$(\gamma_1, \dots, \gamma_a, \rho_1, \dots, \rho_c, ((T_1; \dots; T_m) \circ (U_1; \dots; U_n); (T_{m+1}; \dots; T_b; U_{n+1}; \dots; U_d))).$$

Let us verify the required three conditions. First, we verify the variable condition. For γ_i, we proceed as follows. By $\varphi \in \mathsf{Sub}(\Gamma_1, \Pi_1, \Delta_1, \Sigma_1)$,

$$\mathsf{Prop}(\gamma_i) \subseteq \mathsf{Prop}(\Gamma_1, \Delta_1, \varphi) \cap \mathsf{Prop}(\Gamma_2, \Delta_2) \subseteq \mathsf{Prop}(\Gamma_1, \Pi_1, \Delta_1, \Sigma_1) \cap \mathsf{Prop}(\Gamma_2, \Delta_2)$$

As for ρ_j, it is shown from $\varphi \in \mathsf{Sub}(\Gamma_1, \Pi_1, \Delta_1, \Sigma_1)$ as:

$$\mathsf{Prop}(\rho_j) \subseteq \mathsf{Prop}(\varphi, \Pi_1, \Sigma_1) \cap \mathsf{Prop}(\Pi_2, \Sigma_2) \subseteq \mathsf{Prop}(\Gamma_1, \Pi_1, \Delta_1, \Sigma_1) \cap \mathsf{Prop}(\Pi_2, \Sigma_2).$$

Second, we proceed as follows for the closure condition. By Lemma 5.3 and the closedness of $(U_1; \dots; U_d)$, we can derive T_j from $(T_jU_1; \dots; T_jU_n)$ and $(U_{n+1}; \dots; U_d)$ by (Cut) and contraction rules, for every j such that $1 \leqslant j \leqslant m$. Then, we can derive $\Rightarrow$ from just obtained $T_1; \dots; T_m$ and $(T_{m+1}; \dots; T_b)$ since $(T_1; \dots; T_b)$ is closed. To sum up, the empty sequent $\Rightarrow$ is derivable from $(T_1; \dots; T_m) \circ (U_1; \dots; U_n)$; $(T_1; \dots; T_b)$; $(U_{n+1}; \dots; U_d)$ by definition of the composition $\circ$.

For the derivability condition, we first show that $(\Gamma_1, \Pi_1 \Rightarrow \Delta_1, \Sigma_1)(T_iU_j)^{*\star}$ ("$*\star$" is the concatenation or union of two substitutions, recall that $(T_1; \dots; T_b)$

and $(U_1; \ldots ; U_d)$ have no common propositional variable. So, $(T_i U_j)^{**} = T_i^* U_j^\star)$ is derivable as follows:

$$\dfrac{(\Gamma_1 \Rightarrow \Delta_1, \varphi)T_i^* \quad (\varphi, \Pi_1 \Rightarrow \Sigma_1)U_j^\star}{(\Gamma_1, \Pi_1 \Rightarrow \Delta_1, \Sigma_1)(T_i U_j)^{**}}\ (Cut)^a,$$

where the upper sequents are derivable by assumption. Second, sequents $(\Gamma_2, \Pi_2 \Rightarrow \Delta_2, \Sigma_2)T_i^*$ and $(\Gamma_2, \Pi_2 \Rightarrow \Delta_2, \Sigma_2)U_j^\star$ are derivable in terms of weakening rules from the derivability of sequents $(\Gamma_2 \Rightarrow \Delta_2)T_i^*$ and $(\Pi_2 \Rightarrow \Sigma_2)U_j^\star$, respectively. This finishes establishing the derivability condition. □

We can prove Lemma 5.15 similarly to Lemma 5.14.

Lemma 5.15 *Interpolation transfers across applications of inference rules* $(\rightarrow\Rightarrow)$*,* $(\Rightarrow \prec)$*,* $(\Rightarrow \wedge)$ *and* $(\Rightarrow \vee)$*.*

Let us discuss the remaining one premise rules below.

