Global neighbourhood completeness of the provability logic GLP

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In memoriam Sergei Ivanovich Adian
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Abstract

The provability logic GLP introduced by G. Japaridze is a propositional polymodal logic with important applications in proof theory, specificially, in ordinal analysis of arithmetic. Though being incomplete with respect to any class of Kripke frames, the logic GLP is complete for its neighbourhood interpretation. This completeness result, established by L. Beklemishev and D. Gabelaia, implies strong neighbourhood completeness of this system for the case of the so-called local semantic consequence relation. In the given article, we consider Hilbert-style non-well-founded derivations in the provability logic GLP and establish that GLP with the obtained derivability relation is strongly neighbourhood complete in the case of the global semantic consequence relation.

Keywords: provability logic, algebraic semantics, neighbourhood semantics, global consequence relations, non-well-founded derivations.

1 Introduction

The provability logic GLP introduced by G. Japaridze [6] is a propositional modal logic in a language with infinitely many modal connectives $\Box_0, \Box_1, \dots$. It is sound and complete with respect to a natural provability semantics, where the modal connective $\Box_n$ corresponds to the provability predicate "*... is provable from the axioms of Peano arithmetic together with all true arithmetical* Π^0_n*-sentences*". This system has important applications in proof theory, specificially, in ordinal analysis of arithmetic [1]. In the given article, we consider non-well-founded derivations in the provability logic GLP and study algebraic

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and neighbourhood semantics of the system GLP with the obtained derivability relation.

Neighbourhood semantics is an interesting generalization of Kripke semantics independently developed by D. Scott and R. Montague in [9] and [7]. The logic GLP is incomplete with respect to any class of Kripke frames. At the same time GLP is complete for its neighbourhood interpretation [3]. Notice that this completeness result implies strong neighbourhood completeness of this system for the case of the so-called local semantic consequence relation. Over neighbourhood GLP-models, a formula φ is a local semantic consequence of Γ if for any neighbourhood GLP-model $\mathcal{M}$ and any world x of $\mathcal{M}$

$$(\forall \psi \in \Gamma \ \ \mathcal{M}, x \vDash \psi) \Rightarrow \mathcal{M}, x \vDash \varphi.$$

A formula φ is a global semantic consequence of Γ if for any neighbourhood GLP-model $\mathcal{M}$

$$(\forall \psi \in \Gamma \ \ \mathcal{M} \vDash \psi) \Rightarrow \mathcal{M} \vDash \varphi.$$

Recently, global neighbourhood completeness of the Gödel-Löb provability logic GL with non-well-founded derivations was established in [10,11]. In the given article, we obtain an analogous result for the provability logic GLP.

2 Non-well-founded derivations in GLP

In this section we recall the provability logic GLP and define Hilbert-style non-well-founded derivations for the given system.

The provability logic GLP is a propositional modal logic in a language with infinitely many modal connectives $\Box_0, \Box_1, \dots$. In other words, formulas of the logic are built from the countable set of variables $PV = \{p, q, \dots\}$ and the constant $\bot$ using propositional connectives $\to$ and $\Box_i$ for each $i \in \mathbb{N}$. We treat other Boolean connectives and modal connectives $\Diamond_i$ as abbreviations:

$$\neg\varphi := \varphi \to \bot, \qquad \top := \neg\bot, \qquad \varphi \wedge \psi := \neg(\varphi \to \neg\psi),$$
$$\varphi \vee \psi := \neg\varphi \to \psi, \qquad \varphi \leftrightarrow \psi := (\varphi \to \psi) \wedge (\psi \to \varphi), \qquad \Diamond_i \varphi := \neg \Box_i \neg\varphi.$$

By Fm, we denote the set of formulas of GLP.

The provability logic GLP is defined by the following axiom schemes and inference rules.

Axiom schemes:

(i) the tautologies of classical propositional logic;

(ii) $\Box_i(\varphi \to \psi) \to (\Box_i\varphi \to \Box_i\psi)$;

(iii) $\Box_i(\Box_i\varphi \to \varphi) \to \Box_i\varphi$;

(iv) $\Diamond_i\varphi \to \Box_{i+1} \Diamond_i \varphi$;

(v) $\Box_i\varphi \to \Box_{i+1}\varphi$.

Inference rules:

$$\mathsf{mp}\ \frac{\varphi \qquad \varphi \to \psi}{\psi}\ , \qquad \mathsf{nec}\ \frac{\varphi}{\Box_0\varphi}\ .$$

We remark that transitivity of the modal connectives $\Box_i$ is provable in GLP, i.e. $\mathsf{GLP} \vdash \Box_i\psi \to \Box_i \Box_i \psi$ for any formula ψ and any $i \in \mathbb{N}$.

Now we define non-well-founded derivations in GLP. An *∞-derivation* is a (possibly infinite) tree whose nodes are marked by formulas of GLP and that is constructed according to the rules (mp) and (nec). In addition, any infinite branch in an ∞-derivation must contain infinitely many applications of the rule (nec). An *assumption leaf* of an ∞-derivation is a leaf that is not marked by an axiom of GLP.

The *main fragment* of an ∞-derivation is a finite tree obtained from the ∞-derivation by cutting every infinite branch at the nearest to the root application of the rule (nec). The *local height* $|\pi|$ *of an ∞-derivation* π is the length of the longest branch in its main fragment. An ∞-derivation consisting of a single formula only has height 0.

For example, consider the following ∞-derivation

$$\mathsf{mp}\,\dfrac{\mathsf{nec}\,\dfrac{\mathsf{mp}\,\dfrac{\mathsf{nec}\,\dfrac{\mathsf{mp}\,\dfrac{\overset{\vdots}{\Box_0 p_3} \qquad \Box_0 p_3 \to p_2}{p_2}}{\Box_0 p_2} \qquad \Box_0 p_2 \to p_1}{p_1}}{\Box_0 p_1} \qquad \Box_0 p_1 \to p_0}{p_0}\ ,$$

where assumption leaves are marked by formulas of the form $\Box_0 p_{i+1} \to p_i$. The local height of this ∞-derivation equals to 1 and its main fragment has the form

$$\mathsf{mp}\,\dfrac{\Box_0 p_1 \qquad \Box_0 p_1 \to p_0}{p_0}\ .$$

Definition 2.1 We set $\Gamma \vdash_g \varphi$ if there is an ∞-derivation with the root marked by φ in which all assumption leaves are marked by some elements of Γ.

Proposition 2.2 *For any formula φ, we have*

$$\mathsf{GLP} \vdash \varphi \Longleftrightarrow \varnothing \vdash_g \varphi.$$

We give a proof of this proposition in the Appendix since this statement is not essential for the global neighbourhood completeness result of the final section.

3 Algebraic semantics

In this section we consider algebraic semantics for the provability logic GLP enriched with non-well-founded derivations.

A *Magari algebra* (or a *diagonalizable algebra*) $\mathcal{A} = (A, \wedge, \vee, \to, 0, 1, \Box)$ is a Boolean algebra $(A, \wedge, \vee, \to, 0, 1)$ together with a unary map $\Box \colon A \to A$ satisfying the identities:

$$\Box 1 = 1, \quad \Box(x \wedge y) = \Box x \wedge \Box y, \quad \Box(\Box x \to x) = \Box x.$$

For any Magari algebra $\mathcal{A}$, the mapping $\Box$ is monotone with respect to the order (of the Boolean part) of $\mathcal{A}$. Indeed, if $a \leqslant b$, then $a \wedge b = a$, $\Box a \wedge \Box b =$

$\Box(a \wedge b) = \Box a$, and $\Box a \leqslant \Box b$. In addition, we remark that an inequality $\Box x \leqslant \Box\Box x$ holds in any Magari algebra.

We call a Magari algebra $\Box$-*founded (or Pakhomov-Walsh-founded)*[2] if, for every sequence of its elements $(a_i)_{i\in\mathbb{N}}$ such that $\Box a_{i+1} \leqslant a_i$, we have $a_0 = 1$. Note that, for any such sequence $(a_i)_{i\in\mathbb{N}}$, all elements a_i are equal to 1 in any $\Box$-founded Magari algebra.

