Modal inverse correspondence via ALBA

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Abstract

We reformulate Kracht's theory of internal descriptions in the algebraic language of the correspondence algorithm ALBA and, within this language, we characterize (modulo standard translation) the class of first-order correspondents of modal inductive formulas as a suitable subclass of Kracht formulas for tense logic. Our result provides an alternative strategy to Kikot's generalization to the inductive (or 'generalized Sahlqvist') modal formulas of Kracht's inverse correspondence theorem for Sahlqvist formulas. This highlights and makes explicit the order-theoretic mechanisms underlying Kracht's algorithm and thereby paves the way to a generalization of inverse correspondence to modal logics on non-classical base including polyadic intuitionistic, distributive and non-distributive modal logics.

Keywords: Inverse correspondence, Unified correspondence, Kracht's Theorem, ALBA, Classical modal logic.

1 Introduction

Sahlqvist correspondence theory effectively connects a large, syntactically defined class of modal formulas with the first-order conditions their validity impose on Kripke frames. This is an immensely useful and powerful results when one's starting point is a logic axiomatized by modal axioms. However, when seeking an axiomatization for a first-order definable class of frames, one needs a result that goes in the other way, namely one that identifies a large class of first-order conditions which are modally definable and effectively associates them with their modal definitions. This is precisely what Kracht's 'inverse correspondence' theorem [17], based on his calculus of internal descriptions [16,18], provides. The MSQIN second-order quantifier introduction algorithm [9] constitutes and alternative, top-down approach to obtaining finding the modal Sahlqvist equivalents to Kracht formulas. In [12] Goranko and Vakarelov introduce the class of inductive formulas which essentially extends the class of Sahlqvist formulas, and in [14] Kikot establishes the corresponding generalisation of Kracht's theorem. The research programme of 'unified correspondence' (see e.g. [4,5,6]) has greatly generalised Goranko and Vakarelov's result and has established a definition of inductive formulas which can be applied to arbitrary logics algebraically captured by classes of lattice expansions (LE logics) and any relational semantics linked to these classes via an appropriate duality. This definition is based purely on the order-theoretic properties of the algebraic operations interpreting the connectives. A core tool in this research programme is the algorithm ALBA, which applies a set of equivalence preserving rewrite rules to transform modal formulas into 'pure' ones (in an extended language with the adjoints an residuals of all connectives) by eliminating the propositional variables in in favour of special variables, called nominals and co-nominals, which are constrained to range over the join and meet irreducible elements of the algebras which correspond (via duality) to first-order definable subsets of the relational semantics. This brings about a modularization of the correspondence theory, where correspondents are computed in the pure extended algebraic language, independent of any particular choice of relational semantics, and can thence be translated into first-order formulas via standard translations appropriate to particular choices of dual relational semantics.

In this paper, we initiate a line of research aimed at reformulating and extending inverse correspondence from classical modal logic to general LE logics. Key to this extension is a reformulation of the main engine of Kracht's result in the environment of unified correspondence which gives us access to conceptual and algorithmic tools developed there which, as mentioned above, apply across signatures and relational semantics. Specifically, in this paper we focus on the original setting of classical modal logic, where we formulate and prove an inverse correspondence result which characterizes the class of pure formulas in the extended modal language which can effectively be shown to correspond to inductive formulas. The utility of this is two-fold. Firstly, this reformulation helps to distil the order-theoretic information underlying the Kracht-Kikot model-theoretic results, thus paving the way to the required generalization to inverse correspondence for general LE logics.

Secondly, our strategy further modularizes the characterization of pure modal (and thence first-order) correspondents of the inductive formulas: we first characterize the pure correspondents of the Sahlqvist formulas in tense logic and then rely on the fact that every inductive formula in the language of classical modal logic is semantically equivalent to some scattered *very simple Sahlqvist* formula in the language of *tense* logic (cf. Lemma 3.6). We characterize the syntactic shape of such tense formulas, and suitably *restrict* the class of Kracht formulas that target tense Sahlqvist formulas. Thus, our proposed definition also features backward-looking restricted quantifiers.

Structure of the paper. In Section 2 we collect some brief preliminaries on the languages used in the paper, Sahlqvist and inductive formulas, Kracht formulas and the ALBA algorithm. Section 3 presents a characterization of the very simple Sahlqvist formulas in the language of tense logic which are equivalent to inductive formulas in the basic modal language. Our main results are presented in Section 4 where we define Kracht ML^K formulas in an extended pure hybrid modal language and show that they correspond on frames exactly to the very simple scattered Sahlqvist formulas in the language of tense logic, and then to inductive modal formulas.

2 Preliminaries

2.1 Modal languages

The *basic classical modal language* ML is defined using a set of propositional variables AtProp; its well formed formulas ϕ are given by the rule

$$\phi ::= p \mid \neg p \mid \phi \wedge \phi \mid \phi \vee \phi \mid \phi \rightarrow \phi \mid \phi \succ\!\!- \phi \mid \Diamond\phi \mid \Box\phi,$$

where p ranges over AtProp, and $\succ\!\!-$ is co-implication. It will be convenient to consider all the connectives as primitives. This is interpreted with the standard Kripke semantics, and the standard algebraic semantics is based on Boolean algebras with operators; furthermore it can be naturally expanded into the language ML^* with the two additional unary connectives $\blacklozenge$ and $\blacksquare$, which should be interpreted as the adjoints of $\Box$ and $\Diamond$ respectively.

The language ML^+ expands ML^* with two sorts of variables: nominals (usually denoted by $\mathbf{h}, \mathbf{i}, \mathbf{j}, \mathbf{k}$) and conominals (usually denoted by $\mathbf{l}, \mathbf{m}, \mathbf{n}, \mathbf{o}$). In perfect BAOs, (co)nominals are interpreted as (co)atoms. In what follows, we will denote ML^+-terms with the lower case letters s and t.

We will often consider ML^+-inequalities $s \leq t$; the language of such inequalities is $\mathrm{ML}_{\leq}$. Finally, the language for correspondence ML^K is built upon $\mathrm{ML}_{\leq}$ through the following rules:

$$\xi ::= s \leq t \mid \xi \;\&\; \xi \mid \xi \;⅋\; \xi \mid {\sim}\xi \mid \xi \Rightarrow \xi \mid \forall \mathbf{j}\; \xi \mid \forall \mathbf{m}\; \xi \mid \exists \mathbf{j}\; \xi \mid \exists \mathbf{m}\; \xi,$$

where $\&$ denotes conjunction, $⅋$ disjunction, $\Rightarrow$ implication, and $\sim$ negation.

2.2 Kracht's inverse correspondence

In what follows, FO denotes the frame correspondence language of classical modal logic. An FO-formula is *clean* (cf. [1, Chapter 3]) if no variable occurs both free and bound, and no two distinct (occurrences of) quantifiers bind the same variable. The definition of Kracht FO-formulas relies on the concept of *restricted quantifier*, i.e., quantifiers of the form $(\forall x \rhd y)\beta \equiv \forall x(yRx \rightarrow \beta)$ and $(\exists x \rhd y)\beta \equiv \exists x(yRx \wedge \beta)$. When we wish to suppress the restrictor, we will write $\forall^R x\ \beta$ and $\exists^R x\ \beta$.

Definition 2.1 [Kracht formulas] A *Kracht formula* [1] is a clean FO-formula in prenex normal form with a single free variable x_0 and shape:

$$\forall^R x_1 \cdots \forall^R x_n Q_1^R y_1 \cdots Q_m^R y_m\ \beta(x_0, x_1, \ldots, x_n, y_1, \ldots, y_m)\,,$$

where $Q_i \in \{\forall, \exists\}$ (for $1 \leq i \leq m$), variables in $X = \{x_1, \ldots, x_n\}$ and $Y = \{y_1, \ldots, y_m\}$ are called *inherently universal* and *non-inherently universal* respectively; β is an unquantified formula in DNF whose atoms are of the

[1] The definition we present is commonly referred to as *type 1 Kracht formula*. As is well known (cf. [1]), type 1 Kracht formulas are Kracht formulas in prenex normal form where the matrix is rewritten in DNF.

form: $\top$, $\bot$, uRx, xRu, $x = u$ where $x \in X \cup \{x_0\}$ and $u \in X \cup Y$.

Theorem 2.2 ([1]) *Any Kracht formula can be effectively shown to be the first order correspondent of some Sahlqvist formula.*

2.3 Inductive and very simple Sahlqvist inequalities

Inductive formulas are introduced by Goranko and Vakarelov in [10,11,12],and are referred to as generalized Sahlqvist formulas by Kikot [14]. We present an alternative definition that will be convenient for the results in this paper.

Definition 2.3 [**Signed Generation Tree**] The *positive* (resp. *negative*) *generation tree* of any ML-formula s is defined by labelling the root node of the generation tree of s with the sign $+$ (resp. $-$), and then propagating the labelling on each remaining node as follows: for any

- node labelled with $\vee$, $\wedge$, $\Diamond$ or $\Box$ assign the same sign to its children nodes,
- $\neg$-node, assign the opposite sign to its child,
- $\rightarrow$-node, assign the opposite (resp. same) sign to the left (resp. right) child,
- $>\!\!-$-node, assign the opposite (resp. same) sign to the right (resp. left) child.

Nodes in signed generation trees are *positive* (resp. *negative*) if they are signed $+$ (resp. $-$).

Signed generation trees will be used in the context of formula inequalities $s \leq t$. In this context we will typically consider the positive generation tree $+s$ for the left-hand side and the negative one $-t$ for the right-hand side. In this case we will speak of *signed generation trees of inequalities.* An *order type* over $p_1, \ldots, p_n$ is a map $\varepsilon : \{p_1, \ldots, p_n\} \rightarrow \{1, \partial\}$. A term-inequality $s \leq t$ is *uniform* in a given variable p if all occurrences of p in both $+s$ and $-t$ have the same sign, and $s \leq t$ is ε-*uniform* in a (sub)array $\overline{p}$ of its variables if $s \leq t$ is uniform in p, occurring with the sign indicated by ε, for every p in $\overline{p}$. Given $\rho \in \{1, \partial\}$, and terms s and t, the notation $s \leq_\rho t$ indicates the inequality $s \leq t$ when $\rho = 1$, and $t \leq s$ otherwise.

For any term $s(p_1, \ldots p_n)$, any order type ε over n, and any $1 \leq i \leq n$, an ε-*critical node* in a signed generation tree of s is a leaf node $+p_i$ with $\varepsilon_i = 1$ or $-p_i$ with $\varepsilon_i = \partial$. An ε-*critical branch* in the tree is a branch ending in an ε-critical node.

We will write $\phi(!x)$ (resp. $\phi(!\overline{x})$) to indicate that the variable x (resp. each variable x in $\overline{x}$) occurs exactly once in ϕ. Accordingly, we will write $\phi(\gamma/!x)$ (resp. $\phi(\overline{\gamma}/!\overline{x})$) to indicate the formula obtained from ϕ by substituting γ (resp. each variable γ in $\overline{\gamma}$) for the unique occurrence of (its corresponding variable) x in ϕ.

Definition 2.4 [Inductive inequality] For any order type ε, and any strict order $<_\Omega$ on the variables (called *dependency order*), a formula is (Ω, ε)-inductive if:

- every ε-critical branch is a concatenation of two (possibly empty) paths P_1 and P_2 from leaf to root, such that, excluding the leaf, P_1 consists of *PIA* nodes, i.e. nodes in $\{-\wedge, +\vee, -\Diamond, +\Box, +\rightarrow\}$; and P_2 consists of *skeleton*

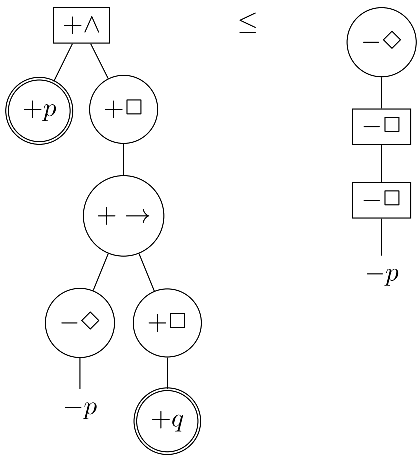
nodes, i.e. nodes in $\{+\wedge, -\vee, +\Diamond, -\Box, -\rightarrow\}$ [2].
- each subtree rooted in a $+\rightarrow$, $-\wedge$, or $+\vee$ node contains at most one ε-critical variable p and all the other variables q in the subtree are such that $q <_\Omega p$.

