Bisimulational Categoricity

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Abstract

We introduce and study the notion of *bisimulational categoricity* – the property of having a unique model *up to bisimulation.* We show that: (1) a complete modal theory (i.e. a maximal consistent set of formulae) t has a unique model up to bisimulation iff it has an image-finite model.
We further prove two analogous characterisations: (2) a complete theory t in transitive modal logic (EF-logic) has a unique model up to transitive bisimulation (EF-bisimulation) iff it has a finite model; and (3) a complete theory t in two-way modal logic has a unique model up to two-way bisimulation iff it has a model where every point has finite in- and out-degree.

Keywords: modal logic, model theory, categoricity, bisimulation.

1 Introduction

One of the central notions of classical model theory is that of *categoricity* – a theory is called *categorical* if it has a unique model *up to isomorphism.* In the context of modal logic, bisimilarity seems more appropriate than the isomorphism. One may therefore ask about *bisimulational categoricity*, i.e. the property of having a unique model *up to bisimulation.* [2]

It turns out that the notion of bisimulational categoricity for theories expressed in modal logic is indeed well-behaved and can be characterised in terms of image-finiteness. [3] We show that a complete theory in modal logic has a unique model up to bisimulation iff it has an image-finite model. While the right-to-left implication is (an easy folklore strengthening of) the well-known

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[2] Somewhat similar idea of finding modal analogues of classical results can be found in Chapter 6 of [12], where the author investigates the number of *non-bisimilar* models of a given modal fixpoint formula – analogically to the result of [9], where the number of *non-isomorphic* models of an MSO formula is considered. Nevertheless, both the result and the involved tools of the mentioned dissertation are rather far from the content of this paper.

[3] Note that, due to the obvious limitations given by the Skolem-Löwenheim Theorem, the classical notion of categoricity of first-order theories is only interesting when models of fixed cardinality are considered. However, unlike with isomorphism, structures of different sizes may still be bisimilar – and so there is no need to relativise *bisimulational* categoricity.

Hennessy-Milner Theorem [6], the left-to-right one requires adaptation of some classical model-theoretic tools and a simple topological argument. As such, our characterization can be thought of both as a completion of the Hennessy-Milner Theorem and as a modal version of the Ryll-Nardzewski Theorem (proven independently by Ryll-Nardzewski [10] Svenonius [11] and Engeler [3]).

Apart from standard modal logic, we provide analogous characterisations for two other interesting logics: transitive modal logic (sometimes known as the *EF-logic* in the context of computer science) and two-way modal logic (i.e. modal logic with both forward and backward modalities). We show that: (i) a complete theory in two-way modal logic has a unique model up to two-way bisimulation iff it has a model where every point has finite in- and out-degree and (ii) a complete theory in the transitive modal logic has a model unique up to transitive bisimulation (also called *EF*-bisimulation) iff it has a finite model.

In the proof we adapt standard model-theoretic tools to the modal framework and introduce new concepts of *induced modal logics* and *induced bisimulations*, which allow us to uniformly describe a wide range of modal-like logics and their corresponding bisimulations. We also discuss a simple example showing limitations of our method: modal logic enriched with the universal modality fails to have analogous characterisation.

The paper is organised as follows. After this introduction, in Section 2 we recall the basic notions and facts of modal logic and state our first main result, Theorem 2.8. Then, in Section 3 we formally introduce the notion of an inducing assignment, establish some simple related facts and prepare model-theoretic tools for the proof. Finally, in Section 4 we state the other two main theorems – Theorem 4.1 and Theorem 4.2 – and give proofs for all three of them. We conclude with a discussion of limitations of our method.

2 Modal Logic and Bisimulations

We assume the reader to be familiar with basic notions of modal logic ([1] is a good reference). However, for the sake of completeness and to fix notation, we recall the most basic definitions and facts.

Fix a finite set Σ of atomic propositions.

Definition 2.1 A (Kripke) *model* $\mathcal{M}$ for a signature $R = \{R_1, R_2, ..., R_l\}$ of binary relational symbols consists of: a universe M; an interpretation $R_k^{\mathcal{M}} \subseteq M \times M$ for every relation $R_k \in R$; and a valuation $\mathsf{val}^{\mathcal{M}} : \Sigma \to \mathcal{P}(M)$. A *pointed model* is a model with distinguished point – called its *root*. We will usually abuse terminology and call both non-pointed and pointed models just *models* whenever it does not lead to confusion. Moreover, following the notational traditions of modal logic we will skip parentheses and denote pointed models by $\mathcal{M}, p$ instead of $(\mathcal{M}, p)$.

The class of all models over signature R will be denoted $\mathsf{Krip(R)}$. We will typically identify a model with its universe and write $p \in \mathcal{M}$ instead of $p \in M$. Moreover, for the sake of simplicity we write R_k and val, skipping the superscripts whenever the model $\mathcal{M}$ is clear from the context.

Recall the standard syntax and semantics of (poly)modal logic $\mathsf{ML}(R)$ over signature R.

Definition 2.2 The set of formulae of modal logic Φ_R for binary signature R is given by the following grammar:

$$\varphi \mapsto \varphi \vee \varphi \mid \neg\varphi \mid \Diamond_k\varphi \mid a$$

for $a \in \Sigma$ and k such that $R_k \in R$. We use the standard syntactic sugar: $\Box_k\varphi = \neg\Diamond_k\neg\varphi$ and $\varphi \wedge \psi = \neg(\neg\varphi \vee \neg\psi)$. The modal depth of a formula is the maximal nesting of (possibly different) "$\Diamond_k$" operators. In case there is only one operator in R, we skip the subscript and write "$\Diamond$" instead of "$\Diamond_1$".

Definition 2.3 Given a model $\mathcal{M} \in \mathsf{Krip(R)}$, the *semantics map* $[\![\_]\!]^{\mathcal{M}} : \Phi_R \to \mathcal{P}(M)$ is defined inductively as follows:

$$\begin{aligned} [\![a]\!]^{\mathcal{M}} &= \mathsf{val}^{\mathcal{M}}(a); \\ [\![\varphi \vee \psi]\!]^{\mathcal{M}} &= [\![\varphi]\!]^{\mathcal{M}} \cup [\![\psi]\!]^{\mathcal{M}}; \\ [\![\neg\varphi]\!]^{\mathcal{M}} &= M - [\![\varphi]\!]^{\mathcal{M}}; \\ [\![\Diamond_k\varphi]\!]^{\mathcal{M}} &= \{p \in M \mid \exists_{q \in [\![\varphi]\!]^{\mathcal{M}}} pR_k^{\mathcal{M}}q\}. \end{aligned}$$

Definition 2.4 A *bisimulation* between two (not necessarily distinct) models $\mathcal{M}, \mathcal{M}' \in \mathsf{Krip(R)}$ is a relation $Z \subseteq M \times M'$ that satisfies, for every $a \in \Sigma$, $R_k \in R$ and pZp':

- (base condition) $p \in \mathsf{val}(a) \iff p' \in \mathsf{val}(a)$;
- (forth condition) if pR_kq then there exists q' s.t. $p'R_kq'$ and qZq';
- (back condition) if $p'R_kq'$ then there exists q s.t. pR_kq and qZq'.