We give a series of examples of $\Box$-founded Magari algebras. A Magari algebra is called σ-*complete* if its underlying Boolean algebra is σ-complete, that is, each of its countable subsets S has the least upper bound $\bigvee S$. An equivalent condition is that every countable subset S has the greatest lower bound $\bigwedge S$.

Proposition 3.1 *Any σ-complete Magari algebra is $\Box$-founded.*

Proof. Assume we have a σ-complete Magari algebra $\mathcal{A}$ and a sequence of its elements $(a_i)_{i\in\mathbb{N}}$ such that $\Box a_{i+1} \leqslant a_i$. We shall prove that $a_0 = 1$.

Put $b = \bigwedge\limits_{i\in\mathbb{N}} a_i$. For any $i \in \mathbb{N}$, we have $b \leqslant a_{i+1}$ and $\Box b \leqslant \Box a_{i+1} \leqslant a_i$. Hence,

$$\Box b \leqslant b, \qquad \Box b \to b = 1, \qquad \Box b = \Box(\Box b \to b) = \Box 1 = 1, \qquad b = 1.$$

We obtain that $a_0 = 1$. $\Box$

Remark 3.2 Let us additionally mention an arithmetical example of $\Box$-founded Magari algebra without going into details. If we consider the second-order arithmetical theory $\Sigma^1_1 - \mathsf{AC}_0$ extended with all true Σ^1_1-sentences, then its provability algebra forms a $\Box$-founded Magari algebra. This observation can be obtained following the lines of Theorem 3.2 from [8].

The notion of $\Box$-founded Magari algebra $\mathcal{A}$ can be defined in terms of the binary relation $\prec_{\mathcal{A}}$ on $\mathcal{A}$:

$$a \prec_{\mathcal{A}} b \Longleftrightarrow \Box a \leqslant b.$$

Proposition 3.3 (see Proposition 3.1 from [11]) *For any Magari algebra $\mathcal{A} = (A, \wedge, \vee, \to, 0, 1, \Box)$, the relation $\prec_{\mathcal{A}}$ is a strict partial order on $A \smallsetminus \{1\}$.*

Proposition 3.4 (see Proposition 3.2 from [11]) *For any Magari algebra $\mathcal{A} = (A, \wedge, \vee, \to, 0, 1, \Box)$, the algebra $\mathcal{A}$ is $\Box$-founded if and only if the partial order $\prec_{\mathcal{A}}$ on $A \smallsetminus \{1\}$ is well-founded.*

A Boolean algebra $(A, \wedge, \vee, \to, 0, 1)$ together with a sequence of unary mappings $\Box_0, \Box_1, \dots$ is called a GLP-*algebra* if it satisfies the following conditions for each $i \in \mathbb{N}$:

(i) $(A, \wedge, \vee, \to, 0, 1, \Box_i)$ is a Magari algebra;

(ii) $\Diamond_i a \leqslant \Box_{i+1} \Diamond_i a$ for any $a \in A$;

(iii) $\Box_i a \leqslant \Box_{i+1} a$ for any $a \in A$.

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[2] This notion has been inspired by an article of F. Pakhomov and J. Walsh [8].

A GLP-algebra $\mathcal{A} = (A, \wedge, \vee, \to, 0, 1, \Box_0, \Box_1, \dots)$ is called *$\Box$-founded* if the Magari algebra $\mathcal{A}_0 = (A, \wedge, \vee, \to, 0, 1, \Box_0)$ is $\Box$-founded. In the same way, we apply notions defined for the Magari algebra $\mathcal{A}_0$ to $\mathcal{A}$. From Proposition 3.1, we immediately see that any σ-complete GLP-algebra is $\Box$-founded.

Now we define a semantic consequence relation over $\Box$-founded GLP-algebras, which, we shall see, corresponds to the derivability relation $\vdash_g$. A *valuation in a* GLP*-algebra* $\mathcal{A} = (A, \wedge, \vee, \to, 0, 1, \Box_0, \Box_1, \dots)$ is a function $v\colon Fm \to A$ such that $v(\bot) = 0$, $v(\varphi \to \psi) = v(\varphi) \to v(\psi)$, and $v(\Box_i \varphi) = \Box_i v(\varphi)$.

Definition 3.5 Given a set of formulas Γ and a formula φ, we set $\Gamma \Vdash_g \varphi$ if for any $\Box$-founded GLP-algebra $\mathcal{A}$ and any valuation v in $\mathcal{A}$

$$(\forall \psi \in \Gamma \;\; v(\psi) = 1) \Rightarrow v(\varphi) = 1.$$

Lemma 3.6 *For any set of formulas Γ and any formula φ, we have*

$$\Gamma \vdash_g \varphi \Longrightarrow \Gamma \Vdash_g \varphi.$$

Proof. Assume π is an ∞-derivation with the root marked by φ in which all assumption leaves are marked by some elements of Γ. In addition, assume we have a $\Box$-founded GLP-algebra $\mathcal{A} = (X, \wedge, \vee, \to, 0, 1, \Box_0, \Box_1, \dots)$ together with a valuation v in $\mathcal{A}$ such that $v(\psi) = 1$ for any $\psi \in \Gamma$. We shall prove that $v(\varphi) = 1$.

For any node w of the ∞-derivation π, let π_w be the subtree of π with the root w. Also, put $r(w) = |\pi_w|$. In addition, let φ_w be the formula of the node w. A node w belongs to the $(n+1)$*-th slice of* π if there are precisely n applications of the rule (nec) on the path from this node to the root of π. By c_n, we denote the element $\bigwedge\{v(\varphi_w) \mid w \text{ belongs to the } (n+1)\text{-th slice of } \pi\}$.

We claim that $\Box_0 c_{n+1} \leqslant c_n$ for any $n \in \mathbb{N}$. It is sufficient to prove that $\Box_0 c_{n+1} \leqslant v(\varphi_w)$ whenever w belongs to the $(n+1)$-th slice of π. The proof is by induction on $r(w)$.

If φ_w is an axiom of GLP or an element of Γ, then we immediately obtain the required statement. Otherwise, φ_w is obtained by an application of an inference rule in π.

If φ_w is obtained by the rule (nec), then this formula has the form $\Box_0 \varphi_u$, where u is the premise of w. We see that u belongs to the $(n+2)$-th slice of π. Consequently $c_{n+1} \leqslant v(\varphi_u)$ and $\Box_0 c_{n+1} \leqslant v(\varphi_w)$.

Suppose φ_w is obtained by the rule (mp). Consider the premises u_1 and u_2 of w. We have $r(u_1) < r(w)$ and $r(u_2) < r(w)$. By our induction hypotheses, we obtain $\Box_0 c_{n+1} \leqslant v(\varphi_{u_1}) \wedge v(\varphi_{u_2}) \leqslant v(\varphi_w)$.

This proves the claim that $\Box_0 c_{n+1} \leqslant c_n$ for any $n \in \mathbb{N}$. Applying $\Box$-foundedness of $\mathcal{A}$, we note that $c_0 = 1$. Since the root of the ∞-derivation π belongs to the first slice of π, we conclude that $c_0 \leqslant v(\varphi)$ and $v(\varphi) = 1$. $\Box$

Theorem 3.7 (Algebraic completeness) *For any set of formulas Γ and any formula φ, we have*

$$\Gamma \vdash_g \varphi \Longleftrightarrow \Gamma \Vdash_g \varphi.$$

Proof. The left-to-right implication follows from Lemma 3.6. We prove the converse. Assume $\Gamma \Vdash_g \varphi$. Consider the theory $T = \{\theta \in Fm \mid \Gamma \vdash_g \theta\}$. We see that T contains all axioms of GLP and is closed under the rules (mp) and (nec). We define an equivalence relation $\sim_T$ on the set of formulas Fm by putting $\mu \sim_T \rho$ if and only of $(\mu \leftrightarrow \rho) \in T$. Let us denote the equivalence class of μ by $[\mu]_T$. Applying the Lindenbaum-Tarski construction, we obtain a GLP-algebra $\mathcal{L}_T$ on the set of equivalence classes of formulas, where $[\mu]_T \wedge [\rho]_T = [\mu \wedge \rho]_T$, $[\mu]_T \vee [\rho]_T = [\mu \vee \rho]_T$, $[\mu]_T \to [\rho]_T = [\mu \to \rho]_T$, $0 = [\bot]_T$, $1 = [\top]_T$ and $\Box_i[\mu] = [\Box_i \mu]$.