An *inductive* inequality is (Ω, ε)-inductive for some ε and $<_\Omega$. In what follows, we will refer to a formula χ such that $+\chi$ (resp. $-\chi$) consists only of skeleton nodes as a *positive* (resp. *negative*) *skeleton*; and we dub formulas ζ as positive (resp. negative) PIA if there is a path from a leaf to the root of $+\zeta$ (resp. $-\zeta$) consisting only of PIA nodes. For every positive (definite) PIA formula [3] $\varphi = \varphi(!x, \overline{z})$ and negative PIA formula $\psi = \psi(!x, \overline{z})$ where x is a leaf of a PIA-path to the root, we define the formulas $\mathsf{LA}(\varphi)(u, \overline{z})$ and $\mathsf{RA}(\psi)(u, \overline{z})$ (with u a new fresh variable) by simultaneous recursion:

$$
\begin{array}{cc}
\mathsf{LA}(x) = u & \mathsf{RA}(x) = u \\
\mathsf{LA}(\Box\varphi(x,\overline{z})) = \mathsf{LA}(\varphi)(\blacklozenge u, \overline{z}) & \mathsf{RA}(\Diamond\psi, \overline{z}) = \mathsf{RA}(\psi)(\blacksquare u, \overline{z}) \\
\mathsf{LA}(\psi(\overline{z}) \rightarrow \varphi(x,\overline{z})) = \mathsf{LA}(\varphi)(\psi(\overline{z}) \wedge u, \overline{z}) & \mathsf{RA}(\psi(x,\overline{z}) \succ\!\!- \varphi(\overline{z})) = \mathsf{RA}(\psi)(\varphi(\overline{z}) \vee u, \overline{z}) \\
\mathsf{LA}(\psi(x,\overline{z}) \rightarrow \varphi(\overline{z})) = \mathsf{RA}(\psi)(u \rightarrow \varphi(\overline{z}), \overline{z}) & \mathsf{RA}(\psi(\overline{z}) \succ\!\!- \varphi(x,\overline{z})) = \mathsf{LA}(\varphi)(\psi(\overline{z}) \succ\!\!- u, \overline{z}) \\
\mathsf{LA}(\psi_1(x,\overline{z}) \vee \psi_2(\overline{z})) = \mathsf{LA}(\psi_1)(u \succ\!\!- \psi_2(\overline{z})) & \mathsf{RA}(\varphi_1(x,\overline{z}) \wedge \varphi_2(\overline{z})) = \mathsf{RA}(\varphi_1)(\varphi_2(\overline{z}) \rightarrow u)
\end{array}
$$

The definition of inductive inequality is expanded to ML* by adding $-\blacklozenge$ and $+\blacksquare$ as PIA nodes and adding $+\blacklozenge$ and $-\blacksquare$ as Skeleton nodes.

Example 2.5 The formula $p \wedge \Box(\Diamond p \rightarrow \Box q) \leq \Diamond\Box\Box q$ is inductive for $\varepsilon_p = \varepsilon_q = 1$ and $p <_\Omega q$. Its signed generation tree is the following



where nodes inside a box are skeleton, nodes inside circles are PIAs, and nodes inside double-circles are the critical occurrences of the variables. The left

[2] This definition of inductive inequality is not the most general one and, in fact, refers to *definite* inductive inequalities (cf. [13]). However, modulo exhaustively distributing $\vee$ and $\wedge$ over the other connectives, every (general) inductive inequality is equivalent to a conjunction of such inequalities.

[3] Here definite refers to the fact that $\vee$ and $\wedge$ have been exhaustively distributed over the other connectives; therefore no $-\vee$, nor $+\wedge$ node occurs in the PIA formula.

adjoint of the maximal PIA formula $\varphi(q,p) \equiv \Box(\Diamond p \to q)$ is $\mathsf{LA}(\varphi(q,p)) = \blacklozenge(\Diamond p \wedge \blacklozenge u)$.

The notion of a *Sahlqvist* inequality is obtained by restricting the nodes that are allowed in the PIA parts of critical branches, while that of a *very simple Sahlqvist* formula eliminates the PIA parts altogether:

Definition 2.6 [(Very simple) Sahlqvist inequalities - version 1] A *Sahlqvist* inequality (in ML or ML*) is any inductive inequality where the P_1 parts of all ε-critical branches consists only of nodes in $\{+\wedge, -\vee, -\Diamond, +\Box\}$. A *very simple Sahlqvist* inequality is any inductive inequality where all ε-critical branches consists only of skeleton nodes (i.e. the P_1 parts are empty).

Following [2], we will often represent (Ω, ε)-inductive inequalities as follows:

$$(\varphi \leq \psi)[\overline{\alpha}/!\overline{x}, \overline{\beta}/!\overline{y}, \overline{\gamma}/!\overline{z}, \overline{\delta}/!\overline{w}],$$

where $(\varphi \leq \psi)[!\overline{x}, !\overline{y}, !\overline{z}, !\overline{w}]$ contains only skeleton nodes, is positive (resp. negative) in $!\overline{x}$ and $!\overline{z}$ (resp. $!\overline{y}$ and $!\overline{w}$), and it is *scattered*, i.e. each variable occurs only once; each α in $\overline{\alpha}$ (resp. β in $\overline{\beta}$) is a positive (resp. negative) PIA.

Definition 2.7 [(Very simple) Sahlqvist inequality] An inductive inequality $(\varphi \leq \psi)[\overline{\alpha}/!\overline{x}, \overline{\beta}/!\overline{y}, \overline{\gamma}/!\overline{z}, \overline{\delta}/!\overline{w}]$, is *Sahlqvist* [4] if every α in $\overline{\alpha}$ and β in $\overline{\beta}$ contains only unary connectives. It is *very simple Sahlqvist* if every α and β is a propositional variable.

2.4 ALBA

ALBA is a calculus for correspondence that is based on the Ackermann lemma and which successfully reduces all inductive inequalities [4,5,6]. (Please see Appendix A for some more details on ALBA.) One can prove (cf. [7]) the output of ALBA on an inductive inequality $(\varphi \leq \psi)[\overline{\alpha}/!\overline{x}, \overline{\beta}/!\overline{y}, \overline{\gamma}/!\overline{z}, \overline{\delta}/!\overline{w}]$ is

$$\forall \overline{\mathbf{j}} \forall \overline{\mathbf{m}} \left((\varphi \leq \psi)[!\overline{\mathbf{j}}/!\overline{x}, !\overline{\mathbf{m}}/!\overline{y}, \overline{\gamma}\left[\overline{\bigvee \mathsf{Mv}(p)}/\overline{p}, \overline{\bigwedge \mathsf{Mv}(q)}/\overline{q}\right]/!\overline{z}, \overline{\delta}\left[\overline{\bigvee \mathsf{Mv}(p)}/\overline{p}, \overline{\bigwedge \mathsf{Mv}(q)}/\overline{q}\right]/!\overline{w}] \right),$$

where $\overline{p}$ (resp. $\overline{q}$) are the variables occuring in positive (resp. negative) position, $\mathsf{Mv}(p)$ and $\mathsf{Mv}(q)$ are defined by recursion on the dependency order as follows:

(i) for $<_\Omega$-minimal variables p and q,
- $\mathsf{Mv}(p) := \{\mathsf{LA}(\alpha_p)[\mathbf{j}_k/u], \mathsf{RA}(\beta_p)[\mathbf{m}_h/u] \mid 1 \leq k \leq n_{i_1}, 1 \leq h \leq n_{i_2}\}$
- $\mathsf{Mv}(q) := \{\mathsf{LA}(\alpha_q)[\mathbf{j}_h/u], \mathsf{RA}(\beta_q)[\mathbf{m}_k/u \mid 1 \leq h \leq m_{j_1}, 1 \leq k \leq m_{j_2}\}$

where, n_{i_1} (resp. n_{i_2}) is the number of occurrences of p in αs (resp. in βs) for every $p \in \overline{p}$, and m_{j_1} (resp. m_{j_2}) is the number of occurrences of q in αs (resp. in βs) for every $q \in \overline{q}$; the subscript p in α_p denotes the only critical occurrence under α.

(ii) for non $<_\Omega$-minimal variables p and q,

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[4] Analogously, this definition of Sahlqvist inequality is not the most general one and it refers to *definite* Sahlqvist inequalities (cf. [13]). Again, modulo exhaustively distributing $\vee$ and $\wedge$ over the other connectives, every (general) Sahlqvist inequality is equivalent to a conjunction of such inequalities.

- $\mathsf{Mv}(p) := \{\mathsf{RA}(\alpha_p)[\mathbf{j}_k/u, \overline{\mathsf{mv}(p)}/\overline{p}, \overline{\mathsf{mv}(q)}/\overline{q}], \mathsf{RA}(\beta_p)[\mathbf{m}_h/u, \overline{\mathsf{mv}(p)}/\overline{p}, \overline{\mathsf{mv}(q)}/\overline{q}] \mid 1 \leq k \leq n_{i_1}, 1 \leq h \leq n_{i_2}, \overline{\mathsf{mv}(p)} \in \overline{\mathsf{Mv}(p)}, \overline{\mathsf{mv}(q)} \in \overline{\mathsf{Mv}(q)}\}$
- $\mathsf{Mv}(q) := \{\mathsf{LA}(\alpha_q)[\mathbf{j}_h/u, \overline{\mathsf{mv}(p)}/\overline{p}, \overline{\mathsf{mv}(q)}/\overline{q}], \mathsf{RA}(\beta_q)[\mathbf{m}_k/u, \overline{\mathsf{mv}(p)}/\overline{p}, \overline{\mathsf{mv}(q)}/\overline{q}) \mid 1 \leq h \leq m_{j_1}, 1 \leq k \leq m_{j_2}, \overline{\mathsf{mv}(p)} \in \overline{\mathsf{Mv}(p)}, \overline{\mathsf{mv}(q)} \in \overline{\mathsf{Mv}(q)}\}$

where, n_{i_1} (resp. n_{i_2}) is the number of occurrences of p in αs (resp. in βs) for every $p \in \overline{p}$, and m_{j_1} (resp. m_{j_2}) is the number of occurrences of q in αs (resp. in βs) for every $q \in \overline{q}$.

3 Inductive formulas in ML as very-simple Sahlqvist with residuals

Definition 3.1 A branch in a signed generation tree $\pm s$ is called *splittable* if it is the concatenation of two paths Q_1 and Q_2, one of which may possibly be of length 0, such that Q_1 is a path from the leaf consisting (apart from variable nodes) only of nodes in $\{+\blacksquare, -\blacklozenge, +\vee, +\rightarrow, -\wedge, -\succ\!\!-\}$ and Q_2 consists of (any) ML-nodes.

Definition 3.2 Given an order type ε, a strict partial order Ω on propositional variables, a signed generation tree $\pm\phi$ is called (Ω, ε)-*unpackable* if $\varepsilon^\partial(\pm\phi)$ and

(i) ϕ is a propositional variable or constant, or

(ii) If p_0 is maximal in $var(\phi)$ with respect to Ω, then

  (a) the path Q in $\pm\phi$ ending in p_0 is splittable, and

  (b) wherever Q passes through a node in $\{+\vee, +\rightarrow, -\wedge, -\succ\!\!-\}$, the subtree $\pm\gamma$ corresponding to the argument through which Q does *not* pass is (Ω, ε)-unpackable.

Definition 3.3 An ML*-inequality $\phi \leq \psi$ is called a *crypto $\mathcal{L}$-inductive* if it is a very simple ε-Sahlqvist inequality in ML* and in the signed generation trees $+\phi$ and $-\psi$:

(i) All ε-critical branches contain only signed connectives from ML,

(ii) There exists a strict partial order Ω on the propositional variables occurring in $\phi \leq \psi$, such that for every ε-non-critical branch the signed subtree rooted at the topmost (closest to the root) node on the branch properly belonging to ML* is (Ω, ε)-unpackable.

Proposition 3.4 *Every crypto* ML**-inductive inequality is frame-equivalent to an inductive inequality in* ML.

Proof. Suppose $\phi \leq \psi$ crypto ML*-inductive and let ε and Ω be an order type and a strict partial order satisfying Definition 3.3. Suppose that $var(\phi \leq \psi) = \{p_1, \ldots p_n\}$. We may assume w.l.o.g. that $p_i <_\Omega p_j$ implies $i < j$. Starting from propositional variables p_i minimal with respect to Ω, apply the inverse Ackermann rules to extract the subformulas corresponding to the subtree rooted at the topmost (closest to the root) node on the branch properly belonging to ML*. This transforms $\phi \leq \psi$ into a quasi-inequality of the form

$$q_1 \leq \alpha_1, \ldots, q_m \leq \alpha_m, \beta_{m+1} \leq r_{m+1}, \ldots, \beta_\ell \leq r_\ell \Rightarrow (\phi' \leq \psi')[\overline{q}/!\overline{x}, \overline{r}/!\overline{y}]$$

where $\phi' \leq \psi'$ contains only connectives from ML, the q_i and r_i are new variables, each α_i contains exactly one variable among $p_1, \ldots p_n$ which was Ω-maximal in the extracted subtree from which α originates. Applying adjunction and residuation rules this can be transformed into

$$\mathsf{LA}(\alpha_1)(q_1) \leq_{\varepsilon(p_{i_1})} p_{i_1}, \ldots, \mathsf{LA}(\alpha_m)(q_m) \leq_{\varepsilon(p_{i_m})} p_{i_m},$$
$$p_{i_{m+1}} \leq_{\varepsilon(p_{i_{m+1}})} \mathsf{RA}(\beta_1)(r_{m+1}), \ldots, p_{i_\ell} \leq_{\varepsilon(p_{i_\ell})} \mathsf{RA}(\beta_\ell)(r_\ell) \Rightarrow (\phi' \leq \psi')[\overline{q}/!\overline{x}, \overline{r}/!\overline{y}]$$

Note that each $\mathsf{LA}(\alpha_i)(q_i)$ and $\mathsf{RA}(\beta_i)(r_i)$ is a ML-formula.

This is now in Ackermann-shape w.r.t. the variables $p_1, \ldots p_n$. Applying the Ackermann rules produces $(\phi' \leq \psi')[\overline{q}/!\overline{x}, \overline{r}/!\overline{y}, \overline{\xi}/\overline{p}]$, which is an (Ω', ε) inductive inequality in ML where $q_\ell <_{\Omega'} q_j$ iff $p_{i_\ell} <_\Omega p_{i_j}$, for each ℓ and j.

