Intuitionistic Modality and Beth Semantics

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Abstract

One of the standard methods to understand intuitionistic logic is to see it through its semantical counterpart. Kripke semantics in particular offers an insightful interpretation in terms of the growth of knowledge. This interpretation is extended to modal operators in the case of intuitionistic modal logics. In the framework of M. Božić and K. Došen, the four intuitionistic modalities (necessity, possibility, impossibility and non-necessity) are characterised in a uniform manner, suggesting that they share a type of assumption on how modal notions interact with the growth of knowledge. On the other hand, there is another intuitionistic semantics called Beth semantics, which supports a different perspective on the notion of the growth of knowledge. A natural question then is how the four modalities appear from this alternative perspective. The main observation of this paper is how the above-mentioned uniformity breaks down in Beth semantics, which hints that the modalities can be seen to be based on different conceptions of the growth of knowledge. In addition, we look at the Beth correspondence theory of some modal principles, which is then applied to obtain a Beth completeness of a paraconsistent system by R. Sylvan.

Keywords: Beth semantics, Intuitionistic Logic, Modal Logic, Negation, Paraconsistent Logic.

1 Introduction

Relational semantics is a valuable tool to provide intuitive interpretations to logical systems. An important example of this type of semantics is *Kripke semantics* for intuitionistic logic [22]. This semantics captures the validity of formulas using the pictures of the growth of knowledge (of an agent), depicted with a partially ordered set of worlds (or information states). A characteristic feature of intuitionistic Kripke semantics is the valuation of implication; in order to establish that an implication holds at a world, one has to look at

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not only the world in question, but also all later worlds. This feature plays an essential role in invalidating constructively unacceptable logical principles such as the law of excluded middle.

Kripke semantics is one of the standard semantics for intuitionistic logic. There is, however, another type of relational semantics that is worthy of attention. This semantics, introduced by E.W. Beth [3], is accordingly called *Beth semantics*. Like Kripke semantics, Beth semantics can also be understood to depict the growth of knowledge, but as the main differences, a model of Beth semantics (a) typically involves infinitely many worlds, (b) has a stronger constraint on the valuation of propositional variables, and (c) uses a different condition for the forcing of disjunction, which allows a disjunction to hold at a world without either of the disjuncts being so. The last point is analysed in depth by W.H. Holliday [20].

Beth semantics is known to be more general than Kripke semantics (see [4,31]), and so we may embed a Kripke model into a Beth model, whereas the converse is not always possible. On the other hand, as far as logic is concerned, the two semantics capture the same logic, namely intuitionistic logic. The situation can change, however, when we add an additional operator. For instance, it is observed in [24] that when a type of alternative negation called *empirical negation* [7,8] is considered, an identical forcing condition (falsity at the root of a model) ends up in two different logics, depending on which of the semantics is used. In other words, it is reflected in the logics that the two semantics offer different philosophical interpretations of the notion of 'growth of knowledge'.

Since empirical negation can be seen as a kind of modal operator in intuitionistic setting, this gives a motivation to investigate modal operators in Beth semantics. We note there already exists a related investigation by R. Goldblatt [19], which established that the lax operator [15] can be captured by the forcing condition of (classical) possibility operator in cover semantics, which is a generalisation of Beth semantics. Analogous observations are also established in [19] for bimodal systems **CK** [23] and **CS4** [1]. D. Rogozin [27] also applies the framework for the systems of intuitionistic epistemic logic by S. Artemov and T. Protopopescu [2].

Intuitionistic modal logics have been studied by various authors since Fitch [17] (see e.g. [18,28,34] for overviews of early approaches). Among these, we shall base our enquiry on the systems investigated by K. Došen and M. Božić [5,10,11,12,13]. In particular, we shall mainly focus on (i) the system $\mathbf{HK}\Box$ for necessity operator, and (ii) its negative counterpart, the system $\mathbf{HK}\Box'$ for non-necessity operator. We shall also consider operators for possibility and impossibility, and the corresponding systems $\mathbf{HK}\Diamond$ and $\mathbf{HK}\Diamond'$ [2]. These systems can all be seen as intuitionistic analogues of the classical modal logic **K**. Kripke semantics for the logics are defined in a uniform way, by (a) using the same forcing condition as the classical case, and (b) employing a frame condition specifying the interaction of the intuitionistic ordering and the modal acces-

[2] See [14] for an investigation of further operators definable in these systems.

sibility relation. Our central observation will be that this uniformity breaks down once we move to Beth semantics.

The structure of this paper is as follows. Firstly, we shall look at the necessity operator: after introducing the axiomatisation and the corresponding Kripke semantics, we shall formulate the Beth semantics for **HK**$\Box$ and show the soundness and completeness with respect to the axiomatic system. Secondly, we shall look at the non-necessity operator; we shall analogously formulate the Beth semantics for **HK**$\Box'$ using a different forcing condition from the Kripke one, and show the soundness and completeness. The third section treats the cases for **HK**$\Diamond$ and **HK**$\Diamond'$. We shall point out that these cases are different from the first two cases in that they require a new condition for the soundness. This is followed by a step into the correspondence theory of a couple of modal principles in the Beth semantics of necessity and non-necessity. The result is then used in the final section to obtain a Beth semantics for a paraconsistent logic of R. Sylvan [29].

2 Beth semantics for intuitionistic necessity

In this section, we shall give a complete Beth semantics for Božić and Došen [5]'s system **HK**$\Box$, which is an intuitionistic analogue of the classical modal logic **K** (with $\Box$ primitive). We first specify the language to be used.

Definition 2.1 [$\mathcal{L}_\Box$] We shall use the following language $\mathcal{L}_\Box$:

$$A ::= p \mid \bot \mid A \wedge A \mid A \vee A \mid A \to A \mid \Box A.$$

We shall adopt $A \leftrightarrow B$ as an abbreviation for $(A \to B) \wedge (B \to A)$.

As an inessential difference from [5], we take $\bot$ as primitive and define a negation $\neg A$ as $A \to \bot$.

2.1 HK$\Box$: axiomatisation and Kripke semantics

Next, we give a Hilbert-style axiomatisation of **HK**$\Box$.

Definition 2.2 [**HK**$\Box$] We define the system **HK**$\Box$ in $\mathcal{L}_\Box$ by the following axiom schemata and rules.

$$(A \to (B \to C)) \to ((A \to B) \to (A \to C)) \quad \text{(S)} \qquad \bot \to A \quad \text{(EFQ)}$$

$$A \to (B \to A) \quad \text{(K)} \qquad (\Box A \wedge \Box B) \to \Box(A \wedge B) \quad \text{(P1)}$$

$$(A \to B) \to ((A \to C) \to (A \to B \wedge C)) \quad \text{(CI)} \qquad \Box(A \to A) \quad \text{(P2)}$$

$$A_1 \wedge A_2 \to A_i \quad \text{(CE)} \qquad \frac{A \quad A \to B}{B} \quad \text{(MP)}$$

$$A_i \to A_1 \vee A_2 \quad \text{(DI)}$$

$$(A \to C) \to ((B \to C) \to (A \vee B \to C)) \quad \text{(DE)} \qquad \frac{A \to B}{\Box A \to \Box B} \quad \text{(RM)}$$

where $i \in \{1, 2\}$. A formula A is called a *theorem* of **HK**$\Box$ if there is a finite sequence $B_1, \ldots, B_n \equiv A$ of formulas such that each B_i is either an instance of one of the axiom schemata, or obtained from the preceding formulas by one of the rules. A *proof* of A from a set of formulas Γ in **HK**$\Box$ is a finite sequence $B_1, \ldots, B_n \equiv A$, where each B_i is either an element of Γ, a theorem of $\mathbf{HK}\Box$, or obtained from previous items in the sequence by (MP). We shall write $\Gamma \vdash_{\mathbf{HK}\Box} A$ to denote the provability. In particular, when $\Gamma = \emptyset$ we shall write $\vdash_{\mathbf{HK}\Box} A$. Similar conventions apply to later systems.

We have the next Kripke semantics corresponding to the proof system.

Definition 2.3 [Kripke semantics for $\mathbf{HK}\Box$] A *Kripke frame* $\mathcal{F}$ is a triple $(K, \leq, R)$, where $(K, \leq)$ is a non-empty partially ordered set, and $R \subseteq K \times K$ such that $\leq R \subseteq R \leq$.[3] A *Kripke model* $\mathcal{M}$ then is a pair $(\mathcal{F}, \mathcal{V})$ where $\mathcal{V}$ is a mapping assigning to each propositional variable p a set $\mathcal{V}(p) \subseteq K$. We require each $\mathcal{V}(p)$ to be *upward closed*, i.e. $k \in \mathcal{V}(p)$ and $k' \geq k$ implies $k' \in \mathcal{V}(p)$. $\mathcal{V}$ is uniquely extended to the forcing $\Vdash_{kp}$ of formulas by the clauses below.

$$
\begin{aligned}
k \Vdash_{kp} \bot &\text{ iff } \textit{never}.\\
k \Vdash_{kp} p &\text{ iff } k \in \mathcal{V}(p).\\
k \Vdash_{kp} A \wedge B &\text{ iff } k \Vdash_{kp} A \text{ and } k \Vdash_{kp} B.\\
k \Vdash_{kp} A \vee B &\text{ iff } k \Vdash_{kp} A \text{ or } k \Vdash_{kp} B.\\
k \Vdash_{kp} A \to B &\text{ iff } \text{ for all } k' \geq k(k' \Vdash_{kp} A \text{ implies } k' \Vdash_{kp} B).\\
k \Vdash_{kp} \Box A &\text{ iff } \text{ for all } k' R^{-1} k(k' \Vdash_{kp} A).
\end{aligned}
$$

where $R^{-1} = \{(k', k) : kRk'\}$. We shall write $\mathcal{M} \vDash_{kp} A$ if $k \Vdash_{kp} A$ for any k in $\mathcal{M}$. We write $\mathcal{F} \vDash_{kp} A$ if $\mathcal{M} \vDash_{kp} A$ for any $\mathcal{M}$ with $\mathcal{F}$ as the frame. Finally, we write $\vDash_{kp} A$ if $\mathcal{M} \vDash_{kp} A$ for all $\mathcal{M}$.

The next propositions are established in [5, Lemma 2, Theorem 1].