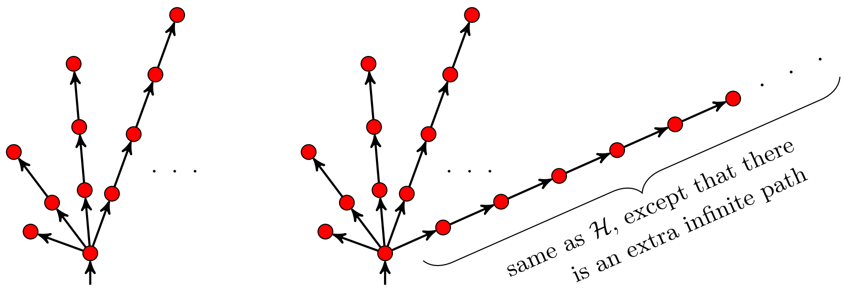
Pointed models $\mathcal{M}, p$ and $\mathcal{M}', p'$ are said to be *bisimilar* if there exists a bisimulation Z between them s.t. pZp' (notation $\mathcal{M}, p \leftrightarroweq \mathcal{M}', p'$). A *functional bisimulation* is a function whose graph is a bisimulation. We will also use the standard characterization of bisimilarity in terms of a bisimulation game between players ∃ve and ∀dam.

It is widely known that modal logic is invariant under bisimulation, i.e. bisimilar points are always logically indistinguishable. The converse may require an additional assumption of image-finiteness.

Definition 2.5 A model $\mathcal{M} \in \mathsf{Krip(R)}$ is *image-finite* if every point $p \in \mathcal{M}$ has only finitely many R_k-children for every $R_k \in R$.

The classical result of Hennessy and Milner [6] states that, in image-finite models, points that are logically indistinguishable have to be bisimilar. The following example shows that without the assumption of image-finiteness this does not have to be the case.

Example 2.6 The Hedgehogs: $\mathcal{H}, \mathsf{root}_{\mathcal{H}}$ and $\mathcal{H}', \mathsf{root}_{\mathcal{H}'}$ [4]:



The two models are not bisimilar, as one of them is well-founded but the other is not. However, it is easy to show that they cannot be distinguished by ML formulae. [5]

As it turns out, the above example is an illustration of a general phenomenon, which is that among infinitely many behaviours one can always find a *limit* one that: (i) can be either included or removed from the model but (ii) our local logical means are too weak to tell the difference. This will be the key intuition underlying our characterisation of bisimulational categoricity (i.e. the property of having a unique model up to bisimulation). Roughly, the characterization says that the requirement of image-finiteness – treated up to bisimulation – is not only sufficient but also necessary.

In order to formulate the theorem, we first formally introduce the notion of a *type* – i.e. a maximal consistent set of formulae – analogous to types in first-order model theory (here by *type* we always mean a *complete* one). For the sake of simplicity, let us confine ourselves to the case when the signature consists of a single relation "$\to$" (the symbol should not be confused with implication: "$\Rightarrow$").

Definition 2.7 Given a point $p \in \mathcal{M} \in \mathsf{Krip}(\{\to\})$, its *modal type* – denoted $\mathbf{tp}^{\mathcal{M}}(p)$ – is the set $\{\varphi \in \Phi_{\{\to\}} \mid p \in [\![\varphi]\!]^{\mathcal{M}}\}$ of all modal formulae it satisfies. The set of all modal types will be denoted $\mathbb{T}$.

We are now ready to formulate our first main theorem.

Theorem 2.8 *For every type $t \in \mathbb{T}$, the following are equivalent:*

(1) *t has a unique model up to $\leftrightarroweq$;*

(2) *every model of t is bisimilar to an image-finite model;*

(3) *t has a model which is image-finite.*

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[4] Here the valuation is not important – for the sake of this example assume $\Sigma = \emptyset$.

[5] In fact, even the full first-order logic cannot distinguish the models, as can be shown using Ehrenfeucht-Fraïssé games.

We will moreover show two analogous characterisations involving two other logics and their corresponding equivalence relations. In order to neatly extract the common part of the structure of the logics we investigate, and because it is interesting in its own right, we formally introduce the notion of induced relations.

3 Induced Relations

Various modal-like logics and bisimilarity-like congruences can be obtained by considering some relation *induced* by the original accessibility relation.

Definition 3.1 Given two binary signatures S, R (*source* and *result*), an *inducing assignment* is an assignment

$$\mathsf{ind} : \mathsf{Krip(S)} \to \mathsf{Krip(R)}$$

such that every $\mathcal{M} \in \mathsf{Krip(S)}$ has the same universe and valuation as its image $\mathsf{ind}(\mathcal{M})$.

3.1 Induced Logic and Bisimulations

Every inducing assignment gives rise to the induced logic.

Definition 3.2 Given an inducing assignment $\mathsf{ind} : \mathsf{Krip(S)} \to \mathsf{Krip(R)}$, we define the *induced modal logic* $\mathsf{ML_{ind}}$ interpreted over $\mathsf{Krip(S)}$. Formulae $\Phi_{\mathsf{ind}} = \Phi_R$ are standard modal formulae over signature R. The semantics map $[\![\_]\!]^{\mathcal{M}}_{\mathsf{ind}} : \Phi_{\mathsf{ind}} \to \mathcal{P}(M)$ is defined with respect to the induced model – on every $\mathcal{M} \in \mathsf{Krip(S)}$ we put:

$$[\![\varphi]\!]^{\mathcal{M}}_{\mathsf{ind}} = [\![\varphi]\!]^{\mathsf{ind}(\mathcal{M})}$$

We say that model $\mathcal{M}, p$ *satisfies* formula φ (notation: $\mathcal{M}, p \models \varphi$) if $\varphi \in [\![\varphi]\!]^{\mathcal{M}}_{\mathsf{ind}}$. Models $\mathcal{M}, p$ and $\mathcal{N}, q$ are *equivalent* (denoted $\mathcal{M}, p \equiv_{\mathsf{ML_{ind}}} \mathcal{N}, q$) if they satisfy the same $\mathsf{ML_{ind}}$ formulae.