Lemma 5.16 *Interpolation transfers across applications of* $(\Rightarrow\rightarrow)$ *and* $(\prec \Rightarrow)$*.*

Proof. First, let us consider the following application of the rule $(\prec \Rightarrow)$:

$$\dfrac{\varphi \Rightarrow \psi, \Delta_1, \Delta_2}{\varphi \prec \psi \Rightarrow \Delta_1, \Delta_2}\ (\prec \Rightarrow)$$

All possible partitions are $(\varphi \prec \psi : \Delta_1); (\varnothing : \Delta_2)$ and $(\varnothing : \Delta_1); (\varphi \prec \psi : \Delta_2)$. First, we consider the former partition. By induction hypothesis and Lemma 5.9, there is a *Craig interpolant* ρ of $(\varphi : \psi, \Delta_1); (\varnothing : \Delta_2)$. We show that $(\rho, (\Rightarrow q; q \Rightarrow))$ is an interpolant. The variable condition and closure conditions are trivial. The derivability of $\rho \Rightarrow \Delta_2$ is immediate. Moreover, from the derivability of $\varphi \Rightarrow \psi, \Delta_1, \rho$, we conclude by $(\prec \Rightarrow)$ that $\varphi \prec \psi \Rightarrow \Delta_1, \rho$ is derivable. Next, we consider the partition $(\varnothing : \Delta_1); (\varphi \prec \psi : \Delta_2)$. By induction hypothesis and Lemma 5.9, there is a Craig interpolant ρ of $(\varphi : \psi, \Delta_2); (\varnothing : \Delta_1)$. We show that $(\rho, (q \Rightarrow; \Rightarrow q))$ is an interpolant for $(\varnothing : \Delta_1); (\varphi \prec \psi : \Delta_2)$ (be careful about the positions of q). All the conditions are verified similarly.

Second, we can apply a similar argument for the rule $(\Rightarrow\rightarrow)$ but we need Lemma 5.8 instead of Lemma 5.9. □

Lemma 5.17 *Interpolation transfers across applications of* $(\Box)$ *and* $(\blacklozenge)$*.*

Proof. First, we check the following application of the rule $(\blacklozenge)$:

$$\dfrac{\varphi \Rightarrow \Delta_1, \Delta_2, \Box\Theta_1, \Box\Theta_2}{\blacklozenge\varphi \Rightarrow \blacklozenge\Delta_1, \blacklozenge\Delta_2, \Theta_1, \Theta_2}\ (\blacklozenge).$$

All possible partitions have the following two forms:

$$(\blacklozenge\varphi : \blacklozenge\Delta_1, \Theta_1); (\varnothing : \blacklozenge\Delta_2, \Theta_2) \text{ or } (\varnothing : \blacklozenge\Delta_1, \Theta_1); (\blacklozenge\varphi : \blacklozenge\Delta_2, \Theta_2).$$

We focus on the partition of the form $(\varnothing : \blacklozenge\Delta_1, \Theta_1); (\blacklozenge\varphi : \blacklozenge\Delta_2, \Theta_2)$ below, because we can find an interpolant for the other partition by a similar argument. Let $((\gamma_1, \ldots, \gamma_a), T_1; \ldots ; T_b)$ be an interpolant of $(\varphi : \Delta_2, \Box\Theta_2); (\varnothing : \Delta_1, \Box\Theta_1)$.

Then, there exists a *Craig interpolant* for $(\varphi : \Delta_2, \Box\Theta_2); (\varnothing : \Delta_1, \Box\Theta_1)$ by Lemma 5.9. Then, we can find a formula γ such that $\varphi \Rightarrow \Delta_2, \gamma, \Box\Theta_2$ and $\gamma \Rightarrow \Delta_1, \Box\Theta_1$ are derivable, and $\mathsf{Prop}(\gamma) \subseteq \mathsf{Prop}(\varphi, \Delta_2, \Box\Theta_2) \cap \mathsf{Prop}(\Delta_1, \Box\Theta_1)$. By the following applications of the rule $(\blacklozenge)$:

$$\frac{\varphi \Rightarrow \Delta_2, \gamma, \Box\Theta_2}{\blacklozenge\varphi \Rightarrow \blacklozenge\Delta_2, \blacklozenge\gamma, \Theta_2}\ (\blacklozenge) \qquad \frac{\gamma \Rightarrow \Delta_1, \Box\Theta_1}{\blacklozenge\gamma \Rightarrow \blacklozenge\Delta_1, \Theta_1}\ (\blacklozenge),$$

we can conclude that $(\blacklozenge\gamma; (q \Rightarrow; \Rightarrow q))$ is our desired interpolant for the partition $(\varnothing : \blacklozenge\Delta_1, \Theta_1); (\blacklozenge\varphi : \blacklozenge\Delta_2, \Theta_2)$ because $\mathsf{Prop}(\blacklozenge\gamma) \subseteq \mathsf{Prop}(\blacklozenge\varphi, \blacklozenge\Delta_2, \Theta_2) \cap \mathsf{Prop}(\blacklozenge\Delta_1, \Theta_1)$ holds easily.

Second, we can also verify the case of $(\Box)$ similarly as above except we need Lemma 5.8 instead of Lemma 5.9. $\Box$

Lemma 5.18 *If* $\Gamma \Rightarrow \Delta$ *is derivable in* $\mathsf{G}^*(\mathbf{BiSKt})$*, then every partition* $(S; S')$ *for* $\Gamma \Rightarrow \Delta$ *has an interpolant in* $\mathsf{G}(\mathbf{BiSKt})$*.*

Proof. By induction on derivation of $\Gamma \Rightarrow \Delta$ in $\mathsf{G}^*(\mathbf{BiSKt})$. For the base case, we can use Lemma 5.10 and for the inductive step, it suffices to apply Lemmas 5.12, 5.14, 5.15, 5.16, and 5.17. $\Box$

Theorem 5.19 (Craig Interpolation) *If* $\Rightarrow \varphi \rightarrow \psi$ *is derivable in* $\mathsf{G}(\mathbf{BiSKt})$ *then there exists a formula* γ *such that* $\Rightarrow \varphi \rightarrow \gamma$ *and* $\Rightarrow \gamma \rightarrow \psi$ *are derivable in* $\mathsf{G}(\mathbf{BiSKt})$ *and* $\mathsf{Prop}(\gamma) \subseteq \mathsf{Prop}(\varphi) \cap \mathsf{Prop}(\psi)$*.*

Proof. Suppose that $\Rightarrow \varphi \rightarrow \psi$ is derivable in $\mathsf{G}(\mathbf{BiSKt})$. By

$$\dfrac{\dfrac{}{\Rightarrow \varphi \rightarrow \psi} \qquad \dfrac{\varphi \Rightarrow \varphi \quad \psi \Rightarrow \psi}{\varphi \rightarrow \psi, \varphi \Rightarrow \psi}\ (\rightarrow\Rightarrow)}{\varphi \Rightarrow \psi}\ (Cut),$$

we get the derivability of $\varphi \Rightarrow \psi$ in $\mathsf{G}(\mathbf{BiSKt})$. By Corollary 4.8, $\varphi \Rightarrow \psi$ is derivable in $\mathsf{G}^*(\mathbf{BiSKt})$. By Lemma 5.18, there is an interpolant for $(\varphi : \varnothing); (\varnothing : \psi)$ in $\mathsf{G}(\mathbf{BiSKt})$. By Lemma 5.8 (or Lemma 5.9), there is a *Craig interpolant* γ of $(\varphi : \varnothing); (\varnothing : \psi)$ in $\mathsf{G}(\mathbf{BiSKt})$. Therefore, $\Rightarrow \varphi \rightarrow \gamma$ and $\Rightarrow \gamma \rightarrow \psi$ are derivable in $\mathsf{G}(\mathbf{BiSKt})$ and the variable condition is also satisfied. $\Box$

6 Conclusion

This paper established the Craig interpolation theorem for bi-intuitionistic stable tense logic [30]. A novelty of this paper is in revealing that Mints' symmetric interpolation method [16] works properly with analytic cuts. Since our argument is modular, it also provides a simplification of Kowalski and the first author's argument in [7] for the Craig interpolation theorem of bi-intuitionistic logic in two respects. First, the restriction of applications of (Cut) to analytic ones for bi-intuitionistic logic is proved more directly than in [7]. Second, Mints' symmetric interpolation method [16] enables us to prove the interpolation theorem in a simpler manner than in [7].