Proposition 2.4 (Upward closure) *$k \Vdash_{kp} A$ and $k' \geq k$ implies $k' \Vdash_{kp} A$.*

Theorem 2.5 (Kripke soundness and completeness of $\mathbf{HK}\Box$)
$\vdash_{\mathbf{HK}\Box} A$ if and only if $\vDash_{kp} A$.

2.2 $\mathbf{HK}\Box$: Beth semantics

We now set out to formulate a Beth semantics for $\mathbf{HK}\Box$. An important point to note is that Beth semantics uses a stronger condition than the upward closure of valuation. This means we have to generalise the condition $\leq R \subseteq R \leq$, which is put in place in order to preserve upward closure.

Beth semantics can be based either on trees (e.g. [31]) or posets (e.g. [32]). In this paper we take the former approach as it appears simpler and perhaps more loyal to the intuitionistic picture. At the same time, the conditions we will see are not necessarily optimised for trees, in view of possible generalisations into posets.

We define a *tree* $\mathcal{T} = (T, \preceq)$ to be a poset s.t. there is the minimum element $g \in T$ and for each t, $\{t' : t' \preceq t\}$ is linearly ordered. A *path* of T is then a maximal linearly ordered subset of T.[4] We will use $\alpha, \beta, \ldots$ to denote a path in $\mathcal{T}$.

[3] where $R_1 R_2 = \{(x, z) : \exists y(xR_1 y \wedge yR_2 z)\}$.

[4] For a more constructive approach, we can also take a tree to be a certain collection of sequences of natural numbers [30, p.186].

Definition 2.6 [Beth semantics for $\mathbf{HK}\Box$] We define a *Beth frame* $\mathcal{F}$ for $\mathbf{HK}\Box$ as a triple $(B, \preceq, S)$, where $(B, \preceq)$ is a tree s.t. $\forall b \in B \exists b' \in B(b' \succ b)$ (i.e. it is a *spread*), and $S \subseteq B \times B$. S also has to satisfy the next condition:

$$\forall b, b' \in B(b \preceq S\ b' \Rightarrow \exists \alpha \ni b \forall c \in \alpha(c\ S \preceq\ b'))$$

A *Beth model* $\mathcal{M}$ for $\mathbf{HK}\Box$ is then a pair $(\mathcal{F}, \mathcal{V})$ such that $\mathcal{V}$ is a mapping assigning to each propositional variable p a set $\mathcal{V}(p)$ subject to the following condition (*covering property*):

$$b \in \mathcal{V}(p) \text{ if and only if } \forall \alpha \ni b \exists b' \in \alpha(b' \in \mathcal{V}(p)).$$

$\mathcal{V}$ is now extended to the forcing $\Vdash_{bp}$ by the clauses below.

$$\begin{aligned}
b \Vdash_{bp} \bot &\text{ iff } \textit{never.}\\
b \Vdash_{bp} p &\text{ iff } b \in \mathcal{V}(p).\\
b \Vdash_{bp} A \wedge B &\text{ iff } b \Vdash_{bp} A \text{ and } b \Vdash_{bp} B.\\
b \Vdash_{bp} A \vee B &\text{ iff } \forall \alpha \ni b \exists b' \in \alpha(b' \Vdash_{bp} A \text{ or } b' \Vdash_{bp} B).\\
b \Vdash_{bp} A \to B &\text{ iff } \forall b' \succeq b(b' \Vdash_{bp} A \text{ implies } b' \Vdash_{bp} B).\\
b \Vdash_{bp} \Box A &\text{ iff } \forall b' S^{-1} b(b' \Vdash_{bp} A).
\end{aligned}$$

We shall write $\mathcal{M} \vDash_{bp} A$ if $b \Vdash_{bp} A$ for any b in $\mathcal{M}$. We write $\mathcal{F} \vDash_{bp} A$ if $\mathcal{M} \vDash_{bp} A$ for any $\mathcal{M}$ with $\mathcal{F}$ as the frame, and $\vDash_{bp} A$ if $\mathcal{M} \vDash_{bp} A$ for all $\mathcal{M}$.

The conditions for disjunction are changed from those of Kripke semantics and satisfy the following properties.

Proposition 2.7 (Covering property)
(i) $b \Vdash_{bp} A$ *if and only if* $\forall \alpha \ni b \exists b' \in \alpha(b' \Vdash_{bp} A)$.
(ii) $b \Vdash_{bp} A$ *and* $b' \succeq b$ *implies* $b' \Vdash_{bp} A$.

Proof. We proceed by induction on the complexity of A, and treat (i) and (ii) simultaneously. Note that the cases for (ii) follows from the cases for (i), because if $b' \succeq b$, then any $\alpha \ni b'$ must pass through b as well.[5] Hence $b \Vdash_{bs} A$ implies $b' \Vdash_{bs} A$ from the case of A for (i).

As for (i), the left-to-right direction is always immediate. For the right-to-left direction, the case $A \equiv p$ follows from the covering property of $\mathcal{V}$. The case for conjunction is straightforward, and the case for disjunction is analogous to the case for non-necessity we shall look at in the next section. For $A \equiv B \to C$, if $\forall \alpha \ni b \exists b' \in \alpha(b' \Vdash_{bp} B \to C)$, then $c \succeq b$ implies $\forall \alpha \ni c \exists c' \in \alpha(c' \Vdash_{bp} B \to C)$. Hence if $c \Vdash_{bp} B$, then for any α s.t. $c \in \alpha$ there is $c' \in \alpha$ with $c' \Vdash_{bp} B \to C$. We have either $c \preceq c'$ or $c \succeq c'$, but in each case, the later world must (using (ii) for B in one of the cases) force C. So $\forall \alpha \ni c \exists c' \in \alpha(c' \Vdash_{bp} C)$. Thus by I.H. again $c \Vdash_{bp} C$. Therefore $b \Vdash_{bp} B \to C$.

When $A \equiv \Box B$, then if $\forall \alpha \ni b \exists b' \in \alpha(b' \Vdash_{bp} \Box B)$, suppose bSc. Then $b \preceq S\ c$, and so there is $\alpha \ni b$ s.t. $\forall c' \in \alpha(c'\ S \preceq\ c)$, by the frame condition of

[5] It is essential here that we are considering trees and not posets.

S. Thus in particular, $b'\ S{\preceq}\ c$ for some b' s.t. $b' \Vdash_{bp} \Box B$. Hence there exists c' such that $b'Sc'$ and $c' \preceq c$, which implies $c' \Vdash_{bp} B$. So by (ii) for B, we infer $c \Vdash_{bp} B$. Therefore $b \Vdash_{bp} \Box B$. $\Box$

Theorem 2.8 (Beth soundness of HK□) *If* $\vdash_{\mathbf{HK}\Box} A$ *then* $\vDash_{bp} A$.

Proof. See Appendix. $\Box$

In order to prove the Beth completeness, we shall employ an embedding of Kripke models to Beth models. This method is similar to the one given in [31] for the Beth completeness of intuitionistic logic, except that we put an extra world to guarantee the existence of the root in the constructed Beth model.

Theorem 2.9 (Beth completeness of HK□) *If* $\vDash_{bp} A$ *then* $\vdash_{\mathbf{HK}\Box} A$.

Proof. See Appendix. $\Box$

3 Beth semantics for non-necessity

In this section, we continue the investigation of Beth semantics for intuitionistic modality. We now turn our attention to the non-necessity operator, denoted by $\Box'$ in [10]. Also, $\bot$ turns out to be definable in the system we will consider; we shall however include $\bot$ in the language for uniformity.

Definition 3.1 [$\mathcal{L}_{\Box'}$] We shall use the following language $\mathcal{L}_{\Box'}$:

$$A ::= p \mid \bot \mid A \wedge A \mid A \vee A \mid A \to A \mid \Box' A.$$

3.1 HK□′: axiomatisation and Kripke semantics

The next system gives [6] an axiomatisation of the system **HK□′** in [10].

Definition 3.2 [**HK□′**] We define the system **HK□′** in $\mathcal{L}_{\Box'}$ with the axiom schemata (S)–(EFQ), the rule (MP) in addition to the following axiom schemata and a rule.

$$\Box'(A \wedge B) \to (\Box' A \vee \Box' B) \quad \text{(N1)}$$
$$\Box'(A \to A) \to B \quad \text{(N2)}$$
$$\frac{A \to B}{\Box' B \to \Box' A} \quad \text{(RC)}$$

The notion of a proof is then defined as in **HK□**.

The following Kripke semantics is given to **HK□′**.

Definition 3.3 [Kripke semantics for **HK□′**] A *Kripke frame* $\mathcal{F}$ for **HK□′** is a triple $(K, \leq, R)$, where $(K, \leq)$ is as before, and $R \subseteq K \times K$ is such that $\geq R \ \subseteq\ R\leq$. A *Kripke model* $\mathcal{M}$ for **HK□′** then is a pair $(\mathcal{F}, \mathcal{V})$ where $\mathcal{V}$ is again an upward closed assignment of propositional variables to worlds. The forcing $\Vdash_{kn}$ is defined similarly to $\Vdash_{kp}$; for non-necessity, we use the next clause.

$$k \Vdash_{kn} \Box' A \text{ iff } \text{ for some } k' R^{-1} k (k' \nVdash_{kn} A).$$

We will use $\vDash_{kn}$ for the validity with respect to this semantics.

[6] Strictly speaking, there is a difference in that Došen's system again has $\neg$ as primitive.

The next propositions are established in [10, Lemma 10, Theorem 3].

Proposition 3.4 (Upward closure) *$k \Vdash_{kn} A$ and $k' \geq k$ implies $k' \Vdash_{kn} A$.*

Theorem 3.5 (Kripke soundness and completeness of HK□′)
$\vdash_{\mathbf{HK}\Box'} A$ *if and only if* $\vDash_{kn} A$.

3.2 HK□′: Beth semantics

We now move on to the definition of Beth semantics for **HK□′**. In Kripke semantics, the clauses for necessity and non-necessity are in a sense dual of each other, as we have seen. In comparison, the clauses will turn out to be quite different in Beth semantics, a situation comparable to those of conjunction and disjunction. This is because the condition on the accessibility relation, even when generalised for Beth semantics, guarantees only the upward closure of forcing, and not the covering property. On the other hand, once we have a forcing condition for non-necessity that assures covering property, there is no longer a need for a condition on the accessibility relation: upward closure follows automatically from the covering property. As a result, Beth semantics for non-necessity will have simpler frames, but models are not necessarily so.