Similarly to the induced logic, we also define an *induced bisimulation*, where we ignore the original relations and only take the induced ones into account.

Definition 3.3 Given an assignment $\mathsf{ind} : \mathsf{Krip(S)} \to \mathsf{Krip(R)}$, a relation $Z \subseteq M \times N$ between models $\mathcal{M}, \mathcal{N} \in \mathsf{Krip(S)}$ is an ind-*bisimulation* if it is a bisimulation between $\mathsf{ind}(\mathcal{M})$ and $\mathsf{ind}(\mathcal{N})$. Induced bisimilarity is defined accordingly and denoted $\leftrightarrows_{\mathsf{ind}}$.

The standard characterization of bisimilarity in terms of a two-player game carries over to the induced setting. Moreover, it follows immediately from invariance of modal logic under bisimulation that for any ind, $\mathsf{ML_{ind}}$ is invariant under $\leftrightarrows_{\mathsf{ind}}$:

Proposition 3.4 *For any pair of models $\mathcal{M}, \mathcal{N} \in \mathsf{Krip(S)}$, if $\mathcal{M}, p \leftrightarrows_{\mathsf{ind}} \mathcal{N}, q$ then $\mathcal{M}, p \equiv_{\mathsf{ML_{ind}}} \mathcal{N}, q$.*

As it was mentioned, several interesting logics and bisimilarity relations fit well into our induced framework. Let us show a few examples.

Example 3.5 A trivial example is the identity assignment Id. Logic induced by $\mathsf{Id} : \mathsf{Krip}(\mathsf{R}) \to \mathsf{Krip}(\mathsf{R})$ is the same as the original one, i.e. $\mathsf{ML}_{\mathsf{Id}} = \mathsf{ML}(R)$. Likewise, $\leftrightarrows_{\mathsf{Id}}$ equals $\leftrightarrows$.

Example 3.6 Let $\mathsf{ind}_{\leftrightarrows} : \mathsf{Krip}(\{\to\}) \to \mathsf{Krip}(\{\to, \leftarrow\})$ be the assignment that keeps the relation "$\to$" unchanged and additionally introduces its inverse (i.e. a fresh relation "$\leftarrow$" s.t. $p \leftarrow q$ iff $p \to q$ for any two points p and q in the model). Then, $\mathsf{ML}_{\mathsf{ind}_{\leftrightarrows}}$ is the modal logic with *forward* and *backward* (or *future* and *past*) modalities and $\leftrightarrows_{\mathsf{ind}_{\leftrightarrows}}$ is a two-way bisimilarity – where a two-way bisimulation is a relation that is a bisimulation w.r.t. both the accessibility relation and its converse.

Example 3.7 Consider the assignment $\mathsf{ind}_{+} : \mathsf{Krip}(\{\to\}) \to \mathsf{Krip}(\{\to^{+}\})$ that maps a relation to its transitive closure. That way we obtain the transitive modal logic $\mathsf{ML}_{\mathsf{ind}_{+}}$ and transitive bisimilarity $\leftrightarrows_{\mathsf{ind}_{+}}$ – also known as *EF-logic* and *EF-bisimilarity* in the context of computer science (see e.g. [2]).

Example 3.8 Let $\mathsf{ind}_{\forall} : \mathsf{Krip}(\{\to\}) \to \mathsf{Krip}(\{\to, \langle\exists\rangle\})$ be the assignment that keeps "$\to$" and adds a new relation "$\langle\exists\rangle$" which is the full relation on the model's universe. This gives raise to logic $\mathsf{ML}_{\mathsf{ind}_{\forall}}$ being the modal logic with universal modalities and to $\leftrightarrows_{\mathsf{ind}_{\forall}}$ being global bisimilarity.

It is worth to emphasize that the term "logic" as we use it denotes a *set of formulae* together with an appropriate *satisfaction relation* between formulae and models. In particular, it is something different from what is known as *normal modal logic* which is just a *sets of formulae*. For example, *the set of all tautologies* of the transitive modal logic $\mathsf{ML}_{\mathsf{ind}_{+}}$ is precisely *the normal modal logic K4*.

The next example shows that one has to be careful, as in general ind could encode an oracle for arbitrary class of models:

Example 3.9 Let $\mathcal{C}$ be an arbitrary class of pointed models over signature S. The assignment $\mathsf{ind}_{\mathcal{C}} : \mathsf{Krip}(\mathsf{S}) \to \mathsf{Krip}(\mathsf{S} \cup \{\mathsf{R}_{\mathcal{C}}\})$ takes a model $\mathcal{M} \in \mathsf{Krip}(\mathsf{S})$, keeps all the relations from S unchanged and sets $pR_{\mathcal{C}}q$ iff $p = q$ and $\mathcal{M}, p \in \mathcal{C}$ – i.e. $\mathsf{ind}_{\mathcal{C}}$ adds a self-loop labelled by "$\mathcal{C}$" to precisely these points p for which $\mathcal{M}, p \in \mathcal{C}$. Then, the formula $\Diamond_{\mathcal{C}}\top$ is true in $\mathcal{M}, p$ iff $\mathcal{M}, p \in \mathcal{C}$.

3.2 Model Theory – The Space of Types

The notion of modal type can be adapted to the induced setting in a natural way.

Definition 3.10 Given a logic $\mathsf{ML}_{\mathsf{ind}}$, we define an $\mathsf{ML}_{\mathsf{ind}}$-*type* of a point $p \in \mathcal{M} \in \mathsf{Krip}(\mathsf{S})$ – denoted $\mathbf{tp}^{\mathcal{M}}(p)$ – to be the set $\{\varphi \in \mathsf{ML}_{\mathsf{ind}} \mid \mathcal{M}, p \models \varphi\}$. The set of all $\mathsf{ML}_{\mathsf{ind}}$-types will be denoted $\mathbb{T}_{\mathsf{ind}}$.

Along the same lines as in the classical model theory for first-order logic, our types can be equipped with a topology turning it into a Hausdorff space.

Definition 3.11 For any $\varphi \in \mathsf{ML}_{\mathsf{ind}}$, we take the set $\langle\varphi\rangle = \{t \in \mathbb{T}_{\mathsf{ind}} \mid \varphi \in t\}$ of all types containing it. Then, the set $\{\langle\varphi\rangle \mid \varphi \in \mathsf{ML}_{\mathsf{ind}}\}$ is a basis of clopen sets generating a topology on $\mathbb{T}_{\mathsf{ind}}$.

Alternatively, one could obtain the same topology by first picking any enumeration of $\mathsf{ML}_{\mathsf{ind}}$ formulae and then defining a metric $d(t, t') = \frac{1}{n}$ for n being the number of the first formula on which t and t' differ (and 0 if $t = t'$). The underlying intuition is that types which are similar – i.e. hard to distinguish – should be close to each other.