Let us check that the algebra $\mathcal{L}_T$ is $\Box$-founded. Assume we have a sequence of formulas $(\mu_i)_{i\in\mathbb{N}}$ such that $\Box_0[\mu_{i+1}]_T \leqslant [\mu_i]_T$. We have $[\Box_0\mu_{i+1} \to \mu_i]_T = 1$ and $(\Box_0\mu_{i+1} \to \mu_i) \in T$. For every $i \in \mathbb{N}$, there exists an ∞-derivation π_i for the formula $\Box_0\mu_{i+1} \to \mu_i$ such that all assumption leaves of π_i are marked by some elements of Γ. We obtain the following ∞-derivation for the formula μ_0:

$$\text{mp}\,\dfrac{\text{nec}\,\dfrac{\text{mp}\,\dfrac{\text{nec}\,\dfrac{\text{mp}\,\dfrac{\begin{matrix}\vdots\\ \Box_0\mu_3\end{matrix}\qquad \begin{matrix}\pi_2\\ \vdots\\ \Box_0\mu_3 \to \mu_2\end{matrix}}{\mu_2}}{\Box_0\mu_2}\qquad \begin{matrix}\pi_1\\ \vdots\\ \Box_0\mu_2 \to \mu_1\end{matrix}}{\mu_1}}{\Box_0\mu_1}\qquad \begin{matrix}\pi_0\\ \vdots\\ \Box_0\mu_1 \to \mu_0\end{matrix}}{\mu_0}\ ,$$

where all assumption leaves are marked by some elements of Γ. Hence, $\mu_0 \in T$ and $[\mu_0]_T = [\top]_T = 1$. We conclude that the GLP-algebra $\mathcal{L}_T$ is $\Box$-founded.

Consider the valuation $v\colon \theta \mapsto [\theta]_T$ in the GLP-algebra $\mathcal{L}_T$. Since $\Gamma \subset T$, we have $v(\psi) = 1$ for any $\psi \in \Gamma$. From the assumption $\Gamma \Vdash \varphi$, we obtain that $v(\varphi) = 1$. Consequently $\varphi \in T$ and $\Gamma \vdash_g \varphi$. $\Box$

4 Neighbourhood semantics

In this section we recall neighbourhood semantics of the provability logic GLP.

An *Esakia frame* (or a *Magari frame*) $\mathcal{X} = (X, \Box)$ is a set X together with a mapping $\Box : \mathcal{P}(X) \to \mathcal{P}(X)$ such that the powerset Boolean algebra $\mathcal{P}(X)$ with the mapping $\Box$ forms a Magari algebra.

We briefly recall a connection between scattered topological spaces and Esakia frames (cf. [4]). Note that we allow Esakia frames and topological spaces to be empty.

In a topological space, an open set U containing a point x is called a *neighbourhood* of x. A set U is a *punctured neighbourhood* of x if $x \notin U$ and $U \cup \{x\}$ is open. For a topological space (X, τ) and a subset V the *derived set* $d_\tau(V)$ *of* V is the set of limit points of V:

$$x \in d_\tau(V) \Longleftrightarrow \forall U \in \tau\ (x \in U \Rightarrow \exists y \neq x\ (y \in U \cap V))\ .$$

The *co-derived set* $cd_\tau(V)$ *of* V is defined as $X \smallsetminus d_\tau(X \smallsetminus V)$. By definition, $x \in cd_\tau(V)$ if and only if there is a punctured neighbourhood of x entirely contained in V.

In a topological space, a point having an empty punctured neighbourhood is called *isolated*. A topological space (X,τ) is *scattered* if each non-empty subset of X (as a topological space with the inherited topology) has an isolated point.

Proposition 4.1 (L. Esakia [5]) *If* $(X,\Box)$ *is an Esakia frame, then* X *bears a unique topology* τ *for which* $\Box = cd_\tau$. *Moreover, the space* (X,τ) *is scattered.*

Proposition 4.2 (H. Simmons [12], L. Esakia [5]) *If* (X,τ) *is a scattered topological space, then* (X, cd_τ) *is an Esakia frame.*

A *neighbourhood* GLP*-frame* $\mathcal{X} = (X,\Box_0,\Box_1,\dots)$ is a set X together with a sequence of unary mappings $\Box_0,\Box_1,\dots$ on $\mathcal{P}(X)$ such that the powerset Boolean algebra $\mathcal{P}(X)$ with the given mappings forms a GLP-algebra. Elements of X are called *worlds* of the frame $\mathcal{X}$. A *neighbourhood* GLP*-model* is a pair $\mathcal{M} = (\mathcal{X}, v)$, where $\mathcal{X}$ is a neighbourhood GLP-frame and v is a valuation in the powerset GLP-algebra of $\mathcal{X}$. A formula φ is *true at a world* x *of a model* $\mathcal{M}$, written as $\mathcal{M}, x \vDash \varphi$, if $x \in v(\varphi)$. A formula φ is called *true in* $\mathcal{M}$, written as $\mathcal{M} \vDash \varphi$, if φ is true at all worlds of $\mathcal{M}$.

A GLP*-space* is a polytopological space $(X,\tau_0,\tau_1,\dots)$, where, for each $i \in \mathbb{N}$, τ_i is scattered, $\tau_i \subset \tau_{i+1}$, and $d_{\tau_i}(V) \in \tau_{i+1}$ for any $V \in \mathcal{P}(X)$.

Proposition 4.3 (see Proposition 4 from [4])

(i) *If* $(X,\Box_0,\Box_1,\dots)$ *is a* GLP*-frame, then* X *bears a unique series of topologies* $\tau_0,\tau_1,\dots$ *such that* $\Box_i = cd_{\tau_i}$ *for every* $i \in \mathbb{N}$. *Moreover, the polytopological space* $(X,\tau_0,\tau_1,\dots)$ *is a* GLP*-space.*

(ii) *If* $(X,\tau_0,\tau_1,\dots)$ *is a* GLP*-space, then* $(X, cd_{\tau_0}, cd_{\tau_1},\dots)$ *is a* GLP*-frame.*

In the sequel, we don't distinguish GLP-frames and corresponding polytopological spaces so that we use the topological terminology referring to $(X,\tau_0,\tau_1,\dots)$ for the frame $(X, cd_{\tau_0}, cd_{\tau_1},\dots)$. For example, we say that a subset U is n*-open* in $(X,\Box_0,\Box_1,\dots)$ if it belongs to the corresponding n-th topology on X (which is equivalent to $U \subset \Box_n U$).

Now we define a global semantic consequence relation over GLP-frames.

Definition 4.4 Given a set of formulas Γ and a formula φ, we set $\Gamma \vDash_g \varphi$ if for any GLP-model $\mathcal{M}$

$$(\forall \psi \in \Gamma \ \ \mathcal{M} \vDash \psi) \Rightarrow \mathcal{M} \vDash \varphi.$$

Let us recall the following neighbourhood completeness result obtained by L. Beklemishev and D. Gabelaia in [3].

Theorem 4.5 *For any formula* φ, *if* GLP $\nvdash \varphi$, *then there is a* GLP*-model* $\mathcal{M}$ *and a world* x *such that* $\mathcal{M}, x \nvDash \varphi$.

We notice that, for any GLP-frame $\mathcal{X}$, the powerset GLP-algebra of $\mathcal{X}$ is σ-complete. Consequently this algebra is $\Box$-founded by Proposition 3.1. Hence we immediately obtain the following proposition.

Proposition 4.6 *For any set of formulas* Γ *and any formula* φ, *we have*

$$\Gamma \Vdash_g \varphi \Longrightarrow \Gamma \vDash_g \varphi.$$

5 Representation of $\Box$-founded Magari algebras

In this section we prove that any $\Box$-founded Magari algebra can be embedded into the powerset Magari algebra of an Esakia frame. We also obtain some related results, which will be applied in the next section.