Definition 3.6 [Beth semantics for **HK□′**] We define a *Beth frame* $\mathcal{F}$ for **HK□′** as a triple $(B, \preceq, S)$, where $(B, \preceq)$ is as before, and $S \subseteq B \times B$ without any other conditions. A *Beth model* $\mathcal{M}$ for **HK□′** is a pair $(\mathcal{F}, \mathcal{V})$ such that $\mathcal{V}$ is an assignment satisfying the covering property. For the forcing $\Vdash_{bn}$, it has the same clauses as $\Vdash_{bp}$, except the clause for non-necessity, which is:

$$b \Vdash_{bn} \Box' A \text{ iff } \forall \alpha \ni b \exists b' \in \alpha \exists c S^{-1} b' (c \nVdash_{bn} A).$$

We shall write $\vDash_{bn}$ for the validity with respect to this semantics.

Remark 3.7 The Beth clause for disjunction may be interpreted as saying that one can assert a disjunction even when one can not assert either of the disjuncts. This perhaps better capture the informal usage of disjunction than the Kripke clause. In a similar manner, the Beth clause for non-necessity appears to be telling that one can assert "It is not necessary that A" even when all the states that can be referred to (accessed) support A. This possibility is however under the condition that a world which does not support A will eventually become accessible in all cases.

Let us check that the forcing condition establishes the covering property.

Proposition 3.8 (Covering property)
$b \Vdash_{bn} A$ if and only if $\forall \alpha \ni b \exists b' \in \alpha (b' \Vdash_{bn} A)$.

Proof. We look at the right-to-left direction for the case of non-necessity. When $A \equiv \Box' B$, then if $\forall \alpha \ni b \exists b' \in \alpha (b' \Vdash_{bn} \Box' B)$, it follows that

$$\forall \alpha \ni b \exists b' \in \alpha (\forall \beta \ni b' \exists c \in \beta \exists c' S^{-1} c (c' \nVdash_{bn} B)).$$

In particular, since $\alpha \ni b'$, we infer $\forall \alpha \ni b \exists b' \in \alpha \exists c' S^{-1} b' (c' \nVdash_{bn} B)$. Hence $b \Vdash_{bn} \Box' B$. □

Corollary 3.9 (Upward closure) *$b \Vdash_{bn} A$ and $b' \succeq b$ implies $b' \Vdash_{bn} A$.*

We leave the proofs of Beth soundness and completeness for $\mathbf{HK}\Box'$ in Appendix.

Theorem 3.10 (Beth soundness of $\mathbf{HK}\Box'$) *If $\vdash_{\mathbf{HK}\Box'} A$ then $\vDash_{bn} A$.*

Theorem 3.11 (Beth completeness of $\mathbf{HK}\Box'$) *If $\vDash_{bn} A$ then $\vdash_{\mathbf{HK}\Box'} A$.*

4 Beth semantics for possibility and impossibility

Having looked at the necessity and non-necessity operators, let us move on to consider their counterparts, possibility and impossibility operators [5,10]. At first glance, one may expect that the Beth semantics for these operators are identical to the semantics for non-necessity and necessity modulo a simple rewriting of the accessibility and forcing conditions. However, as we shall find out, there is a subtle difference for these operators, which is perhaps not visible in Kripke semantics. It thus points to a divergence of the two semantics for intuitionistic modal logics at this basic level.

Let us again start with specifying the languages.

Definition 4.1 $[\mathcal{L}_\Diamond, \mathcal{L}_{\Diamond'}]$ The languages $\mathcal{L}_\Diamond$ and $\mathcal{L}_{\Diamond'}$ are defined as follows.

$$A ::= p \mid \bot \mid A \wedge A \mid A \vee A \mid A \to A \mid \Diamond A.$$
$$A ::= p \mid \bot \mid A \wedge A \mid A \vee A \mid A \to A \mid \Diamond' A.$$

4.1 $\mathbf{HK}\Diamond, \mathbf{HK}\Diamond'$: axiomatisation and Kripke semantics

The proof systems $\mathbf{HK}\Diamond$ and $\mathbf{HK}\Diamond'$ quite resemble $\mathbf{HK}\Box$ and $\mathbf{HK}\Box'$.

Definition 4.2 [$\mathbf{HK}\Diamond$, $\mathbf{HK}\Diamond'$] We define the system $\mathbf{HK}\Diamond$ in $\mathcal{L}_\Diamond$ and $\mathbf{HK}\Diamond'$ in $\mathcal{L}_{\Diamond'}$ with the axiom schemata (S)–(EFQ), rule (MP) in addition to the following axiom schemata and a rule.
For $\mathbf{HK}\Diamond$:

$$\Diamond(A \vee B) \to (\Diamond A \vee \Diamond B) \quad \text{(Q1)} \qquad \neg\Diamond\neg(A \to A) \quad \text{(Q2)} \qquad \frac{A \to B}{\Diamond A \to \Diamond B} \quad \text{(RM2)}$$

For $\mathbf{HK}\Diamond'$:

$$(\Diamond' A \wedge \Diamond' B) \to \Diamond'(A \vee B) \quad \text{(O1)} \qquad \Diamond'\neg(A \to A) \quad \text{(O2)} \qquad \frac{A \to B}{\Diamond' B \to \Diamond' A} \quad \text{(RC2)}$$

The notions of a proof are then defined as in $\mathbf{HK}\Box$.

Definition 4.3 [Kripke semantics for $\mathbf{HK}\Diamond$, $\mathbf{HK}\Diamond'$] The Kripke semantics for $\mathbf{HK}\Diamond$ and $\mathbf{HK}\Diamond'$ are defined similarly to those of $\mathbf{HK}\Box$ and $\mathbf{HK}\Box'$. The only difference for the frames is the frame condition for R, given respectively as $\geq R \subseteq R \geq$ and $\leq R \subseteq R \geq$. The forcing conditions $\Vdash_{kq}$ and $\Vdash_{ko}$ have the next conditions for the respective modality.

$$k \Vdash_{kq} \Diamond A \text{ iff for some } k' R^{-1} k (k' \Vdash_{kq} A).$$
$$k \Vdash_{ko} \Diamond' A \text{ iff for all } k' R^{-1} k (k' \nVdash_{ko} A).$$

We will use $\vDash_{kq}, \vDash_{ko}$ for the validity in the semantics, respectively.

Proposition 4.4 (Upward closure) *For* $x \in \{q, o\}$*,* $k \vDash_{kx} A$ *and* $k' \geq k$ *implies* $k' \Vdash_{kx} A$*.*

Proof. See [5, Lemma 16] and [10, Lemma 2]. $\square$

Theorem 4.5 (Kripke soundness and completeness of HK$\Diamond$, HK$\Diamond'$)
(i) $\vdash_{\mathbf{HK}\Diamond} A$ *if and only if* $\vDash_{kq} A$*.*
(ii) $\vdash_{\mathbf{HK}\Diamond'} A$ *if and only if* $\vDash_{ko} A$*.*

Proof. See [5, Theorem 4] and [10, Theorem 1]. $\square$

4.2 HK$\Diamond$, HK$\Diamond'$: Beth semantics

Definition 4.6 [Beth semantics for **HK**$\Diamond$] Beth semantics for **HK**$\Diamond$ is mostly identical to that of **HK**$\Box'$. We need an extra condition that $\forall b, b' \in B(bSb' \Rightarrow \exists \alpha \ni b' \forall c \in \alpha(c \succeq b' \Rightarrow bSc))$. Possibility has the next condition in $\Vdash_{bq}$:

$$b \Vdash_{bq} \Diamond A \text{ iff } \forall \alpha \ni b \exists b' \in \alpha \exists c S^{-1} b' (c \Vdash_{bn} A).$$

We shall write $\vDash_{bq}$ for the validity with respect to this semantics.

Definition 4.7 [Beth semantics for **HK**$\Diamond'$] We define Beth semantics for **HK**$\Diamond'$ in a similar way as that of **HK**$\Box$. We have the following condition on the accessibility relation:

$$\forall b, b' \in B(b \preceq S\ b' \Rightarrow \exists \alpha \ni b \forall c \in \alpha(c\ S \succeq b'))$$

In addition, we again require that $\forall b, b' \in B(bSb' \Rightarrow \exists \alpha \ni b' \forall c \in \alpha(c \succeq b' \Rightarrow bSc))$. The forcing $\Vdash_{bo}$ has the next clause for impossibility.

$$b \Vdash_{bo} \Diamond' A \text{ iff } \forall b' S^{-1} b (b' \nVdash_{bo} A).$$

We shall then denote the validity in the semantics by $\vDash_{bo}$.

Remark 4.8 As we shall see, the condition we added for the above two semantics is required to show the Beth soundness. The additional condition ensures that all Beth frames behave like the ones obtained from Kripke frames; this allows us to overcome the difference in the two semantics.

Proposition 4.9 (Covering property) *For* $x \in \{q, o\}$*, the following hold.*
(i) $b \Vdash_{bx} A$ *if and only if* $\forall \alpha \ni b \exists b' \in \alpha(b' \Vdash_{bx} A)$*.*
(ii)$b \Vdash_{bx} A$ *and* $b' \succeq b$ *implies* $b' \Vdash_{bx} A$*.*

Proof. Analogous to the cases for **HK**$\Box'$ and **HK**$\Box$, respectively. $\square$

Let us denote by $\vDash_{bq'}$ and $\vDash_{bo'}$ the validity with respect to Beth semantics for **HK**$\Diamond$ and **HK**$\Diamond'$ minus the condition that $\forall b, b' \in B(bSb' \Rightarrow \exists \alpha \ni b' \forall c \in \alpha(c \succeq b' \Rightarrow bSc))$. Then we observe that they fail one of the axiom schemata of the corresponding system.

Proposition 4.10
(i) $\nvDash_{bq'} \Diamond(A \vee B) \rightarrow (\Diamond A \vee \Diamond B)$.
(ii) $\nvDash_{bo'} (\Diamond' A \wedge \Diamond' B) \rightarrow \Diamond'(A \vee B)$.