Proposition 3.12 *Analogously to the first-order case, we have that:*

- *the space* $\mathbb{T}_{\mathsf{ind}}$ *is always Hausdorff;*
- *the logic* $\mathsf{ML}_{\mathsf{ind}}$ *is compact (i.e. if any finite fragment of a set of formulae* t *is satisfiable, then so is the entire* t*)* $\iff$ *the space* $\mathbb{T}_{\mathsf{ind}}$ *is compact;*
- *given* $T \subseteq \mathbb{T}_{\mathsf{ind}}$*,* $t \in \mathbb{T}_{\mathsf{ind}}$ *is isolated in* $T \iff$ *there exists a single* $\mathsf{ML}_{\mathsf{ind}}$ *formula* $\varphi \in t$ *s.t.* $\varphi \notin t'$ *for every other* $t' \in T$*.*

Proof. Observe that by identifying a type with its characteristic function, we can view the space $\mathbb{T}_{\mathsf{ind}}$ as a subspace of $2^{\Phi_{\mathsf{ind}}}$. Since the later is Hausdorff, so is $\mathbb{T}_{\mathsf{ind}}$. Moreover, a subspace of a compact Hausdorff space is compact iff it is closed – and it is easy to check that closedness of $\mathbb{T}_{\mathsf{ind}}$ is the same as logical compactness of $\mathsf{ML}_{\mathsf{ind}}$. The last item follows from the observation that in any topological space, a point is isolated iff it is isolated by a basic open set. □

An important notion that can be generalised to the induced setting is that of modal saturation (also called m-saturation). Our topology on types allows us to capture it in an elegant way.

Definition 3.13 We say that a point p in a model $\mathcal{M} \in \mathsf{Krip}(\mathsf{S})$ is $\mathsf{ML}_{\mathsf{ind}}$-*saturated* if for every $R_k \in R$, the set of types of its R_k-children $\{\mathbf{tp}^{\mathcal{M}}(q) |\, pR_kq\}$ is closed. We call $\mathcal{M}$ $\mathsf{ML}_{\mathsf{ind}}$-*saturated* if all its points are $\mathsf{ML}_{\mathsf{ind}}$-saturated.

In more concrete terms (the way modal saturation is usually defined): $\mathsf{ML}_{\mathsf{ind}}$-saturation means that if any finite fragment of t is realised in some R_k-child of p, then there exists a p's R_k-child realising the entire t. The following is an immediate consequence of an analogous fact for the standard case of $\mathsf{ML}(R)$ and $\leftrightarrows$:

Theorem 3.14 *Given any two* $\mathsf{ML}_{\mathsf{ind}}$*-saturated models* $\mathcal{M}, \mathcal{M}' \in \mathsf{Krip}(\mathsf{S})$*:*

$$\mathcal{M}, p \equiv_{\mathsf{ML}_{\mathsf{ind}}} \mathcal{M}', p' \qquad \underline{\textit{implies}} \qquad \mathcal{M}, p \leftrightarrows_{\mathsf{ind}} \mathcal{M}', p'$$

for any $p \in \mathcal{M}, p' \in \mathcal{M}'$*.*

Note that it is immediate that $\mathsf{ML}_{\mathsf{ind}}$-saturation generalises the notion of image-finiteness (w.r.t. the induced relations), as in a Hausdorff space finite sets are always closed.

4 The Main Theorem: Bisimulational Categoricity

After collecting all the necessary notions and tools, we are now ready to state and prove three theorems being the main contribution of this paper (including the already mentioned Theorem 2.8).

Theorem 2.8 *For every type $t \in \mathbb{T}$, the following are equivalent:*

(1) *t has a unique model up to $\leftrightarroows$;*

(2) *every model of t is bisimilar to an image-finite model;*

(3) *t has a model which is image-finite.*

Theorem 4.1 *For every type $t \in \mathbb{T}_{\mathsf{ind}_{\leftrightarrows}}$, the following are equivalent:*

(1) *t has a unique model up to $\leftrightarroows_{\mathsf{ind}_{\leftrightarrows}}$;*

(2) *every model of t is $\mathsf{ind}_{\leftrightarrows}$-bisimilar to a model where every point has finite in- and out-degree;*

(3) *t has a model where every point has finite in- and out-degree.*

Theorem 4.2 *For every type $t \in \mathbb{T}_{\mathsf{ind}_+}$, the following are equivalent:*

(1) *t has a unique model up to $\leftrightarroows_{\mathsf{ind}_+}$;*

(2) *every model of t is ind_+-bisimilar to a finite model;*

(3) *t has a finite model.*

Note that in light of Proposition 4.9, the last theorem implies that when it comes to defining models up to transitive bisimulation, the expressive power of the transitive modal logic does not increase when we move from single formulae to entire theories.

Let us now prove the theorems. Most of the proof is the same in all three cases of Theorems 2.8, 4.1 and 4.2.

4.1 (2) $\Rightarrow$ (3)

In all the three cases, the implication (2) $\Rightarrow$ (3) is immediate, as by definition every type has a model.

4.2 (3) $\Rightarrow$ (1)

Let us now prove a generalisation of the Hennessy-Milner Theorem [6] for $\mathsf{ML}_{\mathsf{ind}}$. It strengthens the standard formulation of Hennessy-Milner-like results in that we only require one of the models to be image-finite (which, in the context of usual modal logic ML, is a well-known folklore strengthening of the original Hennessy-Milner Theorem). It does not require any assumptions on ind and the proof is essentially the same as in the standard case.

Theorem 4.3 (à la Hennessy-Milner) *Assume $\mathcal{M} \in \mathsf{Krip}(\mathsf{S})$ and the induced model $\mathsf{ind}(\mathcal{M})$ is image-finite. Then, for every $\mathcal{M}' \in \mathsf{Krip}(\mathsf{S})$ and every $p \in \mathcal{M}, p' \in \mathcal{M}'$:*

$$\mathcal{M}, p \equiv_{\mathsf{ML}_{\mathsf{ind}}} \mathcal{M}', p' \qquad \underline{\textit{implies}} \qquad \mathcal{M}, p \leftrightarroows_{\mathsf{ind}} \mathcal{M}', p'.$$

Proof. It suffices to show that the relation $\equiv_{\mathsf{ML}} \subseteq M \times M'$ of modal equivalence is itself an ind-bisimulation. The base condition is immediate.