We comment on possible directions of further research. First, as we noted in the proof of Lemma 5.10, there is no Craig interpolant for a partition $(\varnothing :$

$p); (p : \varnothing)$ in the multi-succedent calculus of intuitionistic propositional logic (see [16, p.226]). On the other hand, Lemmas 5.8 and 5.9 tell us that we can obtain Craig interpolants from interpolants for particular forms of partitions. From this respect, it would be nice to have a comprehensive study of the relationship between the Mints interpolation for analytic cut and the Craig interpolation for analytic mix.

Second, our calculation of an interpolant for the rule ($\blacklozenge$) (Lemma 5.17) depends on Lemma 5.9, in particular, the existence of coimplication $\prec$. So, it would be interesting to consider if the Craig interpolation theorem holds for the fragment of **G**(**BiSKt**) without coimplication, given that the subformula property of the fragment holds by our argument in Section 4. It is noted that this fragment is different from *intuitionistic* tense logic of Ewald [3] (see also [4,9] for the recent studies).

Third, to emphasize the effectiveness of a combination of analytic cut and Mints' symmetric interpolation method, we may apply our argument in this paper also to other non-classical logics, e.g., axiomatic extensions of bi-intuitionistic stable tense logic [26], bi-intuitionistic tense logic **BiKt** [5] (having future possibility operator $\Diamond$ and past necessity operator $\blacksquare$ in addition to $\blacklozenge$ and $\Box$), intuitionistic modal logic $\mathbf{L}_4$ [19] (the strongest **S5**-type intuitionistic modal logic in [19]), and bi-intuitionistic stable tense logic with universal modality [29]. Luppi [10] established the Craig interpolation theorem of $\mathbf{L}_4$ algebraically, but a proof-theoretic argument has not been provided. By a slight modification of Lemma 5.8, we have already confirmed that our argument is still applicable to $\mathbf{L}_4$. A semantic connection between **BiSKt** and **BiKt** [5] stated in [30, pp.517-8] allows us to apply our argument also to **BiKt** (see Appendix A for the detail). It may be also interesting to consider if we can generalize our argument of this paper to cover substructural modal or tense logics, though our argument depends on the existence of structural rules (e.g., in the proofs of Lemmas 5.3, 5.8 and 5.9).

Finally, algebras of the bi-intuitionistic stable tense logic are double Heyting algebras (cf. [7, pp.259-60]), which are distributive, equipped with a residuated pair of unary modal operators. It is well-known that, for normal modal logics based on classical logic, Craig interpolation corresponds to an algebraic property called superamalgamability [14,13]. It is an open problem to find the corresponding algebraic notion to Mints' symmetric interpolation method. [3]

Appendix

A An Application to Bi-intuitionistic Tense Logic BiKt

The syntax of bi-intuitionistic tense logic **BiKt** by Goré et al. [5] is an expansion of the syntax for bi-intuitionistic stable tense logic **BiSKt** with future possibility operator $\Diamond$ and past necessity operator $\blacksquare$. Kripke semantics for

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BiKt [5, Section 6] is slightly different from the one for **BiSKt** as follows. Let us say that $(W, \leqslant, R, S)$ is a **BiKt**-*frame* if $(W, \leqslant)$ is a preorder and R and S are binary relations on W such that the following two condition hold:

$(F\Box)$ $R \circ \leqslant \subseteq \leqslant \circ R$,

$(F\Diamond)$ $S^{-1} \circ \leqslant \subseteq \leqslant \circ S^{-1}$,

where S^{-1} is the converse relation of S. A **BiKt**-*model* is a pair of a **BiKt**-frame and a valuation assigning $\leqslant$-closed sets to propositional variables. The satisfaction relation is the same as for the bi-intuitionistic stable tense logic except:

$$
\begin{array}{lll}
M, u \models \blacklozenge\varphi & \text{iff} & \text{For some } v \in W \ (vRu \text{ and } M, v \models \varphi), \\
M, u \models \Box\varphi & \text{iff} & \text{For all } v \in W \ (u(\leqslant \circ R)v \text{ implies } M, v \models \varphi), \\
M, u \models \Diamond\varphi & \text{iff} & \text{For some } v \in W \ (vS^{-1}u \text{ and } M, v \models \varphi), \\
M, u \models \blacksquare\varphi & \text{iff} & \text{For all } v \in W \ (u(\leqslant \circ S^{-1})v \text{ implies } M, v \models \varphi).
\end{array}
$$