$\square$

Example 3.5 The ML*-inequality $p_1 \wedge p_2 \leq \Diamond\Box\Box\blacklozenge(\blacklozenge p_2 \wedge \Diamond p_1)$ is crypto-inductive for $\varepsilon(p_1, p_2) = (1, 1)$ and $p_1 <_\Omega p_2$, and is equivalent to $p_1 \leq q_1, \blacklozenge(\blacklozenge p_2 \wedge \Diamond q_1) \leq q_2 \implies p_1 \wedge p_2 \leq \Diamond\Box\Box q_2$, which is equivalent to $p_1 \leq q_1, p_2 \leq \Box(\Diamond q_1 \to \Box q_2) \implies p_1 \wedge p_2 \leq \Diamond\Box\Box q_2$, which becomes $q_1 \wedge \Box(\Diamond q_1 \to \Box q_2) \leq \Diamond\Box\Box q_2$.

Lemma 3.6 *Every inductive inequality* $(\varphi \leq \psi)[\overline{\alpha}/!\overline{x}, \overline{\beta}/!\overline{y}, \overline{\gamma}/!\overline{z}, \overline{\delta}/!\overline{w}]$ *is equivalent to some crypto-*ML**-inductive inequality.*

Proof. Given a definite inductive formula $(\varphi \leq \psi)[\overline{\alpha}/\overline{x}, \overline{\beta}/\overline{y}, \overline{\gamma}/\overline{z}, \overline{\delta}/\overline{w}]$, an ALBA run on it yields

$$\forall\overline{\mathbf{j}}\forall\overline{\mathbf{m}}\forall\overline{\mathbf{i}}\forall\overline{\mathbf{n}}\Big(\overline{\mathbf{i}} \leq \overbrace{\overline{\gamma}\left[\overline{\bigvee \mathsf{Mv}(p)}/\overline{p}, \overline{\bigwedge \mathsf{Mv}(q)}/\overline{q}\right]}^{\overline{\gamma}^{mv}} \ \&\ \overbrace{\overline{\delta}\left[\overline{\bigvee \mathsf{Mv}(p)}/\overline{p}, \overline{\bigwedge \mathsf{Mv}(q)}/\overline{q}\right]}^{\overline{\delta}^{mv}} \leq \overline{\mathbf{n}} \Rightarrow (\varphi \leq \psi)[!\overline{\mathbf{j}}/!\overline{x}, !\overline{\mathbf{m}}/!\overline{y}, !\overline{\mathbf{i}}/!\overline{z}, !\overline{\mathbf{n}}/!\overline{w}]\Big),$$

Consider now the inequality

$$\Big((\varphi \leq \psi)[!\overline{\mathbf{j}}/!\overline{x}, !\overline{\mathbf{m}}/!\overline{y}, !\overline{\gamma}^{mv}/!\overline{z}, !\overline{\delta}^{mv}/!\overline{w}]\Big)\,[\overline{p_j}/\overline{\mathbf{j}}, \overline{p_m}/\overline{\mathbf{m}}], \tag{1}$$

with $\overline{p_j}$ (resp. $\overline{q_m}$) fresh variables, one for each nominal in $\overline{\mathbf{j}}$ (resp. conominal in $\overline{\mathbf{m}}$). Clearly, the inequality is very simple Sahlqvist in ML* for ε such that each for each p_j (resp. q_m), $\varepsilon(p_j) = 1$ (resp. $\varepsilon(q_m) = \partial$) and some inductive order type $<_\Omega$. More precisely, in the ALBA run each PIA in $\overline{\alpha}$ (resp. $\overline{\beta}$) is approximated by some nominal in $\overline{\mathbf{j}}$ (resp. $\overline{\mathbf{m}}$), let τ be the map that given a variable in $\overline{\mathbf{j}}$ and $\overline{\mathbf{m}}$, yields the critical variable in the corresponding PIA formula. Let $<_{\Omega'}$ the inductive order used in the ALBA run. The inequality (1) is very simple Sahlqvist for the inductive order $<_\Omega$ such that for every $r, t \in \{p, q\}$ and $\mathbf{u}, \mathbf{v}$ in $\overline{\mathbf{j}}$ or $\overline{\mathbf{m}}$, $r_u \leq t_v$ iff $\tau(u) \leq \tau(v)$. It is clear how the ε-critical branches contain only connectives in ML, as the only connectives found there are the ones found in the skeleton of the original inductive inequality. Hence, it remains to show condition (2) of Definition 3.3. We show that every signed subtree rooted at the topmost connective properly in ML* is (ε, Ω)-unpackable for the ε and Ω defined above. Since the operators properly belonging to ML* can only occur in the minimal valuations, any of such nodes has to occur in some formula in $\overline{\gamma}^{mv}$ or $\overline{\delta}^{mv}$ inside some $\mathsf{Mv}(p)$ (resp. $\mathsf{Mv}(q)$) for some variable p (resp. q); hence the paths passing through these nodes ending in $<_\Omega$-maximal variables are splittable. Suppose that one of such paths passes through the jth coordinate of some m-ary SRR node in a non ε-critical branch $\circledast(\gamma_1, \ldots, \gamma_{j-1}, \beta, \gamma_{j+1} \ldots, \gamma_m)$, and let i be any index in $\{1, \ldots, m\} \setminus \{j\}$. The subformula γ_i is (ε, Ω)-unpackable as it is of course splittable, and, inductively, every topmost node not properly in ML* is part of some $\mathsf{Mv}(p')$ (resp. $\mathsf{Mv}(q')$) for some p' (resp. q') preceding p (resp. q). $\square$

4 Inverse correspondence in ALBA

This section presents the main results of the paper. We define the Kracht ML^K formulas and show that they correspond on frames exactly to the very simple scattered Sahlqvist formulas in the language of tense logic and, via the results of the previous section, to inductive modal formulas. Any proofs not given in this section can be found in the appendix.

Through the remainder of this section we will treat literals $\neg\mathbf{m}$ (resp. $\neg\mathbf{j}$) as nominals (resp. conominals). Let NL (resp. CNL) be the collection of nominals (resp. conominals) and negated conominals (resp. nominals).

4.1 Original Kracht formulas in ALBA's language

Definition 4.1 [Flat and restricting inequalities] *Flat inequalities* are ML$^+$-inequalities of the following form:

$$\mathbf{i} \leq \Diamond\mathbf{v}, \quad \mathbf{i} \leq \blacklozenge\mathbf{v}, \quad \Box\mathbf{v} \leq \mathbf{m}, \quad \blacksquare\mathbf{v} \leq \mathbf{m}, \quad \mathbf{u} \leq \mathbf{v}, \quad \mathbf{u} \to \mathbf{v} \leq \mathbf{n}, \quad \mathbf{i} \leq \mathbf{u} \succ\!\!- \mathbf{v}$$

where $\mathbf{u}, \mathbf{v} \in \mathrm{NL} \cup \mathrm{CNL}$, and the two variables in the inequality are different. *Restricting inequalities* are flat inequalities of the form

$$\mathbf{j} \leq \Diamond\mathbf{i}, \quad \mathbf{j} \leq \blacklozenge\mathbf{i}, \quad \Box\mathbf{n} \leq \mathbf{m}, \quad \blacksquare\mathbf{n} \leq \mathbf{m}, \quad \mathbf{i} \leq \mathbf{j}, \quad \mathbf{n} \leq \mathbf{m},$$
$$\mathbf{j} \leq \mathbf{i} \succ\!\!- \mathbf{n}, \qquad \mathbf{i} \to \mathbf{n} \leq \mathbf{m}$$

The nominals $\mathbf{j}$ and conominals $\mathbf{m}$ are the *restricting* pure variables, while $\mathbf{i}$ and $\mathbf{n}$ are the *restricted* pure variables.

Restricting inequalities encode the type of atoms that can appear in the matrix of a Kracht formula. Indeed, nominals can be thought of as worlds of the frame $x, y, z, \ldots$, and conominals as their complements $x^c, y^c, z^c, \ldots$. Then, with some abuse of notation, including omitting curly braces when writing singletons and their complements, and reading the order $\leq$ as set-theoretic inclusion, the following equivalences hold in (complex algebras of) Kripke frames:

$$\begin{array}{r||l}
xRy & x \leq \Diamond y \ \text{ iff } \ \Box y^c \leq x^c \ \text{ iff } \ y \leq \blacklozenge x \ \text{ iff } \ \blacksquare x^c \leq y^c \\
x = y & x \leq y \ \text{ iff } \ y^c \leq x^c \ \text{ iff } \ x \not\leq y^c \\
x = y = z & x \to y^c \leq z^c \ \text{ iff } \ x \leq y \succ\!\!- z^c
\end{array}$$

Example 4.2 The inequality $p \wedge \Box(\Diamond p \to \Box q) \leq \Diamond\Box\Box q$ (cf. [12]) is not Sahlqvist for any order type, but it is inductive w.r.t. the order-type $\varepsilon(p, q) = (1, 1)$ and $p <_\Omega q$. Running ALBA on it yields

$$\begin{array}{ll} & \forall p \forall q (p \wedge \Box(\Diamond p \to \Box q) \leq \Diamond\Box\Box q) \\ \text{iff} & \forall \mathbf{j} \forall \mathbf{m} [\Diamond\Box\Box\blacklozenge(\Diamond \mathbf{j} \wedge \blacklozenge \mathbf{j}) \leq \mathbf{m} \Rightarrow \mathbf{j} \leq \mathbf{m}] \\ \text{iff} & \forall \mathbf{j} [\mathbf{j} \leq \Diamond\Box\Box\blacklozenge(\Diamond \mathbf{j} \wedge \blacklozenge \mathbf{j})] \end{array}$$

As the nominal $\mathbf{j}$ represents a world x of the Kripke frame, it is equivalent to:

$$\begin{array}{ll} & \forall x (x \in [\![\Diamond\Box\Box\blacklozenge(\Diamond \mathbf{j} \wedge \blacklozenge \mathbf{j})]\!][\mathbf{j} := x]) \\ \text{iff} & \forall x \exists y (xRy \ \&\ y \in [\![\Box\Box\blacklozenge(\Diamond \mathbf{j} \wedge \blacklozenge \mathbf{j})]\!][\mathbf{j} := x]) \\ \text{iff} & \forall x \exists y (xRy \ \&\ \forall z (yR^2 z \Rightarrow z \in [\![\blacklozenge(\Diamond \mathbf{j} \wedge \blacklozenge \mathbf{j})]\!][\mathbf{j} := x])) \\ \text{iff} & \forall x \exists y (xRy \ \&\ \forall z (yR^2 z \Rightarrow \exists w (wRz \ \&\ wRx \ \&\ xRw))). \end{array}$$

This last condition can equivalently be rewritten in three ways:

$$\begin{array}{ll} & \forall x (\exists y \rhd x)(\forall z_1 \rhd y)(\forall z \rhd z_1)(\exists w \blacktriangleright z)(wRx \ \&\ xRw) \\ \text{iff} & \forall x (\exists y \rhd x)(\forall z_1 \rhd y)(\forall z \rhd z_1)(\exists w \rhd x)(wRz \ \&\ wRx) \\ \text{iff} & \forall x (\exists y \rhd x)(\forall z_1 \rhd y)(\forall z \rhd z_1)(\exists w \blacktriangleright x)(wRz \ \&\ xRw), \end{array}$$

where $(\exists w \blacktriangleright z)$ quantifies w over the *predecessors* of z. The second and third ones are not Kracht formulas, as the atom wRz has no inherently universal variables in it (the only inherently universal is x). The first one is a *tense* Kracht formula. This consideration suggests that in order to express first-order conditions in Kracht shape for an inductive formula, we need to admit the presence of operators in the fully residuated language ML^*, thus allowing for backwards looking restricted quantifiers.

The example above hints at the need to expand the notation for restricted quantifiers to include the residuals $\blacksquare$ and $\blacklozenge$. *Restricted quantifiers* in ML^* are defined in the following way (β being any ML^+-formula):

$$\begin{array}{rcl@{\qquad}rcl} (\forall \mathbf{i} \rhd \mathbf{j})\beta & \equiv & \forall \mathbf{i}(\mathbf{j} \leq \Diamond \mathbf{i} \Rightarrow \beta) & (\forall \mathbf{i} \blacktriangleright \mathbf{j})\beta & \equiv & \forall \mathbf{i}(\mathbf{j} \leq \blacklozenge \mathbf{i} \Rightarrow \beta) \\ (\exists \mathbf{i} \rhd \mathbf{j})\beta & \equiv & \exists \mathbf{i}(\mathbf{j} \leq \Diamond \mathbf{i} \ \&\ \beta) & (\exists \mathbf{i} \blacktriangleright \mathbf{j})\beta & \equiv & \exists \mathbf{i}(\mathbf{j} \leq \blacklozenge \mathbf{i} \ \&\ \beta) \\ (\forall \mathbf{n} \rhd \mathbf{m})\beta & \equiv & \forall \mathbf{n}(\Box \mathbf{n} \leq \mathbf{m} \Rightarrow \beta) & (\forall \mathbf{n} \blacktriangleright \mathbf{m})\beta & \equiv & \forall \mathbf{n}(\blacksquare \mathbf{n} \leq \mathbf{m} \Rightarrow \beta) \\ (\exists \mathbf{n} \rhd \mathbf{m})\beta & \equiv & \exists \mathbf{n}(\Box \mathbf{n} \leq \mathbf{m} \ \&\ \beta) & (\exists \mathbf{n} \blacktriangleright \mathbf{m})\beta & \equiv & \exists \mathbf{n}(\blacksquare \mathbf{n} \leq \mathbf{m} \ \&\ \beta) \end{array}$$

We also consider binary restricted quantifiers whose corresponding restricting inequalities (cf. Definition 4.1) contain a (co)implication.

$$(\forall \mathbf{i}, \mathbf{n} \rhd \mathbf{m})\beta \equiv \forall \mathbf{i} \forall \mathbf{n}(\mathbf{i} \to \mathbf{n} \leq \mathbf{m} \ \Rightarrow\ \beta)$$
$$(\exists \mathbf{i}, \mathbf{n} \rhd \mathbf{m})\beta \equiv \exists \mathbf{i} \exists \mathbf{n}(\mathbf{i} \to \mathbf{n} \leq \mathbf{m} \ \&\ \beta)$$
$$(\forall \mathbf{i}, \mathbf{n} \rhd \mathbf{j})\beta \equiv \forall \mathbf{i} \forall \mathbf{n}(\mathbf{j} \leq \mathbf{i} \succ\!\!- \mathbf{n} \ \Rightarrow\ \beta) \qquad (\exists \mathbf{i}, \mathbf{n} \rhd \mathbf{j})\beta \equiv \exists \mathbf{i} \exists \mathbf{n}(\mathbf{j} \leq \mathbf{i} \succ\!\!- \mathbf{n} \ \&\ \beta)$$

4.2 Kracht MLK-formulas

Definition 4.3 [Kracht disjunct] A *Kracht disjunct* is a formula $\theta(\mathbf{w})$ in ML^K defined inductively together with its *main pure variable* $\mathbf{w} \in \mathrm{NL} \cup \mathrm{CNL}$. It either is:

- a flat inequality (cf. Definition 4.1) $s \leq \mathbf{w}$ or $\mathbf{w} \leq t$;
- $(\exists\ \mathbf{u} \rhd \mathbf{w})\theta(\mathbf{u})$, $(\forall\ \mathbf{u} \rhd \neg\mathbf{w})\theta(\neg\mathbf{u})$, $(\exists\ \mathbf{u} \blacktriangleright \mathbf{w})\theta(\mathbf{u})$, or $(\forall\ \mathbf{u} \blacktriangleright \neg\mathbf{w})\theta(\neg\mathbf{u})$, where $\theta(\mathbf{u})$ is a Kracht disjunct where $\mathbf{w}$ does not occur [5];

[5] Note that the types of $\mathbf{u}$ and $\mathbf{w}$ (nominal or conominal) in these clauses are governed by the typing conventions in the definitions of the restricted quantifiers.