For the back and the forth conditions, let us take $q \equiv_{\mathsf{ML}_{\mathsf{ind}}} q'$, and any $R_k \in R$. By our assumption, q can only have a finite number of R_k-children (in

$\mathsf{ind}(\mathcal{M})$). In particular, they have only a finite number of distinct modal types $t_1, ..., t_n$ – and since $\mathbb{T}_{\mathsf{ind}}$ is a Hausdorff space, we can find pairwise mutually exclusive formulae $\varphi_1, ..., \varphi_n$ s.t. $\varphi_i \in t_i$ but $\varphi_i \notin t_j$ for $i \neq j$. Both q – and by equivalence also q' – satisfy:

$$\Box_k(\bigvee_{i\in\{1,...,n\}} \varphi_i); \qquad \bigwedge_{i\in\{1,...,n\}} \Diamond_k\varphi_i; \qquad \Box_k(\varphi_i \Rightarrow \psi) \text{ for any } \psi \in t_i$$

It follows that the types of R_k-children of q' are exactly $t_1, ..., t_n$. But this implies both the forth and the back conditions, as it means that for every R_k-child of q (or q', respectively) there exists an equivalent R_k-child of q' (resp. q). □

4.3 (1) ⇒ (2)

The last (and hardest to prove) implication is from (1) to (2). Before we proceed, let us recall an elementary topological fact. Since any infinite compact space has to contain a non-isolated point and closed subspaces of a compact space are always compact, it follows that:

Lemma 4.4 *If Y is a closed infinite subset of a compact topological space X, then it contains a point $y \in Y$ that is not isolated in Y.*

As in the classical model theory, we would like to use some good properties based on compactness of the considered logic. However, as shown by Example 3.9, an inducing assignment can encode arbitrary properties and thus in general the logic $\mathsf{ML}_{\mathsf{ind}}$ does not have to be compact. Fortunately, we may overcome this difficulty thanks to additional good properties of the considered assignments.

Lemma 4.5 *Assume that the image of* ind *is axiomatized by a set of sentences A expressed in first-order logic, i.e.:*

$$\mathsf{ind}[\mathsf{Krip}(\mathsf{S})] = \{\mathcal{M} \in \mathsf{Krip}(\mathsf{R}) \mid \mathcal{M} \text{ satisfies } A\}.$$

Then:

- *the logic* $\mathsf{ML}_{\mathsf{ind}}$ *is compact;*
- *every* $t \in \mathbb{T}_{\mathsf{ind}}$ *has an* $\mathsf{ML}_{\mathsf{ind}}$*-saturated model* $\mathcal{M}, r \models t$.

Proof. For the first item, take any set of formulae $t \subseteq \mathsf{ML}_{\mathsf{ind}}$ and translate it to equivalent set t^{FO} of formulae in first-order logic over the signature $R \cup \{a(x) \mid a \in \Sigma\}$. Observe that t is satisfiable w.r.t. the induced semantics iff $A \cup t^{\mathsf{FO}}$ is satisfiable in $\mathsf{Krip}(\mathsf{R})$ in the standard sense. Hence, compactness of $\mathsf{ML}_{\mathsf{ind}}$ follows from compactness of the first-order logic.

The second item can be proven in a similar way, using the model-theoretic method of elementary saturated extensions. The proof is just a straightforward modification of the standard one (e.g. in [1]) and as such is skipped. [6] □

[6] In fact, if one defines induced first-order logic $\mathsf{FO}_{\mathsf{ind}}$ analogously to the induced modal logic – by interpreting it via ind – the assumption of first-order axiomatizability of $\mathsf{ind}[\mathsf{Krip}(\mathsf{S})]$ allows for a generalisation of van Benthem's theorem saying that $\mathsf{ML}_{\mathsf{ind}}$ is precisely the $\leftrightarrows_{\mathsf{ind}}$-invariant fragment of $\mathsf{FO}_{\mathsf{ind}}$.

Note that all the assignments $\mathsf{Id}, \mathsf{ind}_{\leftrightarrows}, \mathsf{ind}_{+}$ and $\mathsf{ind}_{\forall}$ satisfy the assumptions of the above lemma.

Before we proceed, let us adapt two basic constructions related to the notion of a bisimulation to our context – generated submodels and bisimulation quotients (also called bisimulation contractions):

Proposition 4.6 (generated submodels) *Let* ind *be either* Id*,* $\mathsf{ind}_{\leftrightarrows}$ *or* ind_{+}*. Given a model* $\mathcal{M} \in \mathsf{Krip}(\mathsf{S})$ *and a point* $p \in \mathcal{M}$*, the model generated by* p*, denoted* $\langle p \rangle_{\mathcal{M}}$*, is just the submodel of* $\mathcal{M}$ *consisting of points reachable from* p *by a finite path in* $\mathsf{ind}(\mathcal{M})$ *(including* p *itself). Then,* $\mathcal{M}, q \leftrightarrows_{\mathsf{ind}} \langle p \rangle_{\mathcal{M}}, q$ *for any* $q \in \langle p \rangle_{\mathcal{M}}$*.*

Proposition 4.7 (quotients) *Let* ind *be either* Id*,* $\mathsf{ind}_{\leftrightarrows}$ *or* ind_{+}*. For an* ind*-bisimulation* $Z \subseteq \mathcal{M} \times \mathcal{M}$ *being an equivalence relation, there is a model structure on the set of all equivalence classes of* Z *s.t. the projection map* $p \overset{\pi_Z}{\longmapsto} [p]_{/Z}$ *is a functional* ind*-bisimulation. We call that model a quotient of* $\mathcal{M}$ *by* Z *– and denote it* $\mathcal{M}_{/Z}$*.*[7]

Proof. Both constructions are the same as in the standard case – except for quotients by transitive bisimulations.

Given a model $\mathcal{M}$ and a transitive bisimulation Z being an equivalence relation on M, we can first take the model $\mathcal{M}^{+} = (M, \to^{\mathcal{M}^{+}}, \mathsf{val}^{\mathcal{M}})$ with $\to^{\mathcal{M}^{+}}$ being the transitive closure of $\to^{\mathcal{M}}$.

Observe that $\mathsf{ind}_{+}(\mathcal{M}) = \mathsf{ind}_{+}(\mathcal{M}^{+})$ and hence: (*) the identity map $\mathsf{Id} : M \to M$ can be seen as a functional transitive bisimulation $\mathsf{Id} : \mathcal{M} \to \mathcal{M}^{+}$. Moreover, transitivity of $\to^{\mathcal{M}^{+}}$ implies that: (**) on $\mathcal{M}^{+}$, transitive bisimulations are the same as standard bisimulations.

