Medvedev logic is the logic of finite distributive lattices without top element

Gianluca Grilletti [1]

Munich Center for Mathematical Philosophy
Geschwister-Scholl-Platz 1
D-80539 München

Abstract

Medvedev logic ML (also known as the logic of finite problems) is an intermediate logic firstly introduced by Yu.T. Medvedev in 1962. ML can be characterized as the logic of the class of intuitionistic Kripke frames corresponding to finite Boolean algebras (regarded as partially ordered structures) without their top element. Several fundamental questions about this logic still remain open to this day, most notably whether the set of its validities is decidable. In this work we provide an alternative characterization of ML in terms of finite distributive lattices without top element, in the same spirit as the characterization in terms of Boolean algebras.

Keywords: Medvedev logic, Finite distributive lattices, Meet-prime elements.

1 Introduction

Yu.T. Medvedev introduced in [10] the logic of finite problems ML, building on the informal interpretation of intuitionistic logic as "the calculus of a constructive solution to problems" proposed by Kolmogorov [6]. In [11] he also provided several results on the logic, enough to easily infer the following semantic characterization: ML is the logic of the intuitionistic Kripke frames obtained by removing the topmost element from a finite Boolean algebra—the so-called Medvedev frames.

Medvedev further developed this framework in [12], where he proposed a theory of "informational types and their transformations" based on the same models used for ML (see also [20] for further discussion). These works also led to other quite interesting logics, building on the same fundamental ideas. Among these we find the *logic of infinite problems* ML_1 proposed by D.P. Skvortsov [19], a family of logics based on "informational types" proposed by V.B. Shehtman

[1] Email: g.grilletti@lmu.de
Gefördert durch die Deutsche Forschungsgemeinschaft (DFG) - Projektnummer 446711878.
Funded by the German Research Foundation (DFG) - Project number 446711878.
We would like to thank I. Ciardelli and V. Punčochář for discussions on the result here presented. Moreover, we would like to thank the anonymous reviewers for the feedback provided.

and Skvortsov [18] and some tense modal logics built over Medvedev frames developed by W.H. Holliday [5].

Although the theory behind ML was a source of inspiration, not much is known about the properties of the original logic, and the few results available in the literature have been achieved over the span of five decades. In 1976 T. Prucnal showed that ML is structurally complete [13]. In 1979 L.L. Maksimova, Skvortsov and Shehtman showed that ML is not finitely axiomatizable [9]. In 1986 Maksimova showed that ML is maximal among the intermediate logics with the disjunction property [8]. In 1990 Shehtman showed that, not only the logic itself, but also its modal counterparts are not finitely axiomatizable [17]. In 2013, more than 20 years later, M. Łazarz showed that ML can also be characterized as the logic of the so-called *Kubiński frames* [7], obtained by removing the topmost element from a *Kubiński lattice*. And these, as far as the author knows, are the most salient results currently available.

It is surprising that, despite the amount of effort spent to study ML, several fundamental questions about the logic still remain open to this day. For example, the semantics characterization in terms of finite Boolean algebras readily implies that the set of non-valid formulas of the logic is recursively axiomatizable, but it is still not known whether the logic is decidable. A way to settle this issue in the positive would be to prove that ML (the logic of finite problems) coincides with ML_1 (the logic of infinite problems); however, whether this is the case is yet another open question. It is also worth noticing that some of the results about ML are obtained by studying the bounded-morphic images of Medvedev frames (compare with [9,17]), but we currently do not have a satisfying characterization of this class of bounded-morphic images.

In this paper we make a novel contribution to the study of Medvedev logic, by showing that ML can be characterized as the logic of the intuitionistic Kripke frames obtained by removing the topmost element from a finite *distributive lattice.*[2] This characterization is very similar in spirit to the original one presented by Medvedev in [11], as the frames considered in both cases are obtained by removing the topmost element from a class of finite algebraic structures. Moreover, both Medvedev frames and Kubiński frames are examples of distributive lattices without the topmost elements, which means that we found a family extending the ones considered in [7,11], again with ML as its logic.

This characterization is an addition to the study of ML, that might help to uncover new properties of the logic and to simplify the proofs of the results currently available. In fact, the class if finite distributive lattices without topmost element seems more stable under basic transformations than the class of Medvedev frames, and at the same time the structure of finite distributive lattices is well-understood and characterized. This characterization also opens venues for future research: following the approach of [19], we can consider the class of distributive lattices without topmost elements, leaving aside the

---

[2] As far as the author knows, the question whether the logic of this class of frames is ML was firstly proposed by V. Punčochář in 2018.

finiteness requirement. We expect the logic of this class to be ML_1, the logic of infinite problems. If this were the case, the new semantic approach to the study of ML and ML_1 based on distributive lattices could shed some light on the relation between the two logics. Another potential application of our result relates to inquisitive semantics [3], a formalism shown to have tight connections with Medvedev logic (see for example [2, Section 3.4]). This novel characterization of ML could reveal new properties of different inquisitive logics, in particular those explicitly built on algebraic structures, as for example the ones presented in [14,15,16]. [3]

The structure of the paper is as follows: In Section 2 we fix some notation and present the basic notions used in the rest of the paper. In Section 3 we prove the characterization of Medvedev logic in terms of finite distributive lattices, that is, our main result. In Section 4 we provide two examples showcasing the salient passages of the proof from Section 3.

2 Preliminaries

We assume the reader to be familiar with the basic notions on order theory, lattices, intermediate logics and intuitionistic Kripke frames. In this section we limit ourselves to fix some notations and recall the results used throughout the paper. For a basic introduction on order theory and lattices see [4]. And for a basic introduction on intermediate logics and intuitionistic Kripke frames see [1].

2.1 Distributive lattices

Throughout the paper we use the term *lattice* to indicate a *bounded lattice*. In particular, given a lattice $\langle \mathrm{L}, \leq \rangle$, we indicate with $\top$ and $\bot$ respectively its greatest and smallest elements. Given two elements $a, b \in \mathrm{L}$, we indicate with $a \wedge b$ and $a \vee b$ respectively the *meet* and the *join* of a and b. Likewise, given a finite set $S \subseteq \mathrm{L}$ we indicate with $\bigwedge S$ and $\bigvee S$ respectively the *meet* and the *join* of the elements of S (as customary, we define $\bigwedge \emptyset = \top$ and $\bigvee \emptyset = \bot$). With a slight abuse of notation, we indicate with L both the ordered structure $\langle \mathrm{L}, \leq \rangle$ and its underlying set of elements.

We call a lattice L *distributive* if, for every choice of $a, b, c \in \mathrm{L}$ the following two identities hold:

$$a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c) \qquad a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$$

We indicate with $\mathbf{DL}$ the class of distributive lattices and with $\mathbf{DL}_{\mathrm{fin}}$ the class of *finite* distributive lattices.