- $\theta(\mathbf{m}) := (\exists\ \mathbf{i}, \mathbf{n} \rhd \mathbf{m})(\theta_1(\mathbf{i})\ \&\ \theta_2(\mathbf{n}))$ or $\theta(\neg\mathbf{m}) := (\forall\ \mathbf{i}, \mathbf{n} \rhd \mathbf{m})(\theta_1(\neg\mathbf{i})\ \parr\ \theta_2(\neg\mathbf{n}))$, where $\theta_1(\mathbf{i})$ and $\theta_2(\mathbf{n})$ are Kracht disjuncts such that $\mathbf{m}$ does not occur in them;
- $\theta(\mathbf{m}) := (\exists\ \mathbf{i}, \mathbf{n} \rhd \mathbf{j})(\theta_1(\mathbf{i})\ \&\ \theta_2(\mathbf{n}))$ or $\theta(\neg\mathbf{j}) := (\forall\ \mathbf{i}, \mathbf{n} \rhd \mathbf{j})(\theta_1(\neg\mathbf{i})\ \parr\ \theta_2(\neg\mathbf{n}))$, where $\theta_1(\mathbf{i})$ and $\theta_2(\mathbf{n})$ are Kracht disjuncts such that $\mathbf{m}$ does not occur in them;
- $\theta_1(\mathbf{w})\ \&\ \theta_2(\mathbf{w})\ \&\cdots\ \&\theta_n(\mathbf{w})$ where all the θ_i (with $1 \leq i \leq n$) are Kracht disjuncts
- $\theta_1(\mathbf{w}) \parr \theta_2(\mathbf{w}) \parr \cdots \parr \theta_n(\mathbf{w})$ where all the θ_i (with $1 \leq i \leq n$) are Kracht disjuncts.

Furthermore, in the generation trees of all the flat inequalities of $\theta(\mathbf{w})$, each nominal (resp. conominal) different from $\mathbf{w}$ occurs in negative (resp. positive) polarity if it is under the scope of an even number of universal quantifiers, the opposite otherwise.

Definition 4.4 A *Kracht antecedent* in ML^K is a an ML^K-formula $\eta(\mathbf{j}, \mathbf{m})$ which is a conjunction of inequalities of the form $\mathbf{i} \leq \mathbf{h}$ and $\mathbf{o} \leq \mathbf{n}$, plus a single negated inequality $\mathbf{j} \nleq \mathbf{m}$ called a *pivotal inequality*; the variables $\mathbf{j}$ and $\mathbf{m}$ are the *pivotal pure variables* of the antecedent.

Next we introduce the notion of a Kracht ML^K-formula. The reader might find it useful to refer to Example 4.9 while reading this definition.

Definition 4.5 [Kracht ML^K-formula] A closed ML^K-formula is *Kracht* if it is of the following shape:

$$\forall\mathbf{j}\forall\mathbf{m}\forall\overline{\mathbf{h}}\forall\overline{\mathbf{o}}\forall^R\overline{\mathbf{i}}, \overline{\mathbf{n}}(\eta(\mathbf{j}, \mathbf{m}) \Rightarrow \theta_1(\mathbf{w}_1)\ \parr\ \cdots\ \parr\ \theta_n(\mathbf{w}_n)), \tag{2}$$

where $\eta(\mathbf{j}, \mathbf{m})$ is a Kracht antecedent, each θ_i is a Kracht disjunct, and $\forall^R\overline{\mathbf{i}}, \overline{\mathbf{n}}$ denotes a sequence of restricted universal quantifiers introducing the (co)nominals in $\overline{\mathbf{i}}$ and $\overline{\mathbf{n}}$. The variables quantified in the prefix are *inherently universal* variables. The formula also has to satisfy the following conditions:

(i) Each nominal in $\overline{\mathbf{h}}$ (or, resp., conominal in $\overline{\mathbf{o}}$) must appear on the right (resp. left) hand side of exactly one non-pivotal inequality in $\eta(\mathbf{j}, \mathbf{m})$ (and nowhere else in $\eta(\mathbf{j}, \mathbf{m})$).

(ii) the non-main variables (cf. Definition 4.3) in each atom in the consequent are all inherently universal.

(iii) Quantifiers in $\forall^R\overline{\mathbf{i}}, \overline{\mathbf{n}}$ must be of either of the following types: *type 1* quantifiers bind variables occurring in the consequent, but not in the antecedent or as restrictors in the prefix; *type 2* quantifiers bind variables that occur either in the antecedent or as restrictors (exactly once) in the prefix, but not in the consequent.

Remark 4.6 Note that the pivotal inequality in the antecedent $\mathbf{j} \nleq \mathbf{m}$ is not technically necessary as it just translates to $\mathbf{j} = \neg\mathbf{m}$; hence it is possible to apply the same arguments to formulas with just a single pivotal nominal $\mathbf{j}$ and substituting every occurrence of the related pivotal conominal with $\neg\mathbf{j}$. Nevertheless, we shall keep both the pivotal variables in the definition to simplify some of the proofs in the remainder. It follows from Definitions 4.4 and 4.5(i) that the variables in $\overline{\mathbf{h}}$ and $\overline{\mathbf{o}}$ provide alternative names for either pivotal variables or restricted variables in the prefix. This is why we will sometimes refer to them as *aliases*.

Henceforth, we will refer to formulas defined in Definition 4.5 as *Kracht formulas.*

Lemma 4.7 *Every Kracht formula is equivalent to some Kracht formula where the pivotal variables do not occur in the consequent.*

Proof. Any Kracht formula has the following form

$$\forall \mathbf{j}\forall \mathbf{m}\forall \overline{\mathbf{h}}\forall \overline{\mathbf{o}}\forall^R \overline{\mathbf{i}}, \overline{\mathbf{n}}(\eta' \ \& \ \mathbf{j} \not\leq \mathbf{m} \Rightarrow \theta_1(\mathbf{w}_1) \ \parr \ \cdots \ \parr \ \theta_n(\mathbf{w}_n)),$$

and hence it can be equivalently rewritten as the following Kracht formula

$$\forall \mathbf{j}'\forall \mathbf{m}'\forall \mathbf{j}\forall \mathbf{m}\forall \overline{\mathbf{h}}\forall \overline{\mathbf{o}}\forall^R \overline{\mathbf{i}}, \overline{\mathbf{n}}(\eta' \ \& \ \mathbf{j}' \leq \mathbf{j} \ \& \ \mathbf{m} \leq \mathbf{m}' \ \& \ \mathbf{j}' \not\leq \mathbf{m}' \Rightarrow \theta_1(\mathbf{w}_1) \ \parr \ \cdots \ \parr \ \theta_n(\mathbf{w}_n)),$$

where $\mathbf{j}'$ and $\mathbf{m}'$ are fresh variables, and therefore they do not occur in the consequent. The variables $\mathbf{j}$ and $\mathbf{m}$ become part of the $\overline{\mathbf{h}}$ and $\overline{\mathbf{o}}$ respectively of the new formula. ◻

Lemma 4.8 *Any Kracht formula is equivalent to some Kracht formula such that each alias variable occurs in the consequent.*

Proof. Suppose that an alias nominal $\mathbf{h}$ (resp. conominal $\mathbf{o}$) does not occur in the consequent. By definition of Kracht formulas, it occurs exactly once in the antecedent in an inequality of shape $\mathbf{k_h} \leq \mathbf{h}$ (resp. $\mathbf{o} \leq \mathbf{l_o}$). As it does not occur in the consequent, the universal quantifier that introduces it can be rewritten as an existential quantifier in the antecedent. Now the formula $\exists \mathbf{h}(\mathbf{k_h} \leq \mathbf{h})$ (resp. $\exists \mathbf{o}(\mathbf{o} \leq \mathbf{l_o})$) is equivalent to $\top$, and, therefore it can be eliminated from the antecedent. ◻

Thanks to Lemmas 4.7 and 4.8, we will henceforth consider only Kracht formulas where the pivotal variables do not occur in the consequent and whose unrestricted non-pivotal variables occur in the consequent. We will also assume that the variables introduced by *type 1* restricted quantifier occur in the consequent, since, otherwise, the formula would be equivalent to the same formula without those quantifiers. We refer to such formulas as *refined Kracht formulas.*

Example 4.9 The following formula

$$\forall \mathbf{j}\forall \mathbf{m}\forall \mathbf{h}_1\forall \mathbf{h}_2[\mathbf{j} \leq \mathbf{h}_1 \ \& \ \mathbf{j} \leq \mathbf{h}_2 \ \& \ \mathbf{j} \not\leq \mathbf{m} \Rightarrow (\exists \mathbf{i}_1 \rhd \neg \mathbf{m})(\forall \mathbf{n}_1 \rhd \neg \mathbf{i}_1)(\forall \mathbf{n}_2 \rhd \mathbf{n}_1)(\exists \mathbf{i}_2 \blacktriangleright \neg \mathbf{n}_2)(\mathbf{i}_2 \leq \Diamond \mathbf{h}_1 \ \& \ \mathbf{i}_2 \leq \blacklozenge \mathbf{h}_2)]$$

is Kracht with pivotal variables $\mathbf{j}$ and $\mathbf{m}$, aliases $\mathbf{h}_1$ and $\mathbf{h}_2$, and a single Kracht disjunct. Indeed, in $\mathbf{i}_2 \leq \Diamond \mathbf{h}_1$ and $\mathbf{i}_2 \leq \blacklozenge \mathbf{h}_2$, $\mathbf{h}_1$ and $\mathbf{h}_2$ are inherently universal and they occur in negative polarity while being under the scope of an even number of universal quantifiers. By Lemma 4.7 and by renaming $\mathbf{m}$ to $\mathbf{o}_1$, it is equivalent to the following refined Kracht formula:

$$\forall \mathbf{j} \forall \mathbf{m} \forall \mathbf{h}_1 \forall \mathbf{h}_2 \forall \mathbf{o}_1 [\mathbf{j} \leq \mathbf{h}_1 \ \& \ \mathbf{j} \leq \mathbf{h}_2 \ \& \ \mathbf{o}_1 \leq \mathbf{m} \ \& \ \mathbf{j} \nleq \mathbf{m} \Rightarrow (\exists \mathbf{i}_1 \rhd \neg \mathbf{o}_1)(\forall \mathbf{n}_1 \rhd \neg \mathbf{i}_1)(\forall \mathbf{n}_2 \rhd \mathbf{n}_1)(\exists \mathbf{i}_2 \blacktriangleright \neg \mathbf{n}_2)(\mathbf{i}_2 \leq \Diamond \mathbf{h}_1 \ \& \ \mathbf{i}_2 \leq \blacklozenge \mathbf{h}_2)]$$

The Kracht formula

$$\forall \mathbf{j} \forall \mathbf{m} \forall \mathbf{h}_1 \forall \mathbf{h}_2 (\forall \mathbf{i}_1 \rhd \mathbf{j})(\forall \mathbf{n}_1 \rhd \mathbf{m})[\mathbf{i}_1 \leq \mathbf{h}_1 \ \& \ \mathbf{i}_1 \leq \mathbf{h}_2 \ \& \ \mathbf{j} \nleq \mathbf{m} \Rightarrow \neg \mathbf{m} \leq \mathbf{h}_2 \ \parr \ (\exists \mathbf{i}_2 \rhd \neg \mathbf{n}_1)(\forall \mathbf{n}_2 \rhd \neg \mathbf{i}_2)(\neg \mathbf{n}_2 \leq \Diamond \mathbf{h}_1)]$$

is equivalent to the following refined Kracht formula (introducing an alias for $\mathbf{m}$)

$$\forall \mathbf{j} \forall \mathbf{m} \forall \mathbf{h}_1 \forall \mathbf{h}_2 \forall \mathbf{o}_1 (\forall \mathbf{i}_1 \rhd \mathbf{j})(\forall \mathbf{n}_1 \rhd \mathbf{m})[\mathbf{i}_1 \leq \mathbf{h}_1 \ \& \ \mathbf{i}_1 \leq \mathbf{h}_2 \ \& \ \mathbf{o}_1 \leq \mathbf{m} \ \& \ \mathbf{j} \nleq \mathbf{m} \Rightarrow \neg \mathbf{o}_1 \leq \mathbf{h}_2 \ \parr \ (\exists \mathbf{i}_2 \rhd \neg \mathbf{n}_1)(\forall \mathbf{n}_2 \rhd \neg \mathbf{i}_2)(\neg \mathbf{n}_2 \leq \Diamond \mathbf{h}_1)]$$

4.3 From Kracht to very simple Sahlqvist with residuals

In this section, we introduce an algorithm that takes refined Kracht formulas as input, and, using ALBA rules, computes very simple Sahlqvist ML*-formulas of which they are first order correspondents, and which are equivalent to inductive ML-inequalities, as discussed in Section 3.