Proof. Let $(B, \preceq)$ be a tree defined from two paths $\alpha_1 = (g, b_1, b_2, \ldots)$ and $\alpha_2 = (g, b'_1, b'_2, \ldots)$ branching at g.

For (i), we set a Beth frame $\mathcal{F} = (B, \preceq, S)$ with $S = \{(g, g)\}$. Then define a Beth model $\mathcal{M} = (\mathcal{F}, \mathcal{V})$ where $\mathcal{V}(p) = \{b_i : i \in \mathbb{N}\}$ and $\mathcal{V}(q) = \{b'_i : i \in \mathbb{N}\}$. Then it is straightforward to check that $\mathcal{M}$ satisfies the covering property, and so is well-defined. Now, since $g \Vdash_{bq'} p \vee q$ and gSg it holds that $g \Vdash_{bq'} \Diamond(p \vee q)$. On the other hand, for any $b \in \alpha_1$ we have $b \nVdash_{bq'} \Diamond p$ and $b \nVdash_{bq'} \Diamond q$, because the only world accessible by S is g, which neither forces p nor q. Hence $g \nVdash_{bq'} \Diamond p \vee \Diamond q$ and so $g \nVdash_{bq'} \Diamond(p \vee q) \to (\Diamond p \vee \Diamond q)$.

For (ii), we use the same tree and $\mathcal{V}$ but define $\mathcal{F}$ with $S = \{(b, g) : b \in B\}$. Then we have to check that $b \preceq S\ b' \Rightarrow \exists \alpha \ni b \forall c \in \alpha (c\ S \succeq\ b')$. This is not difficult to see, as the premise holds precisely for arbitrary b and $b' = g$, and taking any $\alpha \ni b$ allows us to reach the conclusion. Thus the model is again well-defined. Now we readily observe that $g \Vdash_{bo'} \Diamond' p \wedge \Diamond' q$, but as $g \Vdash_{bo'} p \vee q$ we have to conclude that $g \nVdash_{bo'} \Diamond'(p \vee q)$. Therefore $g \nVdash_{bo'} (\Diamond' p \wedge \Diamond' q) \to \Diamond'(p \vee q)$. $\Box$

The same thing does not happen for the models we defined earlier. For soundness and completeness, again we leave them to Appendix.

Theorem 4.11 (Beth soundness of HK$\Diamond$, HK$\Diamond'$)
(i) *If* $\vdash_{\mathbf{HK}\Diamond} A$ *then* $\vDash_{bq} A$.
(ii) *If* $\vdash_{\mathbf{HK}\Diamond'} A$ *then* $\vDash_{bo} A$.

Theorem 4.12 (Beth completeness of HK$\Diamond$, HK$\Diamond'$)
(i) *If* $\vDash_{bq} A$ *then* $\vdash_{\mathbf{HK}\Diamond} A$.
(ii) *If* $\vDash_{bo} A$ *then* $\vdash_{\mathbf{HK}\Diamond'} A$.

Remark 4.13 The acceptability of $\Diamond(A \vee B) \to (\Diamond A \vee \Diamond B)$ in intuitionistic modal logic has been questioned by some authors [1,33] on computational grounds. K. Kojima [21] points out that whether to admit the formula depends on the viewpoint one takes for a Kripke model. One viewpoint is that of an *external observer*, who can check all worlds in a model. For such an observer, assuming $\Diamond(A \vee B)$ at a world gives an accessible world in which the disjunction holds. Then he can look at the world in question to find out which of the disjuncts holds there, and thereby conclude $\Diamond A \vee \Diamond B$. Another viewpoint is that of an *internal observer*, who is confined to a world and only has incomplete information for other worlds. Such an agent cannot ascertain which of the disjuncts is the case in the accessible world, and so cannot assert $\Diamond A \vee \Diamond B$. Hence the formula is not acceptable for an internal observer.

5 Beth frame conditions for HK$\Box$ and HK$\Box'$

In this section, we shall consider a few modal principles for necessity and non-necessity, as a first step to see what kind of frame conditions correspond to them in Beth semantics. Correspondence theory for Beth semantics is already undertaken for some intermediate logics by B. de Beer [9]. He worked with the version of Beth semantics based on posets, and the frame conditions are quite complex compared with the frame conditions for Kripke semantics. Our version of Beth semantics is based on trees, which should simplify the matter to a certain extent: nonetheless, as we shall see, the conditions are still more complex [7] than those of intuitionistic Kripke semantics in [10,11,12,13].

We shall focus on relatively simple frame conditions in Kripke semantics, namely the reflexivity and symmetry for the relation $R_{\leq}$. They correspond to different formulas in $\mathbf{HK}\Box$ and $\mathbf{HK}\Box'$. We shall show how even in these simple cases, the frame conditions for the formulas diverge in Beth semantics.

In what follows, $S_{\preceq}$ in some of the conditions can in fact be replaced with S; nonetheless we use the first relation to make the comparison clearer.

5.1 Correspondence for HK□

Let us start with the case for necessity. It is shown in [11] that the reflexivity and symmetry for $R_{\leq}$ correspond to the validity of $\Box A \to A$ and $A \vee \Box\neg\Box A$, respectively. For the first schema, the corresponding Beth-frame condition is still relatively straightforward. The condition may be seen to generalise the notion of reflexivity to each path.

Proposition 5.1 *Let $\mathcal{F} = (B, \preceq, S)$ be a Beth frame for* $\mathbf{HK}\Box$. *Then the following are equivalent.*
(i) $\mathcal{F} \vDash_{bp} \Box A \to A$ *for all* A.
(ii) $\forall \alpha \ni b \exists b' \in \alpha (b\ S_{\preceq}\ b')$.

Proof. From (i) to (ii), we argue by contradiction. Suppose for some b, there is $\alpha \ni b$ s.t. for all $b' \in \alpha$, it is not the case that $b\ S_{\preceq}\ b'$ [8]. Then let $\mathcal{V}(p) = \{b' : \forall \beta \ni b' \exists c \in \beta (b\ S_{\preceq}\ c)\}$. In order to see that $\mathcal{V}$ satisfies the covering property, assume $\forall \alpha \ni c \exists c' \in \alpha (c' \in \mathcal{V}(p))$. Then $\forall \alpha \ni c \exists c' \in \alpha \forall \beta \ni c' \exists d \in \beta (b\ S_{\preceq}\ d)$. So taking $\beta = \alpha$ allows us to conclude $c \in \mathcal{V}(p)$. Now if bSb' then $\forall \beta \ni b' \exists c \in \beta (b\ S_{\preceq}\ c)$. So $b' \Vdash_{bp} p$ and thus $b \Vdash_{bp} \Box p$. On the other hand, if $b \Vdash_{bp} p$ then $\forall \beta \ni b \exists c \in \beta (b\ S_{\preceq}\ c)$, contradicting our initial supposition. Thus $b \nVdash_{bp} p$ and so $\mathcal{F} \nvDash_{bp} \Box p \to p$.

From (ii) to (i), if for some b and $b' \succeq b$ we have $b' \Vdash_{bp} \Box A$, then $\forall c S^{-1} b' (c \Vdash_{bp} A)$. By the assumption, in any $\alpha \ni b'$ there is $c \in \alpha$ such that $b'Sc'$ and $c' \preceq c$ for some c'. This means $c' \Vdash_{bp} A$ and so $c \Vdash A$. Thus $\forall \alpha \ni b' \exists c \in \alpha (c \Vdash_{bp} A)$. Hence by the covering property, $b' \Vdash_{bp} A$ and so $b \Vdash_{bp} \Box A \to A$ for all $b \in B$. □

The case for the second schema is a bit more involved because it contains a disjunction. Like the previous case, there is a flavour of symmetry in the condition, but one has to come back only to the original path, and not necessarily to the same world.

Proposition 5.2 *Let $\mathcal{F} = (B, \preceq, S)$ be a Beth frame for* $\mathbf{HK}\Box$. *Then the following are equivalent.*
(i) $\mathcal{F} \vDash_{bp} A \vee \Box\neg\Box A$ *for all* A.
(ii) $\forall \alpha \ni b \exists b' \in \alpha \forall c (b'\ S_{\preceq}\ c \Rightarrow \exists d \in \alpha (c\ S_{\preceq}\ d))$.

[7] Note however that **CS4**-modalities in the setting of Goldblatt [19] have simple conditions.
[8] That is to say, $\neg(b\ S_{\preceq}\ b')$.

Proof. From (i) to (ii), we argue by contraposition. Suppose for some b, there exists $\alpha \ni b$ such that $\forall b' \in \alpha \exists c(b'\ S{\preceq}\ c$ and $\forall d \in \alpha(\neg(c\ S{\preceq}\ d)))$. Then define a set $\Sigma = \{c : \exists b' \in \alpha(b'\ S{\preceq}\ c)$ and $\forall d \in \alpha(\neg(c\ S{\preceq}\ d))\}$ and let $\mathcal{V}(p) = \{d : \forall \gamma \ni d \exists d' \in \gamma \exists c \in \Sigma(c\ S{\preceq}\ d')\}$. We can check that $\mathcal{V}$ satisfies the covering property similarly to Proposition 5.1. Now for any $b' \in \alpha$, if $b' \Vdash_{bp} p$ then $\exists d' \in \alpha \exists c \in \Sigma(c\ S{\preceq}\ d')$. However any $c \in \Sigma$ cannot access any $d \in \alpha$ via $S{\preceq}$, a contradiction. So $b' \nVdash_{bp} p$. Also if $b' \Vdash_{bp} \Box\neg\Box p$, then we first infer from our initial supposition that there is c such that $b'\ S{\preceq}\ c$ and $\forall d \in \alpha(\neg(c\ S{\preceq}\ d))$. This means $c \in \Sigma$, which implies that $c \Vdash_{bp} \Box p$. But as $b'\ S{\preceq}\ c$, by the upward closure we have $c \Vdash_{bp} \neg\Box p$ as well, a contradiction. Hence $b' \nVdash_{bp} \Box\neg\Box p$ as well for any $b' \in \alpha$. Therefore $b \nVdash_{bp} p \vee \Box\neg\Box p$ and so $\mathcal{F} \nvDash_{bp} p \vee \Box\neg\Box p$.