From Proposition 3.4, we know that a Magari algebra $\mathcal{A} = (A, \wedge, \vee, \rightarrow, 0, 1, \Box)$ is $\Box$-founded if and only if the binary relation $\prec_{\mathcal{A}}$ is well-founded on $A \setminus \{1\}$, where

$$a \prec_{\mathcal{A}} b \Longleftrightarrow \Box a \leqslant b.$$

Let us recall some basic properties of well-founded relations.

A *well-founded set* is a pair $\mathcal{S} = (S, \prec)$, where $\prec$ is a well-founded relation on S. For any element a of $\mathcal{S}$, its ordinal height in $\mathcal{S}$ is denoted by $ht_{\mathcal{S}}(a)$. Recall that $ht_{\mathcal{S}}$ is defined by transfinite recursion on $\prec$ as follows:

$$ht_{\mathcal{S}}(a) = \sup\{ht_{\mathcal{S}}(b) + 1 \mid b \prec a\}.$$

A *homomorphism* from $\mathcal{S}_1 = (S_1, \prec_1)$ to $\mathcal{S}_2 = (S_2, \prec_2)$ is a function $f\colon S_1 \to S_2$ such that $f(b) \prec_2 f(c)$ whenever $b \prec_1 c$.

Proposition 5.1 *Suppose $f\colon \mathcal{S}_1 \to \mathcal{S}_2$ is a homomorphism of well-founded sets and a is an element of $\mathcal{S}_1$. Then $ht_{\mathcal{S}_1}(a) \leqslant ht_{\mathcal{S}_2}(f(a))$.*

For well-founded sets $\mathcal{S}_1 = (S_1, \prec_1)$ and $\mathcal{S}_2 = (S_2, \prec_2)$, their product $\mathcal{S}_1 \times \mathcal{S}_2$ is defined as the set $S_1 \times S_2$ together with the following relation

$$(b_1, b_2) \prec (c_1, c_2) \Longleftrightarrow b_1 \prec_1 c_1 \text{ and } b_2 \prec_2 c_2.$$

Clearly, $\prec$ is a well-founded relation on $S_1 \times S_2$.

Proposition 5.2 *Suppose a and b are elements of well-founded sets $\mathcal{S}_1$ and $\mathcal{S}_2$ respectively. Then $ht_{\mathcal{S}_1 \times \mathcal{S}_2}((a, b)) = \min\{ht_{\mathcal{S}_1}(a), ht_{\mathcal{S}_2}(b)\}$.*

For an element a of a $\Box$-founded Magari algebra $\mathcal{A}$, define $ht_{\mathcal{A}}(a)$ as the ordinal height of a with respect to $\prec_{\mathcal{A}}$. We put $ht_{\mathcal{A}}(a) = \infty$ if $a = 1$.

Lemma 5.3 *Suppose a and b are elements of a $\Box$-founded Magari algebra $\mathcal{A}$. Then $ht_{\mathcal{A}}(a \wedge b) = \min\{ht_{\mathcal{A}}(a), ht_{\mathcal{A}}(b)\}$ and $ht_{\mathcal{A}}(a) + 1 \leqslant ht_{\mathcal{A}}(\Box a)$, where we define $\infty + 1 := \infty$.*

Proof. Assume we have a $\Box$-founded Magari algebra $\mathcal{A} = (A, \wedge, \vee, \rightarrow, 0, 1, \Box)$ and two elements a and b of $\mathcal{A}$.

First, we prove that $ht_{\mathcal{A}}(a \wedge b) = \min\{ht_{\mathcal{A}}(a), ht_{\mathcal{A}}(b)\}$. If $a = 1$ or $b = 1$, then the equality immediately holds. Suppose $a \neq 1$ and $b \neq 1$. Let $\mathcal{S}$ be the set $A \setminus \{1\}$ together with the well-founded relation $\prec_{\mathcal{A}}$. We have $a \wedge b \neq 1$, $ht_{\mathcal{A}}(a) = ht_{\mathcal{S}}(a)$, $ht_{\mathcal{A}}(b) = ht_{\mathcal{S}}(b)$ and $ht_{\mathcal{A}}(a \wedge b) = ht_{\mathcal{S}}(a \wedge b)$. The mapping

$$f\colon (c, d) \mapsto c \wedge d$$

is a homomorphism from $\mathcal{S} \times \mathcal{S}$ to $\mathcal{S}$. From Proposition 5.2 and Proposition 5.1, we have

$$\min\{ht_{\mathcal{S}}(a), ht_{\mathcal{S}}(b)\} = ht_{\mathcal{S} \times \mathcal{S}}((a, b)) \leqslant ht_{\mathcal{S}}(a \wedge b).$$

Consequently,

$$\min\{ht_{\mathcal{A}}(a), ht_{\mathcal{A}}(b)\} \leqslant ht_{\mathcal{A}}(a \wedge b).$$

On the other hand, $ht_{\mathcal{A}}(a \wedge b) \leqslant ht_{\mathcal{A}}(a)$ since

$$\{e \in A \setminus \{1\} \mid e \prec_{\mathcal{A}} (a \wedge b)\} \subset \{e \in A \setminus \{1\} \mid e \prec_{\mathcal{A}} a\}.$$

Analogously, we have $ht_{\mathcal{A}}(a \wedge b) \leqslant ht_{\mathcal{A}}(b)$. It follows that

$$ht_{\mathcal{A}}(a \wedge b) = \min\{ht_{\mathcal{A}}(a), ht_{\mathcal{A}}(b)\}.$$

Now we prove $ht_{\mathcal{A}}(a) + 1 \leqslant ht_{\mathcal{A}}(\Box a)$. If $\Box a = 1$, then the inequality immediately holds. Suppose $\Box a \neq 1$. Then $a \neq 1$. We see $a \prec_{\mathcal{A}} \Box a$. The required inequality holds from the definition of $ht_{\mathcal{A}}$. □

For a □-founded Magari algebra $\mathcal{A} = (A, \wedge, \vee, \to, 0, 1, \Box)$ and an ordinal γ, put $M_{\mathcal{A}}(\gamma) = \{a \in A \mid \gamma \leqslant ht_{\mathcal{A}}(a)\}$. We see that $M_{\mathcal{A}}(0) = A$ and $M_{\mathcal{A}}(\delta) \supset M_{\mathcal{A}}(\gamma)$ whenever $\delta \leqslant \gamma$.

Lemma 5.4 *For any □-founded Magari algebra $\mathcal{A}$ and any ordinal γ, the set $M_{\mathcal{A}}(\gamma)$ is a filter in $\mathcal{A}$.*

Proof. Suppose a and b belong to $M_{\mathcal{A}}(\gamma)$. Then $\gamma \leqslant ht_{\mathcal{A}}(a)$ and $\gamma \leqslant ht_{\mathcal{A}}(b)$. We have $\gamma \leqslant \min\{ht_{\mathcal{A}}(a), ht_{\mathcal{A}}(b)\} = ht_{\mathcal{A}}(a \wedge b)$ by Lemma 5.3. Consequently $a \wedge b$ belongs to $M_{\mathcal{A}}(\gamma)$.

Now suppose c belongs to $M_{\mathcal{A}}(\gamma)$ and $c \leqslant d$. We shall show that $d \in M_{\mathcal{A}}(\gamma)$. We have $\gamma \leqslant ht_{\mathcal{A}}(c) = ht_{\mathcal{A}}(c \wedge d) = \min\{ht_{\mathcal{A}}(c), ht_{\mathcal{A}}(d)\} \leqslant ht_{\mathcal{A}}(d)$ by Lemma 5.3. Hence $d \in M_{\mathcal{A}}(\gamma)$. □

Let $\mathit{Ult}\,\mathcal{A}$ be the set of all ultrafilters of (the Boolean part of) a Magari algebra $\mathcal{A} = (A, \wedge, \vee, \to, 0, 1, \Box)$. Put $\widehat{a} = \{u \in \mathit{Ult}\,\mathcal{A} \mid a \in u\}$ for $a \in A$. We recall that the mapping $\widehat{\cdot}\colon a \mapsto \widehat{a}$ is an embedding of the Boolean algebra $(A, \wedge, \vee, \to, 0, 1)$ into the powerset Boolean algebra $\mathcal{P}(\mathit{Ult}\,\mathcal{A})$ by Stone's representation theorem.