We define the notion of validity of a sequent as for **BiSKt**. We use $[\![\varphi]\!]_M$ to mean $\{ u \in W \mid M, u \models \varphi \}$. We can prove that $[\![\varphi]\!]$ is $\leqslant$-closed by induction. As noted in [5], however, we need the conditions $(F\Box)$ and $(F\Diamond)$ for showing that $[\![\blacklozenge\varphi]\!]$ and $[\![\Diamond\varphi]\!]$ are $\leqslant$-closed, respectively.

To define sequent calculi for **BiKt**, it suffices to consider the following two additional rules for $\Diamond$ and $\blacksquare$ and their analytic variants:

$$
\frac{\varphi \Rightarrow \Delta, \blacksquare\Theta}{\Diamond\varphi \Rightarrow \Diamond\Delta, \Theta}\ (\Diamond) \qquad \frac{\Diamond\Theta, \Gamma \Rightarrow \varphi}{\Theta, \blacksquare\Gamma \Rightarrow \blacksquare\varphi}\ (\blacksquare)
$$

where the side conditions for analytic applications of $(\blacklozenge)$ and $(\Box)$ are similarly defined as in Table 1. In what follows, we use $\mathsf{G}(\textbf{BiKt})$, $\mathsf{G}^*(\textbf{BiKt})$, and $\mathsf{G}^a(\textbf{BiKt})$ as the full calculus, the calculus where the rule of cut are analytic, and the analytic calculus, respectively (those are also similarly defined as in Table 1). Once we establish for **BiKt** a similar result to Corollary 4.8, we can apply the same argument to obtain the Craig interpolation theorem for **BiKt**, since the shape of rules $(\blacklozenge)$ and $(\Box)$ are the same as $(\Diamond)$ and $(\blacksquare)$. So, in what follows, we comment on the soundness and the subformula property of $\mathsf{G}(\textbf{BiKt})$.

Theorem A.1 *If* $\Gamma \Rightarrow \Delta$ *is derivable in* $\mathsf{G}(\textbf{BiKt})$ *then it is valid in all* **BiKt**-*models.*

Proof. We only prove that the rule $(\blacksquare)$ preserves the validity on a **BiKt**-model $M := (W, \leqslant, R, S, V)$. Suppose that $\Diamond\Theta, \Gamma \Rightarrow \varphi$ is valid on M and fix any state $u \in W$ such that $M, u \models \bigwedge(\Theta, \blacksquare\Gamma)$. To show $M, u \models \blacksquare\varphi$, let us also fix any state v such that $u(\leqslant \circ S^{-1})v$, i.e., $u \leqslant w$ and $wS^{-1}v$ for some $w \in W$. We fix such a state w. Our goal is to show $M, v \models \varphi$. By the supposition of $M, u \models \bigwedge(\Theta, \blacksquare\Gamma)$ and $u(\leqslant \circ S^{-1})v$, we obtain $M, v \models \bigwedge\Gamma$. By $u \leqslant w$ and the same supposition, we get $M, w \models \bigwedge\Theta$, since the truth set for $\bigwedge\Theta$ is $\leqslant$-closed. It follows from $wS^{-1}v$ (i.e., vSw) that $M, v \models \bigwedge\Diamond\Theta$ hence $M, v \models \bigwedge(\Diamond\Theta, \Gamma)$. Since $\Diamond\Theta, \Gamma \Rightarrow \varphi$ is valid on M, we conclude that $M, v \models \varphi$, as desired. $\Box$

The following proposition, which is extracted from [30, pp.517-8], is useful for relating the semantic completeness of $\mathsf{G}^a(\mathbf{BiSKt})$ with that of $\mathsf{G}^a(\mathbf{BiKt})$ below.