From (ii) to (i), suppose for $\alpha \ni b$ we have $b' \nVdash_{bp} A$ for any $b' \in \alpha$. By assumption, there is $c \in \alpha$ s.t. for all c', $c\ S{\preceq}\ c'$ implies $c'\ S{\preceq}\ d$ for some $d \in \alpha$. We claim $c \Vdash_{bp} \Box\neg\Box A$. If $c\ S{\preceq}\ c'$, then $c' \nVdash_{bp} \Box A$ because $d \in \alpha$ accessible from c' via $S{\preceq}$ does not force A. It then follows that $c \Vdash_{bp} \Box\neg\Box A$. Therefore $b \Vdash_{bp} A \vee \Box\neg\Box A$ for all b. □

5.2 Correspondence for HK□′

We now move on to the cases for non-necessity. It does not seem to be treated explicitly by Došen, but it is apparent from the results in [13,29] for related systems that the reflexivity of $R{\leq}$ in Kripke semantics should correspond to an analogue of the *law of excluded middle* for non-necessity.

Proposition 5.3 *Let $\mathcal{F} = (K, \leq, R)$ be a Kripke frame for* **HK□′**. *Then the following are equivalent.*
(i) $\mathcal{F} \vDash_{kn} A \vee \Box' A$ *for all* A.
(ii) $k\ R{\leq}\ k$.

Proof. See Appendix. □

Now, the frame condition for Beth semantics can be calculated as follows. It can still be seen as a generalisation of reflexivity, but is more complex than the condition for necessity.

Proposition 5.4 *Let $\mathcal{F} = (B, \preceq, S)$ be a Beth frame for* **HK□′**. *Then the following are equivalent.*
(i) $\mathcal{F} \vDash_{bn} A \vee \Box' A$ *for all* A.
(ii) $\forall \alpha \ni b \exists b' \in \alpha \forall \beta \ni b' \exists c \in \beta \exists c' \in \alpha(c\ S{\preceq}\ c')$.

Proof. From (i) to (ii), we show the contrapositive. Suppose for some $b \in B$,

$$\exists \alpha \ni b \forall b' \in \alpha \exists \beta \ni b' \forall c \in \beta \forall c' \in \alpha(\neg(c\ S{\preceq}\ c'))$$

Then we let $\mathcal{M} = (\mathcal{F}, \mathcal{V})$ be a model such that $b' \in \mathcal{V}(p) \Leftrightarrow b' \notin \alpha$. We need to check that $\mathcal{V}$ satisfies the covering property. If $\forall \beta \ni b' \exists c \in \beta(c \in \mathcal{V}(p))$, then we have that $b' \in \alpha$ implies $\exists c \in \alpha(c \notin \alpha)$, a contradiction. Hence $b' \notin \alpha$, and so $b' \in \mathcal{V}(p)$. Thus the model satisfies the covering property.

Now for $b' \in \alpha$, on one hand if $b' \Vdash_{bn} p$ then $b' \notin \alpha$, a contradiction. Hence $b' \nVdash_{bn} p$. On the other hand, $b' \Vdash_{bn} \Box' p$ implies $\forall \beta \ni b' \exists c \in \beta \exists c' S^{-1} c(c' \in \alpha)$.

But by our supposition, $\exists\beta \ni b'\forall c \in \beta\forall c' \in \alpha(\neg(c\ S{\preceq}\ c'))$, a contradiction. Thus $\forall b' \in \alpha(b' \nVdash_{bn} p$ and $b' \nVdash_{bn} \Box' p)$. So $b \nVdash_{bn} p \vee \Box' p$. Therefore $\mathcal{F} \nvDash_{bn} p \vee \Box' p$.

From (ii) to (i), assume that $\mathcal{F}$ satisfies (ii) and let $\mathcal{M}$ be a model and suppose $b \nVdash_{bn} A \vee \Box' A$. Then

$$\exists\alpha \ni b\forall b' \in \alpha(b' \nVdash_{bn} A \text{ and } b' \nVdash_{bn} \Box' A).$$

The latter conjunct means $\forall b' \in \alpha\exists\beta \ni b'\forall c \in \beta\forall c' S^{-1} c(c' \Vdash_{bn} A)$. But as $c' \in \alpha \Rightarrow c' \nVdash_{bn} A$ by the first conjunct, it follows that

$$\forall b' \in \alpha\exists\beta \ni b'\forall c \in \beta\forall c' S^{-1} c(c' \notin \alpha).$$

However, this contradicts the assumption that

$$\exists b' \in \alpha\forall\beta \ni b'\exists c \in \beta\exists c' \in \alpha(c\ S{\preceq}\ c'),$$

because $d \preceq c'$ and $c' \in \alpha$ implies $d \in \alpha$. Thus $b \Vdash_{bn} A \vee \Box' A$ and as the model was arbitrary, we conclude $\mathcal{F} \vDash_{bn} A \vee \Box' A$. $\Box$

The second principle we shall consider is the analogue of *double negation elimination* $\Box'\Box' A \to A$ which corresponds Kripke-semantically to the symmetry of $R{\leq}$.

Proposition 5.5 *Let $\mathcal{F} = (K, \leq, R)$ be a Kripke frame for* **HK**$\Box'$. *Then the following are equivalent.*
(i) $\mathcal{F} \vDash_{kn} \Box'\Box' A \to A$ *for all* A.
(ii) $k\ R{\leq}\ k' \Rightarrow k'\ R{\leq}\ k$.

Proof. From (i) to (ii), we argue by contradiction. If there are k, k' with $k\ R{\leq}\ k'$ but not $k'\ R{\leq}\ k$, then take $\mathcal{V}$ s.t. $\mathcal{V}(p) = \{l : k'\ R{\leq}\ l\}$. Then $k \Vdash_{kn} \Box'\Box' p$ but $k \nVdash_{kn} p$. Hence $\mathcal{F} \nvDash_{kn} \Box'\Box' p \to p$. From (ii) to (i), if $k' \Vdash_{kn} \Box'\Box' A$ for some k and $k' \geq k$, then there is a world l accessible from k', all of whose accessible worlds force A. But by the frame condition, there must be a world l' s.t. lRl' and $l' \leq k'$. Thus $k' \Vdash_{kn} A$. Therefore $\mathcal{F} \vDash_{kn} \Box'\Box' A \to A$. $\Box$

The frame condition for Beth semantics is on the other hand rather involved.

Proposition 5.6 *Let $\mathcal{F} = (B, \preceq, S)$ be a Beth frame for* **HK**$\Box'$. *Then the following are equivalent.*
(i) $\mathcal{F} \vDash_{bn} \Box'\Box' A \to A$ *for all* A.
(ii) $\forall\alpha \ni b\exists\beta \ni b\forall b' \in \beta\forall c(b'\ S{\preceq}\ c \Rightarrow \forall\gamma \ni c\exists c' \in \gamma\exists d \in \alpha(c'\ S{\preceq}\ d))$.

Proof. From (i) to (ii), we show the contrapositive. Assume for some $b \in B$,

$$\exists\alpha \ni b\forall\beta \ni b\exists b' \in \beta\exists c(b'\ S{\preceq}\ c \text{ and } \exists\gamma \ni c\forall c' \in \gamma\forall d \in \alpha(\neg(c'\ S{\preceq}\ d))).$$

We then fix, for each $\beta \ni b$, worlds $b'_\beta \in \beta$, $c_\beta \succeq S^{-1}\ b'_\beta$ and a path $\gamma_\beta \ni c_\beta$ s.t. $\forall c' \in \gamma_\beta\forall d \in \alpha(\neg(c'\ S{\preceq}\ d))$. We define a model $\mathcal{M} = (\mathcal{F}, \mathcal{V})$ by choosing $\mathcal{V}$ such that:

$$x \in \mathcal{V}(p) \Leftrightarrow \forall\delta \ni x\exists\beta \ni b\exists y \in \gamma_\beta\exists z \in \delta(y\ S{\preceq}\ z).$$

In order to check that $\mathcal{V}$ satisfies the covering property, suppose $\forall\alpha \ni x \exists y \in \alpha(y \in \mathcal{V}(p))$ but $x \notin \mathcal{V}(p)$. Then there is $\delta \ni x$ such that for all $\beta \ni b$, $y \in \gamma_\beta$ and $z \in \delta$, it is not the case that $y \; S\!\preceq \; z$. But by our supposition, there is $x' \in \delta$ such that $x' \in \mathcal{V}(p)$. So for some $\beta_0 \ni b$, $y_0 \in \gamma_{\beta_0}$ and $z_0 \in \delta$, we have $y_0 \; S\!\preceq \; z_0$. Hence we obtain a contradiction, which allows us to conclude $x \in \mathcal{V}(p)$, as desired.

Now, for each $\beta \ni b$, if $c'Sd$ for some $c' \in \gamma_\beta$, then $d \Vdash_{bn} p$ because for each $\delta \ni d$, d itself is an element accessible from c'. Hence $c_\beta \nVdash_{bn} \Box' p$ for each $\beta \ni b$. Therefore $b \Vdash_{bn} \Box'\Box' p$. On the other hand, by assumption, for any $\beta \ni b$ and $c' \in \gamma_\beta$, there is no $d \in \alpha$ such that $c' \; S\!\preceq \; d$. So $b \nVdash_{bn} p$ and thus $\mathcal{F} \nvDash_{bn} \Box'\Box' p \to p$.