Lemma 5.5 *For any □-founded Magari algebra $\mathcal{A}$, there exists a scattered topology τ on $\mathit{Ult}\,\mathcal{A}$ such that $\widehat{\Box a} = cd_{\tau}(\widehat{a})$ for any element a of $\mathcal{A}$.*

Proof. Assume we have a □-founded Magari algebra $\mathcal{A} = (A, \wedge, \vee, \to, 0, 1, \Box)$. Let $ht(\mathcal{A}) = \sup\{ht_{\mathcal{A}}(a) + 1 \mid a \in A \setminus \{1\}\}$. We see that $M_{\mathcal{A}}(ht(\mathcal{A})) = \{1\}$. For an ultrafilter u of $\mathcal{A}$, set $rk(u) := \min\{\delta \leqslant ht(\mathcal{A}) \mid M_{\mathcal{A}}(\delta) \subset u\}$. Also, for an ordinal γ, put $I(\gamma) := \{u \in \mathit{Ult}\,\mathcal{A} \mid rk(u) < \gamma\}$.

Set $\tau = \{V \subset \mathit{Ult}\,\mathcal{A} \mid \forall u \in V\ \exists a \in A\ \ (\Box a \in u) \wedge (\widehat{\boxdot a} \cap I(rk(u)) \subset V)\}$, where $\boxdot a = a \wedge \Box a$.

Let us check that τ is a topology on $\mathit{Ult}\,\mathcal{A}$. Trivially, $\varnothing \in \tau$ and τ is closed under arbitrary unions. For any $u \in \mathit{Ult}\,\mathcal{A}$, we see that $\Box 1 = 1 \in u$ and $\widehat{\boxdot 1} \cap I(rk(u)) \subset \mathit{Ult}\,\mathcal{A}$. Consequently $\mathit{Ult}\,\mathcal{A} \in \tau$. Assume $S_0 \in \tau$ and $S_1 \in \tau$. Consider an arbitrary $u \in S_0 \cap S_1$. By definition of τ, there exist elements b and c of A such that $\Box b \in u$, $\Box c \in u$, $\widehat{\boxdot b} \cap I(rk(u)) \subset S_0$ and $\widehat{\boxdot c} \cap I(rk(u)) \subset S_1$. We have $\Box(b \wedge c) = (\Box b \wedge \Box c) \in u$ and $\widehat{\boxdot(b \wedge c)} \cap I(rk(u)) = \widehat{\boxdot a} \cap \widehat{\boxdot c} \cap I(rk(u)) \subset S_0 \cap S_1$. Therefore $S_0 \cap S_1 \in \tau$. This shows that τ is a topology on $\mathit{Ult}\,\mathcal{A}$.

It easily follows from the definition of τ that $\widehat{\boxdot a} \in \tau$, for any $a \in A$, and $I(\gamma) \in \tau$, for any ordinal γ. Now we claim that τ is scattered. Consider any non-empty subset S of $\mathit{Ult}\,\mathcal{A}$. There is an ultrafilter $h \in S$ such that $rk(h) = \min\{rk(u) \mid u \in S\}$. We see that a set $\{h\} \cup I(rk(h))$ is a τ-neighbourhood of h and $S \cap (\{h\} \cup I(rk(h))) = \{h\}$. Hence the ultrafilter h is an isolated point in S. This proves that τ is a scattered topology.

It remains to show that $\widehat{\Box a} = cd_\tau(\widehat{a})$ for any $a \in A$. First, we check that $\widehat{\Box a} \subset cd_\tau(\widehat{a})$. For any ultrafilter d, if $d \in \widehat{\Box a}$, then $\widehat{\boxdot a} \cap I(rk(d))$ is a punctured neighbourhood of d. Also, $\widehat{\boxdot a} \cap I(rk(d)) \subset \widehat{a}$. By definition of the co-derived-set operator, $d \in cd_\tau(\widehat{a})$. Consequently $\widehat{\Box a} \subset cd_\tau(\widehat{a})$.

Now we claim that $cd_\tau(\widehat{a}) \subset \widehat{\Box a}$. Consider any ultrafilter d such that $d \notin \widehat{\Box a}$. Let W be an arbitrary punctured neighbourhood of d. It is sufficient to show that W is not included in $\widehat{a}$.

By definition of τ, there exists an element e of A such that $\Box e \in d$ and $\widehat{\boxdot e} \cap I(rk(d)) \subset W$. From the conditions $\Box e \in d$ and $\Box a \notin d$, it follows that $\Box(\boxdot e \to a) \notin d$. Hence $\Box(\boxdot e \to a) \notin M_{\mathcal{A}}(rk(d)) \subset d$ and $ht_{\mathcal{A}}(\Box(\boxdot e \to a)) < rk(d)$. Note that $(\boxdot e \to a) \notin M_{\mathcal{A}}(ht_{\mathcal{A}}(\boxdot e \to a)+1)$. By the Boolean ultrafilter theorem, there exists an ultrafilter w of $\mathcal{A}$ such that $(\boxdot e \to a) \notin w$ and $M_{\mathcal{A}}(ht_{\mathcal{A}}(\boxdot e \to a)+1) \subset w$. We see that $\boxdot e \in w$, $a \notin w$ and $rk(w) \leqslant ht_{\mathcal{A}}(\boxdot e \to a) + 1$. From Lemma 5.3, we have $ht_{\mathcal{A}}(\boxdot e \to a) + 1 \leqslant ht_{\mathcal{A}}(\Box(\boxdot e \to a)) < rk(d)$. Thus $rk(w) < rk(d)$, $w \in \widehat{\boxdot e} \cap I(rk(d))$ and $w \notin \widehat{a}$. Consequently w is an element of W, which does not belong to $\widehat{a}$.

We obtain that none of the punctured neighbourhoods of d are included in $\widehat{a}$. In other words, $d \notin cd_\tau(\widehat{a})$ for any $d \notin \widehat{\Box a}$. We conclude that $cd_\tau(\widehat{a}) \subset \widehat{\Box a}$. Hence $\widehat{\Box a} = cd_\tau(\widehat{a})$. □

Theorem 5.6 *A Magari algebra is* $\Box$*-founded if and only if it is embeddable into the powerset Magari algebra of an Esakia frame.*

Proof. (if) Suppose a Magari algebra $\mathcal{A}$ is isomorphic to a subalgebra of the powerset Magari algebra of an Esakia frame $\mathcal{X}$. The powerset Magari algebra of $\mathcal{X}$ is σ-complete. Hence, by Proposition 3.1, it is $\Box$-founded. Since any subalgebra of a $\Box$-founded Magari algebra is $\Box$-founded, the algebra $\mathcal{A}$ is $\Box$-founded.

(only if) Suppose a Magari algebra $\mathcal{A}$ is $\Box$-founded. By Lemma 5.5, there exists a scattered topology τ on $\mathit{Ult}\,\mathcal{A}$ such that $\widehat{\Box a} = cd_\tau(\widehat{a})$ for any element a of $\mathcal{A}$. We know that $\mathcal{X} = (\mathit{Ult}\,\mathcal{A}, cd_\tau)$ is an Esakia frame by Proposition 4.2. We see that the mapping $\widehat{\cdot} : a \mapsto \widehat{a}$ is an injective homomorphism from $\mathcal{A}$ to the powerset Magari algebra of the frame $\mathcal{X}$. Therefore the algebra $\mathcal{A}$ is embeddable into the powerset Magari algebra of an Esakia frame. □

For a Magari algebra $\mathcal{A}$, by $\mathit{Top}\,\mathcal{A}$, we denote the set of all scattered topologies τ on $\mathit{Ult}\,\mathcal{A}$ such that $\widehat{\Box a} = cd_\tau(\widehat{a})$ for any element a of $\mathcal{A}$.

Lemma 5.7 *Suppose* $\mathcal{A}$ *is a Magari algebra and* $\tau \in \mathit{Top}\,\mathcal{A}$*. Then there is a maximal with respect to inclusion element of* $\mathit{Top}\,\mathcal{A}$ *that extends* τ*.*

Proof. Consider the set $P = \{\sigma \in \mathit{Top}\,\mathcal{A} \mid \tau \subset \sigma\}$, which is a partially ordered set with respect to inclusion. We claim that any chain in P has an upper bound.