In the following section we focus on two special families of elements of finite distributive lattices: *meet-prime elements* and *coatoms*. Given a distributive lattice L we indicate with $\mathfrak{M}_{\mathrm{L}}$ the set of its meet-prime elements, that is, the elements $p \in \mathrm{L} \setminus \{\top\}$ satisfying the condition that, for every $a, b \in \mathrm{L}$, if $a \wedge b \leq p$, then $a \leq p$ or $b \leq p$. And we indicate with $\mathfrak{C}_{\mathrm{L}}$ the set of coatoms of L, that is,

[3] Thanks to V. Punčochář for pointing out this very interesting connection.

the maximal elements of the set $\mathrm{L} \setminus \{\top\}$. Notice that coatoms are in particular meet-prime elements, so $\mathfrak{C}_\mathrm{L} \subseteq \mathfrak{M}_\mathrm{L}$. Moreover, a non-trivial finite lattice always contains coatoms, thus for $\mathrm{L} \in \mathbf{DL}_{\mathrm{fin}}$ we have $\mathfrak{C}_\mathrm{L} \neq \emptyset$.

Meet-prime elements play a special role in the study of distributive lattices. In this paper we use some well-known technical results from the literature. The first is the following lemma. [4]

Lemma 2.1 ([4, Lemma 5.11]) *Consider a lattice* $\mathrm{L} \in \mathbf{DL}$. *For an element* $a \in \mathrm{L} \setminus \{\top\}$ *the following conditions are equivalent:*

- *a is meet-prime;*
- *for any $b_1, b_2 \in \mathrm{L}$, if $a = b_1 \wedge b_2$ then $a = b_1$ or $a = b_2$;*
- *for any $b_1, \ldots, b_k \in \mathrm{L}$, if $a \geq b_1 \wedge \cdots \wedge b_k$ then $a \geq b_i$ for some $1 \leq i \leq k$.*

A fundamental result about finite distributive lattices—another essential ingredient for the results of this paper—is *Birkhoff's representation theorem.* Let us indicate with $\mathcal{P}$ the powerset functor.

Theorem 2.2 ([4, Theorem 5.12] Birkhoff's representation for $\mathbf{DL}_{\mathrm{fin}}$) *Consider a lattice* $\mathrm{L} \in \mathbf{DL}_{\mathrm{fin}}$. *For an element* $a \in \mathrm{L}$ *define the set*

$$\mathfrak{M}_a := \{\ p \in \mathfrak{M}_\mathrm{L} \mid p \geq a\ \}$$

Then for every $a, b \in \mathrm{L}$ it holds that $a = \bigwedge \mathfrak{M}_a$ and that $a \leq b$ iff $\mathfrak{M}_a \supseteq \mathfrak{M}_b$.

We refer to the set $\mathfrak{M}_a$ as the set of meet-prime elements *above a*.

2.2 Intuitionistic Kripke frames and bounded morphisms

An *intuitionistic Kripke frame* (henceforth, just *frame*) is a pair $\mathcal{F} = \langle F, \leq \rangle$ where F is a set (the points of the frame) and $\leq$ is a partial order (the accessibility relation). Frames are commonly used to provide a semantics for intuitionistic and intermediate logics. We do not spell out the definition of this semantics—which can be found in [1, Chapter 2]—since we do not need it to present the results of this paper. We limit ourselves to define some basic notions about frames and their logics and to recall some basic results.

Henceforth we indicate with AP a fixed infinite set of atomic propositions, and we focus exclusively on propositional formulas over AP, that is, formulas generated by the following grammar:

$$\phi \ ::= \ p \in \mathrm{AP} \mid \bot \mid \phi \wedge \phi \mid \phi \vee \phi \mid \phi \to \phi$$

Given a frame $\mathcal{F}$, the *logic of* $\mathcal{F}$ is the set $\mathrm{Log}(\mathcal{F})$ of formulas valid over $\mathcal{F}$. Similarly, given a class of frames $\mathcal{C}$, the *logic of the class* $\mathcal{C}$ is the collection of formulas valid over *every* frame of the class, that is, the set $\mathrm{Log}(\mathcal{C}) := \bigcap_{\mathcal{F} \in \mathcal{C}} \mathrm{Log}(\mathcal{F})$.

[4] The statements in [4, Lemma 5.11, Theorem 5.12] are formulated in terms of the set $\mathcal{J}(\mathrm{L})$ of *join-prime elements* of L. The formulation we employ here using meet-prime elements is obtained by applying the *duality principle for lattices* (Statement 1.20, *ibidem*).

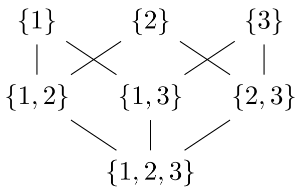

Figure 1. The Medvedev frame $\mathcal{M}_3$, represented using its Hasse diagram.

In this paper we focus on a particular family of frames, that is, *Medvedev frames*. For $n \in \mathbb{N}$ a positive natural number, the Medvedev frame of order n is defined as

$$\mathcal{M}_n \;:=\; \langle\, \mathcal{P}_0(\{1, \dots, n\}),\, \supseteq \rangle$$

where $\mathcal{P}_0(\{1, \dots, n\})$ indicates the non-empty subsets of $\{1, \dots, n\}$. Notice that these frames—modulo isomorphism—are obtained by removing the topmost element from a finite Boolean algebra (considered as a partial order). *Medvedev logic* ML is the logic of the family of Medvedev frames, or in the notation introduced above: $\mathsf{ML} = \mathrm{Log}(\{\mathcal{M}_n \,|\, n \geq 1\})$ (compare with [10,11]). We give a graphical representation of a Medvedev frame in Figure 1.

We also introduce *bounded morphisms* (also known as *reductions* or *p-morphisms*), a powerful tool to study the logics of frames and their families.

Definition 2.3 [Bounded morphism [1, Section 2.3]] Given two frames $\mathcal{F}$ and $\mathcal{F}'$ a *bounded morphism* from $\mathcal{F} = \langle F, \leq \rangle$ to $\mathcal{F}' = \langle F', \preceq \rangle$ is a function $f : F \to F'$ satisfying the following properties.

Forth condition For every $a, b \in F$, if $a \leq b$ then $f(a) \preceq f(b)$.

Back condition For every $a \in F$ and $b' \in F'$, if $f(a) \preceq b'$ then there exists $b \in F$ such that $a \leq b$ and $f(b) = b'$.

We indicate with $f : \mathcal{F} \twoheadrightarrow \mathcal{F}'$ that f is a *surjective* bounded morphism from $\mathcal{F}$ to $\mathcal{F}'$.

Bounded morphisms allow us to compare the logics of two frames, as the following result shows.