Compaction of the non-inherently universals. By exhaustively applying Ackermann eliminations and inverse splitting, a Kracht disjunct $\theta(\mathbf{w})$ is shown to be equivalent to some inequality that has $\mathbf{w}$ on display.

Algorithm 1 Compaction of a Kracht disjunct $\theta(\mathbf{w})$.

procedure DisjunctCompaction(θ)
  if θ is a flat inequality **then return** θ
  else
    Let $\{\theta_1, \ldots, \theta_n\}$ be the set of all the direct sub-disjuncts of θ
    Let $I = [I_1, \ldots, I_n]$ a list of inequalities
    for all the direct sub-disjuncts θ_i in θ **do**
      $I_i \leftarrow$ DisjunctCompaction(θ_i)
    end for
    if θ is a (dis/con)junction of disjuncts $\theta_i(\mathbf{j})$ **then**
      Let $s_1, \ldots, s_n$ be formulas such that I_i is $\mathbf{j} \leq s_i$ for $i = 1, \ldots, n$
      return $\mathbf{j} \leq s_1 \wedge \cdots \wedge s_n$ if conjunction, $\mathbf{j} \leq s_1 \vee \cdots \vee s_n$ otherwise
    else if θ is a (dis/con)junction of disjuncts $\theta_i(\mathbf{m})$ **then**
      Let $s_1, \ldots, s_n$ be formulas such that I_i is $s_i \leq \mathbf{m}$ for $i = 1, \ldots, n$
      return $s_1 \vee \cdots \vee s_n \leq \mathbf{m}$ if conjunction, $s_1 \wedge \cdots \wedge s_n \leq \mathbf{m}$ otherwise
    else if θ has form $(Q \ \mathbf{u} \ r \ \mathbf{v})(\theta_1)$ with $Q \in \{\forall, \exists\}$ and $r \in \{\rhd, \blacktriangleright\}$ **then**
      return Eliminate $\mathbf{u}$ via Inverse Approximation Rules
    else if θ has form $(\exists \mathbf{i}, \mathbf{n} \rhd \mathbf{m})(\theta_1(\mathbf{i}) \ \& \ \theta_2(\mathbf{n}))$ or $(\forall \mathbf{i}, \mathbf{n} \rhd \mathbf{m})(\theta_1(\neg \mathbf{i}) \ \parr \ \theta_2(\neg \mathbf{n}))$ **then**
      return Eliminate $\mathbf{i}$ and $\mathbf{n}$ via Inverse Approximation Rules
    end if
  end if
end procedure

Lemma 4.10 *When applied to a Kracht disjunct θ, Algorithm 1 outputs an inequality of shape $\mathbf{k} \leq s$ (resp. $s \leq \mathbf{l}$), where $\mathbf{k}$ (resp. $\mathbf{l}$) is the main pure variable of θ, and it does not occur in s.*

Proof. We proceed by induction on the structure of $\theta(\mathbf{w})$. If $\theta(\mathbf{w})$ is a flat inequality, then the statement holds by definition of Kracht disjunct. If $\theta(\mathbf{w}) :=$

$\theta_1(\mathbf{w})$ & $\cdots$ & $\theta_n(\mathbf{w})$ (resp. $\theta(\mathbf{w}) := \theta_1(\mathbf{w}) \parr \cdots \parr \theta_n(\mathbf{w})$), then the algorithm applies inverse splittingin line 11 if $\mathbf{w}$ is a nominal, line 14 if it is a conominal. As, by inductive hypothesis on each θ_i, $\mathbf{w}$ does not occur in s_i, it does not occur in $\bigwedge_i s_i$ (resp. $\bigvee_i s_i$). Assume that $\theta(\mathbf{w}) := (\exists \mathbf{u}\ r\ \mathbf{w})(\theta_1(\mathbf{u}))$ (resp. $\theta(\mathbf{w}) := (\forall \mathbf{u}\ r\ \neg\mathbf{w})(\theta_1(\neg\mathbf{u}))$), with $r \in \{\rhd, \blacktriangleright\}$, i.e. $\theta(\mathbf{w})$ is of the form $\exists\mathbf{u}(\mathbf{w} \leq f(\mathbf{u})\ \&\ \theta_1(\mathbf{u}))$ (resp. $\forall\mathbf{u}(\neg\mathbf{w} \leq f(\mathbf{u}) \Rightarrow \theta_1(\neg\mathbf{u}))$) if $\mathbf{u}$ is a nominal, or as $\exists\mathbf{u}(g(\mathbf{u}) \leq \mathbf{w}\ \&\ \theta_1(\mathbf{u}))$ (resp. $\forall\mathbf{u}(g(\mathbf{u}) \leq \neg\mathbf{w} \Rightarrow \theta_1(\neg\mathbf{u}))$) if it is a conominal, where $f \in \{\Diamond, \blacklozenge\}$ (resp. $g \in \{\Box, \blacksquare\}$). In each such case, the induction hypothesis on θ_1 ensures that $\theta(\mathbf{w})$ is in Ackermann shape w.r.t. $\mathbf{u}$, which can then be eliminated by applying the Ackermann rule. The cases $\theta(\mathbf{w}) := (\exists \mathbf{i}, \mathbf{n} \rhd \mathbf{w})(\theta_1(\mathbf{i})\ \&\ \theta_2(\mathbf{n}))$ and $\theta(\mathbf{w}) := (\forall \mathbf{i}, \mathbf{n} \rhd \neg\mathbf{w})(\theta_1(\neg\mathbf{i}) \parr \theta_2(\neg\mathbf{n}))$ are treated analogously by eliminating $\mathbf{i}$ and $\mathbf{n}$. ◻

Lemma 4.11 *When applied to some Kracht disjunct $\theta(\mathbf{w})$, Algorithm 1 produces an inequality where the nominals (resp. conominals) different from $\mathbf{w}$ occur in negative (resp. positive) polarity.*

Proof. By definition of Kracht disjunct, the polarity of its non-main variables depends on the number of universal quantifiers under which they are nested. Indeed, polarities are preserved by applications of existential inverse approximation and inverse splitting rules, and are reversed by applications of the universal inverse approximation rule. Thus, in the end nominals (resp. conominals) must occur in negative (resp. positive) polarity. ◻

Example 4.12 Let us apply Algorithm 1 to the consequent of the formulas in Example 4.9. The first one is $(\exists \mathbf{i}_1 \rhd \neg\mathbf{o}_1)(\forall \mathbf{n}_1 \rhd \neg\mathbf{i}_1)(\forall \mathbf{n}_2 \rhd \mathbf{n}_1)(\exists \mathbf{i}_2 \blacktriangleright \mathbf{n}_2)(\mathbf{i}_2 \leq \Diamond\mathbf{h}_1\ \&\ \mathbf{i}_2 \leq \blacklozenge\mathbf{h}_2)$. The innermost Kracht disjunct is a conjunction sharing the same main variable, thus we can compact it into one inequality. We then proceed to eliminate the restricted quantifiers: working from the inside out, we first expand them according to their definitions (for the sake of clarity) and then apply appropriate inverse approximation rules:

$$
\begin{array}{lll}
 & (\exists \mathbf{i}_1 \rhd \neg\mathbf{o}_1)(\forall \mathbf{n}_1 \rhd \neg\mathbf{i}_1)(\forall \mathbf{n}_2 \rhd \mathbf{n}_1)(\exists \mathbf{i}_2 \blacktriangleright \neg\mathbf{n}_2)(\mathbf{i}_2 \leq \Diamond\mathbf{h}_1 \wedge \blacklozenge\mathbf{h}_2) & \\
\text{iff} & (\exists \mathbf{i}_1 \rhd \neg\mathbf{o}_1)(\forall \mathbf{n}_1 \rhd \neg\mathbf{i}_1)(\forall \mathbf{n}_2 \rhd \mathbf{n}_1)\exists \mathbf{i}_2(\neg\mathbf{n}_2 \leq \blacklozenge\mathbf{i}_2\ \&\ \mathbf{i}_2 \leq \Diamond\mathbf{h}_1 \wedge \blacklozenge\mathbf{h}_2) & \text{Inv. Splitting} \\
\text{iff} & (\exists \mathbf{i}_1 \rhd \neg\mathbf{o}_1)(\forall \mathbf{n}_1 \rhd \neg\mathbf{i}_1)(\forall \mathbf{n}_2 \rhd \mathbf{n}_1)(\neg\mathbf{n}_2 \leq \blacklozenge(\Diamond\mathbf{h}_1 \wedge \blacklozenge\mathbf{h}_2)) & \text{Inv. Approx.} \\
\text{iff} & (\exists \mathbf{i}_1 \rhd \neg\mathbf{o}_1)(\forall \mathbf{n}_1 \rhd \neg\mathbf{i}_1)\forall \mathbf{n}_2(\Box\mathbf{n}_2 \leq \mathbf{n}_1 \implies \neg\mathbf{n}_2 \leq \blacklozenge(\Diamond\mathbf{h}_1 \wedge \blacklozenge\mathbf{h}_2)) & \\
\text{iff} & (\exists \mathbf{i}_1 \rhd \neg\mathbf{o}_1)(\forall \mathbf{n}_1 \rhd \neg\mathbf{i}_1)\forall \mathbf{n}_2(\blacklozenge(\Diamond\mathbf{h}_1 \wedge \blacklozenge\mathbf{h}_2) \leq \mathbf{n}_2 \implies \neg\mathbf{n}_1 \leq \Box\mathbf{n}_2) & \text{Contrapositive} \\
\text{iff} & (\exists \mathbf{i}_1 \rhd \neg\mathbf{o}_1)(\forall \mathbf{n}_1 \rhd \neg\mathbf{i}_1)(\neg\mathbf{n}_1 \leq \Box\blacklozenge(\Diamond\mathbf{h}_1 \wedge \blacklozenge\mathbf{h}_2)) & \text{Inv. Approx.} \\
\text{iff} & (\exists \mathbf{i}_1 \rhd \neg\mathbf{o}_1)\forall \mathbf{n}_1(\Box\mathbf{n}_1 \leq \neg\mathbf{i}_1 \implies \neg\mathbf{n}_1 \leq \Box\blacklozenge(\Diamond\mathbf{h}_1 \wedge \blacklozenge\mathbf{h}_2)) & \\
\text{iff} & (\exists \mathbf{i}_1 \rhd \neg\mathbf{o}_1)\forall \mathbf{n}_1(\Box\blacklozenge(\Diamond\mathbf{h}_1 \wedge \blacklozenge\mathbf{h}_2) \leq \mathbf{n}_1 \implies \mathbf{i}_1 \leq \Box\mathbf{n}_1) & \text{Contapositive} \\
\text{iff} & (\exists \mathbf{i}_1 \rhd \neg\mathbf{o}_1)(\mathbf{i}_1 \leq \Box\Box\blacklozenge(\Diamond\mathbf{h}_1 \wedge \blacklozenge\mathbf{h}_2)) & \text{Inv. Approx} \\
\text{iff} & \exists \mathbf{i}_1(\neg\mathbf{o}_1 \leq \Diamond\mathbf{i}_1\ \&\ \mathbf{i}_1 \leq \Box\Box\blacklozenge(\Diamond\mathbf{h}_1 \wedge \blacklozenge\mathbf{h}_2)) & \\
\text{iff} & \neg\mathbf{o}_1 \leq \Diamond\Box\Box\blacklozenge(\Diamond\mathbf{h}_1 \wedge \blacklozenge\mathbf{h}_2) & \text{Inv. Approx.}
\end{array}
$$

Now for the consequent of the second formula in Example 4.9, which is

$$\neg\mathbf{o}_1 \leq \mathbf{j}_2 \parr (\exists \mathbf{i}_2 \rhd \neg\mathbf{n}_1)(\forall \mathbf{n}_2 \rhd \neg\mathbf{i}_2)(\neg\mathbf{n}_2 \leq \Diamond\mathbf{j}_1).$$

For the sake of brevity, we will not write out the expansion of bounded quantifiers and contrapositive steps for this example. The first Kracht disjunct is already flat, in the second Kracht disjunct the algorithm yields

$$(\exists \mathbf{i}_2 \rhd \neg \mathbf{n}_1)(\forall \mathbf{n}_2 \rhd \neg \mathbf{i}_2)(\neg \mathbf{n}_2 \leq \Diamond \mathbf{j}_1) \quad \text{iff} \quad (\exists \mathbf{i}_2 \rhd \neg \mathbf{n}_1)(\mathbf{i}_2 \leq \Box \Diamond \mathbf{j}_1) \quad \text{iff} \quad \neg \mathbf{n}_1 \leq \Diamond \Box \Diamond \mathbf{j}_1 \,.$$

Compaction of the antecedent. After compacting the consequent, the (refined) input formula (2) has the following shape:

$$\forall \mathbf{j} \forall \mathbf{m} \forall \overline{\mathbf{h}} \forall \overline{\mathbf{o}} \forall^R \overline{\mathbf{i}}, \overline{\mathbf{n}} \left(\eta(\mathbf{j}, \mathbf{m}) \Rightarrow \bigparr (\overline{\mathbf{k} \leq \delta} \ \parr \ \overline{\gamma \leq \mathbf{l}}) \right), \tag{3}$$

where each $\mathbf{k}$ (resp. $\mathbf{l}$) is either an alias variable, or is bound by some *type 1* quantifier. Furthermore, as the input formula is refined, each alias and each *type 1* variable occurs at least once in the consequent. Let us abbreviate $\mathrm{SUCC} \coloneqq \bigparr(\overline{\mathbf{k} \leq \delta} \ \parr \ \overline{\gamma \leq \mathbf{l}})$.