Assume C is a chain in P. Let ν be the coarsest topology containing τ and $\bigcup C$. Note that the topology ν is scattered as an extension of a scattered topology. For any element a of $\mathcal{A}$, we have $\widehat{\Box a} = cd_\tau(\widehat{a}) \subset cd_\nu(\widehat{a})$, because ν is an extension of τ.

Now assume c is an arbitrary element of $\mathcal{A}$ and $u \in cd_\nu(\widehat{c})$. We check that $u \in \widehat{\Box c}$. By definition of the co-derived-set operator, there is a punctured ν-neighbourhood V of u such that $V \subset \widehat{c}$. Since the set $\tau \cup \bigcup C$ is closed under finite intersections, it is a basis of ν. Consequently there is a subset W of V with $W \cup \{u\} \in \tau \cup \bigcup C$. We see that $W \subset \widehat{c}$ and W is a punctured neighbourhood of u with respect to a topology $\kappa \in \{\tau\} \cup C \subset \mathit{Top}\,\mathcal{A}$. Hence $u \in cd_\kappa(\widehat{c}) = \widehat{\Box c}$.

We obtain that $\widehat{\Box a} = cd_\nu(\widehat{a})$ for any element a of $\mathcal{A}$. Therefore $\nu \in \mathit{Top}\,\mathcal{A}$ and ν is an upper bound for C in P.

We see that any chain in P has an upper bound. By Zorn's lemma, there is a maximal element in P, which is the required maximal extension of τ. □

The following lemma was inspired by Lemma 4.5 from [3].

Lemma 5.8 *Suppose $\mathcal{A}$ is a Magari algebra and τ is a maximal element of $\mathit{Top}\,\mathcal{A}$. Then, for any $u \in \mathit{Ult}\,\mathcal{A}$ and any $V \in \tau$, we have $V \cup \{u\} \in \tau$ or there are a τ-open set W and an element a of $\mathcal{A}$ such that $u \in W$, $\Box a \notin u$ and $V \cap W \subset \widehat{a}$.*

Proof. Assume $u \in \mathit{Ult}\,\mathcal{A}$ and $V \in \tau$. It is sufficient to consider the case when $V \cup \{u\} \notin \tau$. Let σ be the coarsest topology containing τ and the set $V \cup \{u\}$. The topology σ is scattered as an extension of a scattered topology. Since τ is a maximal element of $\mathit{Top}\,\mathcal{A}$, the topology σ does not belong to $\mathit{Top}\,\mathcal{A}$ and there exists an element a of $\mathcal{A}$ such that $\widehat{\Box a} \neq cd_\sigma(\widehat{a})$. Notice that $\widehat{\Box a} = cd_\tau(\widehat{a}) \subset cd_\sigma(\widehat{a})$, because $\tau \subset \sigma$. Thus there is an ultrafilter h such that $h \in cd_\sigma(\widehat{a})$ and $h \notin cd_\tau(\widehat{a}) = \widehat{\Box a}$. Hence there is a punctured σ-neighbourhood of h that is included in $\widehat{a}$. In addition, note that $\tau \cup \{W \cap (V \cup \{u\}) \mid W \in \tau\}$ is a basis of σ. We see that $h \in B$ and $B \smallsetminus \{h\} \subset \widehat{a}$ for some $B \in \tau \cup \{W \cap (V \cup \{u\}) \mid W \in \tau\}$. If $B \in \tau$, then $h \in cd_\tau(\widehat{a})$. This is a contradiction with the condition $h \notin cd_\tau(\widehat{a})$. Therefore B has the form $W \cap (V \cup \{u\})$ for some $W \in \tau$. Since $h \in B = W \cap (V \cup \{u\})$, we have $h \in V$ or $h = u$. If $h \in V$, then $h \in W \cap V$ and $(W \cap V) \smallsetminus \{h\} \subset \widehat{a}$. In this case, we obtain $h \in cd_\tau(\widehat{a})$, which is a contradiction. Consequently $h \notin V$ and $h = u$. It follows that $\Box a \notin u$, $u \in W$ and $W \cap V = (W \cap (V \cup \{u\})) \smallsetminus \{h\} \subset \widehat{a}$.

□

For a scattered topological space (X, τ), the *derivative topology τ^+ on X* is defined as the coarsest topology including τ and $\{d_\tau(Y) \mid Y \subset X\}$. The next lemma was inspired by Lemma 5.1 from [3].

Lemma 5.9 *Suppose $\mathcal{A} = (A, \wedge, \vee, \rightarrow, 0, 1, \Box)$ is a Magari algebra and τ is a maximal element of $\mathit{Top}\,\mathcal{A}$. Then the topology τ^+ is generated by τ and the sets $d_\tau(\widehat{a})$ for $a \in A$.*

Proof. Assume τ is a maximal element of $\mathit{Top}\,\mathcal{A}$. Let τ' be the topology generated by τ and the sets $d_\tau(\widehat{a})$ for $a \in A$. It is clear that $\tau' \subset \tau^+$. We prove the converse. We shall check that $d_\tau(Y)$ is τ'-open for any $Y \subset \mathit{Ult}\,\mathcal{A}$.

Consider any $Y \subset \mathit{Ult}\,\mathcal{A}$ and any $u \in d_\tau(Y)$. We claim that there is a τ'-neighbourhood of u entirely contained in $d_\tau(Y)$. Suppose $\Box\Box a \in u$ and $\Box a \notin u$ for some $a \in A$. In this case, we see $u \notin \widehat{\Box a}$ and

$$\{u\} \cup \widehat{\Box a} \subset \widehat{\Box\Box a} = cd_\tau(\widehat{\Box a}) \subset cd_\tau(\{u\} \cup \widehat{\Box a}).$$

Hence the set $\{u\} \cup \widehat{\Box a}$ is τ-open. In addition, we see

$$u \in (\mathit{Ult}\,\mathcal{A} \setminus \widehat{\Box a}) = (\mathit{Ult}\,\mathcal{A} \setminus cd_\tau(\widehat{a})) = d_\tau(\widehat{\neg a}) \in \tau'.$$

It implies that

$$\{u\} = (\{u\} \cup \widehat{\Box a}) \cap (\mathit{Ult}\,\mathcal{A} \setminus \widehat{\Box a}) \in \tau'.$$

In other words, the ultrafilter u is a τ'-isolated point of $\mathit{Ult}\,\mathcal{A}$.

Now consider the case when, for any $a \in A$, we have $\Box\Box a \notin u$ whenever $\Box a \notin u$. By $int_\tau(X)$, we denote the τ-interior of a set X. Recall that $cd_\tau(X) = cd_\tau(int_\tau(X))$ for any set X in any topological space. Put $X = \mathit{Ult}\,\mathcal{A} \setminus Y$. Since $u \in d_\tau(Y)$ and $u \notin cd_\tau(X) = cd_\tau(int_\tau(X))$, the set $\{u\} \cup int_\tau(X) \notin \tau$. By Lemma 5.8, there are a τ-open set W and an element c of A such that $u \in W$, $\Box c \notin u$ and $int_\tau(X) \cap W \subset \widehat{c}$. Since, for any $a \in A$, $\Box\Box a \notin u$ whenever $\Box a \notin u$, we obtain $\Box\Box c \notin u$. It follows that

$$u \in W \cap (\mathit{Ult}\,\mathcal{A} \setminus \widehat{\Box\Box c}) = W \cap d_\tau(\widehat{\neg\Box c}) \in \tau'.$$

Thus $W \cap (\mathit{Ult}\,\mathcal{A} \setminus \widehat{\Box\Box c})$ is a τ'-neighbourhood of u. It remains to show that

$$W \cap (\mathit{Ult}\,\mathcal{A} \setminus \widehat{\Box\Box c}) \subset d_\tau(Y).$$