Lemma 2.4 ([1, Corollary 2.16]) *Let $\mathcal{F}$ and $\mathcal{F}'$ be two frames, and suppose there exists $f : \mathcal{F} \twoheadrightarrow \mathcal{F}'$. Then* $\mathrm{Log}(\mathcal{F}') \subseteq \mathrm{Log}(\mathcal{F})$.

This is all we need about frames and their logics to prove the results of this paper!

3 A novel class of frames for Medvedev logic

For ease of reading, in the rest of this section we indicate $\mathfrak{M}_{\mathrm{L}}$ and $\mathfrak{C}_{\mathrm{L}}$ simply as $\mathfrak{M}$ and $\mathfrak{C}$ (omitting the subscript). Given a finite distributive lattice L, we indicate with L^- the *partial order* obtained by removing the topmost element from L, that is, the set $\mathrm{L}^- = \mathrm{L} \setminus \{\top\}$ with the induced order. We indicate with $\mathbf{DL}^-_{\mathrm{fin}}$ the collection of all the structures obtained this way $\mathbf{DL}^-_{\mathrm{fin}} := \{\mathrm{L}^- \,|\, \mathrm{L} \in$

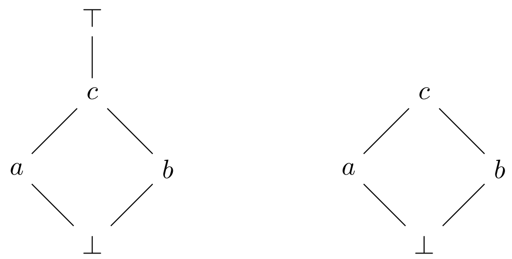

Figure 2. A distributive lattice L (on the left) and the corresponding partial order L^- (on the right), both represented using their Hasse diagrams. Recall that L^- is obtained by removing the topmost element $\top$.

$\mathbf{DL}_{\text{fin}}\}$. In Figure 2 we depict a distributive lattice L and its corresponding partial order L^-. We will use it as a running example through the paper.

The aim of this paper is to provide a proof of the following theorem.

Theorem 3.1 $\mathsf{ML} = \text{Log}(\ \mathbf{DL}^-_{\text{fin}}\ )$.

That is, we want to show that Medvedev logic can be characterized as the logic of finite distributive lattices without their top element. The argument we use mainly relies on the following lemma, and in fact most of this section is devoted to its proof.

Lemma 3.2 *For every finite distributive lattice* L*, there exists a Medvedev frame* $\mathcal{M}_n$ *such that* $\mathcal{M}_n \twoheadrightarrow L^-$*.*

Let us first show how to use Lemma 3.2 to prove Theorem 3.1.

Proof [Proof of Theorem 3.1] Notice that $\langle \mathcal{P}(\{1, \ldots, n\}), \supseteq \rangle$ is a distributive lattice (since it is a boolean algebra) with topmost element $\emptyset$. Thus $\langle \mathcal{P}(\{1, \ldots, n\}), \supseteq \rangle^- = \mathcal{M}_n$. In particular $\{\, \mathcal{M}_n \mid n \geq 1 \,\} \subseteq \mathbf{DL}^-_{\text{fin}}$, so

$$\text{Log}(\ \mathbf{DL}^-_{\text{fin}}\ ) \ \subseteq\ \text{Log}(\ \{\, \mathcal{M}_n \mid n \geq 1 \,\}\ ) \ =\ \mathsf{ML}$$

As for the other inclusion, by Lemma 3.2 for every $L \in \mathbf{DL}_{\text{fin}}$ there exists a Medvedev frame $\mathcal{M}_n$ such that $\mathcal{M}_n \twoheadrightarrow L^-$. In particular by Lemma 2.4 we have that that $\text{Log}(\mathcal{M}_n) \subseteq \text{Log}(L^-)$, and so

$$\mathsf{ML} \ =\ \bigcap_{n \geq 1} \text{Log}(\mathcal{M}_n) \ \subseteq\ \bigcap_{L \in \mathbf{DL}_{\text{fin}}} \text{Log}(L^-) \ =\ \text{Log}(\ \mathbf{DL}^-_{\text{fin}}\ )$$

which concludes the proof. □

We showed how to use Lemma 3.2 to prove Theorem 3.1, so now we are at the hard part: proving the lemma.

To prove Lemma 3.2, given a finite distributive lattice $L \in \mathbf{DL}_{\text{fin}}$ we need to provide two ingredients: a Medvedev frame $\mathcal{M}_n$ and a surjective bounded morphism from $\mathcal{M}_n$ to L^-. For ease of presentation, we work modulo isomorphism: we define a frame $\mathcal{M}$ *isomorphic* to a Medvedev frame and then we construct

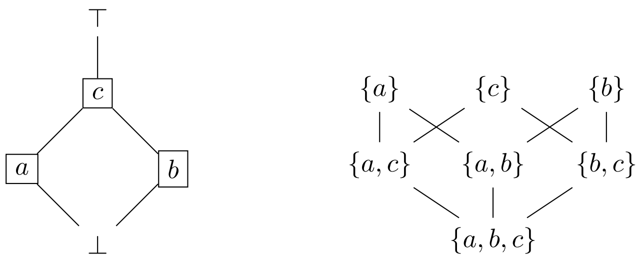

Figure 3. On the left, the lattice L from Figure 2 where we highlighted the elements of $\mathfrak{M}$. On the right, the corresponding frame $\mathcal{M}$.

a surjective bounded morphism $f : \mathcal{M} \twoheadrightarrow \mathrm{L}^-$. The desired bounded morphism can then be obtained by composing f with an automorphism between $\mathcal{M}$ and a Medvedev frame.

The frame we are going to use is $\mathcal{M} = \langle \mathcal{P}_0(\mathfrak{M}), \supseteq \rangle$, that is, the frame whose points are non-empty subsets of meet-prime elements of L ordered by reverse inclusion. This is trivially isomorphic to the Medvedev frame $\mathcal{M}_{|\mathfrak{M}|}$, where $|\mathfrak{M}|$ indicates the cardinality of the set $\mathfrak{M}$. In Figure 3 we depict the frame $\mathcal{M}$ corresponding to the running example of Figure 2.

The construction of the bounded morphism $f : \mathcal{M} \twoheadrightarrow \mathrm{L}^-$ requires the definition of two auxiliary functions $g, h : \mathcal{P}_0(\mathfrak{M}) \to \mathcal{P}(\mathfrak{M})$. We start with the former: given $N \subseteq \mathfrak{M}$ we define the map [5]

$$\begin{aligned} g(N) &= \{ \, p \in \mathfrak{M} \mid \mathfrak{M}_p \subseteq N \, \} \\ &= \{ \, p \in \mathfrak{M} \mid \forall p' \geq p. \; p' \in N \, \} \end{aligned}$$

So $g(N)$ collects all the meet-prime elements $p \in N$ for which *p and its successors* are contained in N. In Figure 4 we represent the function g for our running example. We collect some properties of g in the following proposition, after recalling some basic definitions. We call $U \subseteq \mathfrak{M}$ an *upset* (of $\mathfrak{M}$) if for every $p \in U$ and $q \in \mathfrak{M}$, if $p \leq q$ then $q \in U$. An upset is called *principal* if it contains a minimum element, that is, if it is of the form $\mathfrak{M}_q$ for some $q \in \mathfrak{M}$. Finally, we indicate with $\mathrm{Up}(\mathfrak{M})$ the collection of the upsets of $\mathfrak{M}$ and with $\mathrm{Up}_0(\mathfrak{M})$ the collection of *non-empty* subsets.