Each restricted quantifier binds one nominal $\mathbf{i}$, one conominal $\mathbf{n}$, or one nominal and one conominal; in each case, the quantifier comes equipped with a restricting inequality in the antecedent, which has shape $\mathbf{k_i} \leq \Diamond \mathbf{i}$ in the first case, $\Box \mathbf{n} \leq \mathbf{l_n}$ in the second case, and $\mathbf{i} \to \mathbf{n} \leq \mathbf{l_n}$ or $\mathbf{k_i} \leq \mathbf{i} \succ\!\!- \mathbf{n}$ in the third case, for some nominal $\mathbf{k_i}$ and conominal $\mathbf{l_n}$. By currying, for any formula σ,

$$(\forall \mathbf{i} \rhd \mathbf{k_i})(\sigma \Rightarrow \mathrm{SUCC}) \quad \text{i.e.} \quad \forall \mathbf{i}(\mathbf{k_i} \leq \Diamond \mathbf{i} \Rightarrow (\sigma \Rightarrow \mathrm{SUCC})) \quad \text{iff} \quad \forall \mathbf{i}((\mathbf{k_i} \leq \Diamond \mathbf{i} \ \& \ \sigma) \Rightarrow \mathrm{SUCC}),$$

and similarly for the other two cases. Let us apply this procedure exhaustively, so to rewrite the antecedent of (3) by conjoining it with all the restricting inequalities of *type 2* quantifiers and of *type 1* quantifiers restricted by variables bound by *type 2* quantifiers. The next lemma shows that all *type 2* quantifiers can be eliminated by proceeding from the rightmost to the leftmost via an inverse approximation rule. We suggest that the reader glance at Example 4.16 while reading the following two lemmas.

Lemma 4.13 *After exhaustively currying, the antecedent of* (3) *is in the right shape for the elimination of the rightmost quantifier via inverse approximation, and, after the elimination, it is again in the right shape for the elimination of the successive quantifiers.*

Proof. After currying, the variables $\mathbf{i}$ and/or $\mathbf{n}$ bound by a a *type 2* quantifier can either occur in the antecedent in inequalities $\mathbf{i} \leq \mathbf{h}$ (resp. $\mathbf{o} \leq \mathbf{n}$) for some alias variable $\mathbf{h}$ (resp. $\mathbf{o}$), or in restricting inequalities. Notice that $\mathbf{i}$ (resp. $\mathbf{n}$) occurs negatively (resp. positively) only in the restricting inequality of the quantifier that binds it (let us call it I_1), and occurs in the opposite polarity in any other restricting inequality where it is a restrictor. Therefore, we can merge via inverse splitting all the inequalities involving aliases and the restricting inequalities where $\mathbf{i}$ (resp. $\mathbf{n}$) occur as restrictor, thus obtaining an inequality I_2. When eliminating the rightmost quantifier, we only have to consider two inequalities in the antecedent, I_1 and I_2, which are in Ackermann shape for the elimination of $\mathbf{i}$ and/or $\mathbf{n}$ (considering that they cannot occur in the consequent). The variable $\mathbf{u}$ on display in the resulting inequality is the restrictor of I_1, and, if it is a nominal (resp. conominal) it occurs on the left (resp. right) hand side of the inequality; furthermore it occurs only once in the inequality. The variable $\mathbf{u}$ can either be a pivotal variable or a variable bound by another *type 2* quantifiers. In the latter case, $\mathbf{u}$ will be eliminated in a later stage by repeating the same procedure. At that stage, this inequality will be merged via inverse splitting with the ones where $\mathbf{u}$ is on display on the left (resp. right) hand side if it is a nominal (resp. conominal). □

After eliminating all the variables bound by *type 2* quantifiers, the shape of the antecedent reduces to the inequality $\mathbf{j} \not\leq \mathbf{m}$ in conjunction with inequalities of the form $\mathbf{j} \leq \theta$ or $\eta \leq \mathbf{m}$, and, moreover, the remaining *type 1* quantifiers can only be restricted by $\mathbf{j}$ and $\mathbf{m}$. Hence, by expanding these remaining quantifiers, exhaustively currying, and applying inverse splitting, the antecedent equivalently reduces to the following conjunction of inequalities $\mathbf{j} \leq \theta_1 \wedge \cdots \wedge \theta_n$ & $\eta_1 \vee \cdots \vee \eta_m \leq \mathbf{m}$ & $\mathbf{j} \not\leq \mathbf{m}$.

Lemma 4.14 *After the elimination of* type 2 *restricted quantifiers and the expansion of* type 1 *quantifiers, the antecedent has form*

$$\mathbf{j} \leq \bigwedge_{i=1}^{n} \theta_i \;\;\&\;\; \bigvee_{i=1}^{m} \eta_i \leq \mathbf{m} \;\&\;\; \mathbf{j} \not\leq \mathbf{m}, \tag{4}$$

where $+\bigwedge_{i=1}^{n} \theta_i$ *and* $-\bigwedge_{i=1}^{m} \eta_i$ *are pure scattered Skeleton formulas where* $\mathbf{j}$ *and* $\mathbf{m}$ *do not occur, and any nominal (resp. conominal) occurs in positive (resp. negative) polarity.*

Proof. It is sufficient to show that every $+\theta_i$ and $-\eta_i$ is made of Skeleton nodes and that each variable occurs only once, since $\mathbf{j}$ and $\mathbf{m}$ clearly cannot occur there as they are not in $\overline{\mathbf{h}}$ or $\overline{\mathbf{o}}$ and they are not even restricted variables. The conjuncts that come from the *type 1* restricted quantifiers clearly satisfy the statement, hence it remains to show that the algorithm for the elimination of *type 2* quantifiers produces conjuncts with the same property. We proceed by induction. Let us consider the case in which we eliminate a quantifier of the kind $(\forall \mathbf{i}, \mathbf{n} \rhd \mathbf{l})$. Before the inverse approximation of an iteration is performed, we have a restricting inequality $\mathbf{i} \to \mathbf{n} \leq \mathbf{l}$ and the two inequalities $\mathbf{i} \leq \varphi$ and $\psi \leq \mathbf{n}$. In the base case, the last two are either restricting inequalities or inequalities without connectives nor operators; in both the cases $+\varphi$ and $-\psi$ are skeleton nodes and each variable occurs only once, as the pure variables in $\overline{\mathbf{h}}$ and $\overline{\mathbf{o}}$ occur only once in the antecedent and the variables in a restricting inequalities of a restricted quantifier are all different. Applying inverse approximation we have $\varphi \to \psi \leq \mathbf{l}$, and clearly $-(\varphi \to \psi)$ is a scattered Skeleton formula where the constraints on the polarity of the variables are met. In the inductive case, we proceed in the same way noting that, by applying inductive hypothesis, $\mathbf{i} \leq \varphi$ or $\psi \leq \mathbf{n}$ are scattered Skeleton formulas that meet the constraints on the polarities; hence, as in the base case, also the result of inverse approximation meets the same requirements. The remaining cases are proved similarly. □

Remark 4.15 The variables occurring $+\bigwedge_{i=1}^{n} \theta_i$ and $-\bigwedge_{i=1}^{m} \eta_i$ are exactly all the ones in $\overline{\mathbf{h}}$, $\overline{\mathbf{o}}$ and the ones bound by *type 1* quantifiers. The former variables are indeed captured because each one of them occurs at least (exactly) once in the antecedent, whilst the latter variables are clearly captured by writing the expansion of the quantifier.

Example 4.16 When treating the antecedent of the first formula in Example 4.9 as further processed in Example 4.12, we do not have inherently universal restricted quantifiers; hence this step consists in a straightforward application of the inverse splitting rule

$$\begin{array}{l} \forall \mathbf{j}\forall \mathbf{m}\forall \mathbf{h}_1\forall \mathbf{h}_2\forall \mathbf{o}_1[\mathbf{j} \leq \mathbf{h}_1 \ \&\ \mathbf{j} \leq \mathbf{h}_2 \ \&\ \mathbf{o}_1 \leq \mathbf{m} \ \&\ \mathbf{j} \nleq \mathbf{m} \Rightarrow \neg \mathbf{o}_1 \leq \Box\Box\blacklozenge(\Diamond \mathbf{h}_1 \wedge \blacklozenge \mathbf{h}_2)] \\ \text{iff } \forall \mathbf{j}\forall \mathbf{m}\forall \mathbf{h}_1\forall \mathbf{h}_2\forall \mathbf{o}_1[\mathbf{j} \leq \mathbf{h}_1 \wedge \mathbf{h}_2 \ \&\ \mathbf{o}_1 \leq \mathbf{m} \ \&\ \mathbf{j} \nleq \mathbf{m} \Rightarrow \neg \mathbf{o}_1 \leq \Box\Box\blacklozenge(\Diamond \mathbf{h}_1 \wedge \blacklozenge \mathbf{h}_2)] \end{array}$$

As for the second formula from Examples 4.9 and 4.12, namely

$$\begin{array}{r} \forall \mathbf{j}\forall \mathbf{m}\forall \mathbf{h}_1\forall \mathbf{h}_2\forall \mathbf{o}_1(\forall \mathbf{i}_1 \rhd \mathbf{j})(\forall \mathbf{n}_1 \rhd \mathbf{m})[\mathbf{i}_1 \leq \mathbf{h}_1 \ \&\ \mathbf{i}_1 \leq \mathbf{h}_2 \ \&\ \mathbf{o}_1 \leq \mathbf{m} \ \&\ \mathbf{j} \nleq \mathbf{m} \Rightarrow \\ \neg \mathbf{o}_1 \leq \mathbf{h}_2 \ \parr\ \neg \mathbf{n}_1 \leq \Diamond\Box\Diamond \mathbf{h}_1], \end{array}$$

the quantifier $(\forall \mathbf{n}_1 \rhd \mathbf{m})$ is of *type 1*, while $(\forall \mathbf{i}_1 \rhd \mathbf{j})$ is of *type 2*. We start by eliminating the latter and then we merge the inequalities of the antecedent with the one of the restricted quantifier of *type 1*.

$$\begin{array}{l} \forall \mathbf{j}\forall \mathbf{m}\forall \mathbf{h}_1\forall \mathbf{h}_2\forall \mathbf{o}_1\forall \mathbf{i}_1\forall \mathbf{n}_1[\mathbf{i}_1 \leq \mathbf{h}_1 \ \&\ \mathbf{i}_1 \leq \mathbf{h}_2 \ \&\ \mathbf{j} \leq \Diamond \mathbf{i}_1 \ \&\ \mathbf{o}_1 \leq \mathbf{m} \ \&\ \Box \mathbf{n}_1 \leq \mathbf{m} \ \&\ \mathbf{j} \nleq \mathbf{m} \Rightarrow \\ \hfill \neg \mathbf{o}_1 \leq \mathbf{h}_2 \ \parr\ \neg \mathbf{n}_1 \leq \Diamond\Box\Diamond \mathbf{h}_1] \\ \text{iff } \forall \mathbf{j}\forall \mathbf{m}\forall \mathbf{h}_1\forall \mathbf{h}_2\forall \mathbf{o}_1\forall \mathbf{i}_1\forall \mathbf{n}_1[\mathbf{i}_1 \leq \mathbf{h}_1 \wedge \mathbf{h}_2 \ \&\ \mathbf{j} \leq \Diamond \mathbf{i}_1 \ \&\ \mathbf{o}_1 \leq \mathbf{m} \ \&\ \Box \mathbf{n}_1 \leq \mathbf{m} \ \&\ \mathbf{j} \nleq \mathbf{m} \Rightarrow \\ \hfill \neg \mathbf{o}_1 \leq \mathbf{h}_2 \ \parr\ \neg \mathbf{n}_1 \leq \Diamond\Box\Diamond \mathbf{h}_1] \\ \text{iff } \forall \mathbf{j}\forall \mathbf{m}\forall \mathbf{h}_1\forall \mathbf{h}_2\forall \mathbf{o}_1\forall \mathbf{n}_1[\mathbf{j} \leq \Diamond(\mathbf{h}_1 \wedge \mathbf{h}_2) \ \&\ \mathbf{o}_1 \leq \mathbf{m} \ \&\ \Box \mathbf{n}_1 \leq \mathbf{m} \ \&\ \mathbf{j} \nleq \mathbf{m} \Rightarrow \\ \hfill \neg \mathbf{o}_1 \leq \mathbf{h}_2 \ \parr\ \neg \mathbf{n}_1 \leq \Diamond\Box\Diamond \mathbf{h}_1] \\ \text{iff } \forall \mathbf{j}\forall \mathbf{m}\forall \mathbf{h}_1\forall \mathbf{h}_2\forall \mathbf{o}_1\forall \mathbf{n}_1[\mathbf{j} \leq \Diamond(\mathbf{h}_1 \wedge \mathbf{h}_2) \ \&\ \mathbf{o}_1 \vee \Box \mathbf{n}_1 \leq \mathbf{m} \ \&\ \mathbf{j} \nleq \mathbf{m} \Rightarrow \\ \hfill \neg \mathbf{o}_1 \leq \mathbf{h}_2 \ \parr\ \neg \mathbf{n}_1 \leq \Diamond\Box\Diamond \mathbf{h}_1] \end{array}$$