Since (transitive) bisimulations are closed under compositions, (*) implies that Z is a transitive bisimulation not only on $\mathcal{M}$, but also on $\mathcal{M}^{+}$ – and so by (**) it is also a standard bisimulation on $\mathcal{M}^{+}$. This allows us to quotient (in the standard sense) $\mathcal{M}^{+}$ by Z obtaining $(\mathcal{M}^{+})_{/Z}$. Since the natural projection $\pi_Z : \mathcal{M}^{+} \to (\mathcal{M}^{+})_{/Z}$ is a functional bisimulation and bisimulations are always instances of transitive bisimulations, the graph of the function π_Z – and therefore by (*) also $\pi_Z \circ \mathsf{Id} : \mathcal{M} \to (\mathcal{M}^{+})_{/Z}$ – is a transitive bisimulation. □

We are now ready for the proof.

Case 1: $\mathsf{ind} = \mathsf{Id}$

Let us take a model $\mathcal{M}, r$ that is not bisimilar to any image-finite model – we will construct another model that is equivalent, but non-bisimilar to it. We may combine: (i) Lemma 4.5 to obtain an equivalent model which is $\mathsf{ML}(\{\to\})$-saturated, (ii) Proposition 4.7 to take its quotient by $\leftrightarrows$ where (by Proposition 3.14) no two points satisfy the same formulae and finally (iii) apply Proposition 4.6 to take a submodel accessible from the root. If such model is not bisimilar

[7] Note that in the case of ind_{+} such quotient does not have to be unique. Nevertheless, it is unique *up to* $\leftrightarrows_{\mathsf{ind}}$.

to $\mathcal{M}, r$, we are done – so the remaining case is when $\mathcal{M}, r$ has all the properties listed above.

Since by our assumption $\mathcal{M}, r$ is not image-finite, there must exist a point p reachable from r by a finite path and having infinitely many children. The set $T = \{\mathbf{tp}^{\mathcal{M}}(q) \mid p \to q\}$ is an infinite closed subset of a compact space and so by Lemma 4.4 it contains a non-isolated limit type $t^{\lim}$ realised in some p's child $p^{\lim}$.

Now, in order to construct another model for t we simply remove the arrow leading from p to $p^{\lim}$:

$$\mathcal{N} = (M, \to^{\mathcal{M}} - \{(p, p^{\lim})\}, \mathsf{val}^{\mathcal{M}})$$

We prove by induction on n that any point $q \in \mathcal{M}$ satisfies exactly the same formulae of modal depth n in both $\mathcal{M}$ and $\mathcal{N}$ (and thus in particular $\mathcal{N}, r \models t$). The base case is obvious. For the induction step, the only interesting case is for p, as prima facie it could satisfy less sentences of the form $\Diamond\varphi$. However, since $t^{\lim}$ is not isolated in T, for any $\varphi \in t^{\lim}$ there must be $t' \in T$ s.t. $\varphi \in t'$. By definition of T this means that there is a sibling s of $p^{\lim}$ s.t. $\mathcal{M}, s \models t'$ – and so in particular $\mathcal{M}, s \models \varphi$. But modal depth of φ is smaller than that of $\Diamond\varphi$ – so we know by induction hypothesis that $\mathcal{N}, s \models \varphi$, and hence $\mathcal{N}, p \models \Diamond\varphi$.

On the other hand, we will show that $\mathcal{M}, r \not\rightleftharpoons \mathcal{N}, r$, as ∀dam has the following winning strategy in the bisimulation game: (i) First follow the path to the point p in $\mathcal{M}$. If after that ∃ve responds with a point $q \in \mathcal{N}$ other than p, we know that $\mathcal{M}, p \not\equiv_{\mathsf{ML}} \mathcal{N}, q$ (as no two different points are equivalent in $\mathcal{N}$) and so $\mathcal{M}, p \not\rightleftharpoons \mathcal{N}, q$ – which means that ∀dam can now win the game. (ii) If ∃ve responded with the same point $p \in \mathcal{N}$, ∀dam moves to $p^{\lim}$ in $\mathcal{M}$. Now ∃ve has to respond with some point $q \in \mathcal{N}$ – but by definition of $\mathcal{N}$ we know that she cannot choose $p^{\lim}$, and so again $\mathcal{M}, p^{\lim} \not\equiv_{\mathsf{ML}} \mathcal{N}, q$, meaning that ∀dam can win the game from that point.

Case 2: $\mathsf{ind} = \mathsf{ind}_{\leftrightarrows}$

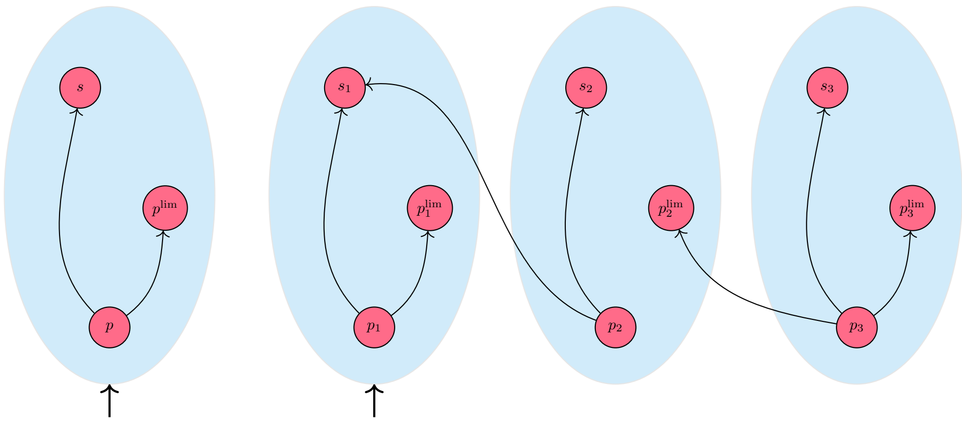
In this case, we need a slight modification of the previous construction due to the fact that we deal with two-way modalities and removing an arrow $q \to q'$ changes both sets: q's successors and q''s predecessors.

As in the previous case, we take an $\mathsf{ML}_{\mathsf{ind}_{\leftrightarrows}}$-saturated model of $t \in \mathbb{T}_{\mathsf{ind}_{\leftrightarrows}}$ where any two different points have different types and any point is accessible by a finite path (possibly using forward and backward moves) from the root – s.t. some point $p \in \mathcal{M}$ has infinitely many successors (the case with infinitely many *predecessors* is entirely symmetric). We take the limit $t^{\lim}$ of $T = \{\mathbf{tp}^{\mathcal{M}}(q) \mid p \to q\}$ realised by some $p^{\lim}$.