Proposition 3.3 *For every $N \subseteq \mathfrak{M}$, $g(N)$ is the greatest (under set-theoretic inclusion)* upset of $\mathfrak{M}$ *contained in N. In particular, the map g is monotone and $g \restriction_{\mathrm{Up}_0(\mathfrak{M})}$ is the identity function.*

Proof By definition, $g(N)$ is the union of all the principal upsets contained in N, which implies that $g(N)$ is the greatest *upset of* $\mathfrak{M}$ contained in N. Given this, it follows trivially that g is monotone and that $g(U) = U$ for an upset U. □

[5] Recall that $\mathfrak{M}_q := \{p \in \mathfrak{M} \,|\, p \geq q\}$, as defined in Theorem 2.2.

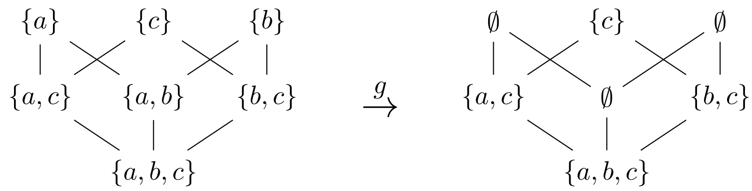

Figure 4. Representation of the function $g : \mathcal{P}_0(\mathfrak{M}) \to \mathcal{P}(\mathfrak{M})$ for $\mathcal{M}$ the running example from Figures 2 and 3. Recall that the points of the frame $\mathcal{M}$ (on the left) are the elements of the set $\mathcal{P}_0(\mathfrak{M})$. On the right, we indicate for each $N \in \mathcal{P}_0(\mathfrak{M})$ the corresponding image $g(N)$.

The function g allows to shift our focus from the collection of non-empty subsets $\mathcal{P}_0(\mathfrak{M})$ to the collection of *upsets* $\mathrm{Up}(\mathfrak{M})$. And this is particularly interesting since there is a natural surjective bounded morphism from the frame $\langle \mathrm{Up}_0(\mathfrak{M}), \supseteq \rangle$ to L^-:

$$\begin{aligned} \bigwedge :\ \mathrm{Up}_0(\mathfrak{M}) &\twoheadrightarrow \mathrm{L}^- \\ U \quad &\mapsto \bigwedge U \end{aligned}$$

Recall that $\bigwedge U$ is the meet of all the elements in U, computed in the lattice L. Notice that this map is well-defined: since U is non-empty, for a meet-prime $p \in U$ we have $\bigwedge U \leq p < \top$. Moreover, the surjectiveness of this map is a direct consequence of Theorem 2.2.

So a naive approach to define a surjective bounded morphism $f : \mathcal{P}_0(\mathfrak{M}) \twoheadrightarrow \mathrm{L}^-$ would be to compose the maps $\bigwedge$ and g, that is, to consider the map $N \mapsto \bigwedge g(N)$. However, there is a complication: the set $g(N)$ might be *empty* (for example when N does not contain any maximal element of $\mathfrak{M}$) and the bounded morphism $\bigwedge : \mathrm{Up}_0(\mathfrak{M}) \to \mathrm{L}^-$ cannot be extended to $\mathrm{Up}(\mathfrak{M})$. To avoid this issue, we need another auxiliary function to *tweak* the set $g(N)$ before we take the meet $\bigwedge g(N)$. Recall that we indicate with $\mathfrak{C}$ the set of coatoms of the lattice L.

Definition 3.4 [link function] Given L a finite distributive lattice, we define a *link function* as a map $h : \mathfrak{M} \to \mathfrak{C}$ such that $h(p) \geq p$ for every $p \in \mathfrak{M}$.

So a link function is an increasing map from $\mathfrak{M}$ to $\mathfrak{C}$. In particular a link function restricted to the set $\mathfrak{C}$ is always the identity. Notice that every element of L^- is smaller than some coatom since L is finite, so under our assumptions link functions always exist.

In the rest of the section we work with a fixed arbitrary link function h. With a slight abuse of notation, we indicate with the same notation also the lifting of h to sets of meet-prime elements: $h(N) = \{\, h(p) \mid p \in N \,\}$. In Figure 5 we represent the function h for our running example. Notice that (in contrast to the map g) for a non-empty subset $N \subseteq \mathfrak{M}$ the image $h(N)$ is not empty. Moreover, h is monotone over $\mathcal{P}_0(\mathfrak{M})$ by definition.

Proposition 3.5 *For $N, N' \in \mathcal{P}_0(\mathfrak{M})$ with $N \subseteq N'$, it holds that $h(N) \subseteq h(N')$.*

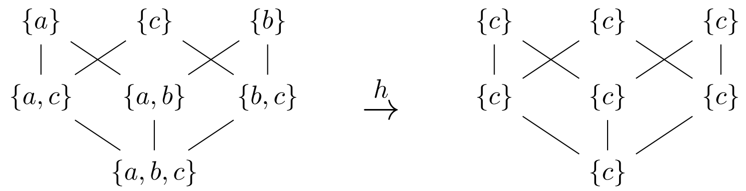

Figure 5. The function $h : \mathcal{P}_0(\mathfrak{M}) \to \mathcal{P}(\mathfrak{M})$ for the running example from Figures 2 and 3. The only link function for the lattice L of Figure 2 is the constant function with value c, that is, the function h used in this example. As in Figure 4, on the left we depict the points of the frame $\mathcal{M}$, that is, the elements of the set $\mathcal{P}_0(\mathfrak{M})$, and on the right their images under h. As the map g of Figure 4, the map h is monotone.

The map h is also well-behaved when restricted to upsets, and this allows to draw an interesting connection between g and h that will be useful later on.