Elimination of pivotal variables. After the elimination of *type 2* quantifiers, the contrapositive of the formula is

$$\forall \mathbf{j}\forall \mathbf{m}\forall \overline{\mathbf{h}}\forall \overline{\mathbf{o}}\forall \overline{\mathbf{i}}'\forall \overline{\mathbf{n}}' \left(\overline{\delta \leq \neg \mathbf{k}} \ \&\ \overline{\neg \mathbf{l} \leq \gamma} \Rightarrow (\mathbf{j} \leq \textstyle\bigwedge_{i=1}^{n} \theta_i \ \ \&\ \ \bigvee_{i=1}^{m} \eta_i \leq \mathbf{m} \Rightarrow \ \mathbf{j} \leq \mathbf{m})\right),$$

where $\overline{\mathbf{i}}'$ and $\overline{\mathbf{n}}'$ are the variables originally introduced by *type 1* restricted quantifiers. By applying inverse approximation to eliminate $\mathbf{j}$ and $\mathbf{m}$, and by putting $\varphi \coloneqq \bigwedge_{i=1}^{n} \theta_i$ and $\psi \coloneqq \bigwedge_{i=1}^{m} \eta_i$, the formula above is equivalent to

$$\forall \overline{\mathbf{h}}\forall \overline{\mathbf{o}}\forall \overline{\mathbf{i}}'\forall \overline{\mathbf{n}}' \left(\overline{\delta \leq \neg \mathbf{k}} \ \&\ \overline{\neg \mathbf{l} \leq \gamma} \Rightarrow \varphi \leq \psi\right).$$

If one of the literals in $\overline{\mathbf{k}}$ (resp. in $\overline{\mathbf{l}}$) is already negated, namely it is of the form $\neg \mathbf{i}$ (resp. $\neg \mathbf{n}$) for some nominal $\mathbf{i}$ (resp. conominal $\mathbf{n}$), we can just apply self adjunction of negation to obtain a formula $\mathbf{i} \leq \theta$. Hence, we apply this procedure obtaining a formula with shape

$$\forall \overline{\mathbf{h}}\forall \overline{\mathbf{o}}\forall \overline{\mathbf{i}'}\forall \overline{\mathbf{n}'} \left(\overline{\mathbf{k}' \leq \gamma'} \ \&\ \overline{\delta' \leq \mathbf{l}'} \Rightarrow \varphi \leq \psi\right). \tag{5}$$

We can assume that each variable in $\overline{\mathbf{k}'}$ (resp. $\overline{\mathbf{l}'}$) is different, since if it occurs in more than one inequality, these two inequalities can be merged via inverse splitting.

Very simple Sahlqvist in ML*. To simplify notation, we drop the apostrophe in $\overline{\mathbf{k}'}$ and $\overline{\mathbf{l}'}$, and we let $\overline{\mathbf{i}}$ (resp. $\overline{\mathbf{n}}$) denote all the other nominals (resp. conominals), i.e. the ones occurring in $\overline{\gamma}$ and $\overline{\delta}$. The formula (5) is thus equivalent to:

$$\forall \overline{\mathbf{k}}\forall \overline{\mathbf{l}}\forall \overline{\mathbf{i}}\forall \overline{\mathbf{n}} \left(\overline{\mathbf{k} \leq \gamma} \ \&\ \overline{\delta \leq \mathbf{l}} \Rightarrow \varphi \leq \psi\right). \tag{6}$$

By Lemma 4.14, we know that the inequality $\varphi \leq \psi$ is a scattered Skeleton inequality containing every variable quantified in the prefix. Furthermore, each nominal in $\bar{\mathbf{i}}$ and $\overline{\mathbf{k}}$ occurs in positive polarity in it, and each conominal in $\overline{\mathbf{n}}$ and $\bar{\mathbf{l}}$ occurs in negative polarity; hence we may eliminate of each $\mathbf{k}$ and $\mathbf{l}$ through inverse approximation (see [5, Section 4.2]). Therefore (6) is equivalent to $\forall\bar{\mathbf{i}}\forall\overline{\mathbf{n}}\left(\varphi[\overline{\gamma}/\overline{\mathbf{k}},\overline{\delta}/\bar{\mathbf{l}}] \leq \psi[\overline{\gamma}/\overline{\mathbf{k}},\overline{\delta}/\bar{\mathbf{l}}]\right)$. For each (co)nominal in $\mathbf{i}$ (resp. in $\mathbf{n}$) we introduce a new variable $p_{\mathbf{i}}$ (resp. $q_{\mathbf{n}}$). Let

$$\varphi' := \left(\varphi\left[\overline{\gamma}/\overline{\mathbf{k}},\overline{\delta}/\bar{\mathbf{l}}\right]\right)\left[\overline{p_{\mathbf{i}}}/\bar{\mathbf{i}},\overline{q_{\mathbf{n}}}/\overline{\mathbf{n}}\right] \qquad \psi' := \left(\psi\left[\overline{\gamma}/\overline{\mathbf{k}},\overline{\delta}/\bar{\mathbf{l}}\right]\right)\left[\overline{p_{\mathbf{i}}}/\bar{\mathbf{i}},\overline{q_{\mathbf{n}}}/\overline{\mathbf{n}}\right].$$

By Lemma 4.11, nominals (resp. conominals) in each $+\gamma$ in $\overline{\gamma}$ and $-\delta$ in $\overline{\delta}$ occur in negative (resp. positive) polarity; hence every $+\gamma$ and $-\delta$ is an ε^{∂}-uniform subtree in $+\varphi'$ and $-\psi'$, where ε is the order type on $\overline{p}_{\mathbf{i}}$ and $\overline{q}_{\mathbf{n}}$ such that $\varepsilon(p_{\mathbf{i}}) = 1$ and $\varepsilon(q_{\mathbf{n}}) = \partial$.

Hence, $\varphi' \leq \psi'$ is a scattered very simple ε-Sahlqvist inequality in ML*, and, moreover, ALBA reduces it to (6), as shown below $\forall\overline{p}_{\mathbf{i}}\forall\overline{q}_{\mathbf{n}}\,(\varphi' \leq \psi')$ is equivalent to

$$\forall\overline{p}_{\mathbf{i}}\forall q_{\mathbf{n}}\forall\bar{\mathbf{j}}\forall\overline{\mathbf{m}}\forall\bar{\mathbf{j}}'\forall\overline{\mathbf{m}}'\Big(\bar{\mathbf{j}} \leq \overline{\gamma}[\overline{p}_{\mathbf{i}}/\bar{\mathbf{i}},\overline{q}_{\mathbf{n}}/\overline{\mathbf{n}}] \;\&\; \overline{\delta}[\overline{p}_{\mathbf{i}}/\bar{\mathbf{i}},\overline{q}_{\mathbf{n}}/\overline{\mathbf{n}}] \leq \overline{\mathbf{m}} \;\&\; \bar{\mathbf{j}}' \leq \overline{p}_{\mathbf{i}} \;\&\; \overline{p}_{\mathbf{n}} \leq \mathbf{m}' \\ \Rightarrow \varphi[\bar{\mathbf{j}}/\overline{\mathbf{k}},\overline{\mathbf{m}}/\bar{\mathbf{l}}] \leq \psi[\bar{\mathbf{j}}/\overline{\mathbf{k}},\overline{\mathbf{m}}/\bar{\mathbf{l}}]\Big),$$

which is equivalent to

$$\forall\bar{\mathbf{j}}\forall\overline{\mathbf{m}}\forall\bar{\mathbf{j}}'\forall\overline{\mathbf{m}}'\Big(\bar{\mathbf{j}} \leq \overline{\gamma}[\bar{\mathbf{j}}'/\bar{\mathbf{i}},\overline{\mathbf{m}}'/\overline{\mathbf{n}}] \;\&\; \overline{\delta}[\bar{\mathbf{j}}'/\bar{\mathbf{i}},\overline{\mathbf{m}}'/\overline{\mathbf{n}}] \leq \overline{\mathbf{m}} \Rightarrow \varphi[\bar{\mathbf{j}}/\overline{\mathbf{k}},\overline{\mathbf{m}}/\bar{\mathbf{l}}] \leq \psi[\bar{\mathbf{j}}/\overline{\mathbf{k}},\overline{\mathbf{m}}/\bar{\mathbf{l}}]\Big)$$

From the above discussion, the main result follows.

Theorem 4.17 *Every (refined) Kracht* ML^K *formula can be effectively associated with a scattered very simple Sahlqvist inequality in* ML* *to which it is equivalent on Kripke frames/complete and atomic BAOs.*

Example 4.18 In Example 4.16 we had

$$\forall\mathbf{j}\forall\mathbf{m}\forall\mathbf{h}_1\forall\mathbf{h}_2\forall\mathbf{o}_1[\mathbf{j} \leq \mathbf{h}_1 \wedge \mathbf{h}_2 \;\&\; \mathbf{o}_1 \leq \mathbf{m} \;\&\; \mathbf{j} \not\leq \mathbf{m}_1 \Rightarrow \neg\mathbf{o}_1 \leq \Box\Box\blacklozenge(\Diamond\mathbf{h}_1 \wedge \blacklozenge\mathbf{h}_2)].$$

After the contrapositive step it becomes $\forall\mathbf{h}_1\forall\mathbf{h}_2\forall\mathbf{o}_1[\Box\Box\blacklozenge(\Diamond\mathbf{h}_1 \wedge \blacklozenge\mathbf{h}_2) \leq \mathbf{o}_1 \Rightarrow \mathbf{h}_1 \wedge \mathbf{h}_2 \leq \mathbf{o}_1]$, which, by the previous discussion, is equivalent to the very simple Sahlqvist formula $\forall p_{h_1}\forall p_{h_2}[p_{h_1} \wedge p_{h_2} \leq \Box\Box\blacklozenge(\Diamond p_{h_1} \wedge \blacklozenge p_{h_2})]$.

Taking the second formula in Example 4.16, i.e.

$$\forall\mathbf{j}\forall\mathbf{m}\forall\mathbf{h}_1\forall\mathbf{h}_2\forall\mathbf{o}_1\forall\mathbf{n}_1[\mathbf{j} \leq \Diamond(\mathbf{h}_1 \wedge \mathbf{h}_2) \;\&\; \mathbf{o}_1 \vee \Box\mathbf{n}_1 \leq \mathbf{m} \;\&\; \mathbf{j} \not\leq \mathbf{m} \Rightarrow \\ \neg\mathbf{o}_1 \leq \mathbf{h}_2 \;⅋\; \neg\mathbf{n}_1 \leq \Diamond\Box\Diamond\mathbf{h}_1],$$

after the contrapositive step we obtain $\forall\mathbf{h}_1\forall\mathbf{h}_2\forall\mathbf{o}_1\forall\mathbf{n}_1[\mathbf{h}_2 \leq \mathbf{o}_1 \;\&\; \Diamond\Box\Diamond\mathbf{h}_1 \leq \mathbf{n}_1 \Rightarrow \Diamond(\mathbf{h}_1 \wedge \mathbf{h}_2) \leq \mathbf{o}_1 \vee \Box\mathbf{n}_1]$, which in turn is equivalent to the very simple Sahlqvist $\forall p_{h_1}\forall q_{o_1}[\Diamond(p_{h_1} \wedge q_{o_1}) \leq q_{o_1} \vee \Box\Diamond\Box\Diamond p_{h_1}]$.

From inductive inequalities to Kracht ML^K formulas. By applying an ALBA inductive formula, taking the contrapositive of the resulting pure quasi-inequality, and then applying approximation rules to obtain flat inequalities in the whole formula, the following result can be proved.

Theorem 4.19 *Every inductive inequality is equivalent to some Kracht* ML^K *formula.*

From Kracht ML^K to inductive. As, by Lemma 3.6 and Proposition 3.4, the class of inductive formulas is equivalent to the one of crypto ML^*-inductive, it is sufficient to restrict ourselves to the class of Kracht ML^K-formulas which correspond to crypto-inductive formulas. To do so, it is sufficient to note that the only condition to enforce is that the Kracht ML-disjuncts starting with an operator in ML^* are reduced (by Algorithm 1) to inequalities whose non-main side is an (ε, Ω)-unpackable formula for some ε and Ω. This is easily achieved by imposing the same restrictions as in Definition 3.2 to the operators in the restricted quantifiers of the branch.

5 Conclusion

We have established an inverse correspondence result between the Sahlqvist formulas in tense logic and the inductive formulas in modal logic on the one hand, and a class of quantified pure hybrid tense formulas on the other. The order-theoretic perspective we have introduced in the present paper lays the groundwork for a generalization of Kracht's and Kikot's inverse correspondence theory to general logics algebraically captured by classes of (distributive) lattice expansions or (D)LE-logics (see [3]). One of the main advantages of the latter is that it is a general, modular result which is largely independent of any particular choice of relational semantics for a (D)LE logic but links to such particular choices (and thence to *first-order* inverse correspondence) via duality and the accompanying standard translation.