From (ii) to (i), let $\mathcal{M} = (\mathcal{F}, \mathcal{V})$ be a model and suppose $b' \Vdash_{bn} \Box'\Box' A$ for some b and $b' \succeq b$. This is equivalent to saying that

$$\forall\beta \ni b' \exists c \in \beta \exists c' S^{-1} c \exists \gamma \ni c' \forall d \in \gamma \forall d' S^{-1} d (d' \Vdash_{bn} A).$$

Now take $\alpha \ni b'$. then by our assumption, there is $\beta \ni b'$ satisfying

$$\circledast : \; \forall c \in \beta \forall c' (c \; S\!\preceq \; c' \Rightarrow \forall \gamma \ni c' \exists d \in \gamma \exists d' \in \alpha (d \; S\!\preceq \; d')).$$

Also by the equivalence above, there is $c_0 \in \beta$, c_0' satisfying $c_0 \; S\!\preceq \; c_0'$ and $\gamma \ni c_0'$ such that $\forall d \in \gamma \forall d' S^{-1} d (d' \Vdash_{bn} A)$. With respect to this γ, it holds from $\circledast$ that $\exists d \in \gamma \exists d' \in \alpha (d \; S\!\preceq \; d')$. Hence $d' \in \alpha$ and $d' \Vdash_{bn} A$ for such d'. Therefore $\forall \alpha \ni b' \exists d' \in \alpha (d' \Vdash_{bn} A)$. Then by the covering property, we infer $b' \Vdash_{bn} A$. Consequently $b \Vdash_{bn} \Box'\Box' A \to A$; so $\mathcal{F} \vDash_{bn} \Box'\Box' A \to A$ for all A. $\Box$

6 An application to paraconsistent logic

In what follows, we shall utilise the frame conditions we discovered to obtain the completeness of a paraconsistent logic called $\mathbf{CC}_\omega$. This system is formulated in [29] as an extension of N.C.A. da Costa's system $\mathbf{C}_\omega$ [6] by (RC). $\mathbf{CC}_\omega$ can be extended to both $\mathbf{TCC}_\omega$ and the logic **daC** of dual negation by G. priest [26]: see [24,25] for more details. Like empirical negation, the forcing conditions for negation in these systems define different systems depending on which semantics (Kripke/Beth) is used. Indeed, empirical negation itself can be obtained as an extension from these systems [8,16]. In order to better understand this phenomenon, it would be desirable to have a characterisation of its basis in terms of Beth semantics. Beth completeness of $\mathbf{CC}_\omega$ is therefore expected to provide us insights into the difference between the semantics.

Let us first look at the axiomatic system for $\mathbf{CC}_\omega$ and its Kripke semantics. For the sake of consistency, we will keep using the symbol $\Box'$ to denote the negation of the system.

Definition 6.1 [$\mathbf{CC}_\omega$] We define the system $\mathbf{CC}_\omega$ by adding to $\mathbf{HK}\Box'$ the next axiom schemata.

$$A \vee \Box' A \qquad \text{(LEM)} \qquad\qquad \Box'\Box' A \to A \qquad \text{(DNE)}$$

System	(a)	(b)	(c)
$\mathbf{HK}\Box$	$+$	$-$	$-$
$\mathbf{HK}\Box'$	$-$	$+$	$-$
$\mathbf{HK}\Diamond$	$-$	$+$	$+$
$\mathbf{HK}\Diamond'$	$+$	$-$	$+$

Table 1
differences among the semantics

It is known that (N1) becomes redundant in $\mathbf{CC}_\omega$. To see this, we note $\Box'(\Box' A \vee \Box' B) \to \Box'\Box' A \wedge \Box'\Box' B$ is derivable in $\mathbf{HK}\Box'$ without appealing to (N1). Then use (DNE), (RC) and (DNE) again to obtain $\Box'(A \wedge B) \to \Box' A \vee \Box' B$.

Definition 6.2 [Kripke semantics for $\mathbf{CC}_\omega$] A *Kripke frame* $\mathcal{F}$ for $\mathbf{CC}_\omega$ is a triple $(K, \leq, R)$ where $(K, \leq)$ is as before, and $R \subseteq K \times K$ is a reflexive and symmetric relation such that $\geq R \subseteq R$. (This corresponds to the 'condensed' relation $R_{\Box'}$ in [10, p.11].) Then the notions of Kripke models and validity are defined analogously to the ones in $\mathbf{HK}\Box'$. We will use $\Vdash_{kc}$ etc. as the notations.

Theorem 6.3 (Kripke soundness and completeness of $\mathbf{CC}_\omega$)
$\vdash_{\mathbf{CC}_\omega} A$ *if and only if* $\vDash_{kc} A$.

Proof. See [29, p.56]. $\Box$

Applying the results from the preceding subsection, we shall define the Beth semantics of $\mathbf{CC}_\omega$ in the following manner.

Definition 6.4 [Beth semantics for $\mathbf{CC}_\omega$] We define a *Beth frame* $\mathcal{F}$ for $\mathbf{CC}_\omega$ as the ones of $\mathbf{HK}\Box'$ which satisfies the following conditions.

- $\forall \alpha \ni b \exists b' \in \alpha \forall \beta \ni b' \exists c \in \beta \exists c' \in \alpha (c \ S_\preceq \ c')$.
- $\forall \alpha \ni b \exists \beta \ni b \forall b' \in \beta \forall c (b' \ S_\preceq \ c \Rightarrow \forall \gamma \ni c \exists c' \in \gamma \exists d \in \alpha (c' \ S_\preceq \ d))$

Otherwise, the notions of Beth models and validity are defined as in the ones for $\mathbf{HK}\Box'$. We shall use the notation $\Vdash_{bc}$ etc.

Then as for Beth completeness, we shall again argue via an embedding of Kripke models into Beth models: see Appendix for the details.

Theorem 6.5 (Beth soundness and completeness of $\mathbf{CC}_\omega$)
$\vdash_{\mathbf{CC}_\omega} A$ *if and only if* $\vDash_{bc} A$.

7 Conclusion

In this paper, we investigated intuitionistic modal logics in terms of Beth semantics. In the Božić-Došen style of Kripke semantics for intuitionistic modal logic, modal operators are incorporated by means of a frame condition expressing the interaction between the intuitionistic ordering and the accessibility relation. We observed that the situation is rather different in Beth semantics. The semantics for the operators differ on whether (a) there is a frame condition ensuring upward closure; (b) the forcing condition is altered to satisfy the covering property; and (c) it has an extra condition to keep the soundness. Table 1 summarises the properties of each semantics according to these criteria.

The divergence suggests that we have to make different types of assumptions on the notion of growth of knowledge in Beth semantics, depending on which modality is considered. Beth semantics therefore appears more advantageous than Kripke semantics in capturing the particularity of each modal notion. In addition, the Beth forcing condition for non-necessity and possibility might be seen as more natural or preferable in that they allow one to assert a non-necessity (possibility) even when currently accessible worlds say otherwise. Furthermore, some people may see the condition in (a) and its Kripke counterpart as rather ad hoc; so it is perhaps more satisfying that the Beth semantics for non-necessity and possibility does not endorse it.

For future works, a natural direction is to explore more fully the correspondence theory of Beth semantics for each of the modal operators. In particular, the frame conditions for the axiom schemata extending $\mathbf{CC}_\omega$ are of interest for gaining more insights into the difference between Kripke and Beth semantics. Another important direction would be to study the precise relationship between the semantics of this paper and that of Goldblatt. Finally, we only considered one operator at a time, but it should offer new insights into Beth semantics to study the interaction of different operators in a system, as Božić and Došen already did for Kripke semantics.

Appendix

Proof of Theorem 2.8

Proof. By induction on the depth of derivation. Here we look at the cases for (P1), (P2) and (RM). For (P1), if $b' \Vdash_{bs} \Box A \wedge \Box B$ for $b' \succeq b$, then for all $cS^{-1}b'$, it holds that $c \Vdash_{bp} A \wedge B$. So, $b' \Vdash_{bp} \Box(A \wedge B)$. Thus $b \Vdash_{bp} (\Box A \wedge \Box B) \to \Box(A \wedge B)$. For (P2), we have $b' \Vdash A \to A$ for any $b'S^{-1}b$. So $b \Vdash_{bp} \Box(A \to A)$.

For (RM), if $b' \Vdash_{bp} \Box A$ for any b and $b' \succeq b$, then $\forall cS^{-1}b'(c \Vdash_{bp} A)$. Also, by I.H. $\vDash_{bp} A \to B$. Thus $\forall cS^{-1}b'(c \Vdash_{bp} B)$. Hence $b' \Vdash_{bp} \Box B$. Consequently $b \Vdash_{bp} \Box A \to \Box B$; so $\vDash_{bp} \Box A \to \Box B$. □

Proof of Theorem 2.9

Proof. We argue via Kripke completeness. If $\nvdash_{\mathbf{HK}\Box} A$, then by Theorem 2.5 there is a Kripke model $\mathcal{M}_k = ((K, \leq, R), \mathcal{V}_K)$ such that $k \nVdash_{kp} A$ for some $k \in K$. Then we construct a Beth model $\mathcal{M}_b = ((B, \preceq, S), \mathcal{V}_B)$ by the following clauses.

- $B = \{(k_1, \ldots, k_n) : k_i \in K \text{ and } k_1 \leq \ldots \leq k_n\} \cup \{g\}$.

(i) $(k_1, \ldots, k_n) \preceq (k'_1, \ldots, k'_m)$ iff $n \leq m$ and $k_i = k'_i$ for $i \leq n$.
(ii) $g \preceq b$ for all $b \in B$ and $b \preceq g \Rightarrow b = g$.

(i) $(k_1, \ldots, k_n)S(k'_1, \ldots, k'_m)$ iff $k_n R k'_m$.
(ii) $gS(k_1, \ldots, k_n)$ iff $k' R k_n$ for some $k' \in K$.

(iii) $\neg(bSg)$ for all $b \in B$.

(i) $(k_1, \ldots, k_n) \in \mathcal{V}_B(p)$ iff $k_n \in \mathcal{V}_K(p)$.
(ii) $g \in \mathcal{V}_B(p)$ iff $k \in \mathcal{V}_K(p)$ for all $k \in K$.

Then it is straightforward to see that $\mathcal{M}_{\mathcal{B}}$ is a Beth model: in particular, if $\forall\alpha \ni b \exists b' \in \alpha(b' \in \mathcal{V}_B(p))$, then for $b = (k_1, \ldots, k_n)$ we look at

$$\alpha = (g, (k_1), \ldots, (k_1, \ldots, k_n), (k_1, \ldots, k_n, k_n), \ldots).$$

Then $k' \in \mathcal{V}_K(p)$ for some $k' \leq k_n$, hence $k_n \in \mathcal{V}_K(p)$ and so $(k_1, \ldots, k_n) \in \mathcal{V}_B(p)$. For $b = g$, for each $k \in K$ we look at $\alpha = (g, (k), (k, k), \ldots)$ to conclude $k \in \mathcal{V}_K(p)$; it then follows that $g \in \mathcal{V}_B(p)$.