Proposition 3.6 *For $U \in \mathrm{Up}(\mathfrak{M})$ it holds $h(U) \subseteq U$. In particular, $g(U) \cup h(U) = U$.*

Proof Firstly we prove that $h(U) \subseteq U$. Given an arbitrary $p \in U$, we have $h(p) \geq p$, and since U is an upset it follows that $h(p) \in U$. Since p was arbitrary, it follows that $h(U) \subseteq U$. Given this, the identity $g(U) \cup h(U) = U$ is a direct consequence of Proposition 3.3. □

Proposition 3.6 and the previous observations on the map $\bigwedge : \mathrm{Up}_0(\mathfrak{M}) \twoheadrightarrow \mathrm{L}^-$ suggest the following definition for the desired bounded morphism $f : \mathcal{P}_0(\mathfrak{M}) \twoheadrightarrow \mathrm{L}^-$.

$$f(N) := \bigwedge(\, g(N) \cup h(N)\,)$$

In Figure 6 we represent the function f for our running example.

We now have our two ingredients for the proof of Lemma 3.2, that is, the frame $\langle \mathcal{P}_0(\mathfrak{M}), \supseteq \rangle$ and the map f. What remains to be proved is that f is a *surjective bounded morphism* between the structures $\mathcal{P}_0(\mathfrak{M})$ and L^-.

Proof [Proof of Lemma 3.2] We want to show that f is well-defined, surjective and a bounded morphism. In particular, the latter amounts to proving that f satisfies the forth and back conditions from Definition 2.3. We show these results separately.

f is well-defined. Given a non-empty subset $N \in \mathcal{P}_0(\mathfrak{M})$, we already noticed that $h(N) \neq \emptyset$. In particular, this implies that for $c \in h(N)$ it holds

$$f(N) \;=\; \bigwedge(\, g(N) \cup h(N)\,) \;\leq\; c \;<\; \top$$

thus $f(N)$ is an element of L^-.

f is surjective. Let $b \in \mathrm{L}^-$ be an arbitrary element and recall that by Theorem 2.2 we have $b = \bigwedge \mathfrak{M}_b$. We claim that $f(\mathfrak{M}_b) = b$. Since $b < \top$ we have that $\mathfrak{M}_b \neq \emptyset$, and so $\mathfrak{M}_b \in \mathcal{P}_0(\mathfrak{M})$. By definition $\mathfrak{M}_b$ is an upset in $\mathcal{P}_0(\mathfrak{M})$, thus by

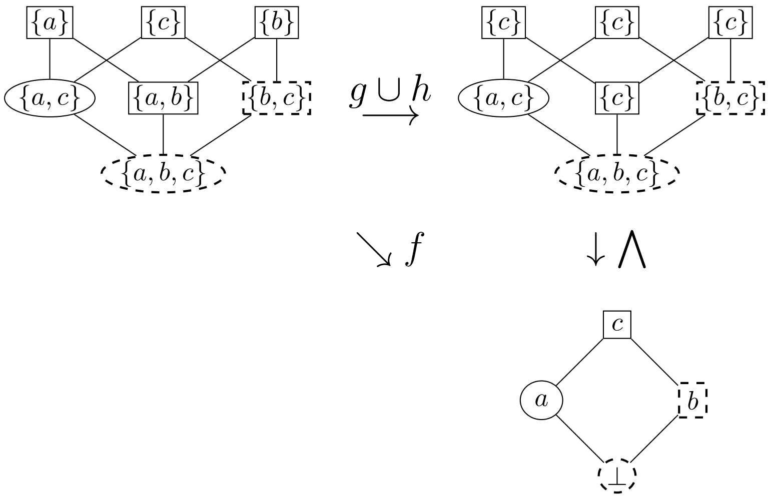

Figure 6. The function $f : \mathcal{P}_0(\mathfrak{M}) \to \mathrm{L}^-$ for the running example, computed as a composition of the functions $g \cup h$ and $\bigwedge$. The image of $g \cup h$ is depicted as in Figures 4 and 5, while the images of the functions $\bigwedge$ and f are coded by the drawing styles of the nodes (e.g., $f(\{b, c\}) = b$ and this is indicated in the picture by the matching rectangular dashed nodes).

Proposition 3.6 it follows that $g(\mathfrak{M}_b) \cup h(\mathfrak{M}_b) = \mathfrak{M}_b$, and so $f(\mathfrak{M}_b) = \bigwedge \mathfrak{M}_b = b$. Since b was an arbitrary element in L^-, this shows that f is surjective.

f respects the forth condition. That is, we want to show that for every $N, N' \in \mathcal{P}_0(\mathfrak{M})$, if $N \supseteq N'$ then $f(N) \leq f(N')$.[6]

Consider arbitrary $N, N' \in \mathcal{P}_0(\mathfrak{M})$ and assume that $N \supseteq N'$. By Propositions 3.3 and 3.5, we know that $g(N) \cup h(N) \supseteq g(N') \cup h(N')$. Thus we have:

$$f(N) \;=\; \bigwedge(\, g(N) \cup h(N)\,) \;\leq\; \bigwedge(\, g(N') \cup h(N')\,) \;=\; f(N')$$

Since N, N' were arbitrary, the forth condition follows.

f respects the back condition. That is, we want to show that for $N \in \mathcal{P}_0(\mathfrak{M})$ and $b \in \mathrm{L}^-$, if $f(N) \leq b$ then there exists $N' \in \mathcal{P}_0(\mathfrak{M})$ such that $N \supseteq N'$ and $f(N') = b$.

Consider arbitrary $N \in \mathcal{P}_0(\mathfrak{M})$ and $b \in \mathrm{L}^-$, and assume that $f(N) \leq b$. We aim to find a non-empty $N' \subseteq N$ such that $f(N') = b$. Recall that $b = \bigwedge \mathfrak{M}_b$ by Theorem 2.2. In particular, for every $p \in \mathfrak{M}_b$ we have that:

$$f(N) \;=\; \bigwedge(\, g(N) \cup h(N)\,) \;\leq\; b \;\leq\; p$$

By Lemma 2.1 and the previous inequality we obtain:

$$\forall p \in \mathfrak{M}_b. \quad \exists p' \in g(N) \cup h(N). \quad p' \leq p \tag{1}$$

[6] Recall that we are considering $\mathcal{P}_0(\mathfrak{M})$ ordered by the relation $\supseteq$.

We proceed by considering the partition of the set $\mathfrak{M}_b = G \cup H$, where G contains all the $p \in \mathfrak{M}_b$ such that $p' \leq p$ for some $p' \in g(N)$, and $H := \mathfrak{M}_b \setminus G$. Using (1), we can give alternative characterizations of the sets G and H.

- **Claim 1:** $p \in G \iff \mathfrak{M}_p \subseteq N$. Firstly suppose that $p \in G$. By definition of G there exists $p' \in g(N)$ such that $p' \leq p$. This implies that $\mathfrak{M}_p \subseteq \mathfrak{M}_{p'} \subseteq N$, where the last containment follows from the definition of $g(N)$. Secondly, suppose that $\mathfrak{M}_p \subseteq N$. Then, by definition of $g(N)$, we have that $p \in g(N)$ and consequently $p \in G$ (since we can take $p' = p$).
- **Claim 2:** $H = \mathfrak{M}_b \cap h(N) \cap (\mathfrak{M} \setminus N)$. We prove the two inclusions $H \subseteq \mathfrak{M}_b \cap h(N) \cap (\mathfrak{M} \setminus N)$ and $\mathfrak{M}_b \cap h(N) \cap (\mathfrak{M} \setminus N) \subseteq H$ separately.