Proposition A.2 *Let $(W, \leqslant)$ be a preorder and $R, S \subseteq W \times W$.*

(i) *Let R be stable. Then, R satisfies the condition $(F\Box)$ and $\leqslant \circ R = R$.*

(ii) *Let S^{-1} be stable. Then, S satisfies the condition $(F\Diamond)$ and $\leqslant \circ S^{-1} = S^{-1}$.*

It is noted that the condition $\leqslant \circ R = R$ implies that the satisfaction relation of $\Box\varphi$ for **BiSKt** is equivalent to that for **BiKt**. So, if we focus on the fragment with $\blacklozenge$ and $\Box$ but not with $\Diamond$ and $\blacksquare$, this proposition implies that if $\Gamma \Rightarrow \Delta$ is valid on the semantics for **BiKt** then it is also valid on the semantics for **BiSKt**. Therefore, our semantic completeness argument for $\mathsf{G}^a(\mathbf{BiSKt})$ can be naturally extended to an argument for $\mathsf{G}^a(\mathbf{BiKt})$ as in the following proof.

Theorem A.3 *If $\Gamma \Rightarrow \Delta$ is valid in all finite* **BiKt**-*models then it is derivable in $\mathsf{G}^a(\mathbf{BiKt})$.*

Proof. Suppose that $\Gamma \Rightarrow \Delta$ is not derivable in $\mathsf{G}^a(\mathbf{BiKt})$. Put $\Xi := \mathsf{Sub}(\Gamma, \Delta)$. Because Lemma 4.2 still holds for $\mathsf{G}^a(\mathbf{BiKt})$ (its proof needs $(Cut)^a$ alone), we can find a Ξ-partial valuation (Γ^+, Δ^+) such that $\Gamma \subseteq \Gamma^+$ and $\Delta \subseteq \Delta^+$. We define the derived **BiKt**-model $M^\Xi := (W^\Xi, \leqslant^\Xi, R^\Xi, S^\Xi, V^\Xi)$ from Ξ in the same way as in Definition 4.3 except that $(\Gamma, \Delta)S^\Xi(\Pi, \Sigma)$ iff the following two conditions hold:

(i) if $\blacksquare\,\psi \in \Pi$ then $\psi \in \Gamma$, for all formulas ψ,

(ii) if $\Diamond\,\psi \in \Delta$ then $\psi \in \Sigma$, for all formulas ψ.

We can prove that $(S^\Xi)^{-1}$ is stable, i.e., $\leqslant^\Xi \circ (S^\Xi)^{-1} \circ \leqslant^\Xi \subseteq (S^\Xi)^{-1}$. By Proposition A.2, it suffices for us to check the corresponding items from (ix) to (xii) of Lemma 4.5 to $\Diamond$ and $\blacksquare$ (we can just replace $\blacklozenge$, $\Box$ and R^Ξ of items from (ix) to (xii) of Lemma 4.5 with $\Diamond$, $\blacksquare$ and $(S^\Xi)^{-1}$, respectively). These items are shown similarly to those for $\mathsf{G}^a(\mathbf{BiSKt})$. Then, we can establish the corresponding lemma to Lemma 4.6 and so conclude that $\Gamma \Rightarrow \Delta$ is not valid in a finite **BiKt**-model M^Ξ. $\Box$

To sum up all arguments in this section, we can obtain the following.

Theorem A.4 *For any sequent $\Gamma \Rightarrow \Delta$, the following are all equivalent:* (i) *it is valid in all* **BiKt**-*models,* (ii) *it is valid in all finite* **BiKt**-*models,* (iii) *it is derivable in* $\mathsf{G}^a(\mathbf{BiKt})$, (iv) *it is derivable in* $\mathsf{G}^*(\mathbf{BiKt})$, (v) *it is derivable in* $\mathsf{G}(\mathbf{BiKt})$. *Therefore,* $\mathsf{G}(\mathbf{BiKt})$ *enjoys the finite model property, the subformula property and the decidability. Moreover,* $\mathsf{G}(\mathbf{BiKt})$ *enjoys the Craig interpolation theorem.*

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