In closing, we will mention only one of the various further directions that remain to be developed in this line of research: We have focused on correspondence between first-order formulas in one free variable and modal formula, but one may generalize this to *n-correspondence* between first-order formulas in n free variables and n-tuples of modal formulas [18]. Inverse n-correspondence is studied in [15], and a characterization is obtained of the first-order formulas built from relational atoms using conjunction and existential quantification for which modal n-correspondents exist. This problem is also considered in [8] from the perspective of description logics and for, what amounts to, a more general class of first-order formulas. In future work we will generalize and apply the methods of the present paper to the problem of n-correspondence for general DLE-languages.

Appendix

A The rules of ALBA

The algorithm ALBA applies a set of invertable re-write rules to transform (while maintaining equivalence on algebras and frames) quasi-inequalities (of formulas in LE-languages, like ML^+) into sets of pure quasi-inequalities, i.e. ones in which all propositional variables have been eliminated in favour of nominals and co-nominals. We refer the reader to [5] and [6] for the full specification of ALBA including its rules. Because they are of particular importance to the present paper, we here recall only the inverses of the splitting and approximation rules:

Inverse splitting rules.

$$\frac{\alpha \leq \beta \wedge \gamma}{\alpha \leq \beta \quad \alpha \leq \gamma} \qquad \frac{\alpha \vee \beta \leq \gamma}{\alpha \leq \gamma \quad \beta \leq \gamma}$$

Inverse approximation rules. The following are special cases of the general approximation rules given in [5] and [6]. Let Γ be an arbitrary conjunction of ML^+ inequalities and $\phi, \psi \in \mathrm{ML}^+$ such that, in each of the following rules, the quantified nominal or co-nominal does not in occur in the conclusion, and $\diamondsuit \in \{\Diamond, \blacklozenge\}$ and $\boxdot \in \{\Box, \blacksquare\}$:

$$\frac{\exists \mathbf{i}(\mathbf{j} \leq \diamondsuit\ \mathbf{i}\ \&\ \mathbf{i} \leq \phi\ \&\ \Gamma)}{\mathbf{j} \leq \diamondsuit\ \phi\ \&\ \Gamma} \qquad \frac{\exists \mathbf{n}(\boxdot \mathbf{n} \leq \mathbf{m}\ \&\ \phi \leq \mathbf{n}\ \&\ \Gamma)}{\boxdot \phi \leq \mathbf{m}\ \&\ \Gamma}$$

$$\frac{\forall \mathbf{i}(\mathbf{i} \leq \phi\ \&\ \Gamma \Rightarrow \diamondsuit\ \mathbf{i} \leq \psi)}{\Gamma \Rightarrow \diamondsuit\ \phi \leq \psi} \qquad \frac{\forall \mathbf{n}(\phi \leq \mathbf{n}\ \&\ \Gamma \Rightarrow \psi \leq \boxdot \mathbf{n})}{\Gamma \Rightarrow \psi \leq \boxdot \phi}$$

$$\frac{\forall \mathbf{j} \forall \mathbf{m}(\mathbf{j} \leq \phi\ \&\ \psi \leq \mathbf{m}\ \&\ \Gamma \Rightarrow \mathbf{j} \leq \mathbf{m})}{\Gamma \Rightarrow \phi \leq \psi}$$

B From inductive in ML to Kracht

It is well known (cf. [7]) that an ALBA run on a definite inductive formula $(\varphi \leq \psi)[\overline{\alpha}, \overline{x}, \overline{\beta}/\overline{y}, \overline{\gamma}/\overline{z}, \overline{\delta}/\overline{w}]$ yields

$$\forall \bar{\mathbf{j}} \forall \overline{\mathbf{m}} \forall \bar{\mathbf{i}} \forall \overline{\mathbf{n}} \Big(\bar{\mathbf{i}} \leq \overbrace{\overline{\gamma}\left[\overline{\bigvee \mathsf{Mv}(p)}/\overline{p}, \overline{\bigwedge \mathsf{Mv}(q)}/\overline{q}\right]}^{\overline{\gamma}^{mv}}\ \&\ \overbrace{\overline{\delta}\left[\overline{\bigvee \mathsf{Mv}(p)}/\overline{p}, \overline{\bigwedge \mathsf{Mv}(q)}/\overline{q}\right]}^{\overline{\delta}^{mv}} \leq \overline{\mathbf{n}} \Rightarrow (\varphi \leq \psi)[!\bar{\mathbf{j}}/!\overline{x}, !\overline{\mathbf{m}}/!\overline{y}, !\bar{\mathbf{i}}/!\overline{z}, !\overline{\mathbf{n}}/!\overline{w}] \Big),$$

which is equivalent to its contrapositive

$$\forall \bar{\mathbf{j}} \forall \overline{\mathbf{m}} \forall \bar{\mathbf{i}} \forall \overline{\mathbf{n}} \left((\varphi \not\leq \psi)[!\bar{\mathbf{j}}/!\overline{x}, !\overline{\mathbf{m}}/!\overline{y}, !\bar{\mathbf{i}}/!\overline{z}, !\overline{\mathbf{n}}/!\overline{w}] \Rightarrow \bigparr_{i=1}^{n} \gamma_i^{mv} \leq \neg \mathbf{i}_i \ \parr\ \bigparr_{i=1}^{m} \neg \mathbf{n}_i \leq \delta_i^{mv} \right).$$

We put $(\varphi' \leq \psi') \equiv (\varphi \leq \psi)[!\bar{\mathbf{j}}/!\overline{x}, !\overline{\mathbf{m}}/!\overline{y}, !\bar{\mathbf{i}}/!\overline{z}, !\overline{\mathbf{n}}/!\overline{w}]$, by approximating the antecedent we have:

$$\forall \mathbf{j}' \forall \mathbf{m}' \forall \bar{\mathbf{j}} \forall \overline{\mathbf{m}} \forall \bar{\mathbf{i}} \forall \overline{\mathbf{n}} \left(\mathbf{j}' \leq \varphi'\ \&\ \psi' \leq \mathbf{m}'\ \&\ \mathbf{j}' \not\leq \mathbf{m}' \Rightarrow \bigparr_{i=1}^{n} \gamma_i^{mv} \leq \neg \mathbf{i}_i \ \parr\ \bigparr_{i=1}^{m} \neg \mathbf{n}_i \leq \delta_i^{mv} \right). \tag{B.1}$$

The variables $\mathbf{j}'$ and $\mathbf{m}'$ will be the pivotal pure variables of the inductive Kracht formula that we will compute.

Lemma B.1 *Each $\gamma_i^{mv} \leq \neg \mathbf{i}_i$ and $\neg \mathbf{n}_i \leq \delta_i^{mv}$ in* (B.1) *is equivalent to some Kracht disjunct.*

Proof. By induction on the structure of γ_i^{mv} (resp. δ_i^{mv}). When $\gamma_i^{mv} \leq \neg\mathbf{i}_i$ (resp. $\neg\mathbf{n}_i \leq \delta_i^{mv}$) is flat inequalities, there is nothing to do. When it is not flat, $\gamma_i^{mv} \leq \neg\mathbf{i}_i$ (resp. $\neg\mathbf{n}_i \leq \delta_i^{mv}$) can be rewritten in one of the following ways:

- $\Box\theta \leq \neg\mathbf{i}_i$ (resp. $\neg\mathbf{n}_i \leq \Diamond\theta$) for some formula θ. In this case we can apply Ackermann lemma to obtain $(\exists\mathbf{l} \rhd \neg\mathbf{i}_i)\theta \leq \mathbf{l}$ (resp. $(\exists\mathbf{k} \rhd \neg\mathbf{n}_i)\mathbf{k} \leq \theta$) with $\mathbf{l}$ (resp. $\mathbf{k}$) fresh.
- $\blacksquare\theta \leq \neg\mathbf{i}_i$ (resp. $\neg\mathbf{n}_i \leq \blacklozenge\theta$) for some formula θ. In this case we can apply Ackermann lemma to obtain $(\exists\mathbf{l} \blacktriangleright \neg\mathbf{i}_i)\theta \leq \mathbf{l}$ (resp. $(\exists\mathbf{k} \blacktriangleright \neg\mathbf{n}_i)\mathbf{k} \leq \theta$) with $\mathbf{l}$ (resp. $\mathbf{k}$) fresh.
- $\theta \to \eta \leq \neg\mathbf{i}_i$ (resp. $\neg\mathbf{n}_i \leq \theta - \eta$) for some formulas θ and η. In this case we can apply Ackermann lemma to obtain $(\exists\mathbf{k},\mathbf{l} \rhd \neg\mathbf{i}_i)(\mathbf{k} \leq \theta \ \&\ \eta \leq \mathbf{l})$ (resp. $(\exists\mathbf{k},\mathbf{l} \rhd \neg\mathbf{n}_i)(\mathbf{k} \leq \theta \ \&\ \eta \leq \mathbf{l})$) with $\mathbf{k}$ and $\mathbf{l}$ fresh.
- $\Diamond\theta \leq \neg\mathbf{i}_i$ (resp. $\neg\mathbf{n}_i \leq \Box\theta$) for some formula θ. In this case we can apply Ackermann lemma to obtain $(\forall\mathbf{k} \rhd \mathbf{i}_i)\theta \leq \neg\mathbf{k}$ (resp. $(\forall\mathbf{l} \rhd \mathbf{n}_i)\neg\mathbf{l} \leq \theta$) with $\mathbf{k}$ (resp. $\mathbf{l}$) fresh.
- $\blacklozenge\theta \leq \neg\mathbf{i}_i$ (resp. $\neg\mathbf{n}_i \leq \blacksquare\theta$) for some formula θ. In this case we can apply Ackermann lemma to obtain $(\forall\mathbf{k} \blacktriangleright \mathbf{i}_i)\theta \leq \neg\mathbf{k}$ (resp. $(\forall\mathbf{l} \blacktriangleright \mathbf{n}_i)\neg\mathbf{l} \leq \theta$) with $\mathbf{k}$ (resp. $\mathbf{l}$) fresh.
- $\theta - \eta \leq \neg\mathbf{i}_i$ (resp. $\neg\mathbf{n}_i \leq \theta \to \eta$) for some formulas θ and η. In this case we can apply Ackermann lemma to obtain $(\forall\mathbf{l},\mathbf{k} \rhd \mathbf{i}_i)(\neg\mathbf{l} \leq \theta \ \parr\ \eta \leq \neg\mathbf{k})$ (resp. $(\forall\mathbf{l},\mathbf{k} \rhd \mathbf{n}_i)(\neg\mathbf{l} \leq \theta \ \parr\ \eta \leq \neg\mathbf{k})$) with $\mathbf{k}$ and $\mathbf{l}$ fresh.
- $\theta \wedge \eta \leq \neg\mathbf{i}_i$ (resp. $\neg\mathbf{n}_i \leq \theta \vee \eta$) for some formulas θ and η. In this case we can apply Inverse splitting to obtain $\theta \leq \neg\mathbf{i}_i \ \parr\ \eta \leq \neg\mathbf{i}_i$ (resp. $\neg\mathbf{n}_i \leq \theta \ \parr\ \neg\mathbf{n}_i \leq \eta$).
- $\theta \vee \eta \leq \neg\mathbf{i}_i$ (resp. $\neg\mathbf{n}_i \leq \theta \wedge \eta$) for some formulas θ and η. In this case we can apply Inverse splitting to obtain $\theta \leq \neg\mathbf{i}_i \ \&\ \eta \leq \neg\mathbf{i}_i$ (resp. $\neg\mathbf{n}_i \leq \theta \ \&\ \neg\mathbf{n}_i \leq \eta$).

As the inductive hypothesis holds on the subformulae generated either by inverse splitting or Ackermann lemma, the statement holds. □

Lemma B.2 *The algorithm in Lemma B.1 applied to* $\mathbf{j}' \leq \varphi'$ *and* $\psi' \leq \mathbf{m}'$ *yields only existential quantifiers and (conjunctions) of restricting inequalities.*

Proof. By induction on the structure of φ' (resp. ψ'). If either φ' (resp. ψ') is a pure variable or has one single operator/connective, the statement is true as, being positive Skeleton, the operator would either $\Diamond$, $\blacklozenge$, or $\vee$ (resp. $\Box$, $\blacksquare$, $\to$, $\wedge$). The same thing applies to the uppermost connective in more complex formula, which, when simplified via Ackermann lemma, yield smaller skeleton formulas while outputting a single existential quantifier (it is sufficient to check that the possible cases output existential quantifier). Note that the cases that produce (meta) disjunctions can never occur, as they would require either a $\wedge$ on the left or a $\vee$ on the right hand side of the inequality. □

After both the lemmas are applied, we have a formula with a universal prefix, an antecedent with just restricted universal quantifiers, restricting inequalities, and a pivotal inequality; and an inductive Kracht consequent. The existential quantifiers in the antecedent can be rewritten as universal ones in the prefix: as they have been added during the procedure in Lemma B.2, they cannot occur in the consequent. Indeed, they are exactly the *type 2* quantifiers of the formula. The remaining restricting inequalities in the antecedent can be written directly in their corresponding quantifiers: they make up for the *type 1* quantifiers of the formula as they occur in the consequent, but not in the antecedent (since the skeleton in scattered, the restricting inequality originating the quantifier can be the only one containing it). The remaining inequalities in the antecedent do not contain operators/connectives, and the non-pivotal variables occur exactly once (always because the skeleton is scattered). The requirements on polarity do also hold as in the skeleton formula $\varphi' \leq \psi'$ nominals occur in positive position and conominals in negative position due to how the first approximation step of ALBA works. The requirement on the polarities in the consequent is respected as universal quantifiers, as can be verified by looking at the cases in Lemma B.1, flip the polarities of their maximal subformulae (the θs and the ηs in the lemma). Therefore, the claim that every inductive inequality is equivalent to some inductive Kracht formula readily follows from the above discussion.

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