  $\underline{H \subseteq \mathfrak{M}_b \cap h(N) \cap (\mathfrak{M} \setminus N)}$. We have that $H \subseteq \mathfrak{M}_b$ by definition of H, so we only have to show that $H \subseteq h(N)$ and that $H \subseteq \mathfrak{M} \setminus N$.

  We first prove that $H \subseteq h(N)$. By Claim 1 we have that $g(N) \subseteq G$, and so by (1) and the definition of H it follows that:

  $$\forall c \in H. \quad \exists c' \in h(N) \setminus g(N). \quad c' \leq c$$

  In particular, since the elements of $h(N)$ are coatoms, the previous property boils down to $H \subseteq h(N) \setminus g(N) \subseteq \mathfrak{C}$, which in particular implies $H \subseteq h(N)$.

  Finally, we prove that $H \subseteq \mathfrak{M} \setminus N$. Assume towards a contradiction that there exists $c \in H \cap N$. Since c is a coatom, it follows that $\mathfrak{M}_c = \{c\} \subseteq N$. Thus by definition of $g(N)$ we have $c \in g(N)$, which contradicts $H \subseteq h(N) \setminus g(N)$. So we conclude that $H \cap N = \emptyset$, that is, $H \subseteq \mathfrak{M} \setminus N$.

  $\underline{\mathfrak{M}_b \cap h(N) \cap (\mathfrak{M} \setminus N) \subseteq H}$. Consider an arbitrary element $c \in \mathfrak{M}_b \cap h(N) \cap (\mathfrak{M} \setminus N)$. $c \in h(N)$ is a coatom, so we have $\mathfrak{M}_c = \{c\} \nsubseteq N$ which in turn implies $c \notin g(N)$. Since $g(N)$ us an upset it follows that there is no $p \in g(N)$ such that $c \geq p$, that is, $c \notin G$. In particular, we have that $c \in \mathfrak{M}_b \setminus G = H$. Since c was an arbitrary element of $\mathfrak{M}_b \cap h(N) \cap (\mathfrak{M} \setminus N)$, we conclude that $\mathfrak{M}_b \cap h(N) \cap (\mathfrak{M} \setminus N) \subseteq H$.

Now that we have these characterizations of the sets G and H, we are ready to define the set N' we are looking for. Fix an enumeration of the sets $G = \{p_1, \ldots, p_k\}$ and $H = \{c_1, \ldots, c_l\}$. By Claim 2, for every $j \in \{1, \ldots, l\}$ we can fix an element $p'_j \in N$ such that $h(p'_j) = c_j$. We define the set N' as follows:

$$N' := \left(\bigcup_{i=1}^{k} \mathfrak{M}_{p_i} \right) \cup \{p'_1, \ldots, p'_l\}$$

We need to show that $N' \subseteq N$ and that $f(N') = b$. We start with the former condition. By Claim 1, for $i \in \{1, \ldots, k\}$ we have $\mathfrak{M}_{p_i} \subseteq N$. And for $j \in \{1, \ldots, l\}$ we have $p'_j \in N$ by definition of p'_j. Combining these facts we obtain $N' \subseteq N$.

We now show that $f(N') = b$. We firstly prove that $f(N') \leq b$. Notice that $G = \{p_1, \ldots, p_k\} \subseteq g(N')$ and that $H = \{c_1, \ldots, c_l\} = \{h(p'_1), \ldots, h(p'_l)\} \subseteq$

$h(N')$. So in particular we have:

$$f(N') \;=\; \bigwedge(\,g(N') \cup h(N')\,) \;\leq\; \bigwedge(G \cup H) \;=\; \bigwedge \mathfrak{M}_b \;=\; b$$

As for the other inequality, consider an element $q \in g(N') \cup h(N')$. We first show that $q \geq b$. We consider two (non mutually exclusive) cases:

- If $q \in g(N')$, by definition of $g(N')$ we have that $\mathfrak{M}_q \subseteq N' \subseteq N$. We want to show that $q \in \cup_{i=1}^{k} \mathfrak{M}_{p_i}$. We can do it reasoning by contradiction: assume that $q = p'_j$ for some $j \in \{1, \ldots, l\}$. Since $c_j \geq p'_j$ and $\mathfrak{M}_q$ is an upset, it follows that $c_j \in \mathfrak{M}_q \subseteq N$, which contradicts Claim 2. So it follows that $q \neq p'_j$ for every $j \in \{1, \ldots, l\}$, and so $q \in \bigcup_{i=1}^{k} \mathfrak{M}_{p_i}$. In particular, this implies that there exists $i \in \{1, \ldots, k\}$ such that $q \geq p_i \geq b$.
- If $q \in h(N')$, by definition of $h(N')$ there exists $q' \in N'$ such that $h(q') = q$. If $q' \in \mathfrak{M}_{p_i}$ for some $i \in \{1, \ldots, k\}$ then $q = h(q') \geq h(p_i) \geq p_i \geq b$. Otherwise, if $q' = p'_j$ for some $j \in \{1, \ldots, l\}$ then $q = h(q') = h(p'_j) = c_j \geq b$. In both cases we have $q \geq b$.

So for every choice of $q \in g(N') \cup h(N')$ we have that $q \geq b$. From this it follows that:

$$f(N') \;=\; \bigwedge(\,g(N') \cup h(N')\,) \;\geq\; b$$

Thus we conclude that $f(N') = b$. Since $N \in \mathcal{P}_0(\mathfrak{M})$ and $b \in \mathrm{L}^-$ were arbitrary elements, this shows that the back condition holds for f, thus concluding the proof of the lemma. □

4 Examples

In this section we showcase the construction presented in the proof of Lemma 3.2 with two examples. For reasons of layout, we move the figures to the end of the manuscript.

We build the bounded morphisms f_1 and f_2 corresponding to distinct link functions h_1 and h_2 over the same lattice L, depicted in Figure 7. We firstly compute the function g, which is independent from the link functions. The values of g are depicted in Figure 8, following the representation given in Figure 4. Henceforth, we use the same notational conventions used in the running example from the previous section without mentioning it explicitly.

4.1 First example

Consider the link function $h_1 : \mathfrak{M} \to \mathfrak{C}$ defined as $h_1(a) = d$, $h_1(b) = d$, $h_1(d) = d$ and $h_1(e) = e$. We depict the lifting of the function h_1 in Figure 9 and the resulting bounded morphism $f_1 : \mathcal{M} \twoheadrightarrow \mathrm{L}^-$ in Figure 10.