We define $\mathcal{N}$ as follows. First take the disjoint union $\mathcal{N}' = \mathcal{M}_1 + \mathcal{M}_2 + \mathcal{M}_3$, where each $\mathcal{M}_i$ is a copy of $\mathcal{M}$. We will denote the element of $\mathcal{M}_i$ corresponding to $q \in \mathcal{M}$ by q_i. Let us also pick any child $s \in \mathcal{M}$ of p different than $p^{\lim}$. Then, our model $\mathcal{N}$ is just $\mathcal{N}'$ without the arrow $p_2 \to p_2^{\text{lim}}$ and with two additional arrows $p_2 \to s_1$ and $p_3 \to p_2^{\text{lim}}$:

$$\mathcal{N} = (N', \to^{\mathcal{N}'} - \{(p_2, p_2^{\text{lim}})\} \cup \{(p_2, s_1), (p_3, p_2^{\text{lim}})\}, \mathsf{val}^{\mathcal{N}'})$$

A picture of $\mathcal{M}, r$ and $\mathcal{N}, r_1$:



The rest of the proof is analogous to the previous case. We first prove by induction on n that for every $q \in \mathcal{M}$, $\mathcal{M}, q$ and $\mathcal{N}, q_i$ satisfy the same $\mathsf{ML}_{\mathsf{ind}_{\leftrightharpoons}}$-formulae of modal depth n. This boils down to checking several straightforward cases (the one in which we use the fact that t_{lim} was not isolated is that with p_2's successors).

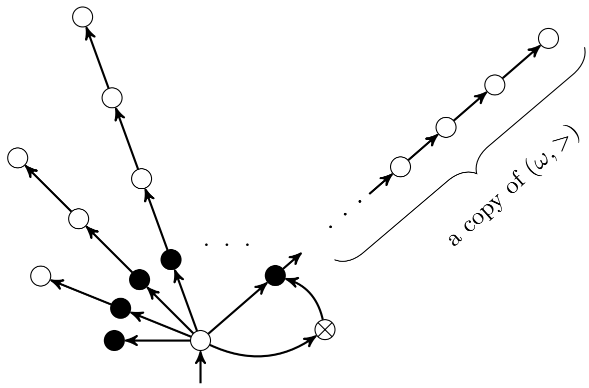
The winning strategy for ∀dam witnessing $\mathcal{M}, r \not\rightleftharpoons_{\mathsf{ind}_{\leftrightharpoons}} \mathcal{N}, r_1$ is as follows: (i) First follow the path from r_1 to $p_2 \in \mathcal{N}$.[8]. In order not to loose, ∃ve has to respond in $\mathcal{M}$ with the only point that is equivalent to p_2, namely p. (ii) Then, ∀dam moves to p^{lim} in $\mathcal{M}$ and ∃ve has to respond in $\mathcal{N}$ with a point non-equivalent with it – thus loosing the game.

Case 3: $\mathsf{ind} = \mathsf{ind}_{+}$

This is the most involved case. The key difficulty is that it does not suffice to simply remove arrows from the model to remove them from its *transitive closure*. Consider the following example.

[8] Note that since in this context *accessibility* means *two-way accessibility*, after removing the arrow $p_2 \to p_2^{\text{lim}}$, p_2 does not have to be accessible from r_2. Indeed, it could actually happen that $\mathcal{M}, r \rightleftharpoons_{\mathsf{ind}_{\leftrightharpoons}} \mathcal{N}, r_2$. However, we know that s_1 is accessible from r_1 and from there we can move backwards to p_2.

Example 4.8 In the model below, the rightmost black point has a copy of ω (with the reverse order as the accessibility relation) as its children.



One can check that the type $t_{\lim}$ of the rightmost black point *is not isolated* among the types of its black siblings. However, it *is isolated* from the perspective of the crossed point – which in turn is isolated from the perspective of the root. Basing on that observation, it is not hard to show that any model $\mathsf{ML}_{\mathsf{ind}_+}$-equivalent to the one above must realise $t_{\lim}$ in a *descendant* (not necessarily a *child*) of its root. In particular, this demonstrates that not every isolated type can be omitted. Nevertheless, we will show that in the presence of a non-isolated type it is always possible to find *some* (possibly different) type that can be omitted.

Let us start with recalling the following well-known fact:

Proposition 4.9 *If $\mathcal{M}, r$ is a finite model, then it is definable in $\mathsf{ML}_{\mathsf{ind}_+}$ up to $\leftrightarroows_{\mathsf{ind}_+}$, i.e. there is an $\mathsf{ML}_{\mathsf{ind}_+}$-formula s.t. every its model is ind_+-bisimilar to $\mathcal{M}, r$. In particular, finite models only realize types isolated in $\mathbb{T}_{\mathsf{ind}^+}$.*

Proof. Since $M = q_1, ..., q_n$ is finite, it realises only finitely many types $t_1, ..., t_n$ (w.l.o.g. all distinct, as otherwise we may quotient the model). Since $\mathbb{T}_{\mathsf{ind}_+}$ is a Hausdorff space, there are mutually exclusive sentences $\varphi_i \in t_i$ for every i. First, define ψ_i to be the formula that describes which atomic propositions belong to t_i and which other types it sees:

$$\begin{gathered}\bigwedge\{a \in \Sigma \mid a \in t_i\} \wedge \\ \Box(\bigvee\{\varphi_j \mid q_i \to^+ q_j\}) \wedge \\ \bigwedge\{\Diamond\varphi_j \mid q_i \to^+ q_j\}\end{gathered}$$

Then, we put:

$$\theta_i = \psi_i \wedge \Box(\bigwedge_{j \in \{1,...,n\}} \{\varphi_j \Rightarrow \psi_j\})$$

It is straightforward that $\theta_i \in t_i$. On the other hand, if $\mathcal{N}, q \models \theta_i$, then already $\mathcal{N}, q \leftrightarrows_{\mathsf{ind}_+} \mathcal{M}, q_i$. Indeed, w.l.o.g. we may assume that such $\mathcal{N}$ is reachable from q and then it is easy to check that: (i) the types of all the points of $\mathcal{N}$ are precisely $\{t_1, ..., t_n\}$, (ii) the map $f : N \to M$ sending a point with type t_i to q_i is a functional bisimulation. It then follows that each type t_i is isolated by its basic neighbourhood $\langle\theta_i\rangle$. □

As in both previous cases, let us take a model $\mathcal{M}, r$ that is infinite, $\mathsf{ML}_{\mathsf{ind}_+}$-saturated, reachable and no two points realise different types – but the model is not bisimilar to a finite one. It follows that the root has infinitely many descendants. We will need the following fact:

Lemma 4.10 *There exists a point $p_\infty \in \mathcal{M}$ s.t. $p_\infty \to^+ p_\infty$ and its type t_∞ is a non-isolated element of $\{\boldsymbol{tp}^{\mathcal{M}}(q) \mid p_\infty \to^+ q\}$.*

Proof. We will inductively construct a sequence of (not necessarily distinct) points, indexed by countable ordinals $(p_\alpha)_{\alpha<\omega_1} \subseteq \mathcal{M}$ with the property that for any $\alpha < \beta$: (i) $p_\alpha \to^+ p_\beta$ and (ii) $\mathbf{tp}^{\mathcal{M}}(p_\beta)$ is not isolated in $\{\mathbf{tp}^{\mathcal{M}}(q) \mid p_\alpha \to^+ q\}$.