We also have to check that S satisfies the condition that for all $b, b' \in B$:

$$b \preceq S\ b' \Rightarrow \exists\alpha \ni b \forall c \in \alpha(c\ S\preceq\ b').$$

If $b \preceq S\ b'$, then note that $b' \neq g$; so suppose $b' = (k'_1, \ldots, k'_m)$. We first consider the case when $b = g$. Then for some c, we have $b \preceq c$ and cSb'. Now, regardless of whether $c = g$, there is a world $(k''_1, \ldots, k''_l)$ such that $k''_l R k'_m$. Take $\alpha = (g, (k''_l), (k''_l, k''_l), \ldots)$. Then if $c' \in \alpha$, either $c' = g$ or c' ends with k''_l. In each case, we have $c'Sb'$. Therefore $\exists\alpha \ni b \forall c \in \alpha(c\ S\preceq\ b')$, as required. Next, for the case when $b = (k_1, \ldots, k_n)$, first we observe that there is a world $(k_1, \ldots, k_n, \ldots, k_l)$ such that $k_l R k'_m$ and consequently gSb'. For the path, we take $\alpha = (g, (k_1), \ldots, (k_1, \ldots, k_n), (k_1, \ldots, k_n, k_n), \ldots)$. Suppose now $c \in \alpha$. If $c = g$, then it is immediate from the above observation that $c\ S\preceq\ b'$. Otherwise, c ends with $k_{n'} \leq k_n$ which satisfies $k_{n'} \leq R\ k'_m$. Then by the frame condition on R, we infer $k_{n'}\ R\leq\ k'_m$. Consequently $c\ S\preceq\ k'_m$. Thus $\exists\alpha \ni b \forall c \in \alpha(c\ S\preceq\ b')$ in this case as well.

We next claim that the following equivalences hold between the two models.

$$k \Vdash_{kp} A \Longleftrightarrow (k_1, \ldots, k) \Vdash_{bp} A.$$
$$\mathcal{M}_k \vDash_{kp} A \Longleftrightarrow g \Vdash_{bp} A.$$

We shall establish this by induction on the complexity of A. The case $A \equiv p$ follows by above. The cases when $A \equiv B \wedge C, B \to C$ are straightforward. The case when $A \equiv B \vee C$ is similar to the case for non-necessity we shall see later.

When $A \equiv \Box B$, then for the first equivalence, if $k \Vdash_{kp} \Box B$ then $\forall k' R^{-1} k(k' \Vdash_{kp} B)$. Then by the I.H. $\forall b S^{-1}(k_1, \ldots, k)(b \Vdash_{bp} B)$, because g is not accessible from any worlds. Hence $(k_1, \ldots, k) \Vdash_{bp} \Box B$. The converse direction is similarly shown. For the second equivalence, note that $\mathcal{M}_k \vDash_{kp} \Box B$ is equivalent to $\forall k(\exists k'(k' Rk) \Rightarrow k \Vdash_{kp} B)$. By the I.H. and the definition of S, this is further equivalent to $\forall b(gSb \Rightarrow b \Vdash_{bp} B)$, and so to $g \Vdash_{bp} \Box B$. (We are again using the fact that g is not accessible from any world.)

Now since $k \nVdash_{kp} A$, the above equivalence implies $(k) \nVdash_{bp} A$. Therefore $\nvDash_{bp} A$. $\Box$

Proof of Theorem 3.10

Proof. By induction on the depth of derivation. Here we look at the cases for (N1),(N2) and (RC). For (N1), if $b' \Vdash_{bn} \Box'(A \wedge B)$ for some b and $b' \succeq b$, then $\forall \alpha \ni b' \exists c \in \alpha \exists c' S^{-1} c(c' \nVdash_{bn} A \wedge B)$. So,

$$\forall \alpha \ni b' \exists c \in \alpha(\exists c' S^{-1} c(c' \nVdash_{bn} A) \text{ or } \exists c' S^{-1} c(c' \nVdash_{bn} B)).$$

Consequently it follows that

$$\forall \alpha \in b' \exists c \in \alpha(\forall \beta \in c \exists c' \in \beta \exists d S^{-1} c'(d \nVdash_{bn} A) \text{ or } \forall \beta \in c \exists c' \in \beta \exists d S^{-1} c'(d \nVdash_{bn} B)).$$

That is to say, $\forall \alpha \ni b' \exists c \in \alpha(c \Vdash_{bn} \Box' A \text{ or } c \Vdash_{bn} \Box' B)$. Thus $b' \Vdash_{bs} \Box' A \vee \Box' B$ and therefore $b \Vdash_{bn} \Box'(A \wedge B) \to (\Box' A \vee \Box' B)$.

For (N2), suppose $b' \Vdash_{bn} \Box'(A \to A)$ for some b and $b' \succeq b$. Consider any $\alpha \ni b'$. Then there has to be $c \in \alpha$ and $c' S^{-1} c$ such that $c' \nVdash_{bn} A \to A$, a contradiction. Thus $b \Vdash_{bn} \Box'(A \to A) \to B$.

For (RC), if $b' \Vdash_{bn} \Box' B$ for some b and $b' \succeq b$, then $\forall \alpha \ni b' \exists c \in \alpha \exists c' S^{-1} c(c' \nVdash_{bn} B)$. Also, by I.H. $\vDash_{bn} A \to B$. Thus $\forall \alpha \ni b' \exists c \in \alpha \exists c' S^{-1} c(c' \nVdash_{bn} A)$. Hence $b' \Vdash_{bn} \Box' A$. Consequently $b \Vdash_{bn} \Box' B \to \Box' A$; so $\vDash_{bn} \Box' B \to \Box' A$. $\Box$

Proof of Theorem 3.11

Proof. We argue via Kripke completeness. If $\nvdash_{\mathbf{HK}\Box'} A$, then by Theorem 3.5 there is an $\mathbf{HK}\Box'$-Kripke model $\mathcal{M}_k = ((K, \leq, R), \mathcal{V}_K)$ such that $k \nVdash_{ks} A$ for some $k \in K$. Then we construct an $\mathbf{HK}\Box'$-Beth model $\mathcal{M}_b = ((B, \preceq, S), \mathcal{V}_B)$ in almost the same way as Theorem 2.9, except that:

(i) $gS(k'_1, \ldots, k'_m)$ iff kRk'_m for all $k \in K$.

(ii) $(k_1, \ldots k_n)Sg$ iff $k_n Rk$ for all $k \in K$.

(iii) (gSg) iff kRk' for all $k, k' \in K$.

Because there is no condition on S this time, it is immediate that $\mathcal{M}_k$ is a Beth model for $\mathbf{HK}\Box'$.

We again need to check that the next equivalences hold between the models.

$$k \Vdash_{kn} A \Longleftrightarrow (k_1, \ldots, k) \Vdash_{bn} A.$$
$$\mathcal{M}_k \vDash_{kn} A \Longleftrightarrow g \Vdash_{bn} A.$$

When $A \equiv \Box' B$, if $k \Vdash_{kn} \Box' B$, then $\exists k' R_k^{-1} k(k' \nVdash_{kn} B)$. By I.H. this is equivalent to $\exists (k'_1, \ldots, k') S^{-1}(k_1, \ldots, k)((k'_1, \ldots, k') \nVdash_{bn} B)$. Thus we can infer that for any $\alpha \ni (k_1, \ldots, k)$ there is a node b in the path such that $\exists (k'_1 \ldots, k') S^{-1} b((k'_1, \ldots, k') \nVdash_{bn} B)$. Hence $(k_1, \ldots, k) \Vdash_{bn} \Box' B$.

For the converse direction, if $(k_1, \ldots, k) \Vdash_{bn} \Box' B$ then it follows that $\forall \alpha \ni (k_1, \ldots k) \exists b \in \alpha \exists b' S^{-1} b(b' \nVdash_{bn} B)$. We then choose the path $\alpha = (g, (k_1), \ldots, (k_1, \ldots, k), (k_1, \ldots, k, k), \ldots)$. Then there are four possibilities, depending on whether $b = g$ and $b' = g$. If $b = g$, then gSb' implies kRk_0 for some k_0 s.t. $k_0 \nVdash_{kn} B$, in both of the cases $b' = g$ and $b' \neq g$. If $b \neq g$, then $b = (k_1, \ldots, k')$ for some $k' \leq k$ such that $k'Rl$ for some l with $l \nVdash_{kn} B$, independent of whether $b' = g$. Then $k \geq R\ l$, so by the frame condition for Kripke frames we infer $k\ R\leq\ l$. Hence there is l' such that kRl' and $l' \nVdash_{kn} B$. Thus in all case $\exists k' R^{-1} k(k' \nVdash_{bn} B)$ and so $k \Vdash_{kn} \Box' B$. The case for $\mathcal{M}_k \vDash_{kn} \Box' B \iff g \Vdash_{bn} \Box' B$ is analogous.

Now, since $k \nVdash_{kn} A$, the above equivalence implies $(k) \nVdash_{bn} A$. Thus $\nvDash_{bn} A$.□

Proof of Theorem 4.11

Proof. For (i), we need to check the cases for (Q1),(Q2) and (RM2). To see that (Q1) is valid, suppose $b' \Vdash_{bq} \Diamond(A \vee B)$ for some b and $b' \succeq b$. Then $\forall \alpha \ni b' \exists c \in \alpha \exists c' S^{-1} c(c' \Vdash_{bq} A \vee B)$. So for any $\beta \ni c'$ there is $d \in \beta$ such that $d \Vdash_{bq} A$ or $d \Vdash_{bq} B$. Note that we can assume $d \succeq c'$ without loss of generality because of the upward closure. Then by the frame condition added for **HK**$\Diamond$, it holds that cSd for such d with respect to some $\beta \ni c'$. Consequently $c \Vdash_{bq} \Diamond A$ or $c \Vdash_{bq} \Diamond B$. Therefore $b' \Vdash_{bq} \Diamond A \vee \Diamond B$; so $b \Vdash_{bq} \Diamond(A \vee B) \to (\Diamond A \vee \Diamond B)$. The cases for (Q2) and (RM2) are straightforward.