4.2 Second example

Consider the link function $h_2 : \mathfrak{M} \to \mathfrak{C}$ defined as $h_2(a) = e$, $h_2(b) = d$, $h_2(d) = d$ and $h_2(e) = e$. We depict the lifting of function h_2 in Figure 11 and the resulting bounded morphism $f_2 : \mathcal{M} \twoheadrightarrow \mathrm{L}^-$ in Figure 12.

References

[1] Chagrov, A. and M. Zakharyaschev, "Modal Logic," Oxford University Press, 1997.

[2] Ciardelli, I., *Inquisitive semantics and intermediate logics* (2009), MSc Thesis, University of Amsterdam.

[3] Ciardelli, I. and F. Roelofsen, *Inquisitive logic*, Journal of Philosophical Logic **40** (2011), pp. 55–94.

[4] Davey, B. and H. Priestley, "Introduction to lattices and order," Cambridge University Press, Cambridge, 1990.
URL `http://www.worldcat.org/search?qt=worldcat_org_all&q=0521367662`

[5] Holliday, W., *On the modal logic of subset and superset: Tense logic over Medvedev frames*, Studia Logica **105** (2017), pp. 13–35.
URL `https://doi.org/10.1007/s11225-016-9680-1`

[6] Kolmogorov, A., *Zur deutung der intuitionistischen logik*, Matematische Zeitschrift **35** (1932), pp. 58–65.

[7] Lazarz, M., *Characterization of Medvedev's logic by means of Kubiński's frames*, Bulletin of the Section of Logic **42** (2013), pp. 83–90.

[8] Maksimova, L., *On maximal intermediate logics with the disjunction property*, Studia Logica **45** (1986), pp. 69–75.

[9] Maksimova, L., V. Shetman and D. Skvortsov, *The impossibility of a finite axiomatization of Medvedev's logic of finitary problems*, Soviet Mathematics Doklady **20** (1979), pp. 394–398.

[10] Medvedev, Y., *Finite problems*, Soviet Math. Dokl. **3** (1962 (Russian)), pp. 227–230.

[11] Medvedev, Y., *Interpretation of logical formulas by means of finite problems*, Soviet Math. Dokl. **7** (1966 (Russian)), pp. 857–860.

[12] Medvedev, Y., *Transformations of information and calculi that describe them: types of information and their possible transformations*, Semiotika i informatika **13** (1979 (Russian)), pp. 109–141.

[13] Prucnal, T., *Structural completeness of Medvedev's propositional calculus*, Reports on Mathematical Logic **6** (1976), pp. 103–105.

[14] Punčochář, V., *Algebras of Information States*, Journal of Logic and Computation **27** (2016), pp. 1643–1675.
URL `https://doi.org/10.1093/logcom/exw021`

[15] Punčochář, V., *Substructural inquisitive logics*, The Review of Symbolic Logic **12** (2019), pp. 296–330.

[16] Punčochář, V., *Inquisitive heyting algebras*, Studia Logica **109** (2021), pp. 995–1017.
URL `https://doi.org/10.1007/s11225-020-09936-9`

[17] Shehtman, V., *Modal counterparts of Medvedev logic of finite problems are not finitely axiomatizable*, Studia Logica: An International Journal for Symbolic Logic **49** (1990), pp. 365–385.
URL `http://www.jstor.org/stable/20015517`

[18] Shehtman, V. and D. Skvortsov, *Logics of some kripke frames connected with Medvedev notion of informational types*, Studia Logica: An International Journal for Symbolic Logic **45** (1986), pp. 101–118.
URL `http://www.jstor.org/stable/20015250`

[19] Skvortsov, D., *Logic of infinite problems and Kripke models on atomic semilattices of sets*, Doklady AN SSSR **245** (1979 (Russian)), pp. 798–801.

[20] Skvortsov, D., *On some propositional logics related to the concept of information types of Yu. T. Medvedev*, Semiotika i informatika **13** (1979 (Russian)), pp. 142–149.

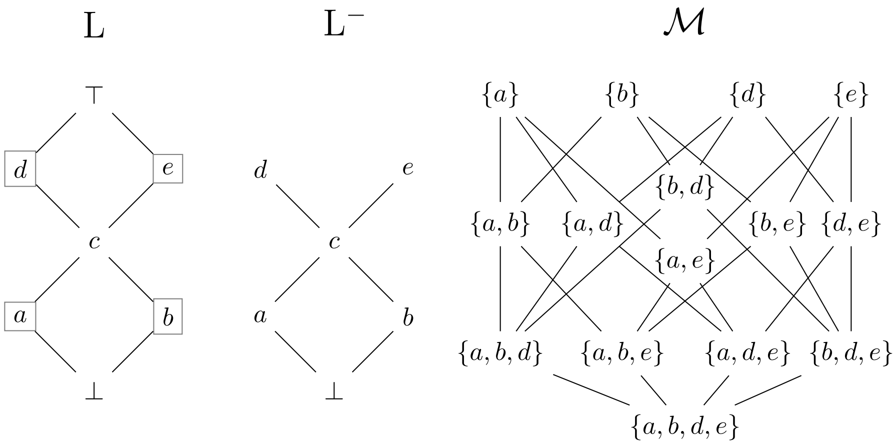

Figure 7. A distributive lattice L, the partial order L^- and the associated Medvedev frame $\mathcal{M}$. As we did in Figure 3, we highlighted the elements of $\mathfrak{M}$.

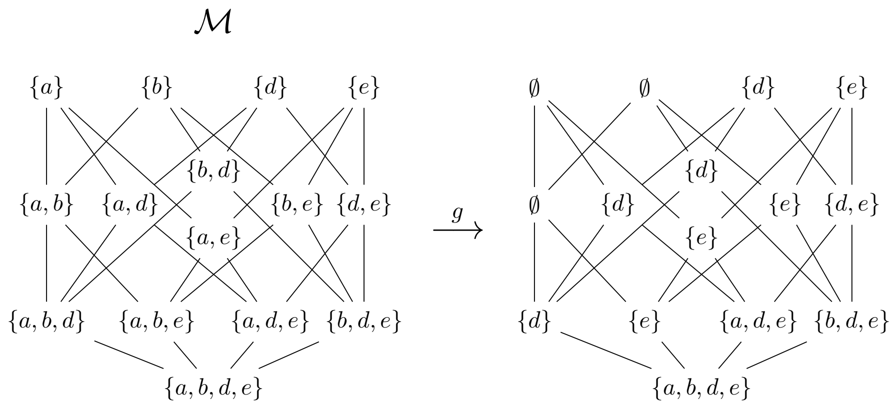

Figure 8. Representation of the function g from the proof of Lemma 3.2, for $\mathcal{M}$ the Medvedev frame in Figure 7.