Indeed, we have

$$\begin{aligned} cd_\tau(X) \cap W \subset\ & cd_\tau(int_\tau(X)) \cap cd_\tau(W) = \\ &= cd_\tau(int_\tau(X) \cap W) \subset cd_\tau(\widehat{c}) \subset cd_\tau(cd_\tau(\widehat{c})) = \widehat{\Box\Box c}, \end{aligned} \quad (1)$$

because W is a τ-open set and $int_\tau(X) \cap W \subset \widehat{c}$. Hence,

$$\begin{aligned} W \cap (\mathit{Ult}\,\mathcal{A} \setminus \widehat{\Box\Box c}) &\subset W \cap (\mathit{Ult}\,\mathcal{A} \setminus (cd_\tau(X) \cap W)) \quad \text{(from 1)} \\ &= W \cap ((\mathit{Ult}\,\mathcal{A} \setminus cd_\tau(X)) \cup (\mathit{Ult}\,\mathcal{A} \setminus W)) \\ &= W \cap (d_\tau(Y) \cup (\mathit{Ult}\,\mathcal{A} \setminus W)) \\ &= (W \cap d_\tau(Y)) \cup (W \cap (\mathit{Ult}\,\mathcal{A} \setminus W)) \\ &= W \cap d_\tau(Y) \\ &\subset d_\tau(Y). \end{aligned}$$

This argument shows that any element of $d_\tau(Y)$ belongs to this set together with a τ'-neighbourhood. We conclude that $d_\tau(Y)$ is τ'-open and $\tau' = \tau^+$.

□

6 Global neighbourhood completeness

In this section we prove that any $\Box$-founded GLP-algebra can be embedded into the powerset algebra of a GLP-frame. As a corollary, we obtain global neighbourhood completeness for GLP w.r.t. non-well-founded derivations.

Analogously to the case of Magari algebras, by $Ult\,\mathcal{A}$, we denote the set of ultrafilters of a GLP-algebra $\mathcal{A}$. For a GLP-algebra $\mathcal{A} = (A, \wedge, \vee, \to, 0, 1, \Box_0, \Box_1, \dots)$, we denote the Magari algebra $(A, \wedge, \vee, \to, 0, 1, \Box_i)$ by $\mathcal{A}_i$. We see $Ult\,\mathcal{A} = Ult\,\mathcal{A}_i$ for any $i \in \mathbb{N}$. We call (maximal with respect to inclusion) elements of $Top\,\mathcal{A}_i$ *(maximal) i-topologies* on $Ult\,\mathcal{A}$.

Lemma 6.1 *For any GLP-algebra $\mathcal{A}$ and any maximal i-topology τ on $Ult\,\mathcal{A}$, there exists a maximal $(i+1)$-topology ν on $Ult\,\mathcal{A}$ such that $\tau \subset \nu$ and $d_\tau(Y)$ is ν-open for each $Y \subset Ult\,\mathcal{A}$.*

Proof. Assume we have a GLP-algebra $\mathcal{A}$ and a maximal i-topology τ on $Ult\,\mathcal{A}$. Consider the coarsest topology τ' containing τ^+ and all sets of the form $\{u\} \cup \widehat{\boxdot_{i+1} a}$, where $u \in Ult\,\mathcal{A}$, $\Box_{i+1} a \in u$ and $\boxdot_{i+1} a = a \wedge \Box_{i+1} a$. We see that $\tau \subset \tau'$ and $d_\tau(Y)$ is τ'-open for each $Y \subset Ult\,\mathcal{A}$. Trivially, the topology τ' is scattered as an extension of a scattered topology. We claim that $\tau' \in Top\,\mathcal{A}_{i+1}$.

We shall show that $\widehat{\Box_{i+1} a} = cd_{\tau'}(\widehat{a})$ for any element a of $\mathcal{A}$. First, we check that $\widehat{\Box_{i+1} a} \subset cd_{\tau'}(\widehat{a})$. For any ultrafilter d, if $d \in \widehat{\Box_{i+1} a}$, then $\widehat{\boxdot_{i+1} a}$ is a punctured τ'-neighbourhood of d. Also, $\widehat{\boxdot_{i+1} a} \subset \widehat{a}$. By definition of the co-derived-set operator, $d \in cd_{\tau'}(\widehat{a})$. Consequently $\widehat{\Box_{i+1} a} \subset cd_{\tau'}(\widehat{a})$.

Now we check that $cd_{\tau'}(\widehat{a}) \subset \widehat{\Box_{i+1} a}$. Consider any ultrafilter d such that $d \notin \widehat{\Box_{i+1} a}$. In addition, let W be an arbitrary punctured τ'- neighbourhood of d. It is sufficient to show that W is not included in $\widehat{a}$.

We have $\Box_{i+1} a \notin d$, $d \notin W$ and $W \cup \{d\} \in \tau'$. From Lemma 5.9, there is a basis of τ' consisting of alls sets of the form

$$V \cap d_\tau(\widehat{b_1}) \cap \dots \cap d_\tau(\widehat{b_n}) \cap (\{u_1\} \cup \widehat{\boxdot_{i+1} c_1}) \cap \dots \cap (\{u_m\} \cup \widehat{\boxdot_{i+1} c_m}),$$

where $V \in \tau$, $\{b_1, \dots, b_n\}$ and $\{c_1, \dots, c_m\}$ are (possibly empty) subsets of A, $\{u_1, \dots, u_m\}$ is a subset of $Ult\,\mathcal{A}$. In addition, $\Box_{i+1} c_k \in u_k$ for $k \in \{1, \dots, m\}$. Hence we have

$$d \in \big(V \cap d_\tau(\widehat{b_1}) \cap \dots \cap d_\tau(\widehat{b_n}) \cap (\{u_1\} \cup \widehat{\boxdot_{i+1} c_1}) \cap \dots \cap (\{u_m\} \cup \widehat{\boxdot_{i+1} c_m})\big) \subset W \cup \{d\}$$

for some element of the basis of τ'. We see that the ultrafilter d contains $\Diamond_i b_1, \dots, \Diamond_i b_n$ and $\Box_{i+1} c_1, \dots, \Box_{i+1} c_m$. Also, $\Diamond_{i+1} \neg a \in d$. In any GLP-algebra, we have

$$\bigwedge\{\Diamond_i b_1, \dots, \Diamond_i b_n\} \leqslant \Box_{i+1} \bigwedge\{\Diamond_i b_1, \dots, \Diamond_i b_n\},$$
$$\bigwedge\{\Box_{i+1} c_1, \dots, \Box_{i+1} c_m\} \leqslant \Box_{i+1} \bigwedge\{\boxdot_{i+1} c_1, \dots, \boxdot_{i+1} c_m\}.$$

Further, we have

$$\begin{aligned}(\Diamond_{i+1}\neg a) \wedge \bigwedge\{\Diamond_i b_1,\dots,\Diamond_i b_n, \Box_{i+1}c_1,\dots,\Box_{i+1}c_m\} \leqslant\\ \leqslant (\Diamond_{i+1}\neg a) \wedge \Box_{i+1}\bigwedge\{\Diamond_i b_1,\dots,\Diamond_i b_n\} \wedge \Box_{i+1}\bigwedge\{\boxdot_{i+1}c_1,\dots,\boxdot_{i+1}c_m\} \leqslant\\ \leqslant \Diamond_{i+1}\left((\neg a) \wedge \bigwedge\{\Diamond_i b_1,\dots,\Diamond_i b_n, \boxdot_{i+1}c_1,\dots,\boxdot_{i+1}c_m\}\right) \leqslant\\ \leqslant \Diamond_i\left((\neg a) \wedge \bigwedge\{\Diamond_i b_1,\dots,\Diamond_i b_n, \boxdot_{i+1}c_1,\dots,\boxdot_{i+1}c_m\}\right)\end{aligned}$$

We obtain $\Diamond_i\left((\neg a) \wedge \bigwedge\{\Diamond_i b_1,\dots,\Diamond_i b_n, \boxdot_{i+1}c_1,\dots,\boxdot_{i+1}c_m\}\right) \in d$ and

$$d \in d_\tau\left(\widehat{\neg a} \cap d_\tau(\widehat{b_1}) \cap \dots \cap d_\tau(\widehat{b_n}) \cap \widehat{\boxdot_{i+1}c_1} \cap \dots \cap \widehat{\boxdot_{i+1}c_m}\right).$$

Since V is a τ-neighbourhood of d, there exists an ultrafilter w such that

$$w \in (V \smallsetminus \{d\}) \cap \widehat{\neg a} \cap d_\tau(\widehat{b_1}) \cap \dots \cap d_\tau(\widehat{b_n}) \cap \widehat{\boxdot_{i+1}c_1} \cap \dots \cap \widehat{\boxdot_{i+1}c_m} \subset W.$$

Consequently w is an element of W, which does not belong to $\widehat{a}$.