For the induction base, we simply take the root $p_0 = r$.

For $\alpha + 1$, we know by induction assumption that $\mathbf{tp}^{\mathcal{M}}(p_\alpha)$ is not isolated, so by Lemma 4.9 we know that the model generated by p_α has to be infinite (except for the case $\alpha = 0$ where we just know that r has infinitely many descendants). Now we look at the infinite set $T_\alpha = \{\mathbf{tp}^{\mathcal{M}}(q) | \; p_\alpha \to^+ q\}$ and pick some its limit – a non-isolated type $t_{\alpha+1} \in \mathbb{T}_{\mathsf{ind}_+}$ which, by $\mathsf{ML}_{\mathsf{ind}_+}$-saturation, is realised in some descendant $p_{\alpha+1}$ of p_α.

For a limit ordinal α, we fix a subsequence $(\alpha_i)_{i\in\omega} \subseteq \alpha$ of shape ω which is cofinal with α (which exists as α is countable). Take any limit t_α of the set $T_\alpha = \{\mathbf{tp}^{\mathcal{M}}(p_{\alpha_i}) \mid i \in \omega\}$. Since t_α is not isolated and $\mathbb{T}_{\mathsf{ind}_+}$ is Hausdorff, every $\varphi \in t_\alpha$ must belong to infinitely many types from T_α. It follows that there are arbitrary big i s.t. $\varphi \in t_{\alpha_i}$, so every p_{α_j} – and hence by cofinality also every p_β – has a descendant satisfying φ. Hence, by $\mathsf{ML}_{\mathsf{ind}_+}$-saturation, each p_β has a descendant realising t_α – and by our assumptions on $\mathcal{M}$ this point p_α is unique.

Now we claim that $p_\alpha = p_\beta$ for some $\alpha \neq \beta$. Indeed, observe that if $p \to^+ q$, then q cannot satisfy more formulae of the form $\Diamond\varphi$ than p. Since there are only countably many formulae, for sufficiently large α all $\mathbf{tp}^{\mathcal{M}}(p_\alpha)$ may only differ on formulae equivalent to boolean combinations of Σ. But $\mathcal{P}(\Sigma)$ is finite, so $p_\alpha = p_\beta$ for some $\alpha < \beta$ and thus we put $p_\infty = p_\alpha$. It then follows from (i) that $p_\infty \to^+ p_\infty$. Finally, (ii) implies that the type t_∞ is not isolated in $\{\mathbf{tp}^{\mathcal{M}}(q) \mid p_\infty \to^+ q\}$, as desired. □

Now, we can define a new model by removing all the arrows leading to p_∞:

$$\mathcal{N} = (M, \to -\{(q, p_\infty) \mid q \in \mathcal{M}\}, \mathsf{val}^{\mathcal{M}}).$$

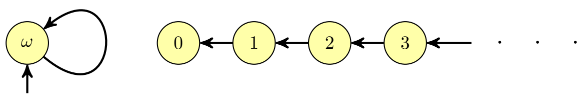
Observe that t_∞ is not isolated in $\{\mathbf{tp}^{\mathcal{M}}(q) \mid p \to^+ q\}$ for any ancestor p of p_∞. This allows us, as in the two previous cases, to prove by induction on modal depth that $\mathcal{M}, q \equiv_{\mathsf{ML}_{\mathsf{ind}_+}} \mathcal{N}, q$ for every $q \in \mathcal{M}$. On the other hand, p_∞ is reachable from the root in $\mathcal{M}$ but not in $\mathcal{N}$ – which gives a winning strategy for $\forall$dam in the bisimulation game. Q.E.D.

4.4 Limitations

We end with two examples illustrating limitations of our method. First of all, let us emphasize that our proofs rely on compactness of the logic under consideration – and it is not hard to come up with an example of a non-compact logic which fails to have analogous characterisation. For instance, consider the mix of ML and $\mathsf{ML}_{\mathsf{ind}_+}$ – i.e. the logic having *both* the standard and the transitive modalities. Such logic is not compact and can describe the infinitely branching Hedgehog (Example 2.6) up to bisimulation – by extending its ML-type with an additional sentence: $\Box(\Box\bot \vee \Diamond^+\Box\bot)$ (i.e. "every child of the root either has no children or has a descendant with no children").

Since non-compact logics seem out of our reach, a natural question is if compactness is *sufficient* for analogous characterisation. Unfortunately, this is not the case. The second example shows that even the stronger assumption of first-order axiomatizability of $\mathsf{ind}[\mathsf{Krip}(\mathsf{S})]$ (which implies compactness of $\mathsf{ML}_{\mathsf{ind}}$ by Lemma 4.5) is not sufficient to generalise our characterization to $\mathsf{ML}_{\mathsf{ind}}$. Recall the universal modality induced by $\mathsf{ind}_\forall$ (Example 3.8). The class $\mathsf{ind}_\forall[\mathsf{Krip}(\{\rightarrow\})]$ is definable by a single first-order sentence: $\forall_{x,y} x\langle\exists\rangle y$. However, consider the following model $\mathcal{M} \in \mathsf{Krip}(\{\rightarrow\})$:

Example 4.11 $M = \omega + 1 = \{0, 1, ..., \omega\}$; $p \rightarrow^{\mathcal{M}} q$ iff $p = q + 1$ or $p = q = \omega$. As in Example 2.6 (The Hedgehogs), we assume $\Sigma = \emptyset$.



Observe that $\mathcal{M}, p \not\equiv_{\mathsf{ML}_{\mathsf{ind}_\forall}} \mathcal{M}, q$ for all $p \neq q$ – and so every point has infinitely many pairwise non-equivalent $\langle\exists\rangle$-children. However, it is not hard to show that any model equivalent to $\mathcal{M}, \omega$ must be $\mathsf{ind}_\forall$-bisimilar to it. The thing is that although the topological part of our reasoning still works and we may find a limit of the types realised in $\mathcal{M}$ (in fact, in this situation there is precisely one such limit type – the type of ω) – it is not possible to omit that limit type.

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