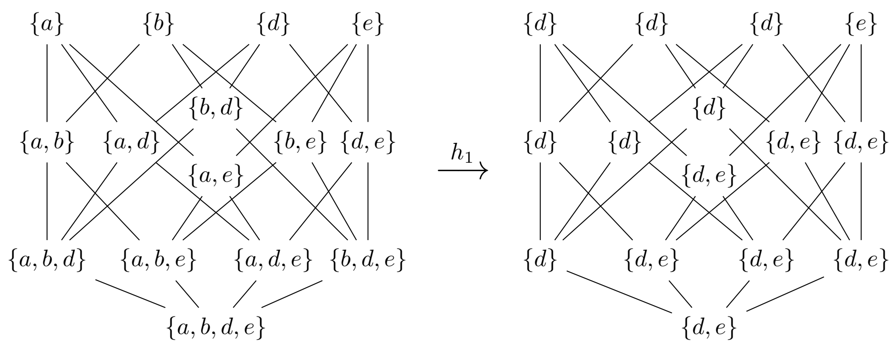

Figure 9. Representation of the function $h_1 : \mathcal{P}_0(\mathfrak{M}) \to \mathcal{P}_0(\mathfrak{C})$.

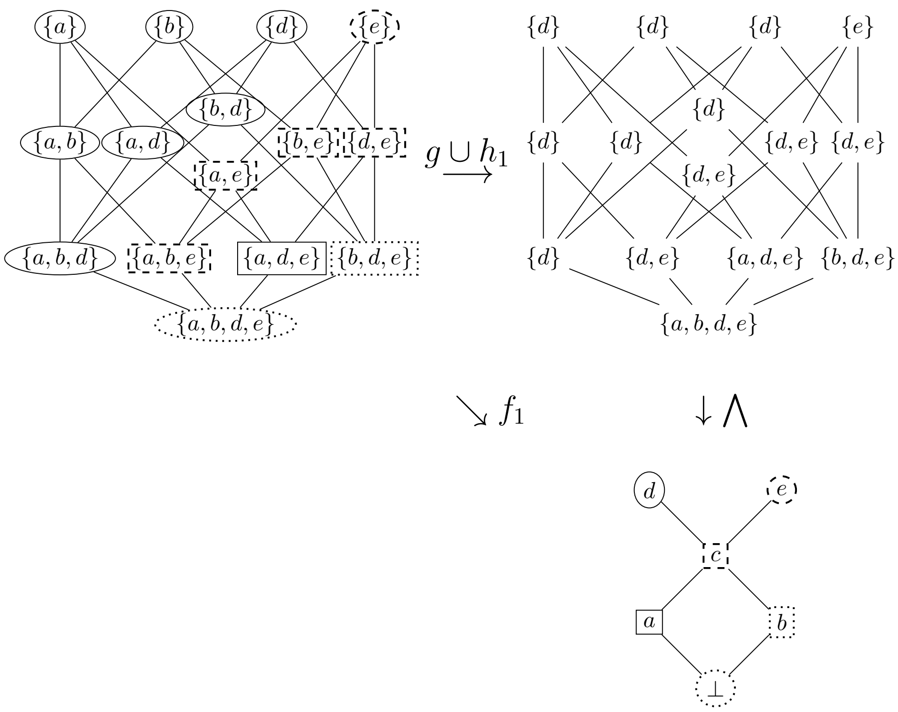

Figure 10. Representation of the bounded morphism $f_1 : \mathcal{P}_0(\mathfrak{M}) \twoheadrightarrow \mathrm{L}^-$.

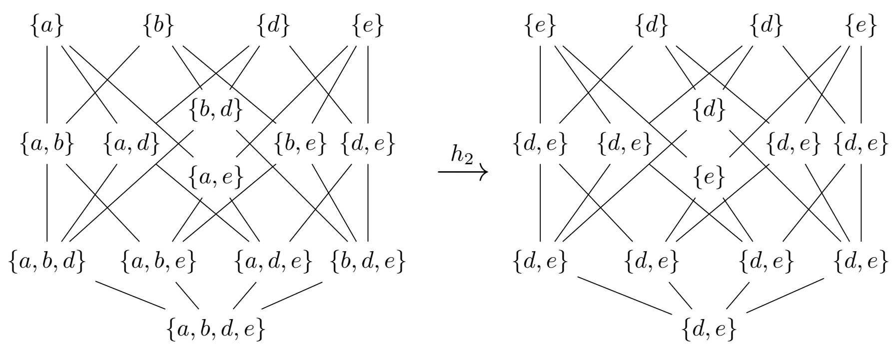

Figure 11. Representation of the function $h_2 : \mathcal{P}_0(\mathfrak{M}) \to \mathcal{P}_0(\mathfrak{C})$.

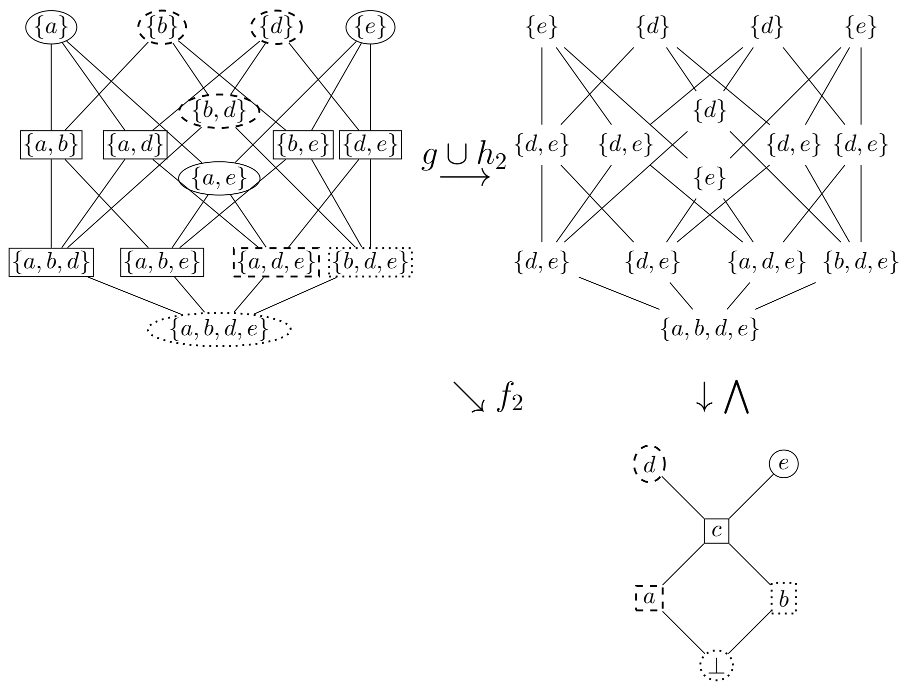

Figure 12. Representation of the bounded morphism $f_2 : \mathcal{P}_0(\mathfrak{M}) \twoheadrightarrow \mathrm{L}^-$.