We obtain that none of the punctured τ'-neighbourhoods of d are included in $\widehat{a}$. In other words, $d \notin cd_{\tau'}(\widehat{a})$ for any $d \notin \widehat{\Box_{i+1}a}$. This argument shows that $cd_{\tau'}(\widehat{a}) \subset \widehat{\Box_{i+1}a}$. Hence $\widehat{\Box_{i+1}a} = cd_{\tau'}(\widehat{a})$. We see $\tau' \in \mathit{Top}\,\mathcal{A}_{i+1}$.

Now we extend the topology τ' applying Lemma 5.7 and obtain the required maximal $(i+1)$-topology ν on $\mathit{Ult}\,\mathcal{A}$.

□

Lemma 6.2 *For any □-founded* GLP*-algebra $\mathcal{A}$, there exists a series of topologies $\tau_0, \tau_1, \dots$ on $\mathit{Ult}\,\mathcal{A}$ such that $(\mathit{Ult}\,\mathcal{A}, \tau_0, \tau_1, \dots)$ is a* GLP*-space and $\tau_i \in \mathit{Top}\,\mathcal{A}_i$ for any $i \in \mathbb{N}$.*

Proof. From Lemma 5.5, there exists a topology $\tau \in \mathit{Top}\,\mathcal{A}_0$. By Lemma 5.7, the topology τ can be extended to a maximal 0-topology τ_0. Applying Lemma 6.1, we obtain a series of topologies $\tau_1, \tau_2, \dots$ on $\mathit{Ult}\,\mathcal{A}$ such that $(\mathit{Ult}\,\mathcal{A}, \tau_0, \tau_1, \dots)$ is a GLP-space and $\tau_i \in \mathit{Top}\,\mathcal{A}_i$ for any $i \in \mathbb{N}$. □

The following theorem is analogous to Theorem 5.6 and is obtained by a similar argument. So we omit the proof.

Theorem 6.3 *A* GLP*-algebra is □-founded if and only if it is embeddable into the powerset* GLP*-algebra of a* GLP*-frame.*

Theorem 6.4 *For any set of formulas Γ and any formula φ, we have*

$$\Gamma \vdash_g \varphi \Longleftrightarrow \Gamma \Vdash_g \varphi \Longleftrightarrow \Gamma \vDash_g \varphi.$$

Proof. From Theorem 3.7 and Proposition 4.6, it remains to show that $\Gamma \Vdash \varphi$ whenever $\Gamma \vDash \varphi$. Assume $\Gamma \vDash \varphi$. Also assume we have a □-founded GLP-algebra $\mathcal{A} = (A, \wedge, \vee, \to, 0, 1, \Box_0, \Box_1, \dots)$ and a valuation v in $\mathcal{A}$ such that $v(\psi) = 1$ for any $\psi \in \Gamma$. We shall prove $v(\varphi) = 1$.

By the previous theorem, there exist a GLP-frame $\mathcal{X} = (X, \Box_0, \Box_1, \dots)$ and a mapping $f\colon A \to \mathcal{P}(X)$ such that f is an embedding of $\mathcal{A}$ into the powerset GLP-algebra of $\mathcal{X}$. We see that $w = f \circ v$ is valuation over $\mathcal{X}$, where $(\mathcal{X}, w) \vDash \psi$ for any $\psi \in \Gamma$. From the assumption $\Gamma \vDash \varphi$, we obtain $(\mathcal{X}, w) \vDash \varphi$. Since f is an embedding, we conclude that $v(\varphi) = 1$. □

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Appendix

Proof. [Proof of Proposition 2.2] First, we recall an important result from [2]. The logic J is obtained from GLP by replacing axiom schemes (iv-v) with the following ones all of which are provable in GLP:

(vi) $\Diamond_i\psi \to \Box_j \Diamond_i \psi$ for $i < j$;

(vii) $\Box_i\psi \to \Box_j \Box_i \psi$ for $i < j$;

(viii) $\Box_i\psi \to \Box_i \Box_j \psi$ for $i < j$.

A Kripke J-frame $(W, R_0, R_1, \dots)$ is a set W together with a sequence of binary relations on W such that

- R_i are transitive and conversely well-founded relations;
- xR_iy and yR_jz implies xR_iz, for $i < j$;
- xR_jy and xR_iz implies yR_iz, for $i < j$;
- xR_jy and yR_iz implies xR_iz, for $i < j$.

A notion of Kripke J-model is defined in the standard way.

L. Beklemishev showed in [2] that the logic J is Kripke complete, i.e. it is complete for its relational interpretation over the class of Kripke J-frames. In addition, he proved the following result: *if* $\mathsf{GLP} \nvdash \psi$*, then there is a* J*-model* $\mathcal{K}$ *such that all theorems of* GLP *are true in* $\mathcal{K}$ *and* $\mathcal{K} \nvDash \psi$ (see Theorem 4 from [2]).

Now we prove that for any formula ξ

$$\mathsf{GLP} \vdash \xi \Longleftrightarrow \varnothing \vdash_g \xi.$$

The left-to-right implication trivially holds. We prove the converse by *reductio ad absurdum*. Assume $\mathsf{GLP} \nvdash \xi$ and there is an ∞-derivation π with the root marked by ξ in which all leaves are marked by some axioms of GLP. Then there exist a J-model $\mathcal{K}$ and its world w such that $\mathcal{K}, w \nvDash \xi$ and all theorems of GLP are true at all worlds of $\mathcal{K}$. For a node x of the ∞-derivation π, let ψ_x be the formula of the node x. We define a sequence of pairs (x_n, w_n), where x_n is a node of π and w_n is a world of $\mathcal{K}$, such that $\mathcal{K}, w_n \nvDash \psi_{x_n}$. Let x_0 be the root of π and $w_0 = w$.

Given a pair (x_n, w_n) such that $\mathcal{K}, w_n \nvDash \psi_{x_n}$, we define (x_{n+1}, w_{n+1}). We see that x_n is not a leaf of π. Indeed, if x_n is a leaf of π, then the formula ψ_{x_n} is an axiom of GLP, which is a contradiction with the assertion that all theorems of GLP are true at all worlds of $\mathcal{K}$. We have that x_n is not a leaf of π and ψ_{x_n} is obtained by an application of an inference rule in π.

Suppose ψ_{x_n} is obtained by the rule (nec). Let x_{n+1} be the premise of x_n. We have $\psi_{x_n} = \Box_0\psi_{x_{n+1}}$ and $\mathcal{K}, w_n \nvDash \Box_0\psi_{x_{n+1}}$. Then there is a world w_{n+1} such that $w_nR_0w_{n+1}$ and $\mathcal{K}, w_{n+1} \nvDash \psi_{x_{n+1}}$.

If ψ_{x_n} is obtained by the rule (mp), then there is a node y such that y is a premise of x_n and $\mathcal{K}, w_n \nvDash \psi_y$. Set $x_{n+1} = y$ and $w_{n+1} = w_n$.

The sequence (x_n, w_n) is well-defined. We see that $x_0, x_1, \dots$ is an infinite branch in π. In addition, the sequence $w_0, w_1, \dots$ satisfies the condition: $w_n R_0 w_{n+1}$ if x_n is a conclusion of the rule (nec) in π, and $w_n = w_{n+1}$, otherwise. Since π is an ∞-derivation, the branch $x_0, x_1, \dots$ contains infinitely many applications of the rule (nec). Consequently, there is an infinite ascending sequence of worlds in $\mathcal{K}$ with respect to the relation R_0, which is a contradiction with the assertion that $\mathcal{K}$ is a J-model. This contradiction concluds the proof. □

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