For (ii), we use the same construction of Beth model as **HK**$\Box'$. We need then to check first that the condition $\forall b, b' \in B(bSb' \Rightarrow \exists \alpha \ni b' \forall c \in \alpha(c \succeq b' \Rightarrow bSc))$ is satisfied. If $b' = (k_1, \ldots, k_m)$, then we can take the path $\alpha = (g, (k_1), \ldots, (k_1, \ldots, k_m), (k_1, \ldots, k_m, k_m), \ldots)$. If $b' = g$, then bSc for any $c \in B$, so we can take an arbitrary α. We furthermore need to show that the equivalences between Kripke and Beth forcings; These can be shown analogously to the cases for **HK**$\Box'$. □

Proof of Theorem 4.12

Proof. For (1), we need to check the cases for (O1),(O2) and (RC2). To see that (O1) is valid, suppose $b' \Vdash_{bo} \Diamond' A \wedge \Diamond' B$ for some b and $b' \succeq b$. Then $b'Sc$ implies $c \nVdash_{bo} A$ and $c \nVdash_{bo} B$. Now if $c' \Vdash_{bo} A \vee B$ for some $c' S^{-1} b'$, then by the added frame condition there is $\alpha \ni c'$ s.t. $b'Sd$ for all $d \in \alpha$ satisfying $d \succeq c'$. So no world in α can force A nor B, a contradiction. Thus $b' \Vdash_{bo} \Diamond'(A \vee B)$ and so $b \Vdash_{bo} (\Diamond' A \wedge \Diamond' B) \to \Diamond'(A \vee B)$. The cases for (O2) and (RC2) are simple.

For (ii), the condition $\forall b, b' \in B(bSb' \Rightarrow \exists \alpha \ni b' \forall c \in \alpha(c \succeq b' \Rightarrow bSc))$ can be checked as in (i), but notice that the case $b' = g$ does not apply. Then we can show $\forall b, b' \in B(b \preceq S\ b' \Rightarrow \exists \alpha \ni b \forall c \in \alpha(c\ S\succeq\ b'))$ analogously to the respective condition for **HK**$\Box$. Similarly for the equivalences of the forcings.□

Proof of Proposition 5.3

Proof. From (i) to (ii), we argue by contradiction. Suppose there is k s.t. $\neg(k\ R\leq\ k)$. Choose $\mathcal{V}$ s.t. $\mathcal{V}(p) = \{k' : k\ R\leq\ k'\}$. $\mathcal{V}(p)$ is upward closed, and $k \nVdash_{kn} p \vee \Box' p$. Hence $\mathcal{F} \nvDash_{kn} p \vee \Box' p$. From (ii) to (i), suppose $k \nVdash_{kn} \Box' A$. Then all worlds accessible from k force A. In particular, by the frame condition there is k' s.t. kRk' and $k' \leq k$. So $k \Vdash_{kn} A$. □

Proof of Theorem 6.5

Proof. The soundness direction follows immediately from Theorem 3.10, Proposition 5.4 and 5.6.

For completeness, the outline is identical to the proof of Theorem 3.11. As we noted, the Kripke semantics for $\mathbf{CC}_\omega$ has the condition $\geq R \subseteq R$ rather than $\geq R \subseteq R\leq$ used for $\mathbf{HK}\Box'$. The latter condition is used in establishing the equivalence of valuation for non-necessity, but it is straightforward to check that the former condition works as well.

We need to check that the Beth model constructed out of a Kripke model satisfies the conditions corresponding to (LEM) and (DNE). For the condition of (LEM), if $b = (k_1, \ldots, k) \neq g$ and $\alpha \ni b$, take $b' = b$. Then given $\beta \ni b'$, we note kRk by reflexivity and so $b\ S\preceq\ b$ and $b \in \alpha$. Thus $\exists c \in \beta \exists c' \in \alpha(c\ S\preceq\ c')$, as desired. If $b = g$, then for $\alpha = (g, (k), \ldots)$ take $b' = (k)$; we can then argue in the same manner to obtain the same conclusion. For the condition of (DNE), given $b = (k_1, \ldots k) \neq g$ and $\alpha \ni b$, we take $\beta = (g, (k_1), \ldots, (k_1, \ldots, k), (k_1, \ldots, k, k), \ldots)$. Suppose $b' \in \beta$, $c \succeq c' S^{-1} b'$ and $\gamma \ni c$. If $b' \neq g$, then $b' = (k_1, \ldots, k')$ for $k' \leq k$. When $c' = (l_1, \ldots, l) \neq g$, we then have $k'Rl$ and so kRl from the frame condition that $\geq R \subseteq R$. So by symmetry lRk, which means $c'S \preceq b$. Consequently, (noting $c' \in \gamma$) we have $\exists c' \in \gamma \exists d \in \alpha(c'\ S\preceq\ d)$. When $c' = g$, then $k'Rl$ for all $l \in K$ and so is k by the frame condition. Thus by symmetry lRk for all $l \in K$, which implies $c'S \preceq b$ again. If $b' = g$ then $b'Sc'$ means either: there is $l \in K$ which is accessible from k (if $c' \neq g$), or every point is accessible from k (if $c' = g$). In both case we can use symmetry to conclude $c'S \preceq b$. Finally, if $b = g$, then for $\alpha = (g, (k), \ldots)$ take $\beta = (g, (k), (k, k), \ldots)$. The rest is analogous. □

References

[1] Alechina, N., M. Mendler, V. de Paiva and E. Ritter, *Categorical and Kripke semantics for constructive S4 modal logic*, in: *International Workshop on Computer Science Logic*, Springer, 2001, pp. 292–307.

[2] Artemov, S. and T. Protopopescu, *Intuitionistic epistemic logic*, The Review of Symbolic Logic **9** (2016), pp. 266–298.

[3] Beth, E. W., *semantic construction of intuitionistic logic*, (Mededelingen van de) Koninklijke Nederlandse Akademie van Wetenschappen (Afdeling Letteren) (Nieuwe Reeks) **19** (1956), pp. 357–388.

[4] Bezhanishvili, G. and W. H. Holliday, *A semantic hierarchy for intuitionistic logic*, Indagationes Mathematicae **30** (2019), pp. 403–469.

[5] Božić, M. and K. Došen, *Models for normal intuitionistic modal logics*, Studia Logica **43** (1984), pp. 217–245.

[6] da Costa, N. C. A., *On the theory of inconsistent formal systems*, Notre Dame Journal of Formal Logic **15** (1974), pp. 497–510.

[7] De, M., *Empirical negation*, Acta Analytica **28** (2013), pp. 49–69.

[8] De, M. and H. Omori, *More on empirical negation*, in: R. Goré, B. Kooi and A. Kurucz, editors, *Advances in Modal Logic*, 10 (2014), pp. 114–133.

[9] de Beer, B., "Frame Properties of Beth Models," B.S. thesis, Utrecht University (2012).

[10] Došen, K., *Negative modal operators in intuitionistic logic*, Publication de l'Instutute Mathematique, Nouv. Ser **35** (1984), pp. 3–14.
[11] Došen, K., *Models for stronger normal intuitionistic modal logics*, Studia Logica **44** (1985), pp. 39–70.
[12] Došen, K., *Negation as a modal operator*, Reports on Mathematical Logic **20** (1986), pp. 15–27.
[13] Došen, K., *Negation on the light of modal logic*, in: D. M. Gabbay and H. Wansing, editors, *What is Negation?*, Kluwer Academic Publishing, 1999 pp. 77–86.
[14] Drobyshevich, S., *On classical behavior of intuitionistic modalities*, Logic and Logical Philosophy **24** (2015), pp. 79–104.
[15] Fairtlough, M. and M. Mendler, *Propositional lax logic*, Information and Computation **137** (1997), pp. 1–33.
[16] Ferguson, T. M., *Extensions of Priest-da Costa Logic*, Studia Logica **102** (2014), pp. 145–174.
[17] Fitch, F. B., *Intuitionistic modal logic with quantifiers*, Portugaliae mathematica **7** (1948), pp. 113–118.
[18] Gabbay, D. M., "Fibring logics," Oxford Logic Guides **38**, Clarendon Press, 1998.
[19] Goldblatt, R., *Cover semantics for quantified lax logic*, Journal of Logic and Computation **21** (2011), pp. 1035–1063.
[20] Holliday, W. H., *Inquisitive intuitinostic logic*, in: S. N. Nicola Olivetti, Rineke Verbrugge and G. Sandu, editors, *Advances in Modal Logic*, 13, College Publications, 2020 pp. 329–348.
[21] Kojima, K., "Semantical study of intuitionistic modal logics," Ph.D. thesis, Kyoto University (2012).
[22] Kripke, S. A., *Semantical analysis of intuitionistic logic I*, in: *Studies in Logic and the Foundations of Mathematics*, 40, Elsevier, 1965 pp. 92–130.
[23] Mendler, M. and V. De Paiva, *Constructive CK for contexts*, Context Representation and Reasoning (CRR-2005) **13** (2005).
[24] Niki, S., *Empirical negation, co-negation and contraposition rule I: Semantical investigations*, Bulletin of the Section of Logic **49** (2020), pp. 231–253.
[25] Osorio, M. and J. A. C. Joo, *Equivalence among RC-type paraconsistent logics*, Logic Journal of the IGPL **25** (2017), pp. 239–252.
[26] Priest, G., *Dualising intuitionictic negation*, Principia: an international journal of epistemology **13** (2009), pp. 165–184.
[27] Rogozin, D., *Categorical and algebraic aspects of the intuitionistic modal logic IEL^- and its predicate extensions*, Journal of Logic and Computation **31** (2021), pp. 347–374.
[28] Simpson, A. K., "The proof theory and semantics of intuitionistic modal logic," Ph.D. thesis, University of Edinburgh (1994).
[29] Sylvan, R., *Variations on da Costa C systems and dual-intuitionistic logics I. analyses of C_ω and CC_ω*, Studia Logica **49** (1990), pp. 47–65.
[30] Troelstra, A. S. and D. van Dalen, "Constructivism in Mathematics: An Introduction I," Studies in Logic and the Foundations of Mathematics, Elsevier, 1988.
[31] Troelstra, A. S. and D. van Dalen, "Constructivism in Mathematics: An Introduction II," Studies in Logic and the Foundations of Mathematics **II**, Elsevier, 1988.
[32] van Dalen, D., *How to glue analysis models*, The Journal of symbolic logic **49** (1984), pp. 1339–1349.
[33] Wijesekera, D., *Constructive modal logics I*, Annals of Pure and Applied Logic **50** (1990), pp. 271–301.
[34] Wolter, F. and M. Zakharyaschev, *Intuitionistic modal logic*, in: *Logic and foundations of mathematics*, Springer, 1999 pp. 227–238.